Chapter 12

Permutation and Combination

INTRODUCTION

Factorial

The important mathematical term "Factorial" has extensively used in this chapter.

The product of first n consecutive **natural numbers** is defined as **factorial of n**. It is denoted by n! or $\underline{|n}$. Therefore,

$$\boxed{n! = 1 \times 2 \times 3 \times .... \times (n-1) \times n}$$

For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Note that :

$$\frac{n!}{r!} \neq \left(\frac{n}{r}\right)!$$

$0! = 1$

The factorials of fractions and negative integers are not defined.

Fundamental Principles of Counting

1. **Principle of Addition :** If an event can occur in 'm' ways and another event can occur in 'n' ways independent of the first event, then either of the two events can occur in (m + n) ways.
2. **Principle of Multiplication :** If an operation can be performed in 'm' ways and after it has been performed in any one of these ways, a second operation can be performed in 'n' ways, then the two operations in succession can be performed in (m × n) ways.

Method of Sampling :

Sampling process can be divided into following forms :

1. The order is IMPORTANT and the repetition is ALLOWED, each sample is then a SEQUENCE.
2. The order is IMPORTANT and the repetition is NOT ALLOWED, each sample is then a PERMUTATION.
3. The order is NOT IMPORTANT and repetition is ALLOWED, each sample is then a MULTISET.
4. The order is NOT IMPORTANT and repetition is NOT ALLOWED, each sample is then a COMBINATION.

PERMUTATION

Each of the arrangements, which can be made by taking, some or all of a number of things is called a PERMUTATION.

For Example : Formation of numbers, word formation, sitting arrangement in a row.

The number of permutations of 'n' things taken 'r' at a time is denoted by ^nP_r. It is defind as, $^nP_r = \dfrac{n!}{(n-r)!}$.

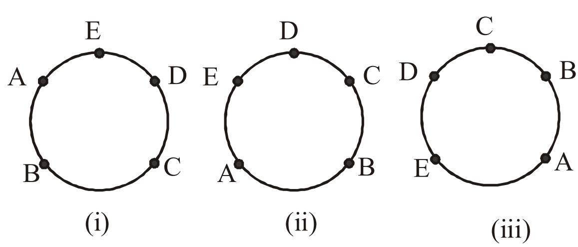
Note that:

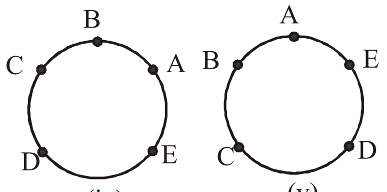
$^nP_n = n!$

Circular permutations:

(i) Arrangements round a circular table :

Consider five persons A, B, C, D and E to be seated on the circumference of a circular table in order (which has no head). Now, shifting A, B, C, D and E one position in anticlockwise direction we will get arrangements as follows:





(iv) (v)

we see that arrangements in all figures are same.

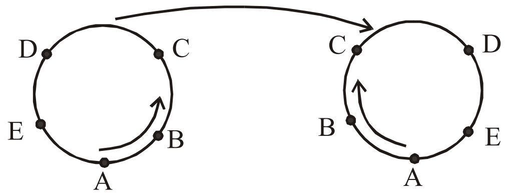
$\therefore$ The number of circular permutations of n different things taken all at a time is $\frac{{}^nP_n}{n} = (n-1)!$, if clockwise and anticlockwise orders are taken as different.

(ii) Arrangements of beads or flowers (all different) around a circular necklace or garland:

Consider five beads A, B, C, D and E in a necklace or five flowers A, B, C and D, E in a garland etc. If the necklace or garland on the left is turned over we obtain the arrangement on the right, i.e., anticlockwise and clockwise order of arrangements are not different.

Thus the number of circular permutations of 'n' different things taken.

all at a time is $\frac{1}{2}(n-1)!$, if clockwise and anticlockwise orders are taken to be some.



Conditional Permutations

Number of permutations of n things taking r at a time, in which a particular thing always occurs $= r.\,{}^{n-1}P_{r-1}$.

Distinguishable Permutations

1. Suppose a set of n objects has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with $n = n_1 + n_2 + n_3 + \ldots + n_k$, Then the number of distinguishable permutations of the n

objects is $\dfrac{n!}{n_1!\,n_2!\,n_3!\,.....\,n_k!}$

2. Number of permutations of n things taking r at a time, in which a particular thing never occurs = ${}^{n-1}P_r$.

3. Number of permutations of n different things taking all at a time, in which m specified things always come together = $m!(n-m+1)!$.

4. Number of permutations of n different things taking all at a time, in which m specified things never come together = $n! - m!(n-m+1)!$

5. The number of permutations of 'n' things taken all at a time, when 'p' are alike of one kind, 'q' are alike of second, 'r' alike of third, and so on

$= \dfrac{n!}{p!\,q!\,r!}$.

6. The number of permutations of 'n' different things, taking 'r' at a time, when each thing can be repeated 'r' times = n^r

☞ ***Shortcut Approach***

Number of ways to declare the result where 'n' match are played = 2^n

See Example : Refer ebook Solved Examples/Ch-12

COMBINATION

Each of the different selections that can be made with a given number of objects taken some or all of them at a time is called a COMBINATION.

The number of combinations of 'n' dissimilar things taken 'r' at a time is denoted by ${}^nC_r$ or C(n, r). It is defined as,

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

REMEMBER

★ ${}^nC_0 = 1,\ {}^nC_n = 1;\ {}^nP_r = r!\,{}^nC_r$

★ ${}^nC_r = {}^nC_{n-r}$

★ ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

★ ${}^nC_x = {}^nC_y \Rightarrow x + y = n$

★ ${}^nC_r = {}^nC_{r+1} = {}^{n+1}C_r$

★ ${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}$

★ ${}^nC_r = \frac{1}{r}(n - r + 1)\,{}^nC_{r-1}$

★ ${}^nC_1 = {}^nC_{n-1} = n$

Conditional Combinations

1. Number of combinations of n distinct things taking $r\ (\le n)$ at a time, when $k\ (0 \le k \le r)$ particular objects always occur $= {}^{n-k}C_{r-k}$.

2. Number of combinations of n distinct objects taking $r (\le n)$ at a time, when $k\,(0 \le k \le r)$ particular objects never occur $= {}^{n-k}C_r$.

3. Number of selections of r things from n things when p particular things are not together in any selection $= {}^nC_r - {}^{n-p}C_{r-p}$

4. Number of selection of r consecutive things out of n things in a row $= n - r + 1$

5. Number of selection of r consecutive things out of n things along a circle

$$= \begin{cases} n, \text{ when } & r < n \\ 1, \text{ when } & r = n \end{cases}$$

6. The number of Combinations of 'n' different things taking some or all at a time

$$= {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots\ldots\ldots + {}^nC_n = 2^n - 1$$

NOTE : *If a person is always there then we have to select only 1 from the remaining 25 – 1 = 24*

☞ Shortcut Approach

Let there are n persons in a hall. If every person shakes his hand with every other person only once, then total number of handshakes

$$= {}^nC_2 = \frac{n(n-1)}{2}$$

Note: *If in place of handshakes each person gives a gift to another person, then formula changes to = n (n – 1)*

See Example : Refer ebook Solved Examples/Ch-12

7. The number of ways of dividing 'm + n' things into two groups containing 'm' and 'n' things respectively

$$= {}^{m+n}C_m \quad {}^nC_n = \frac{(m+n)!}{m!n!}$$

8. The number of ways of dividing 'm + n + p' things into three groups containing 'm', 'n' and 'p' things respectively

$$= {}^{m+n+p}C_m \,.\, {}^{n+p}C_p = \frac{(m+n+p)!}{m!\ n!\ p!}$$

(i) If m = n = p i.e. '3m' things are divided into three equal groups then the number of combinations is $\frac{(3m)!}{m!\ m!\ m!\ 3!} = \frac{(3m)!}{(m!)^3\ 3!}$

(ii) Buf if '3m' things are to be divided among three persons, then the number of divisions is $\frac{(3m)!}{(m!)^3}$

9. If mn distinct objects are to be divided into m groups. Then, the number of combination is

$\frac{(mn)!}{m!\ (n!)^m}$, when the order of groups is not important and

$\frac{(mn)!}{(n!)^m}$, when the order of groups is important

NUMBER OF RECTANGLES AND SQUARES

(a) Number of rectangles of any size in a square of size $n \times n$ is $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} r^2$.

(b) Number of rectangles of any size in a rectangle size $n \times p$ $(n < p)$ is $\frac{np}{4}(n+1)(p+1)$ and number of squares of any size is $\sum_{r=1}^{n}(n+1-r)(p+1-r)$.

Shortcut Approach

- If there are n non-collinear points in a plane, then
  - (i) Number of straight lines formed = ${}^nC_2$
  - (ii) Number of triangles formed = ${}^nC_3$
  - (iii) Number of quadrilaterals formed = ${}^nC_4$
- If there are n points in a plane out of which m are collinear, then
  - (i) Number of straight lines formed = ${}^nC_2 - {}^mC_2 + 1$
  - (ii) Number of triangles formed = ${}^nC_3 - {}^mC_3$
- Number of diagonals in a polygen of n sides = ${}^nC_2 - n$

See Example : Refer ebook Solved Examples/Ch-12

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