5 Continuity and Differentiability

Fastrack Revision

- A function $f(x)$ is continuous at a point $x = a$ in its domain if
$$\lim_{x \to a} f(x) = f(a)$$
or $$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$
- If any function $f(x)$ is not continuous at $x = a$, then that function is called discontinuous at $x = a$.
- If f and g are two real functions which are continuous at $x = a$, then the following are also continuous at $x = a$:
  (i) $f + g$ (ii) $f - g$
  (iii) cf (c is any constant) (iv) fg
  (v) $\frac{1}{f}$, where $f(a) \neq 0$ (vi) $\frac{f}{g}$, where $g(a) \neq 0$
- A function $f(x)$ is called continuous in open interval (a, b) if and only if $f(x)$ is continuous at every point of the interval (a, b).
- A function $f(x)$ is called continuous in closed interval $[a, b]$ if and only if $f(x)$:
  (i) is continuous in the interval (a, b).
  (ii) is continuous from the right of point $x = a$ *i.e.*, $\lim_{x \to a^+} f(x) = f(a)$.
  (iii) is continuous from the left of point $x = b$ *i.e.*, $\lim_{x \to b^-} f(x) = f(b)$.
- The composition of function of two continuous function is continuous.
- If the function f is continuous in domain D, then the function $|f|$ is also continuous in D.
- Constant function, unit function, polynomial function, modulus function, exponential function, logarithmic function, trigonometric function are continuous functions in their domain.
- The function $f(x)$ is called differentiable at point $x = c$ in its domain if and only if $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists.

  The above limit is called the derivative or differential coefficient of the function $f(x)$ at point $x = c$.
- At $x = a$, left-hand derivative of
$$f(x) = Lf'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}.$$
- At $x = a$, right-hand derivative of
$$f(x) = Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$
- A function $f(x)$ is said to be differentiable at $x = a$, if its left hand and right hand derivatives at a exist and are equal.
- If a function is differentiable at any point, then that function is also continuous at that point.
- If a function is continuous at any point, then that function is not necessarily differentiable at that point.

Knowledge BOOSTER

If left and right hand derivatives of a function exist and has a definite value (not necessary equal) at any point, then the function is continuous at that point.

- **Derivatives of Standard Functions**

(i) $\frac{d}{dx}(\text{constant}) = 0$

(ii) $\frac{d}{dx}(x^n) = nx^{n-1}$

(iii) $\frac{d}{dx}(\sin x) = \cos x$

(iv) $\frac{d}{dx}(\cos x) = -\sin x$

(v) $\frac{d}{dx}(\tan x) = \sec^2 x$

(vi) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(vii) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(viii) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

(ix) $\frac{d}{dx}(e^x) = e^x$

(x) $\frac{d}{dx}(a^x) = a^x \log a$, where $a > 0$ and $a \neq 1$

(xi) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$, where $x > 0$

(xii) $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$, where $x > 0, a > 0, a \neq 1$

(xiii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

where $x \in (-1, 1)$

(xiv) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ and $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

where $x \in R$

(xv) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

and $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$, where $x \in R - [-1, 1]$

(xvi) If $y = f(t)$ and $t = g(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

(xvii) If $y = f(t), t = g(u), u = h(x)$, then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}$$

(xviii) If $u = f(x)$ and $v = g(x)$ are two functions in variable x, then $\frac{du}{dv} = \frac{du}{dx} \div \frac{dv}{dx} = \frac{du}{dx} \times \frac{dx}{dv}$

(xix) $\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right), \frac{d^3 y}{dx^3} = \frac{d}{dx}\left(\frac{d^2 y}{dx^2}\right), \ldots$

- **Algebra of Derivative**

(i) $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

(ii) **Product Rule:** $\frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

(iii) **Quotient Rule:** $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

- **Second Order Derivative:** Second order derivative of a function is the derivative of the first order derivative of the function.

Let $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ is called the first derivative of y or $f(x)$ and $\frac{d^2 y}{dx^2} = f''(x)$ is called the second order derivative of y or $f(x)$.

Practice Exercise

Multiple Choice Questions

Q 1. The function $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x, is continuous at: *(CBSE 2023)*

a. $x = 1$ b. $x = 1.5$ c. $x = -2$ d. $x = 4$

Q 2. The value of k $(k < 0)$ for which the function f defined as:

$$(x) = \begin{cases} \frac{1 - \cos kx}{x \sin x}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$$

is continuous at $x = 0$, is: *(CBSE SQP 2021 Term-1)*

a. ± 1 b. -1 c. $\pm \frac{1}{2}$ d. $\frac{1}{2}$

Q 3. The function $f(x) = \begin{cases} \frac{e^{3x} - e^{-5x}}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$ for the value k, as: *(CBSE 2021 Term-1)*

a. 3 b. 5 c. 2 d. 8

Q 4. If a function f defined by

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$$

is continuous at $x = \frac{\pi}{2}$, then the value of k is: *(CBSE 2021 Term-1)*

a. 2 b. 3 c. 6 d. -6

Q 5. If $f(x) = \begin{cases} ax^2 + b, & 0 \le x < 1 \\ 4, & x = 1 \\ x + 3, & 1 < x \le 2 \end{cases}$, then the value of (a, b) for which $f(x)$ cannot be continuous at $x = 1$, is:

a. (2, 2) b. (3, 1) c. (4, 0) d. (5, 2)

Q 6. If $f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2}, & x < 0 \\ a, & x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4}, & x > 0 \end{cases}$ is continuous at $x = 0$, then $a =$

a. 4 b. 6 c. 8 d. 5

Q 7. Let $f(x) = \begin{cases} \frac{\tan x - \cot x}{x - \frac{\pi}{4}}, & x \neq \frac{\pi}{4} \\ a, & x = \frac{\pi}{4} \end{cases}$. Then, the value of a so that $f(x)$ is continuous at $x = \frac{\pi}{4}$, is: *(NCERT EXEMPLAR)*

a. 2 b. 4 c. 3 d. 1

Q 8. If $f(x) = \begin{cases} \frac{x^2 - (a+2)x + a}{x - 2}, & \text{for } x \neq 2 \\ 2, & \text{for } x = 2 \end{cases}$ is continuous at $x = 2$, then a is equal to:

a. 0 b. 1
c. -1 d. None of these

Q 9. If $f(x) = \begin{cases} mx + 1, & \text{if } x \le \frac{\pi}{2} \\ \sin x + n, & \text{if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, then: *(NCERT EXEMPLAR)*

a. $m = 1, n = 0$ b. $m = \frac{n\pi}{2} + 1$
c. $n = \frac{m\pi}{2}$ d. $m = n = \frac{\pi}{2}$

Q 10. The function $f(x) = |x| + |x - 1|$ is: *(NCERT EXEMPLAR)*

a. continuous at $x = 0$ as well as at $x = 1$
b. continuous at $x = 1$ but not at $x = 0$
c. discontinuous at $x = 0$ as well as at $x = 1$
d. continuous at $x = 0$ but not at $x = 1$

Q 11. If the function $f(x)=a[x+1]+b[x-1]$, where $[x]$ is the greatest integer function, then the condition for which $f(x)$ is continuous at $x=1$, is:

a. $a+b=0$
b. $a-b=0$
c. $a=2b$
d. None of these

Q 12. Let $f(x)=\begin{cases} x+a\sqrt{2}\sin x, & 0\le x<\frac{\pi}{4} \\ 2x\cot x+b, & \frac{\pi}{4}\le x\le\frac{\pi}{2} \\ a\cos 2x-b\sin x, & \frac{\pi}{2}<x\le\pi \end{cases}$ be continuous in $[0,\pi]$, then $a+b=$

a. $\frac{\pi}{12}$
b. $\frac{\pi}{6}$
c. $\frac{\pi}{4}$
d. $\frac{\pi}{3}$

Q 13. Let $f(x)=\begin{cases} \frac{\sin(a+1)x+\sin x}{x}, & x<0 \\ c, & x=0 \\ \frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}}, & x>0 \end{cases}$

If $f(x)$ is continuous at $x=0$, then:

a. $a+c=0, b=1$
b. $a+c=1, b\in R$
c. $a+c=-1, b\in R$
d. $a-c=-1, b=-1$

Q 14. If the function $f(x)=\begin{cases} \frac{x^2}{a}, & 0\le x<1 \\ a, & 1\le x<\sqrt{2} \\ \frac{2b^2-4b}{x^2}, & \sqrt{2}\le x<\infty \end{cases}$ is continuous for $0\le x<\infty$, then the most suitable values of a and b are:

a. $a=1, b=-1$
b. $a=-1, b=1+\sqrt{2}$
c. $a=-1, b=1$
d. None of these

Q 15. Let $f(x)=\begin{cases} -2\sin x, & \text{if } x\le-\pi/2 \\ A\sin x+B, & \text{if } -\pi/2<x<\pi/2 \\ \cos x, & \text{if } x\ge\pi/2 \end{cases}$

Then, the values of A and B so that $f(x)$ is continuous everywhere, are:

a. $A=0, B=1$
b. $A=1, B=1$
c. $A=-1, B=1$
d. $A=-1, B=0$

Q 16. Let f be defined on $[-5, 5]$ as $f(x)=\begin{cases} x, \text{ if } x \text{ is rational} \\ -x, \text{ if } x \text{ is irrational} \end{cases}$, then $f(x)$ is:

a. continuous at every x except $x=0$
b. discontinuous at every x except $x=0$
c. continuous everywhere
d. discontinuous everywhere

Q 17. $f+g$ may be a continuous function, if:

a. f is continuous and g is discontinuous
b. f is discontinuous and g is continuous
c. f and g both are discontinuous
d. None of the above

Q 18. The point(s) of discontinuity of the function:

$f(x)=\begin{cases} \frac{1}{5}(2x^2+3), & x\le1 \\ 6-5x, & 1<x<3 \\ x-3 & x\ge3 \end{cases}$ is/are:

a. $x=1$
b. $x=3$
c. $x=1, 3$
d. None of these

Q 19. The function $f(x)=|x|$ is: *(CBSE 2023)*

a. continuous and differentiable everywhere
b. continuous and differentiable nowhere
c. continuous everywhere, but differentiable everywhere except at $x=0$
d. continuous everywhere, but differentiable nowhere

Q 20. If $f(x)=2|x|+3|\sin x|+6$, then the right hand derivative of $f(x)$ at $x=0$ is: *(CBSE 2023)*

a. 6
b. 5
c. 3
d. 2

Q 21. If $f(x)=\begin{cases} 3^x, & -1\le x\le1 \\ 4-x, & 1\le x\le4 \end{cases}$, then $f(x)$ is:

a. continuous as well as differentiable at $x=1$
b. continuous but not differentiable at $x=1$
c. differentiable but not continuous at $x=1$
d. None of the above

Q 22. For what choice of a and b, is the function $f(x)=\begin{cases} x^2, & x\le c \\ ax+b, & x>c \end{cases}$ differentiable at $x=c$?

a. $a=c, b=c$
b. $a=c, b=-c$
c. $a=-c^2, b=2c$
d. $a=2c, b=-c^2$

Q 23. Let $f(x)=|\cos x|$. Then,

a. f is everywhere differentiable
b. f is everywhere continuous but not differentiable at $n=n\pi, n\in Z$
c. f is everywhere continuous but not differentiable at $x=(2n+1)\frac{\pi}{2}, n\in Z$
d. None of the above

Q 24. The set of all points where the function $f(x)=x+|x|$ is differentiable, is: *(CBSE SQP 2023-24)*

a. $(0,\infty)$
b. $(-\infty,0)$
c. $(-\infty,0)\cup(0,\infty)$
d. $(-\infty,\infty)$

Q 25. If $f(x)=\begin{cases} \frac{x}{1+|x|}, & |x|\ge1 \\ \frac{x}{1-|x|}, & |x|<1 \end{cases}$, then $f(x)$ is:

a. discontinuous and non-differentiable at $x=-1, 1$ and 0
b. discontinuous and non-differentiable at $x=-1$, whereas continuous and differentiable at $x=0$ and $x=1$
c. discontinuous and non-differentiable at $x=-1$ and $x=1$ whereas continuous and differentiable at $x=0$
d. None of the above

Q 26. If a function $f(x)$ is defined as

$$f(x)=\begin{cases}-x, & x<0\\ x^2, & 0\le x\le 1\\ x^2-x+1, & x>1\end{cases}$$, then:

a. $f(x)$ is differentiable at $x=0$ and $x=1$
b. $f(x)$ is differentiable at $x=0$ but not at $x=1$
c. $f(x)$ is differentiable at $x=1$ but not at $x=0$
d. $f(x)$ is not differentiable at $x=0$ and $x=1$

Q 27. Differential of $\log[\log(\log x^5)]$ w.r.t. x is: *(CBSE 2021 Term-1)*

a. $\frac{5}{x\log(x^5)\log(\log x^5)}$ b. $\frac{5}{x\log(\log x^5)}$
c. $\frac{5x^4}{\log(x^5)\log(\log x^5)}$ d. $\frac{5x^4}{\log x^5\log(\log x^5)}$

Q 28. If $y=\log(\sin e^x)$, then $\frac{dy}{dx}$ is: *(CBSE 2023)*

a. $\cot e^x$ b. $\operatorname{cosec} e^x$
c. $e^x\cot e^x$ d. $e^x\operatorname{cosec} e^x$

Q 29. If $y=\sin^2(x^3)$, then $\frac{dy}{dx}$ is equal to: *(CBSE 2023)*

a. $2\sin x^3\cos x^3$ b. $3x^3\sin x^3\cos x^3$
c. $6x^2\sin x^3\cos x^3$ d. $2x^2\sin^2(x^3)$

Q 30. If $y=(1+x)(1+x^2)(1+x^4)$, then $\frac{dy}{dx}$ at $x=1$ is:

a. 20 b. 28 c. 1 d. 0

Q 31. If $f(x)=\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$, $0\le x\le\frac{\pi}{2}$, then $f'\left(\frac{\pi}{6}\right)=$

a. $-\frac{1}{4}$ b. $-\frac{1}{2}$ c. $\frac{1}{4}$ d. $\frac{1}{2}$

Q 32. The derivative of x^{2x} w.r.t. x is: *(CBSE 2023)*

a. x^{2x-1} b. $2x^{2x}\log x$
c. $2x^{2x}(1+\log x)$ d. $2x^{2x}(1-\log x)$

Q 33. If $xe^{xy}=y+\sin^2 x$, then at $x=0$, $\frac{dy}{dx}$ is equal to:

a. –1 b. 1
c. 0 d. None of these

Q 34. If $y\cos x+x\cos y=\pi$, then the value of $\frac{dy}{dx}$ at $x=0$ is:

a. 0 b. 1 c. –1 d. 2

Q 35. If $y=x\tan y$, then $\frac{dy}{dx}$ is equal to:

a. $\frac{\tan x}{x-x^2-y^2}$ b. $\frac{y}{x-x^2-y^2}$ c. $\frac{\tan y}{y-x}$ d. $\frac{\tan x}{x-y^2}$

Q 36. If $e^x+e^y=e^{x+y}$, then $\frac{dy}{dx}$ is: *(CBSE SQP 2021 Term-1)*

a. e^{y-x} b. e^{x+y} c. $-e^{y-x}$ d. $2e^{x-y}$

Q 37. If $y=\sqrt{\sin x+y}$, then $\frac{dy}{dx}$ is equal to:

a. $\frac{\cos x}{2y-1}$ b. $\frac{\cos x}{1-2y}$ c. $\frac{\sin x}{1-2y}$ d. $\frac{\sin x}{2y-1}$

Q 38. If $\sin y=x\cos(a+y)$, then $\frac{dx}{dy}$ is: *(CBSE 2021 Term-1)*

a. $\frac{\cos a}{\cos^2(a+y)}$ b. $\frac{-\cos a}{\cos^2(a+y)}$
c. $\frac{\cos a}{\sin^2 y}$ d. $\frac{-\cos a}{\sin^2 y}$

Q 39. If $x^y\cdot y^x=1$, then the value of $\frac{dy}{dx}$ is:

a. $-\frac{(y+x\log y)}{(x+y\log x)}\cdot\frac{y}{x}$ b. $\frac{(y+x\log y)}{(x+y\log x)}\cdot\frac{y}{x}$
c. $-\frac{(x+y\log y)}{(y+x\log x)}\cdot\frac{y}{x}$ d. None of these

Q 40. If $y=x^{1/x}$, then the value of $\frac{dy}{dx}$ at $x=e$, is:

a. 1 b. 0
c. –1 d. None of these

Q 41. If $x=\sin^{-1}(3t-4t^3)$ and $y=\cos^{-1}(\sqrt{1-t^2})$, then $\frac{dy}{dx}$ is equal to:

a. $\frac{1}{2}$ b. $\frac{2}{5}$
c. $\frac{3}{2}$ d. $\frac{1}{3}$

Q 42. If $x=2\cos\theta-\cos 2\theta$ and $y=2\sin\theta-\sin 2\theta$, then $\frac{dy}{dx}$ is:

a. $\frac{\cos\theta+\cos 2\theta}{\sin\theta-\sin 2\theta}$ b. $\frac{\cos\theta-\cos 2\theta}{\sin 2\theta-\sin\theta}$
c. $\frac{\cos\theta-\cos 2\theta}{\sin\theta-\sin 2\theta}$ d. $\frac{\cos 2\theta-\cos\theta}{\sin 2\theta+\sin\theta}$

Q 43. If $x=\frac{1-t^2}{1+t^2}$ and $y=\frac{2t}{1+t^2}$, then $\frac{dy}{dx}$ is equal to:

a. $-\frac{y}{x}$ b. $\frac{y}{x}$ c. $-\frac{x}{y}$ d. $\frac{x}{y}$

Q 44. If $x=a\cos^4\theta$, $y=a\sin^4\theta$, then $\frac{dy}{dx}$ at $\theta=\frac{3\pi}{4}$ is:

a. –1 b. 1 c. $-a^2$ d. a^2

Q 45. If $x=A\cos 4t+B\sin 4t$, then $\frac{d^2x}{dt^2}$ is equal to: *(CBSE 2023)*

a. x b. $-x$ c. $16x$ d. $-16x$

Q 46. If $x=a\sec\theta$, $y=b\tan\theta$, then $\frac{d^2y}{dx^2}$ at $\theta=\frac{\pi}{6}$ is: *(CBSE SQP 2021 Term-1)*

a. $\frac{-3\sqrt{3}b}{a^2}$ b. $\frac{-2\sqrt{3}b}{a}$ c. $\frac{-3\sqrt{3}b}{a}$ d. $\frac{-b}{3\sqrt{3}a^2}$

Q 47. If $y=\log_e\left(\frac{x^2}{e^2}\right)$, then $\frac{d^2y}{dx^2}$ equals: *(CBSE 2020)*

a. $-\frac{1}{x}$ b. $-\frac{1}{x^2}$ c. $\frac{2}{x^2}$ d. $-\frac{2}{x^2}$

Q 48. If $y=(\sin^{-1}x)^2+k\sin^{-1}x$, then $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}$ is equal to:

a. 0 b. 1 c. 2 d. y

Assertion & Reason Type Questions

Directions (Q. Nos. 49-57): *In the following questions, each question contains Assertion (A) and Reason (R). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are:*

a. Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)
b. Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A)
c. Assertion (A) is true but Reason (R) is false
d. Assertion (A) is false but Reason (R) is true

Q 49. Consider the function $f(x)=[\sin x], x \in [0, \pi]$.

Assertion (A): $f(x)$ is not continuous at $x=\frac{\pi}{2}$.

Reason (R): $\lim_{x \to \pi/2} f(x)$ does not exist.

Q 50. Assertion (A): $f(x)=x\left(\frac{1+e^{1/x}}{1-e^{1/x}}\right)(x \neq 0), f(0)=0$ is continuous at $x=0$.

Reason (R): A function is said to be continuous at a if both limits are exists and equal to $f(a)$.

Q 51. Assertion (A): $f(x)=|\log x|$ is differentiable at $x=1$.

Reason (R): Both $\log x$ and $-\log x$ are differentiable at $x=1$.

Q 52. Consider the function $f(x)=\begin{cases} x^2, & x \geq 1 \\ x+1, & x<1 \end{cases}$.

Assertion (A): f is not derivable at $x=1$ as $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$.

Reason (R): If a function f is derivable at a point 'a', then it is continuous at 'a'.

Q 53. Assertion (A): If $e^{xy}+\log(xy)+\cos(xy)+5=0$, then $\frac{dy}{dx}=-\frac{y}{x}$.

Reason (R): $\frac{d}{dx}(xy)=0 \Rightarrow \frac{dy}{dx}=\frac{-y}{x}$

Q 54. Assertion (A): $\frac{d}{dx}\{\tan^{-1}(\sec x+\tan x)\}$
$=\frac{d}{dx}\{\cot^{-1}(\operatorname{cosec} x+\cot x)\}, x \in \left(0, \frac{\pi}{4}\right)$

Reason (R): $\sec^2 x-\tan^2 x=\operatorname{cosec}^2 x-\cot^2 x$

Q 55. Assertion (A): For $x<0, \frac{d}{dx}(\ln|x|)=-\frac{1}{x}$

Reason (R): For $x<0, |x|=-x$

Q 56. Assertion (A): If $y=\log_{10} x+\log_e x$, then $\frac{dy}{dx}=\frac{\log_{10} e}{x}+\frac{1}{x}$.

Reason (R): $\frac{d}{dx}(\log_{10} x)=\frac{\log x}{\log 10}$ and $\frac{d}{dx}(\log_e x)=\frac{\log x}{\log e}$.

Q 57. Assertion (A): If $y=\frac{1}{4}u^4$ and $u=\frac{2}{3}x^3+5$, then $\frac{dy}{dx}=\frac{2}{27}x^2(2x^3+15)^3$.

Reason (R): If y is a function of v and v is a function of x, then $\frac{dy}{dx}=\frac{dy}{dv}\times\frac{dv}{dx}$.

Answers

1. (b)	2. (b)	3. (d)	4. (c)	5. (d)	6. (c)	7. (b)	8. (a)	9. (c)	10. (a)
11. (a)	12. (a)	13. (c)	14. (c)	15. (c)	16. (b)	17. (c)	18. (b)	19. (c)	20. (b)
21. (b)	22. (d)	23. (c)	24. (c)	25. (c)	26. (d)	27. (a)	28. (c)	29. (c)	30. (b)
31. (d)	32. (c)	33. (b)	34. (b)	35. (b)	36. (c)	37. (a)	38. (a)	39. (a)	40. (b)
41. (d)	42. (b)	43. (c)	44. (a)	45. (d)	46. (a)	47. (d)	48. (c)	49. (c)	50. (a)
51. (d)	52. (b)	53. (a)	54. (b)	55. (d)	56. (c)	57. (a)			

Case Study Based Questions

Case Study 1

Ms. Anika Jain, teacher at a well known reputed coaching institute is teaching. Derivatives of function in parametric forms to her students with simple method through video lecture.

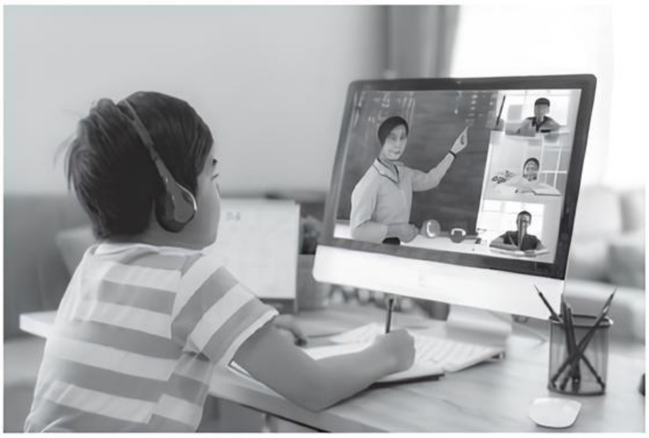
Sometimes the relation between two variables is neither explicit nor implicit, but some link of a third variable with each of the two variables, separately, establishes a relation between the first two variables. In such a situation, we say that the relation between them is expressed *via* a third variable. The third variable is called the parameter, more precisely, a relation expressed between two variables x and y in the form $x=f(t), y=g(t)$ is said to be parametric form with t is a parameter.

In order to find derivative of function in such form, we use chain rule.

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt} \Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}\left\{\text{provided } \frac{dx}{dt} \neq 0\right\}$$

Thus, $\dfrac{dy}{dx} = \dfrac{g'(t)}{f'(t)}$ $\left\{\text{as } \dfrac{dy}{dt} = g'(t) \text{ and } \dfrac{dx}{dt} = f'(t)\right\}$

provided $f'(t) \neq 0$.



Based on the given information, solve the following questions:

Q 1. If $x = t^2, y = t^3$, then $\dfrac{dy}{dx}$ is:

a. $\dfrac{3y}{2x}$ b. $\dfrac{2y}{3x}$ c. $\dfrac{3x}{2y}$ d. $\dfrac{2x}{3y}$

Q 2. If $x = t + \dfrac{1}{t}, y = t - \dfrac{1}{t}$, then $\dfrac{dy}{dx}$ is:

a. $\dfrac{y}{x}$ b. $\dfrac{x}{y}$ c. $-\dfrac{y}{x}$ d. $-\dfrac{x}{y}$

Q 3. If $x = a\sec^2\theta$ and $y = a\tan^2\theta$, then $\dfrac{dy}{dx}$ is:

a. $\sin\theta$ b. $\cos\theta$ c. 1 d. $-\cot\theta$

Q 4. If $\sin x = \dfrac{2t}{1+t^2}$, $\cos y = \dfrac{1-t^2}{1+t^2}$, then $\dfrac{dy}{dx}$ is:

a. $\dfrac{1}{t}$ b. $-t$ c. -1 d. 1

Q 5. If $x = 2\sin t + \sin 2t$ and $y = 2\cos t + \cos 2t$, then $\dfrac{dy}{dx}$ at $t = \dfrac{\pi}{6}$ is:

a. -1 b. $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ c. 1 d. $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

Solutions

1. Given, $x = t^2$ and $y = t^3$

Now differentiate both sides w.r.t. 't', we get

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = 3t^2$$

Now, $y = t^3 = t^2 \cdot t = x \cdot t$ $\quad [\because x = t^2]$

$\therefore$ $t = \dfrac{y}{x}$

We have, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2}{2t} = \dfrac{3}{2}t = \dfrac{3}{2} \times \dfrac{y}{x}$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{3y}{2x}$

So, option (a) is correct.

2. Given, $x = t + \dfrac{1}{t}$ and $y = t - \dfrac{1}{t}$

Now, differentiate both sides w.r.t. 't', we get

$$\frac{dx}{dt} = 1 - \frac{1}{t^2} = \frac{t^2-1}{t^2} = \frac{t^2-1}{t} \cdot \frac{1}{t} = \left(t - \frac{1}{t}\right) \cdot \frac{1}{t}$$

and $\dfrac{dy}{dt} = 1 + \dfrac{1}{t^2} = \dfrac{dy}{dt} = \dfrac{t^2+1}{t^2} = \dfrac{t^2+1}{t} \cdot \dfrac{1}{t} = \left(t + \dfrac{1}{t}\right) \cdot \dfrac{1}{t}$

$\Rightarrow$ $\dfrac{dx}{dt} = \dfrac{y}{t}$ and $\dfrac{dy}{dt} = \dfrac{x}{t}$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{x}{t} \times \dfrac{t}{y} = \dfrac{x}{y}$.

So, option (b) is correct.

3. Given, $x = a\sec^2\theta$ and $y = a\tan^2\theta$

Now, differentiate both sides w.r.t. 'θ', we get

$$\frac{dx}{d\theta} = 2a\sec\theta \cdot \frac{d}{d\theta}\sec\theta$$
$$= 2a\sec\theta \cdot (\sec\theta \cdot \tan\theta)$$
$$= 2a\sec^2\theta \cdot \tan\theta$$

and $\dfrac{dy}{d\theta} = 2a\tan\theta \cdot \dfrac{d}{d\theta}\tan\theta$

$$= 2a\tan\theta \cdot \sec^2\theta$$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{2a\tan\theta \cdot \sec^2\theta}{2a\sec^2\theta \cdot \tan\theta} = 1$

So, option (c) is correct.

4. Given, $\sin x = \dfrac{2t}{1+t^2}$ and $\cos y = \dfrac{1-t^2}{1+t^2}$

$\Rightarrow$ $x = \sin^{-1}\left(\dfrac{2t}{1+t^2}\right)$, and $y = \cos^{-1}\left(\dfrac{1-t^2}{1+t^2}\right)$

$\Rightarrow$ $x = 2\tan^{-1}t$ and $y = 2\tan^{-1}t$

Now, differentiate both sides w.r.t. 't', we get

$$\frac{dx}{dt} = \frac{2}{1+t^2} \text{ and } \frac{dy}{dt} = \frac{2}{1+t^2}$$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2}{1+t^2} \times \dfrac{1+t^2}{2} = 1$

So, option (d) is correct.

5. Given, $x = 2\sin t + \sin 2t$ and $y = 2\cos t + \cos 2t$

Now, differentiate both sides w.r.t. 't', we get

$$\frac{dx}{dt} = 2\cos t + \cos 2t \cdot \frac{d}{dt}(2t)$$
$$= 2\cos t + 2\cos 2t = 2(\cos t + \cos 2t)$$

and $\dfrac{dy}{dt} = -2\sin t - \sin 2t \dfrac{d}{dt}(2t)$

$$= -2\sin t - \sin 2t \cdot 2$$
$$= -2(\sin t + \sin 2t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2(\sin t + \sin 2t)}{2(\cos t + \cos 2t)}$$

$$= -\frac{(\sin t + \sin 2t)}{(\cos t + \cos 2t)}$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{at\left(t=\frac{\pi}{6}\right)} = -\left\{\frac{\sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right)}\right\}$$

$$= -\frac{\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)}{\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)} = -1$$

So, option (a) is correct.

Case Study 2

A function is continuous at $x = c$ if the function is defined at $x = c$ and if the value of the function at $x = c$ equals the limit of the function at $x = c$.

i.e., $$\lim_{x \to c} f(x) = f(c)$$

If f is not continuous at c, we say f is discontinuous at c and c is called a point of discontinuity of f.

Based on the above information, solve the following questions:

Q 1. Suppose f and g be two real functions continuous at a real number 'c', then show that $f+g$ is continuous at $x = c$.

Q 2. Find the value of k so that the given function $f(x)$ is continuous at $x = 5$.

$$f(x) = \begin{cases} kx + 1; & x \le 5 \\ 3x - 5; & x > 5 \end{cases}$$

Solutions

1. $\lim_{x \to c^-} [f(x) + g(x)] = \lim_{x \to c^-} f(x) + \lim_{x \to c^-} g(x)$

$$= f(c) + g(c)$$

$\lim_{x \to c^+} [f(x) + g(x)]$

$$= \lim_{x \to c^+} f(x) + \lim_{x \to c^+} g(x) = f(c) + g(c)$$

$\because$ LHL = RHL

$\therefore$ $(f+g)$ is continuous at $x = c$. **Hence proved.**

2. Since, $f(x)$ is continuous at $x = 5$.

$$\therefore \quad f(5) = \lim_{x \to 5} f(x)$$

$$\Rightarrow \quad 5k + 1 = \lim_{x \to 5} 3x - 5$$

$$\Rightarrow \quad 5k + 1 = 3(5) - 5 = 10$$

$$\Rightarrow \quad 5k = 9$$

$$\Rightarrow \quad k = 9/5$$

Case Study 3

Let $f(x)$ be a real valued function. Then:

Left Hand Derivative (LHD):

$$Lf'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

Right Hand Derivative (RHD):

$$Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Also, a function $f(x)$ is said to be differentiable at $x = a$ if its LHD and RHD at $x = a$ exist and both are equal.

For the function $f(x) = \begin{cases} |x-3|, & x \ge 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$

Based on the above information, solve the following questions: (CBSE 2023)

Q 1. What is RHD of $f(x)$ at $x = 1$?

Q 2. What is LHD of $f(x)$ at $x = 1$?

Q 3. Check whether the function $f(x)$ is differentiable at $x = 1$.

Or

Find $f'(2)$ and $f'(-1)$.

Solutions

1. Given, $f(x) = \begin{cases} |x-3|, & x \ge 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$

$$= \begin{cases} -(x-3), & 1 \le x < 3 \\ x-3, & x \ge 3 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$$

At $x = 1$,

$$\text{RHD} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{-(1+h-3) + (1-3)}{h}$$

$$= \lim_{h \to 0} \frac{-h + 2 - 2}{h} = -1$$

2. At $x = 1$,

$$\text{LHD} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{\frac{(1-h)^2}{4} - 3\frac{(1-h)}{2} + \frac{13}{4} + (1-3)}{-h}$$

$$= \lim_{h \to 0} \frac{\frac{(1^2 + h^2 - 2h)}{4} - \frac{6(1-h) + 13}{4} - 2}{-h}$$

$$= \lim_{h \to 0} \frac{h^2 + 4h + 8 - 8}{-4h} = \lim_{h \to 0} \frac{h(h+4)}{-4h}$$

$$= \frac{4}{-4} = -1$$

3. $\because$ At $x = 1$, RHD $= -1$ and LHD $= -1$

$\therefore$ LHD $=$ RHD $= -1$

Hence, $f(x)$ is differentiable at $x = 1$.

Or

Since, 2 lies in the interval [1, 3), so we consider the function

$$f(x) = -(x-3)$$

Then, $f(x) = -x + 3$

$\Rightarrow$ $f'(x) = -1$

$\Rightarrow$ $f'(2) = -1$

Since, -1 lies in the interval $(\infty, 1)$, so we consider the function, $f(x) = \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}$

$$\Rightarrow \quad f'(x) = \frac{2x}{4} - \frac{3}{2} + 0$$

$$\therefore \quad f'(-1) = \frac{2(-1)}{4} - \frac{3}{2}$$

$$= -\frac{1}{2} - \frac{3}{2} = -\frac{4}{2} = -2$$

Case Study 4

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f[g(x)]$ is a differentiable function of x and $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. This rule is also known as CHAIN RULE.

Based on the above information, solve the following questions.

Q 1. Find the derivative of $\cos\sqrt{x}$ with respect to x.

Q 2. Find the derivative of $7^{x+\frac{1}{x}}$ with respect to x.

Q 3. Find the derivative of $\sqrt{\frac{1-\cos x}{1+\cos x}}$ with respect to x.

Q 4. Find the derivative of $\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) + \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ with respect to x.

Or

Find the derivative of $\sec^{-1} x + \text{cosec}^{-1}\frac{x}{\sqrt{x^2-1}}$ with respect to x.

Solutions

1. Let $y = \cos\sqrt{x}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(\cos\sqrt{x})$$

$$= -\sin\sqrt{x} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= -\sin\sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

2. Let $y = 7^{x+\frac{1}{x}}$

Then $\frac{dy}{dx} = \frac{d}{dx}\left(7^{x+\frac{1}{x}}\right)$

$$= 7^{x+\frac{1}{x}} \cdot \log 7 \cdot \frac{d}{dx}\left(x + \frac{1}{x}\right)$$

$$= 7^{x+\frac{1}{x}} \cdot \log 7 \cdot \left(1 - \frac{1}{x^2}\right)$$

$$= \left(\frac{x^2-1}{x^2}\right) \cdot 7^{x+\frac{1}{x}} \cdot \log 7$$

3. Let $y = \sqrt{\frac{1-\cos x}{1+\cos x}} = \sqrt{\frac{1-1+2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}-1+1}} = \tan\left(\frac{x}{2}\right)$

$$\therefore \quad \frac{dy}{dx} = \sec^2\frac{x}{2} \cdot \frac{1}{2} = \frac{1}{2}\sec^2\frac{x}{2}$$

4. Let $y = \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) + \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{b} \times \frac{1}{1+\frac{x^2}{b^2}} \times \frac{1}{b} + \frac{1}{a} \times \frac{1}{1+\frac{x^2}{a^2}} \times \frac{1}{a}$$

$$= \frac{1}{b^2+x^2} + \frac{1}{a^2+x^2}$$

Or

Let $y = \sec^{-1}x + \text{cosec}^{-1}\frac{x}{\sqrt{x^2-1}}$

Put $x = \sec\theta \Rightarrow \theta = \sec^{-1}x$

$$\therefore \quad y = \sec^{-1}(\sec\theta) + \text{cosec}^{-1}\left(\frac{\sec\theta}{\sqrt{\sec^2\theta - 1}}\right)$$

$$= \theta + \text{cosec}^{-1}\left(\frac{\sec\theta}{\tan\theta}\right)$$

$$= \theta + \text{cosec}^{-1}(\text{cosec}\,\theta) = \theta + \theta$$

$$= 2\theta = 2\sec^{-1}x$$

$$\therefore \quad \frac{dy}{dx} = 2\frac{d}{dx}(\sec^{-1}x)$$

$$= 2 \times \frac{1}{|x|\sqrt{x^2-1}} = \frac{2}{|x|\sqrt{x^2-1}}$$

Very Short Answer Type Questions

Q 1. If the function f defined as

$$f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

is continuous at $x = 3$, find the value of k. *(CBSE 2020)*

Q 2. For what value of 'k' is the function

$$f(x) = \begin{cases} \frac{\sin 5x}{3x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

continuous at $x = 0$? *(CBSE 2017)*

Q 3. Determine the value of constant 'k' so that the function $f(x) = \begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$ is continuous at $x = 0$. *(CBSE 2017; CBSE SQP 2023-24)*

Q 4. Determine the value of 'k' for which the following function is continuous at $x = 3$: *(CBSE 2017)*

$$f(x) = \begin{cases} \frac{(x+3)^2-36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

Q 5. Check the continuity of the function $f(x)=2x+3$ at $x=1$. *(NCERT EXERCISE)*

Q 6. Check that the function $f(x)=\frac{x^2}{2}$ is continuous at $x=0$.

Q 7. Check the continuity of the function $f(x)=2x^2-1$ at $x=3$. *(NCERT EXERCISE)*

Q 8. If the function $f(x)=\frac{\sin(10x)}{x}$, $x\neq 0$ is continuous at $x=0$, find $f(0)$.

Q 9. Show that the function $f(x)=\begin{cases}2x+3, & \text{if } x\le 2\\ 2x-3, & \text{if } x>2\end{cases}$ is not continuous at $x=2$. *(NCERT EXERCISE)*

Q 10. Let $f(x)=x|x|$, for all $x\in R$. Check its differentiability. *(CBSE 2020, 23)*

Q 11. Find the differential coefficient of $e^{\cos(x^2)}$.

Q 12. Find the differential coefficient of the function $3\sqrt{x}+5\log_e x-11\log_a x$ with respect to x.

Q 13. Find the differential coefficient of the function $\cot^3 2x$.

Q 14. If $y=4e^x\sin\frac{\pi}{2}+5^x$, then find $\frac{dy}{dx}$.

Q 15. Find the differential coefficient of $y=\cos^{-1}(\sin x)$ w.r.t. x.

Q 16. Differentiate $x^2\log x\sin x$ with respect to x.

Q 17. Find the differential coefficient of $(e^x\log_a x)$ with respect to x.

Q 18. If $2x+3y=\sin x$, then find the value of $\frac{dy}{dx}$. *(NCERT EXERCISE)*

Q 19. If $y=\sin^{-1}\left[\frac{2x}{1+x^2}\right]$, then find $\frac{dy}{dx}$. *(NCERT EXERCISE)*

Q 20. If $x=at^2$ and $y=2at$, then find $\frac{dy}{dx}$. *(NCERT EXERCISE)*

Short Answer Type-I Questions

Q 1. Find the value(s) of 'λ', if the function

$$f(x)=\begin{cases}\frac{\sin^2\lambda x}{x^2}, & \text{if } x\neq 0\\ 1, & \text{if } x=0\end{cases}$$ is continuous at $x=0$. *(CBSE 2023)*

Q 2. Find the value of k for which the function f given as $f(x)=\begin{cases}\frac{1-\cos x}{2x^2}, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$ is continuous at $x=0$. *(CBSE SQP 2022-23; CBSE 2023)*

Q 3. If $f(x)=\tan^2\frac{\pi x}{6}$, then find $f'(2)$.

Q 4. If $y=\log(\sin x)+\tan x$, then find the value of $\frac{dy}{dx}$ at $x=\frac{\pi}{6}$.

Q 5. Find $\frac{dy}{dx}$ at $x=1, y=\frac{\pi}{4}$, if $\sin^2 y+\cos xy=K$. *(CBSE 2017)*

Q 6. If $e^y=\log(\sin x)$, then find $\frac{dy}{dx}$.

Q 7. Prove that $\frac{d}{dx}\cos^{-1}\left[2x\sqrt{1-x^2}\right]=\frac{-2}{\sqrt{1-x^2}}$.

Q 8. If $f(x)=\begin{cases}x^2, & \text{if } x\ge 1\\ x, & \text{if } x<1\end{cases}$, then show that f is not differentiable at $x=1$. *(CBSE 2023)*

Q 9. If $y\sqrt{1-x^2}+x\sqrt{1-y^2}=1$, then prove that $\frac{dy}{dx}=-\sqrt{\frac{1-y^2}{1-x^2}}$. *(CBSE SQP 2022-23)*

Q 10. If $xy=e^{x-y}$, then show that $\frac{dy}{dx}=\frac{y(x-1)}{x(y+1)}$. *(NCERT EXERCISE; CBSE 2017, 23)*

Q 11. If $(x^2+y^2)^2=xy$, then find $\frac{dy}{dx}$. *(CBSE 2023)*

Q 12. If $y=\sin^{-1}(6x\sqrt{1-9x^2})$, $-\frac{1}{3\sqrt{2}}<x<\frac{1}{3\sqrt{2}}$, then find $\frac{dy}{dx}$. *(CBSE 2017)*

Q 13. Find the differential coefficient of the function $\cot(\cos^{-1}x)$ with respect to x.

Q 14. Find the differential coefficient of the function $\sin^{-1}2x\sqrt{1-x^2}$ with respect to x.

Q 15. Find the differential coefficient of $\sin^{-1}x$ with respect to $\cos^{-1}x$.

Q 16. Find the differential of $\sin^2 x$ w.r.t. $e^{\cos x}$.

Q 17. If $y=x^{x\cos x}$, then find $\frac{dy}{dx}$. *(CBSE 2020)*

Q 18. Find $\frac{dy}{dx}$ at $t=\frac{2\pi}{3}$ when $x=10(t-\sin t)$ and $y=12(1-\cos t)$. *(CBSE 2017)*

Q 19. If $x=a\cos t$ and $y=b\sin t$, then find $\frac{d^2y}{dx^2}$. *(CBSE 2023)*

Or

If $x=a\cos\theta$, $y=b\sin\theta$, then find $\frac{d^2y}{dx^2}$. *(CBSE 2020)*

Short Answer Type-II Questions

Q 1. Prove that the function $f(x)=\begin{cases}\frac{\sin x}{x}, & \text{if } x<0\\ x+1, & \text{if } x\ge 0\end{cases}$ is a continuous function.

Q 2. Examine the continuity of the function

$$f(x)=\begin{cases}\frac{1}{1-e^{1/x}}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$ at point $x=0$.

Q 3. Show that the function f defined by

$$f(x)=\begin{cases}\dfrac{e^{1/x}-1}{e^{1/x}+1}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$ is discontinuous at point $x=0$.

Q 4. Show that the function $f(x)$ which is defined as

$$f(x)=\begin{cases}\dfrac{e^{x}}{1+e^{1/x}}, & x\neq 0\\ 0, & x=0\end{cases}\ \forall\ x\in R,$$ is discontinuous at $x=0$.

Q 5. If $y=(x-1)\log(x-1)-(x+1)\log(x+1)$, then prove that $\dfrac{dy}{dx}=\log\left(\dfrac{x-1}{x+1}\right)$.

Q 6. If $y=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, then prove that $(1-x^2)\dfrac{dy}{dx}=xy+1$.

Q 7. If $f(x)=x+x^2+x^4+x^6+\ldots+x^{30}$, then show that $f'(1)=241$. (NCERT EXEMPLAR)

Q 8. If $x+y=t+\dfrac{1}{t}$ and $x^3+y^3=t^3+\dfrac{1}{t^3}$, then prove that $\dfrac{dy}{dx}=-\dfrac{1}{x^2}$.

Q 9. If $y=x^2+\cfrac{1}{x^2+\cfrac{1}{x^2+\cfrac{1}{x^2+\ldots\infty}}}$, then prove that $\dfrac{dy}{dx}=\dfrac{2xy^2}{1+y^2}$.

Q 10. If $\cos y=x\cos(a+y)$ and $\cos a\neq\pm 1$, then prove that $\dfrac{dy}{dx}=\dfrac{\cos^2(a+y)}{\sin a}$. (NCERT EXEMPLAR)

Q 11. Find $\dfrac{dy}{dx}$ if

$$y=\sqrt{\sin x+\cos x+\sqrt{\sin x+\cos x+\sqrt{\sin x+\cos x+\ldots}}}$$

Q 12. If $y=\sqrt{\cos x+\sqrt{\cos x+\sqrt{\cos x+\ldots\infty}}}$, then find $\dfrac{dy}{dx}$.

Q 13. If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\ldots\infty}}}$, then prove that $(2y-1)\dfrac{dy}{dx}=\cos x$.

Q 14. Find the differential coefficient of $\tan^{-1}\dfrac{2x}{1-x^2}$ with respect to $\cos^{-1}\dfrac{1-x^2}{1+x^2}$.

Q 15. If $y=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)+\tan^{-1}x$, then find $\dfrac{dy}{dx}$ by using the result $\sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}$.

Q 16. If $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$, $0<x<1$, then find $\dfrac{dy}{dx}$. (NCERT EXERCISE)

Q 17. If $y=\sin^{-1}\{x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\}$ and $0<x<1$ then find $\dfrac{dy}{dx}$. (NCERT EXEMPLAR)

Q 18. Find the differential coefficient of the function $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ with respect to $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$.

Q 19. If $y=\tan^{-1}\left[\dfrac{x}{1+\sqrt{1-x^2}}\right]$, then find $\dfrac{dy}{dx}$.

Q 20. If $y=\cot^{-1}\left(\dfrac{\sqrt{1+x^2}+1}{x}\right)$, then find the value of $\dfrac{dy}{dx}$.

Q 21. Find the differential coefficient of $\sin^{-1}\left[\dfrac{1-x}{1+x}\right]$ with respect to $\sqrt{x}$. (NCERT EXERCISE)

Q 22. If $y=e^{\sin^{-1}x^2}$, then find $\dfrac{dy}{dx}$.

Q 23. Find the differential coefficient of the function $\left(4e^{\sin^{-1}x}+\dfrac{\pi}{2}\right)$ with respect to $\left(5\sin^{-1}x+\dfrac{\pi}{4}\right)$.

Q 24. Find the differential coefficient of $x^{\tan x}$ with respect to x.

Q 25. If $y=x^{x^{x^{\cdots\infty}}}$, then find $\dfrac{dy}{dx}$.

Q 26. If $y=(\sin x)^{(\sin x)^{(\sin x)\cdots\infty}}$, then find $\dfrac{dy}{dx}$.

Q 27. Differentiate $y^x=x^y$ with respect to x. (NCERT EXERCISE)

Q 28. If $x=a(t+\sin t)$, $y=a(1+\cos t)$, then find $\dfrac{dy}{dx}$.

Q 29. If $x=a(\cos t+t\sin t)$ and $y=a(\sin t-t\cos t)$, then find $\dfrac{d^2y}{dx^2}$. (NCERT EXERCISE)

Q 30. If $y=\sec x+\tan x$, then prove that $\dfrac{d^2y}{dx^2}=\dfrac{\cos x}{(1-\sin x)^2}$. (NCERT EXEMPLAR; CBSE 2023)

Q 31. If $(a+bx)e^{y/x}=x$, then prove that $x\dfrac{d^2y}{dx^2}=\left(\dfrac{a}{a+bx}\right)^2$. (CBSE SQP 2023-24)

Long Answer Type Questions

Q 1. For what values of a and b, the function defined by

$$f(x)=\begin{cases}5, & \text{if } x\le 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\ge 10\end{cases}$$ is continuous? *(NCERT EXERCISE)*

Q 2. Determine whether the function f defined by

$$f(x)=\begin{cases}x^2\sin\frac{1}{x}, & \text{if } x\ne 0\\ 0, & \text{if } x=0\end{cases}$$ is continuous. *(NCERT EXERCISE)*

Q 3. If $f(x)=\frac{\sqrt{2}\cos x-1}{\cot x-1}$, $x\ne\frac{\pi}{4}$, then find the value of $f\left(\frac{\pi}{4}\right)$ for which $f(x)$ becomes continuous at $x=\frac{\pi}{4}$. *(NCERT EXEMPLAR)*

Q 4. If $\log(x^2+y^2)=2\tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{dy}{dx}=\frac{x+y}{x-y}$. *(CBSE 2019)*

Q 5. If for $-1<x<1$, $x\sqrt{1+y}+y\sqrt{1+x}=0$, then prove that $\frac{dy}{dx}=-\frac{1}{(1+x)^2}$. *(NCERT EXERCISE)*

Q 6. Differentiate $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\tan^{-1}x$. *(NCERT EXEMPLAR)*

Q 7. Differentiate $\cos^{-1}(2x\sqrt{1-x^2})$ with respect to $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$, where $x\in\left(\frac{1}{\sqrt{2}},1\right)$. *(NCERT EXEMPLAR)*

Q 8. If $x^y=e^{x-y}$, then prove that $\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$. *(NCERT EXEMPLAR)*

Q 9. If $y=(\tan x)^{\tan x\ldots\infty}$, then prove that $\frac{dy}{dx}=\frac{2y^2\operatorname{cosec}2x}{1-y\log\tan x}$.

Q 10. Differentiate $y=\sqrt{3x+2}+\frac{1}{\sqrt{2x^2+4}}+(\cos x)^{\tan x}$ with respect to x.

Q 11. Differentiate $y=x^x+(\cos x)^{\tan x}$ with respect to x.

Q 12. Find the differential coefficient of the function $(\sin x)^{\cos x}+(\cos x)^{\sin x}$ with respect to x.

Q 13. If $x^y-y^x=a^b$, find $\frac{dy}{dx}$. *(CBSE 2017, 19)*

Q 14. If $y=(\log x)^x+x^{\log x}$, then find $\frac{dy}{dx}$. *(CBSE 2020)*

Q 15. If $y=(\cos x)^x+\tan^{-1}\sqrt{x}$, find $\frac{dy}{dx}$. *(CBSE 2020)*

Q 16. If $x=a(2\theta-\sin 2\theta)$ and $y=a(1-\cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta=\frac{\pi}{3}$. *(CBSE 2018)*

Q 17. If $x=a\left(\cos t+\log\tan\frac{t}{2}\right)$, $y=a\sin t$, then show that $\frac{dy}{dx}=\tan t$. *(NCERT EXERCISE)*

Q 18. If $e^y(x+1)=1$, then show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$. *(NCERT EXERCISE; CBSE 2017)*

Q 19. If $x^m y^n=(x+y)^{m+n}$, prove that $\frac{d^2y}{dx^2}=0$. *(CBSE 2017)*

Q 20. If $y=\sin(\sin x)$, prove that

$$\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x=0.$$ *(CBSE 2018)*

Q 21. If $y=(\sin^{-1}x)^2$, prove that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-2=0.$$ *(CBSE 2019)*

Q 22. If $x=\cos t+\log\tan\left(\frac{t}{2}\right)$, $y=\sin t$, then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t=\frac{\pi}{4}$. *(CBSE 2019)*

Solutions

Very Short Answer Type Questions

1. It is given that the function f is continuous at $x=3$.

TR!CK
If f is continuous at a, then $\lim_{x\to a}f(x)=f(a)$.

$$\therefore \quad \lim_{x\to 3}f(x)=f(3)$$

$$\Rightarrow \quad \lim_{x\to 3}\frac{x^2-9}{x-3}=k \qquad \left[\left(\frac{0}{0}\right)\text{form}\right]$$

$$\Rightarrow \quad \lim_{x\to 3}\frac{(x+3)(x-3)}{(x-3)}=k$$

$$\Rightarrow \quad \lim_{x\to 3}(x+3)=k$$

$$\Rightarrow \quad 3+3=k$$

$$\Rightarrow \quad k=6$$

Hence, $k=6$

2.

TR!CK
If f is continuous at a, then $\lim_{x\to a}f(x)=f(a)$.

We have, $\lim_{x\to 0}f(x)=\lim_{x\to 0}\left\{\frac{\sin 5x}{3x}+\cos x\right\}$

COMMON ERROR

Some students do not know how to evaluate limits of the form $\frac{0}{0}$.

$$= \lim_{x \to 0} \frac{5}{3}\left(\frac{\sin 5x}{5x}\right) + \lim_{x \to 0} \cos x$$

$$= \frac{5}{3} \times 1 + \cos 0 \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

$$= \frac{5}{3} + 1 = \frac{8}{3}$$

It is given that, $f(x)$ is continuous at $x = 0$.

$$\therefore \quad \lim_{x \to 0} f(x) = f(0)$$

$$\Rightarrow \quad \frac{8}{3} = k \text{ or } k = \frac{8}{3}$$

3. We have, $f(0) = 3$

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$$

$$= \lim_{h \to 0} \frac{k(0-h)}{|0-h|}$$

$$= \lim_{h \to 0} \frac{-kh}{h} = -k$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$$

$$= \lim_{h \to 0} 3 = 3$$

TR!CK

A function f(x) is said to be continuous at a point x = a, if LHL = RHL = f(a).

$\because$ The given function is continuous at $x = 0$.

$$\therefore \quad \text{LHL} = \text{RHL} = f(0)$$

$$\Rightarrow \quad -k = 3$$

$$\text{or} \quad k = -3$$

4. We have,

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{(x+3)^2 - 36}{x - 3} \qquad \left[\left(\frac{0}{0}\right) \text{form}\right]$$

$$= \lim_{x \to 3} \frac{(x+3)^2 - (6)^2}{x - 3}$$

$$= \lim_{x \to 3} \frac{(x+3+6)(x+3-6)}{x-3}$$

$$[\because a^2 - b^2 = (a+b)(a-b)]$$

$$= \lim_{x \to 3} \frac{(x+9)(x-3)}{(x-3)}$$

$$= \lim_{x \to 3} (x+9) = 3 + 9 = 12$$

But given that, $f(x)$ is continuous at $x = 3$.

TR!CK

If f is continuous at a, then $\lim_{x \to a} f(x) = f(a)$.

$$\therefore \quad \lim_{x \to 3} f(x) = f(3)$$

$$\Rightarrow \quad 12 = k \text{ or } k = 12$$

5. Here, at $x = 1$,

$$f(1) = 2(1) + 3 = 2 + 3 = 5$$

i.e., function is defined at $x = 1$.

Now, we find the limit of the function at $x = 1$.

$$\therefore \quad \lim_{x \to 1} f(x) = \lim_{x \to 1} (2x + 3)$$

$$= 2(1) + 3 = 5$$

Therefore, $\lim_{x \to 1} f(x) = 5 = f(1)$

$\therefore$ Function f is continuous at $x = 1$.

6. At $x = 0$, $f(0) = \frac{0^2}{2} = 0$

i.e., function is defined at $x = 0$.

$$\text{Now} \quad \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^2}{2} = (0)^2 = 0$$

$$\therefore \quad \lim_{x \to 0} f(x) = 0 = f(0)$$

So, $f(x)$ is continuous at $x = 0$.

7. At $x = 3$,

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(3-h) = \lim_{h \to 0} \{2(3-h^2) - 1\}$$

$$= \lim_{h \to 0} \{2(9 + h^2 - 6h) - 1\}$$

$$= 2(9 + 0 - 0) - 1$$

$$= 18 - 1 = 17$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(3+h) = \lim_{h \to 0} \{2(3+h)^2 - 1\}$$

$$= \lim_{h \to 0} \{2(9 + h^2 + 6h) - 1\}$$

$$= 2(9 + 0 + 0) - 1 = 18 - 1 = 17$$

$$\text{and} \quad f(3) = 2(3)^2 - 1 = 2 \times 9 - 1$$

$$= 18 - 1 = 17$$

$$\because \quad \text{LHL} = \text{RHL} = f(3)$$

$$\textit{i.e.,} \quad \lim_{x \to 3} f(x) = f(3)$$

Hence, function is continuous at $x = 3$.

8. Given that the function f is continuous at $x = 0$.

$$\therefore \quad \lim_{x \to 0} f(x) = f(0)$$

$$\Rightarrow \quad \lim_{x \to 0} \frac{\sin(10x)}{x} = f(0)$$

$$\Rightarrow \quad \lim_{x \to 0} 10 \cdot \frac{\sin(10x)}{(10x)} = f(0)$$

$$\Rightarrow \quad 10 \cdot \lim_{x \to 0} \frac{\sin(10x)}{(10x)} = f(0)$$

$$\Rightarrow \quad 10 \times 1 = f(0) \qquad \left[\because \lim_{x \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

$$\therefore \quad f(0) = 10$$

Hence, if f is continuous at $x = 0$, then the value of $f(0)$ will be 10.

9. $$\text{LHL} = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} 2(2-h) + 3$$

$$= \lim_{h \to 0} (4 - 2h + 3) = 7 - 0 = 7$$

$$\text{RHL} = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} 2(2+h) - 3$$

$$= \lim_{h \to 0} (4 + 2h - 3) = 1 + 0 = 1$$

and $f(2)=2\times2+3=7$

$\because$ RHL $\neq$ LHL $= f(2)$ *i.e.*, $\lim_{x\to2^+} f(x)\neq f(2)$

Hence, function is not continuous at $x=2$.

Hence proved.

10. Given, $f(x)=x\,|x|$

TiP

Understand the modulus function and its properties.

$$\therefore\quad f(x)=\begin{cases}x(x), & x\geq 0\\ -x\,(x), & x<0\end{cases}$$

TR!CK

A function f(x) is said to be differentiable at a point x = a, if LHD is equal to RHD i.e., Lf′(a) = Rf′(a).

$$\Rightarrow\quad f(x)=\begin{cases}x^2, & x\geq 0\\ -x^2, & x<0\end{cases}$$

Now, RHD $=\lim_{h\to0}\frac{f(0+h)-f(0)}{h}$

$=\lim_{h\to0}\frac{(0+h)^2-(0)^2}{h}$

$=\lim_{h\to0}\frac{h^2}{h}=\lim_{h\to0}h=0$

LHD $=\lim_{h\to0}=\frac{f(0-h)-f(0)}{-h}=\lim_{h\to0}\frac{f(0-h)-f(0)}{-h}$

$=\lim_{h\to0}\frac{-(0-h)^2-(0)^2}{-h}=\lim_{h\to0}\frac{-h^2}{-h}=\lim_{h\to0}h=0$

$\because$ LHD = RHD

So, $f(x)$ is differentiable at $x=0$.

Since, $f(x)$ is a polynomial function, so it also differentiable other than 0. Hence $f(x)$ is differentiable for every real value (*i.e.*, R).

COMMON ERR!R

Mostly students find difficulty in differentiating modulus functions.

11. Let $y=e^{\cos(x^2)}$

Differentiate both sides w.r.t. x, we get

$\frac{dy}{dx}=e^{\cos x^2}\cdot\frac{d}{dx}\cos x^2=e^{\cos x^2}\cdot[-\sin x^2]\frac{d}{dx}x^2$

$=e^{\cos x^2}(-\sin x^2)\,2(x)=-2x\sin x^2\cdot e^{\cos x^2}$

12. Let $y=3\sqrt{x}+5\log_a x-11\log_a x$

Differentiate both sides w.r.t. x, we get

$\frac{dy}{dx}=3\frac{d}{dx}\sqrt{x}+5\frac{d}{dx}\log_a x-11\frac{d}{dx}\log_a x$

$=3\cdot\frac{1}{2\sqrt{x}}+5\cdot\frac{1}{x}-11\cdot\log_a e\cdot\frac{d}{dx}\log_a x$

$=\frac{3}{2\sqrt{x}}+\frac{5}{x}-\frac{11}{x}\cdot\log_a e$

13. Let $y=\cot^3 2x$

Differentiate both sides w.r.t. x, we get

$\frac{dy}{dx}=3\cot^2 2x\cdot\frac{d}{dx}\cot 2x$

$=3\cot^2 2x(-\text{cosec}^2\,2x)\cdot\frac{d}{dx}(2x)$

$\frac{dy}{dx}=3\cot^2 2x\,(-\text{cosec}^2 2x)\cdot 2$

$=-6\cdot\cot^2 2x\cdot\text{cosec}^2 2x$

14. Given, $y=4e^x\sin\frac{\pi}{2}+5^x$

$\Rightarrow\quad y=4e^x(1)+5^x=4e^x+5^x$

Differentiate both sides w.r.t. x, we get

$\frac{dy}{dx}=4\frac{d}{dx}(e^x)+\frac{d}{dx}(5^x)=4e^x+5^x\log 5$

15. Given, $y=\cos^{-1}(\sin x)=\cos^{-1}\left[\cos\left(\frac{\pi}{2}-x\right)\right]$

$=\frac{\pi}{2}-x\qquad[\because\ \cos^{-1}(\cos\theta)=\theta]$

$\Rightarrow\quad\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\pi}{2}-x\right)=0-1=-1$

16. Let $y=x^2\log x\sin x$

$\Rightarrow\frac{dy}{dx}=x^2\log x\frac{d}{dx}(\sin x)+x^2\sin x\frac{d}{dx}(\log x)+\log x\sin x\frac{d}{dx}(x^2)$

$=x^2\log x\cos x+x^2\sin x\cdot\frac{1}{x}+\log x\sin x\cdot 2x$

$=x\,[x\log x\cos x+\sin x+2\log x\sin x]$

COMMON ERR!R

Some students commit errors in u × v × w rule for differentiating the function.

17. Let $y=e^x\log_a x$

$\therefore\quad\frac{dy}{dx}=e^x\times\frac{d}{dx}(\log_a x)+(\log_a x)\times\frac{d}{dx}(e^x)$

$=e^x\times\frac{1}{x}\log_a e+(\log_a x)\times e^x$

$=\frac{e^x}{x}(x\log_a x+\log_a e)$

18. Given, $2x+3y=\sin x$

Differentiate both sides w.r.t. x, we get

$2\times1+3\frac{dy}{dx}=\cos x$

$\Rightarrow\quad\frac{3\,dy}{dx}=\cos x-2$

$\Rightarrow\quad\frac{dy}{dx}=\frac{\cos x-2}{3}$

19. Put $x=\tan A$.

$\therefore\quad y=\sin^{-1}\left[\frac{2x}{1+x^2}\right]=\sin^{-1}\left[\frac{2\tan A}{1+\tan^2 A}\right]$

$=\sin^{-1}(\sin 2A)=2A=2\tan^{-1}x$

$[\because x=\tan A\Rightarrow a=\tan^{-1}x]$

Differentiate both sides w.r.t. x, we get

$\frac{dy}{dx}=\frac{d}{dx}(2\tan^{-1}x)=2\times\frac{1}{1+x^2}=\frac{2}{1+x^2}$

20. Given, $x = at^2$ and $y = 2at$

$\therefore \quad \frac{dx}{dt} = a \times 2t = 2at$

and $\quad \frac{dy}{dt} = 2a \times 1 = 2a$

Now, $\quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t}$

COMMON ERROR

Some students commit errors using $u \times v$ rule derivative function.

Short Answer Type-I Questions

1. Since, $f(x) = \begin{cases} \frac{\sin^2 \lambda x}{x^2}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

$$\therefore \lim_{x \to 0} f(x) = f(0)$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin^2 \lambda x}{x^2} = 1$$

$$\Rightarrow \lim_{x \to 0} \left(\frac{\sin \lambda x}{\lambda x}\right)^2 \times \lambda^2 = 1 \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

$$\Rightarrow 1 \times \lambda^2 = 1 \Rightarrow \lambda^2 = 1$$

$$\Rightarrow \lambda = \pm 1$$

2. Since, $f(x) = \begin{cases} \frac{1 - \cos x}{2x^2}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

$$\therefore \lim_{x \to 0} f(x) = f(0)$$

$$\Rightarrow \lim_{x \to 0} \frac{1 - \cos x}{2x^2} = k$$

$$\Rightarrow \lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{2x^2} = k$$

$$\Rightarrow \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 \times \frac{1}{4} = k$$

$$\Rightarrow 1 \times \frac{1}{4} = k \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

$$\Rightarrow k = \frac{1}{4}$$

3. $f'(x) = \frac{d}{dx}\left[\tan^2 \frac{\pi x}{6}\right] = \frac{d}{dx}\left[\tan \frac{\pi x}{6}\right]^2$

$$= 2\tan \frac{\pi x}{6} \times \frac{d}{dx}\left[\tan \frac{\pi x}{6}\right]$$

$$= 2\tan \frac{\pi x}{6} \cdot \sec^2 \frac{\pi x}{6} \times \frac{d}{dx}\left[\frac{\pi x}{6}\right]$$

$$= 2 \times \frac{\pi}{6} \tan \frac{\pi x}{6} \sec^2 \frac{\pi x}{6}$$

$$= \frac{\pi}{3} \tan \frac{\pi x}{6} \sec^2 \frac{\pi x}{6}$$

$$\therefore f'(2) = \frac{\pi}{3} \tan \frac{2\pi}{6} \sec^2 \frac{2\pi}{6}$$

$$= \frac{\pi}{3} \tan \frac{\pi}{3} \sec^2 \frac{\pi}{3} = \frac{\pi}{3} \times \sqrt{3} \times 2^2 = \frac{4\pi}{\sqrt{3}}$$

4. Given, $y = \log(\sin x) + \tan x$

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) + \sec^2 x$$

$$= \frac{1}{\sin x} \cdot \cos x + \sec^2 x$$

$$= \cot x + \sec^2 x$$

$$\therefore \left[\frac{dy}{dx}\right]_{\left(x = \frac{\pi}{6}\right)} = \cot\left(\frac{\pi}{6}\right) + \sec^2\left(\frac{\pi}{6}\right)$$

$$= \cot 30° + \sec^2 30°$$

$$= \sqrt{3} + \left(\frac{2}{\sqrt{3}}\right)^2 = \sqrt{3} + \frac{4}{3}$$

5. Given, $\sin^2 y + \cos xy = K$

Differentiate both sides w.r.t. x, we get

$$2\sin y \cdot \cos y \frac{dy}{dx} + (-\sin xy)\left\{\frac{d}{dx}(xy)\right\} = 0$$

$$\Rightarrow 2\sin y \cdot \cos y \frac{dy}{dx} - \sin xy\left\{x\frac{dy}{dx} + y\right\} = 0$$

$$\Rightarrow \{2\sin y \cdot \cos y - x\sin xy\}\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin xy}{2\sin y \cdot \cos y - x\sin xy}$$

Put $x = 1$ and $y = \frac{\pi}{4}$.

$$\therefore \frac{dy}{dx} = \frac{\frac{\pi}{4}\sin\left(1 \times \frac{\pi}{4}\right)}{2\sin\frac{\pi}{4} \cdot \cos\frac{\pi}{4} - 1 \cdot \sin\left(1 \cdot \frac{\pi}{4}\right)}$$

$$= \frac{\frac{\pi}{4}\sin\frac{\pi}{4}}{2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - \sin\frac{\pi}{4}} = \frac{\frac{\pi}{4} \times \frac{1}{\sqrt{2}}}{2 \times \frac{1}{2} - \frac{1}{\sqrt{2}}}$$

$$= \frac{\frac{\pi}{4\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2} - 1} \times \frac{\pi}{4\sqrt{2}} = \frac{\pi}{4(\sqrt{2} - 1)}$$

$$= \frac{\pi}{4(\sqrt{2} - 1)} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1} = \frac{\pi(\sqrt{2} + 1)}{4(2 - 1)} = \frac{\pi(\sqrt{2} + 1)}{4}$$

6. Given, $e^y = \log(\sin x)$

Differentiate both sides w.r.t. x, we get

$$\frac{d}{dx}(e^y) = \frac{d}{dx}\{\log(\sin x)\}$$

$$\Rightarrow e^y \cdot \frac{dy}{dx} = \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) = \frac{\cos x}{\sin x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cot x}{e^y} = \frac{\cot x}{\log(\sin x)}$$

7. Put $x = \sin \theta$

$$\therefore \cos^{-1}[2x\sqrt{1 - x^2}] = \cos^{-1}[2\sin\theta\sqrt{1 - \sin^2\theta}]$$

$$= \cos^{-1}(2\sin\theta\cos\theta)$$

$$= \cos^{-1}(\sin 2\theta)$$

TR!CK

$\sin^2\theta + \cos^2\theta = 1 \ \forall \ \theta \in R$

$$= \cos^{-1}\left[\cos\left\{\frac{\pi}{2} - 2\theta\right\}\right] = \frac{\pi}{2} - 2\theta$$

$$= \frac{\pi}{2} - 2\sin^{-1} x \qquad [\because x = \sin\theta \Rightarrow \theta = \sin^{-1} x]$$

$$\therefore \frac{d}{dx}\left[\cos^{-1}\{2x\sqrt{1-x^2}\}\right] = \frac{d}{dx}\left\{\frac{\pi}{2} - 2\sin^{-1} x\right\}$$

$$= \frac{-2}{\sqrt{1-x^2}}$$ **Hence proved.**

8. Given, $f(x) = \begin{cases} x^2, & \text{if } x \geq 1 \\ x, & \text{if } x < 1 \end{cases}$

Now, check the differentiability at $x = 1$.

$$\text{LHD} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{(1-h) - 1^2}{-h} = \lim_{h \to 0} \frac{-h}{-h} = 1$$

and $$\text{RHD} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h}$$

$$= \lim_{h \to 0} \frac{1 + h^2 + 2h - 1}{h}$$

$$= \lim_{h \to 0} \frac{h(h+2)}{h} = \lim_{h \to 0} (h+2)$$

$$= 0 + 2 = 2$$

$\therefore$ LHD $\neq$ RHD

Hence, $f(x)$ is not differentiable at $x = 1$. **Hence proved.**

9. We have, $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$

Differentiating both sides w.r.t. x, we get

$$y \cdot \frac{-2x}{2\sqrt{1-x^2}} + \sqrt{1-x^2} \cdot \frac{dy}{dx} + \sqrt{1-y^2} + x \cdot \frac{-2y}{2\sqrt{1-y^2}} \frac{dy}{dx} = 0$$

$$\Rightarrow \left\{\sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}}\right\}\frac{dy}{dx} = \left\{\frac{xy}{\sqrt{1-x^2}} - \sqrt{1-y^2}\right\}$$

$$\Rightarrow (\sqrt{1-x^2}\sqrt{1-y^2} - xy)\frac{dy}{dx}$$

$$= -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}(\sqrt{1-y^2}\sqrt{1-x^2} - xy)$$

$$\therefore \frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}} \qquad [\because \sqrt{1-x^2}\sqrt{1-y^2} \neq xy]$$

Hence proved.

Alternate Method:

Let $\sin^{-1} x = A$ and $\sin^{-1} y = B$

$\Rightarrow$ $x = \sin A$ and $y = \sin B$

$\because$ $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$

$\therefore$ $\sin B\sqrt{1-\sin^2 A} + \sin A\sqrt{1-\sin^2 B} = 1$

$\Rightarrow$ $\sin B \cdot \cos A + \sin A \cdot \cos B = 1$

$[\because \sin^2\theta + \cos^2\theta = 1]$

$\Rightarrow$ $\sin(A+B) = 1$

$\Rightarrow$ $A + B = \sin^{-1}(1) = \sin^{-1}\left(\sin\frac{\pi}{2}\right)$

$\Rightarrow$ $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$$ **Hence proved.**

10. Given, $xy = e^{x-y}$

Differentiate both sides w.r.t. x, we get

$$x\frac{dy}{dx} + 1 \times y = e^{x-y}\frac{d}{dx}(x-y)$$

$$\Rightarrow x\frac{dy}{dx} + y = e^{x-y}\left(1 - \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}(x + e^{x-y}) = e^{x-y} - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{x-y} - y}{x + e^{x-y}} = \frac{xy - y}{x + xy} = \frac{y(x-1)}{x(1+y)}$$ **Hence proved.**

11. Given, $(x^2 + y^2)^2 = xy$

Differentiate both sides w.r.t. x, we get

$$2(x^2+y^2)\frac{d}{dx}(x^2+y^2) = x\frac{dy}{dx} + y\frac{d}{dx}(x)$$

$$\Rightarrow 2(x^2+y^2)\left(2x + 2y\frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow 4(x^2+y^2)\left(x + y\frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \frac{dy}{dx}[4y(x^2+y^2) - x] = y - 4x(x^2+y^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - 4x^3 - 4xy^2}{4x^2y + 4y^3 - x}$$

12. Given, $y = \sin^{-1}(6x\sqrt{1-9x^2})$

$$= \sin^{-1}(2 \cdot 3x\sqrt{1-(3x)^2})$$

Putting $3x = \sin\theta \Rightarrow \theta = \sin^{-1}(3x)$

$$\therefore y = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$$

$$\left[\because -\frac{1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}} \Rightarrow -\frac{1}{\sqrt{2}} < 3x < \frac{1}{\sqrt{2}}\right.$$

$$\Rightarrow -\frac{1}{\sqrt{2}} < \sin\theta < \frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) < \sin^{-1}(\sin\theta) < \sin^{-1}\frac{1}{\sqrt{2}}$$

$$\left.\Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}\right]$$

$$= \sin^{-1}(2\sin\theta\sqrt{\cos^2\theta}) \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

TR!CK

$\sin^{-1}(\sin x) = x, \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$= \sin^{-1}(2\sin\theta \cdot \cos\theta)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta = 2\sin^{-1}(3x)$$

TiP

Differentiation rule for different functions and terms need attention. A thorough revision is a must.

Now, differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx}=2\cdot\frac{d}{dx}\sin^{-1}(3x)$$

$$\left[\because \frac{d}{dx}\sin^{-1}(ax)=\frac{a}{\sqrt{1-a^2x^2}}\right]$$

$$=2\cdot\frac{1}{\sqrt{1-(3x)^2}}\cdot\frac{d}{dx}(3x)$$

$$=2\cdot\frac{1}{\sqrt{1-9x^2}}\cdot 3=\frac{6}{\sqrt{1-9x^2}}$$

13. Let $y=\cot(\cos^{-1}x)$

TiP

A thorough revision of the formula must be done.

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx}=-\text{cosec}^2(\cos^{-1}x)\frac{d}{dx}(\cos^{-1}x)$$

$$=-\text{cosec}^2(\cos^{-1}x)\times\frac{-1}{\sqrt{1-x^2}}$$

$$=\frac{\text{cosec}^2(\cos^{-1}x)}{\sqrt{1-x^2}}$$

COMMON ERROR

Sometimes students forget the chain rule, so please be careful.

14. Let $y=\sin^{-1}2x\sqrt{1-x^2}$

Put $x=\sin\theta$, so that $y=\sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

TR!CK

$\sin^{-1}(\sin\theta)=\theta$; $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$$=\sin^{-1}(2\sin\theta\cdot\cos\theta)=\sin^{-1}(\sin 2\theta)$$

$$=2\theta=2\sin^{-1}x \qquad [\because \theta=\sin^{-1}x]$$

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx}=2\frac{d}{dx}(\sin^{-1}x)=\frac{2}{\sqrt{1-x^2}}$$

15. Let $u=\sin^{-1}x$ and $v=\cos^{-1}x$

Differentiate both sides w.r.t. x, we get

$$\frac{du}{dx}=\frac{1}{\sqrt{1-x^2}} \text{ and } \frac{dv}{dx}=\frac{-1}{\sqrt{1-x^2}}$$

$$\therefore \frac{du}{dv}=\frac{du}{dx}\times\frac{dx}{dv}=\frac{1}{\sqrt{1-x^2}}\times(-\sqrt{1-x^2})=-1$$

16. Let $u=\sin^2x$ and $v=e^{\cos x}$

Now, $\frac{du}{dx}=2\sin x\cdot\frac{d}{dx}(\sin x)=2\sin x\cdot\cos x$

and $\frac{dv}{dx}=e^{\cos x}\cdot\frac{d}{dx}(\cos x)=-\sin x\cdot e^{\cos x}$

So, differential of $\sin^2x$ w.r.t. $e^{\cos x}$ is

$$\frac{du}{dv}=\frac{du}{dx}\times\frac{dx}{dv}=2\sin x\cdot\cos x\cdot\frac{-1}{\sin x\cdot e^{\cos x}}$$

$$\therefore \frac{d(\sin^2x)}{d(e^{\cos x})}=\frac{-2\cos x}{e^{\cos x}}=-2e^{-\cos x}\cdot\cos x$$

17. Given, $y=x^{x\cos x}$

Taking log on both sides,

$$\log y=\log x^{x\cos x}=x\cos x\log x$$

Differentiate both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=x\cos x\cdot\frac{d}{dx}\log x+\log x\,\frac{d}{dx}(x\cos x)$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx}=x\cos x\cdot\frac{1}{x}+\log x\left\{x\frac{d}{dx}(\cos x)+\cos x\frac{d}{dx}(x)\right\}$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx}=\cos x+\log x\,[-x\sin x+\cos x]$$

$$\Rightarrow \frac{dy}{dx}=y\,\{\cos x-x\sin x\log x+\cos x\cdot\log x\}$$

$$\therefore \frac{dy}{dx}=x^{x\cos x}\{\cos x-x\sin x\log x+\cos x\cdot\log x\}$$

18. Given, $x=10\,(t-\sin t)$ and $y=12\,(1-\cos t)$

Differentiate both sides w.r.t. t, we get

$$\frac{dx}{dt}=10\,(1-\cos t)$$

and $$\frac{dy}{dt}=12\,(0+\sin t)=12\sin t$$

Now, $$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=12\sin t\times\frac{1}{10\,(1-\cos t)}$$

$$=\frac{6}{5}\cdot\frac{\sin t}{1-\cos t}$$

$$\therefore \left[\frac{dy}{dx}\right]_{\text{at } t=\frac{2\pi}{3}}=\frac{6}{5}\cdot\frac{\sin\frac{2\pi}{3}}{1-\cos\frac{2\pi}{3}}=\frac{6}{5}\cdot\frac{\sin 120^\circ}{1-\cos 120^\circ}$$

$$=\frac{6}{5}\times\frac{\cos 30^\circ}{1+\sin 30^\circ}=\frac{6}{5}\times\frac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}}$$

$$=\frac{6}{5}\times\frac{2}{3}\times\frac{\sqrt{3}}{2}=\frac{2\sqrt{3}}{5}$$

COMMON ERROR

Some students commit errors in $u\times v$ rule derivative function.

19. Given, equation of curves are

$x=a\cos t$ and $y=b\sin t$

$$\Rightarrow \frac{dx}{dt}=-a\sin t$$

and $$\frac{dy}{dt}=b\cos t$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=b\cos t\times\frac{-1}{a\sin t}=\frac{-b}{a}\cot t$$

So, $$\frac{d^2y}{dx^2}=-\frac{b}{a}\times-\text{cosec}^2t\,\frac{dt}{dx}=\frac{b}{a}\cdot\frac{1}{\sin^2t}\cdot\left\{\frac{-1}{a\sin t}\right\}$$

$$=-\frac{b}{a^2}\cdot\frac{1}{\sin^3t}=-\frac{b}{a^2}\text{cosec}^3t$$

COMMON ERROR

Some students commit errors in $u\times v$ rule derivative function.

Short Answer Type-II Questions

1. LHL $= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{\sin(0-h)}{0-h}$

TR!CK

$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

$$= \lim_{h\to 0}\frac{-\sin h}{-h} = \lim_{h\to 0}\frac{\sin h}{h} = 1$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}(0+h)+1$$

$$= \lim_{h\to 0}(h+1) = 0+1 = 1$$

and $f(0) = 0+1 = 1$

$\because$ LHL $=$ RHL $= f(0)$

i.e., $\lim_{h\to 0} f(0) = f(0)$

Hence, $f(x)$ is a continuous function. **Hence proved.**

2. LHL $= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$

$$= \lim_{h\to 0}\frac{1}{1-e^{-1/h}}$$

$$= \frac{1}{1-0} = 1 \qquad [\because \lim_{0\to 0} e^{-1/0} = 0]$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0}\frac{1}{1-e^{1/h}}$$

$$= \lim_{h\to 0}\frac{e^{-1/h}}{e^{-1/h}-1} = \frac{0}{0-1} = 0$$

We see that LHL $\neq$ RHL

Hence, $f(x)$ is discontinuous at $x = 0$.

3. At $x = 0$,

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}\frac{e^{\frac{-1}{h}}-1}{e^{\frac{-1}{h}}+1}$$

$$= \frac{0-1}{0+1} = -1 \qquad [\because \lim_{0\to 0} e^{-1/0} = 0]$$

Similarly, $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(h) = \lim_{h\to 0}\frac{e^{1/h}-1}{e^{1/h}+1}$

$$= \lim_{h\to 0}\frac{1-\frac{1}{e^{1/h}}}{1+\frac{1}{e^{1/h}}}$$

$$= \lim_{h\to 0}\frac{1-e^{-1/h}}{1+e^{-1/h}} = \frac{1-0}{1+0} = 1$$

$\because \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

Therefore, $\lim_{x\to 0} f(x)$ does not exist.

Hence, f is discontinuous at $x = 0$. **Hence proved.**

4. Given function is $f(x) = \begin{cases} \frac{e^x}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Here, $f(0) = 0$

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0}\frac{e^{-h}}{1+e^{-1/h}}$$

TR!CK

$e^{\infty} = \infty, e^{-\infty} = 0$

$$= \frac{e^{-0}}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

and RHL $= \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$

$$= \lim_{h\to 0}\frac{e^h}{1+e^{1/h}} = \frac{e^0}{1+e^{\infty}} = \frac{1}{\infty} = 0$$

$\because$ LHL $\neq$ RHL $= f(0)$

Hence, given function $f(x)$ is discontinuous at $x = 0$.

Hence proved.

5. Given, $y = (x-1)\log(x-1) - (x+1)\log(x+1)$

Now, differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx} = (x-1)\frac{d}{dx}\log(x-1) + \log(x-1)\frac{d}{dx}(x-1)$$
$$-\left\{(x+1)\frac{d}{dx}\log(x+1) + \log(x+1)\frac{d}{dx}(x+1)\right\}$$

$$\Rightarrow \frac{dy}{dx} = (x-1)\times\frac{1}{(x-1)}\times(1-0) + \log(x-1)\times(1-0)$$
$$-\left\{(x+1)\times\frac{1}{(x+1)}\times(1+0) + \log(x+1)\times(1+0)\right\}$$

$$\Rightarrow \frac{dy}{dx} = 1+\log(x-1) - (1+\log(x+1))$$

$$= 1+\log(x-1) - 1 - \log(x+1)$$

$$= \log(x-1) - \log(x+1)$$

$$= \log\left(\frac{x-1}{x+1}\right)$$ **Hence proved.**

6. Given, $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$ ...(1)

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\sqrt{1-x^2}\frac{d}{dx}(\sin^{-1}x) - \sin^{-1}x\frac{d}{dx}\sqrt{1-x^2}}{\left(\sqrt{1-x^2}\right)^2}$$

$$= \frac{\sqrt{1-x^2}\times\frac{1}{\sqrt{1-x^2}} - \sin^{-1}x\cdot\frac{(-2x)}{2\sqrt{1-x^2}}}{\left(\sqrt{1-x^2}\right)^2}$$

$$= \frac{1+\frac{x\sin^{-1}x}{\sqrt{1-x^2}}}{(1-x^2)} = \frac{1+xy}{(1-x^2)} \qquad \text{[from eq. (1)]}$$

$$\Rightarrow (1-x^2)\frac{dy}{dx} = 1+xy$$ **Hence proved.**

7. Given, $f(x) = x + x^2 + x^4 + x^6 + \ldots + x^{30}$

Differentiate both sides w.r.t. x, we get

$$f'(x) = 1+2x+4x^3+6x^5+\ldots+30x^{29}$$

Put $x = 1$, $f'(1) = 1+2+4+6+\ldots+30$

$$= 1+[2+4+6+\ldots+30]$$

TR!CK

Sum of 15 terms of AP $= \frac{n}{2}\{a+l\} = \frac{15}{2}(2+30)$.

$$= 1+\frac{15}{2}[2+30]$$

$$= 1+\frac{15}{2}\times 32 = 1+240 = 241$$ **Hence proved.**

8. Given, $x+y=t+\frac{1}{t}$

$\therefore \quad (x+y)^3=\left(t+\frac{1}{t}\right)^3$

$\Rightarrow \quad x^3+y^3+3xy(x+y)=t^3+\frac{1}{t^3}+3t\cdot\frac{1}{t}\left(t+\frac{1}{t}\right)$

$\Rightarrow \quad x^3+y^3+3xy(x+y)=x^3+y^3+3\left(t+\frac{1}{t}\right)$

$\left[\because t^3+\frac{1}{t^3}=x^3+y^3\right]$

$\Rightarrow \quad 3xy(x+y)=3(x+y) \qquad \left[\because t+\frac{1}{t}=x+y\right]$

$\Rightarrow \quad xy=1 \qquad [\because x+y\neq 0,$ so divide by $x+y]$

$\Rightarrow \quad y=\frac{1}{x}=x^{-1}$

Differentiate both sides w.r.t x, we get

$$\frac{dy}{dx}=(-1)\,x^{-2}=-\frac{1}{x^2}$$ **Hence proved.**

9. Given function can be written as:

$$y=x^2+\frac{1}{y}$$

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx}=2x-\frac{1}{y^2}\cdot\frac{dy}{dx}$$

or $\quad \frac{dy}{dx}\left(1+\frac{1}{y^2}\right)=2x$

or $\quad \frac{dy}{dx}\cdot\frac{(y^2+1)}{y^2}=2x$

$\therefore \quad \frac{dy}{dx}=\frac{2xy^2}{1+y^2}$ **Hence proved.**

10. Given, $\cos y=x\cos(a+y)$

TiP

Differentiation rule for different functions and forms need continuous revision practice.

$\Rightarrow \quad x=\frac{\cos y}{\cos(a+y)}$

Differentiate both sides w.r.t. y, we get

$$\frac{dx}{dy}=\frac{\left[\cos(a+y)\frac{d}{dy}(\cos y)-\cos y\frac{d}{dy}\cos(a+y)\right]}{\cos^2(a+y)}$$

$$=\frac{[-\sin y\cos(a+y)+\cos y\sin(a+y)]}{\cos^2(a+y)}$$

$$=\frac{[\sin(a+y)\cos y-\cos(a+y)\sin y]}{\cos^2(a+y)}$$

TR!CK

$\sin(A-B)=\sin A\cdot\cos B-\cos A\cdot\sin B$

$$=\frac{\sin(a+y-y)}{\cos^2(a+y)}$$

$\Rightarrow \quad \frac{dx}{dy}=\frac{\sin a}{\cos^2(a+y)}$

$\therefore \quad \frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$ **Hence proved.**

COMMON ERR!R

Some students commit errors in $\frac{u}{v}$ rule for differentiating the function.

11. We have, $y=\sqrt{\sin x+\cos x+y}$

$\Rightarrow \quad y^2=\sin x+\cos x+y$

Differentiate both sides w.r.t. x, we get

$$2y\frac{dy}{dx}=\cos x-\sin x+\frac{dy}{dx}$$

or $\quad 2y\frac{dy}{dx}-\frac{dy}{dx}=\cos x-\sin x$

or $\quad (2y-1)\frac{dy}{dx}=\cos x-\sin x$ or $\frac{dy}{dx}=\frac{\cos x-\sin x}{2y-1}$

12. Given, $y=\sqrt{\cos x+\sqrt{\cos x+\sqrt{\cos x+\ldots\infty}}}$

$\Rightarrow \quad y=\sqrt{\cos x+y}$

or $\quad y^2=\cos x+y$

Differentiate both sides w.r.t. x, we get

$$\frac{d}{dx}(y^2)=\frac{d}{dx}(\cos x+y)=\frac{d}{dx}(\cos x)+\frac{dy}{dx}$$

or $\quad 2y\frac{dy}{dx}=-\sin x+\frac{dy}{dx}$

or $\quad (2y-1)\frac{dy}{dx}=-\sin x\Rightarrow\frac{dy}{dx}=\frac{-\sin x}{2y-1}$

13. Given, $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\ldots\infty}}}=\sqrt{\sin x+y}$

$\Rightarrow \quad y^2=\sin x+y$

Differentiate both sides w.r.t. x, we get

$$2y\cdot\frac{dy}{dx}=\cos x+\frac{dy}{dx}$$

$\Rightarrow \quad (2y-1)\frac{dy}{dx}=\cos x$ **Hence proved.**

14. Let $u=\tan^{-1}\frac{2x}{1-x^2}$ and $v=\cos^{-1}\frac{1-x^2}{1+x^2}$

Then, $\quad u=2\tan^{-1}x$

and $\quad v=2\tan^{-1}x$

$\Rightarrow \quad \frac{du}{dx}=\frac{2}{1+x^2}$ and $\frac{dv}{dx}=\frac{2}{1+x^2}$

$\therefore \quad \frac{du}{dv}=\frac{du}{dx}\div\frac{dv}{dx}=\frac{2}{1+x^2}\div\frac{2}{1+x^2}=1$

15. Given, $y=\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\tan^{-1}x$

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\tan^{-1}x\right\}$$

$$=\frac{d}{dx}\left\{\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\}+\frac{d}{dx}\tan^{-1}x$$

$$=\frac{d}{dx}(2\tan^{-1}x)+\frac{d}{dx}\tan^{-1}x$$

$$=2\cdot\frac{1}{1+x^2}+\frac{1}{1+x^2}=\frac{3}{1+x^2}$$

16. Given, $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$

TIP
Differential coefficient of constant term is zero.

Put $x=\sin\theta$

$\therefore$ $y=\sin^{-1}(\sin\theta)+\sin^{-1}\sqrt{1-\sin^2\theta}$

$[\because \sin^2\theta+\cos^2\theta=1]$

$\Rightarrow$ $y=\theta+\sin^{-1}(\cos\theta)$

$$=\theta+\sin^{-1}\left[\sin\left(\frac{\pi}{2}-\theta\right)\right]$$

$[\because \sin^{-1}(\sin\theta)=\theta]$

$$=\theta+\frac{\pi}{2}-\theta=\frac{\pi}{2}$$

$\therefore$ $\dfrac{dy}{dx}=0$

17. Given, $y=\sin^{-1}\{x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\}$, where $0<x<1$.

Put $x=\sin A$ and $\sqrt{x}=\sin B$

$\therefore$ $y=\sin^{-1}\{\sin A\sqrt{1-\sin^2 B}-\sin B\sqrt{1-\sin^2 A}\}$

$=\sin^{-1}\{\sin A\cdot\cos B-\sin B\cdot\cos A\}$

$=\sin^{-1}\sin(A-B)=A-B$ $\quad[\because \sin^{-1}(\sin\theta)=\theta]$

$=\sin^{-1}x-\sin^{-1}\sqrt{x}$

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x}}\cdot\frac{d}{dx}\sqrt{x}$$

$$=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}\cdot\frac{1}{2\sqrt{x}}$$

18. Let $u=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ and $v=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$\Rightarrow$ $u=2\tan^{-1}x$

and $v=2\tan^{-1}x$

Differentiate both sides w.r.t. x, we get

$$\frac{du}{dx}=\frac{2}{1+x^2}\quad\text{and}\quad\frac{dv}{dx}=\frac{2}{1+x^2}$$

$$\frac{du}{dv}=\frac{du}{dx}\times\frac{dx}{dv}$$

$$=\frac{2}{1+x^2}\times\frac{1+x^2}{2}=1$$

19. Let $x=\sin\theta \Rightarrow \theta=\sin^{-1}x$

$\therefore$ $y=\tan^{-1}\left[\dfrac{x}{1+\sqrt{1-x^2}}\right]=\tan^{-1}\left(\dfrac{\sin\theta}{1+\sqrt{1-\sin^2\theta}}\right)$

$=\tan^{-1}\left(\dfrac{\sin\theta}{1+\sqrt{\cos^2\theta}}\right)$ $\quad[\because \sin^2\theta+\cos^2\theta=1]$

$$=\tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

$$=\tan^{-1}\left(\frac{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right)$$

$\left[\because \sin\theta=2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\text{ and }\cos\theta=2\cos^2\frac{\theta}{2}-1\right]$

$$=\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$[\because \tan^{-1}(\tan\theta)=\theta]$

$=\dfrac{1}{2}\theta=\dfrac{1}{2}\sin^{-1}x$ $\quad[\because \theta=\sin^{-1}(x)]$

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{2}\cdot\frac{d}{dx}(\sin^{-1}x)=\frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}$$

Hence, $\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{1-x^2}}$

20. Let $x=\tan\theta\Rightarrow\theta=\tan^{-1}x$

Then, $y=\cot^{-1}\left(\dfrac{\sqrt{1+\tan^2 x}+1}{\tan\theta}\right)$

$\Rightarrow$ $y=\cot^{-1}\left(\dfrac{\sec\theta+1}{\tan\theta}\right)$ $\quad[\because 1+\tan^2\theta=\sec^2\theta]$

$\Rightarrow$ $y=\cot^{-1}\left(\dfrac{\frac{1}{\cos\theta}+1}{\frac{\sin\theta}{\cos\theta}}\right)$

$\Rightarrow$ $y=\cot^{-1}\left(\dfrac{1+\cos\theta}{\sin\theta}\right)$

$\Rightarrow$ $y=\cot^{-1}\left(\dfrac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}\right)$

$\left[\because \cos\theta=2\cos^2\frac{\theta}{2}-1\text{ and }\sin\theta=2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\right]$

$\Rightarrow$ $y=\cot^{-1}\left(\dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)=\cot^{-1}\left(\cot\dfrac{\theta}{2}\right)$

$[\because \cot^{-1}(\cot\theta)=\theta]$

$=\dfrac{\theta}{2}=\dfrac{1}{2}\tan^{-1}x$ $\quad[\because \theta=\tan^{-1}x]$

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{2}\times\frac{d}{dx}(\tan^{-1}x)=\frac{1}{2}\cdot\frac{1}{1+x^2}$$

$\therefore$ $\dfrac{dy}{dx}=\dfrac{1}{2(1+x^2)}$

21. Let $u=\sin^{-1}\left[\dfrac{1-x}{1+x}\right]$ and $v=\sqrt{x}$

TIP
Convert inverse trigonometric functions to simplest form before finding the derivatives.

Now, put $x=\tan^2\theta$

$\therefore$ $u=\sin^{-1}\left[\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right]=\sin^{-1}(\cos 2\theta)$

$=\sin^{-1}\left[\sin\left\{\dfrac{\pi}{2}-2\theta\right\}\right]$ $\quad[\because \sin^{-1}(\sin\theta)=\theta]$

$$=\frac{\pi}{2}-2\theta=\frac{\pi}{2}-2\tan^{-1}\sqrt{x}$$

$\therefore \quad \frac{du}{dx} = \frac{-2}{1+(\sqrt{x})^2} \times \frac{d}{dx}(\sqrt{x}) = \frac{-2}{1+x} \times \frac{1}{2\sqrt{x}}$

and $\quad \frac{dv}{dx} = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

Therefore, $\frac{du}{dv} = \frac{du}{dx} \div \frac{dv}{dx} = \frac{-2}{1+x} \times \frac{1}{2\sqrt{x}} \times 2\sqrt{x} = \frac{-2}{1+x}$.

COMMON ERR(!)R

Some students directly apply the formula for derivative of $\sin^{-1} x$ and apply chain rule.

22. Given, $\quad y = e^{\sin^{-1} x^2}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(e^{\sin^{-1} x^2}) = e^{\sin^{-1} x^2} \times \frac{d}{dx}(\sin^{-1} x^2)$$

$$= e^{\sin^{-1} x^2} \times \frac{1}{\sqrt{1-(x^2)^2}} \times \frac{d}{dx}(x^2)$$

$$= e^{\sin^{-1} x^2} \times \frac{1}{\sqrt{1-x^4}} \times 2x = \frac{2xe^{\sin^{-1} x^2}}{\sqrt{1-x^4}}$$

23. Let $u = 4e^{\sin^{-1} x} + \frac{\pi}{2}$ and $v = 5\sin^{-1} x + \frac{\pi}{4}$

Differentiate both sides w.r.t. x, we get

$$\frac{du}{dx} = 4\frac{d}{dx}e^{\sin^{-1} x} + 0 \text{ and } \frac{dv}{dx} = 5\frac{d}{dx}\sin^{-1} x + 0$$

$$\Rightarrow \quad \frac{du}{dx} = 4 \cdot e^{\sin^{-1} x} \cdot \frac{d}{dx}\sin^{-1} x \text{ and } \frac{dv}{dx} = 5 \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{4 \cdot e^{\sin^{-1} x}}{\sqrt{1-x^2}} \text{ and } \frac{dv}{dx} = \frac{5}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv} = \frac{4 \cdot e^{\sin^{-1} x}}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{5}$$

$$= \frac{4}{5}e^{\sin^{-1} x}$$

24. Let $y = x^{\tan x}$, then $\log y = \tan x \log x$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \tan x \cdot \frac{1}{x} + \log x \cdot \sec^2 x$$

$$\therefore \quad \frac{dy}{dx} = y\left(\frac{\tan x}{x} + \log x \cdot \sec^2 x\right)$$

or $\quad \frac{dy}{dx} = x^{\tan x}\left(\frac{\tan x}{x} + \log x \cdot \sec^2 x\right)$

25. Given, $y = x^{x^{x^{\cdots\infty}}}$

$\therefore \quad y = x^y \quad \Rightarrow \quad \log y = y \log x$

[taking logarithm on both sides]

Differentiate both sides w.r.t. x, we get

$$\frac{1}{y} \times \frac{dy}{dx} = y \times \frac{d}{dx}(\log x) + \log x \times \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{y} \times \frac{dy}{dx} - \log x \times \frac{dy}{dx} = y \times \frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{1}{y} - \log x\right] = \frac{y}{x} \Rightarrow \frac{dy}{dx}\left[\frac{1-y\log x}{y}\right] = \frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2}{x(1-y\log x)}.$$

26. Given, $y = (\sin x)^{(\sin x)^{(\sin x)\cdots\infty}} = (\sin x)^y$

Taking logarithm on both sides, we get

$$\log y = \log(\sin x)^y = y \log \sin x$$

$$\Rightarrow \quad \frac{d}{dx}(\log y) = \frac{d}{dx}(y \log \sin x)$$

$$\Rightarrow \quad \frac{d}{dy}(\log y) \times \frac{dy}{dx} = y \times \frac{d}{dx}(\log \sin x) + (\log \sin x) \times \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{y} \times \frac{dy}{dx} = y \times \frac{1}{\sin x} \times \frac{d}{dx}(\sin x) + (\log \sin x) \times \frac{dy}{dx}$$

$$\Rightarrow \left[\frac{1}{y} - \log \sin x\right] \times \frac{dy}{dx} = y \times \frac{1}{\sin x} \times \cos x$$

$$\Rightarrow \quad \frac{1 - y\log \sin x}{y} \times \frac{dy}{dx} = y \cot x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 \cot x}{1 - y \log \sin x}$$

27. Given, $y^x = x^y$

Taking logarithm on both sides, we get

$\log y^x = \log x^y \Rightarrow x \log y = y \log x$

Differentiate both sides w.r.t. x, we get

$$x\frac{d}{dx}(\log y) + \log y \frac{d}{dx}(x) = y\frac{d}{dx}(\log x) + \log x \frac{dy}{dx}$$

$$\Rightarrow \quad x \times \frac{1}{y} \cdot \frac{dy}{dx} + \log y \times 1 = y \times \frac{1}{x} + \log x \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{x}{y} \cdot \frac{dy}{dx} + \log y = \frac{y}{x} + \log x \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{x}{y} \cdot \frac{dy}{dx} - \log x \frac{dy}{dx} = \frac{y}{x} - \log y$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{x}{y} - \log x\right) = \frac{y}{x} - \log y$$

Multiply by xy, $\frac{dy}{dx}(x^2 - xy\log x) = y^2 - xy\log y$

$$\therefore \quad \frac{dy}{dx} = \frac{y^2 - xy\log y}{x^2 - xy\log x}$$

28. Given, $x = a(t + \sin t)$ and $y = a(1 + \cos t)$

Differentiate both sides w.r.t. x, we get

$$\frac{dx}{dt} = a\{1 + \cos t\}$$

and $\quad \frac{dy}{dt} = a(-\sin t) = -a\sin t$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = (-a\sin t) \times \frac{1}{a(1+\cos t)}$$

$$= -\frac{\sin t}{1+\cos t}$$

$$= \frac{2\sin t/2 \cdot \cos t/2}{1 + 2\cos^2 t/2 - 1}$$

$$\left[\because \sin\theta = 2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2} \text{ and } \cos\theta = 2\cos^2\frac{\theta}{2} - 1\right]$$

$$= -\frac{\sin t/2}{\cos t/2}$$

$$\therefore \quad \frac{dy}{dx} = -\tan\frac{t}{2}$$

COMMON ERR(!)R

Some students commit error in $u \times v$ rule derivative function.

29. Given, $x = a(\cos t + t\sin t)$

and $y = a(\sin t - t\cos t)$

Differentiate both sides w.r.t. t, we get

$$\frac{dx}{dt} = a(-\sin t + t\cdot\cos t + \sin t) = at\cos t$$

and $$\frac{dy}{dt} = a[\cos t - \{t(-\sin t) + \cos t\}$$
$$= a[\cos t + t\sin t - \cos t] = a\,t\sin t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\,t\sin t}{a\,t\cos t} = \tan t$$

Again differentiate both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$
$$= \frac{d}{dt}(\tan t) \times \frac{dt}{dx}$$
$$= \sec^2 t \times \frac{1}{a\,t\cos t} \qquad \left[\because \frac{dx}{dt} = a\,t\cos t\right]$$
$$= \frac{\sec^3 t}{at}$$

COMMON ERR(!)R

Some students make mistake while finding second derivative of parametric functions.

30. Given that, $y = \sec x + \tan x$

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sec x) + \frac{d}{dx}(\tan x)$$
$$= \sec x \cdot \tan x + \sec^2 x = \sec x \cdot (\tan x + \sec x)$$

Again differentiate both sides w.r.t. x, we get

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \sec x \times \frac{d}{dx}(\tan x + \sec x) + (\tan x + \sec x) \times \frac{d}{dx}(\sec x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec x \times (\sec^2 x + \sec x \cdot \tan x) + (\tan x + \sec x) \times (\sec x \cdot \tan x)$$
$$= \sec x \cdot \sec x \cdot (\sec x + \tan x) + (\sec x + \tan x) \times (\sec x \cdot \tan x)$$
$$= \sec^2 x \cdot (\sec x + \tan x) + (\sec x + \tan x) \cdot \sec x \cdot \tan x$$
$$= \sec x\,(\tan x + \sec x)(\sec x + \tan x)$$
$$= \sec x \cdot (\sec x + \tan x)^2$$
$$= \frac{1}{\cos x} \times \left[\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right]^2$$
$$= \frac{1}{\cos x} \times \left[\frac{1 + \sin x}{\cos x}\right]^2 = \frac{\cos x}{\cos^2 x} \times \frac{(1 + \sin x)^2}{\cos^2 x}$$
$$= \frac{\cos x \cdot (1 + \sin x)^2}{(\cos^2 x)^2} = \frac{\cos x \cdot (1 + \sin x)^2}{(1 - \sin^2 x)^2}$$
$$[\because \sin^2\theta + \cos^2\theta = 1]$$
$$= \frac{\cos x \cdot (1 + \sin x)^2}{(1 - \sin x)^2 (1 + \sin x)^2} = \frac{\cos x}{(1 - \sin x)^2}$$

Hence proved.

31. Given, $(a + bx)e^{\frac{y}{x}} = x$

Taking logarithm on both sides,

$$\log(a + bx) + \log e^{\frac{y}{x}} = \log x$$
$$\Rightarrow \quad \log(a + bx) + \frac{y}{x} = \log x$$
$$\Rightarrow \quad \frac{y}{x} = \log\frac{x}{(a + bx)}$$
$$\Rightarrow \quad y = x\log\left(\frac{x}{a + bx}\right) \qquad \ldots(1)$$

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx} = 1 \times \log\left(\frac{x}{a + bx}\right) + \frac{x}{x/(a + bx)}\left[\frac{(a + bx) - xb}{(a + bx)^2}\right]$$
$$\Rightarrow \frac{dy}{dx} = \log\left(\frac{x}{a + bx}\right) + \frac{a}{(a + bx)}$$
$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{a}{(a + bx)} \qquad \ldots(2)$$

[$\because$ from eq. (1)]

Again differentiate both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{x\frac{dy}{dx} - y}{x^2} - \frac{ab}{(a + bx)^2}$$
$$= \frac{1}{x}\left[\frac{dy}{dx} - \frac{y}{x}\right] - \frac{ab}{(a + bx)^2}$$
$$= \frac{1}{x}\left[\frac{a}{(a + bx)}\right] - \frac{ab}{(a + bx)^2}$$

[$\because$ from eq. (2)]

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{a^2 + abx - abx}{x(a + bx)^2} = \frac{a^2}{x(a + bx)^2}$$
$$\Rightarrow \quad x\frac{d^2y}{dx^2} = \left(\frac{a}{a + bx}\right)^2$$

Hence proved.

Long Answer Type Questions

1.

TiP

Find LHL (at $x = 2$ and $x = 10$), RHL (at $x = 2$ and $x = 10$), $f(2)$ and $f(10)$ and using these relation to obtain the values of a and b.

Here, $f(x) = \begin{cases} 5, & \text{if} \quad x \le 2 \\ ax + b, & \text{if} \quad 2 < x < 10 \\ 21, & \text{if} \quad x \ge 10 \end{cases}$

At $x = 2$, $f(x) = 5$

$$\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 5 = 5$$
$$\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax + b) = 2a + b$$

and $f(2) = 5$

f is continuous at $x = 2$, if

$$\text{LHL} = \text{RHL} = f(2)$$
$$\therefore \quad 2a + b = 5 \qquad \ldots(1)$$

At $x = 10$, $f(x) = ax + b$

$$\text{LHL} = \lim_{x \to 10^-} f(x) = \lim_{x \to 10^-} (ax + b) = 10a + b$$

$$\text{RHL} = \lim_{x\to 10^+} f(x) = \lim_{x\to 10^+}(21) = 21$$

and $f(10) = 21$

f is continuous at $x = 10$, if

$$\text{LHL} = \text{RHL} = f(10)$$

$$\Rightarrow \quad 10a + b = 21 \quad \ldots(2)$$

Subtracting eq. (1) from eq. (2), we get

$$8a = 21 - 5 = 16$$

$$\therefore \quad a = \frac{16}{8} = 2$$

From eq. (1),

$$2\times 2 + b = 5$$

$$\therefore \quad b = 5 - 4 = 1$$

Hence, f is a continuous function if $a = 2$, $b = 1$.

COMMON ERROR

Many students commit errors in finding the LHL and RHL.

2. Given, $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

Then, $\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x^2\sin\frac{1}{x}\right)$

$$= \lim_{h\to 0}(-h)^2\sin\frac{1}{-h} = -\lim_{h\to 0} h^2\left(\sin\frac{1}{h}\right) = 0$$

$$\left[\because h \to 0, -1 < \sin\frac{1}{h} < 1;\ \text{as } h\to 0, h^2\sin\frac{1}{h} \to 0\right]$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2\sin\frac{1}{x}\right)$$

$$\text{RHL} = \lim_{h\to 0}(0+h)^2\sin\left(\frac{1}{0+h}\right)$$

$$= \lim_{h\to 0}\left(h^2\sin\frac{1}{h}\right) = 0$$

$$\left[\because h \to 0, -1 < \sin\frac{1}{h} < 1;\ \text{as } h\to 0, h^2\sin\frac{1}{h} \to 0\right]$$

and $\quad f(0) = 0$

$\therefore \quad \text{LHL} = \text{RHL} = f(0)$

$\therefore$ At $x = 0$, f is continuous.

At $x = c \neq 0$,

$$\lim_{x\to c} x^2\sin\frac{1}{x} = c^2\sin\frac{1}{c} = f(c)$$

$\Rightarrow$ At $x = c \neq 0$, f is continuous.

Hence, f is continuous at $x \in R$.

3. Given, $f(x) = \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}$, $x \neq \dfrac{\pi}{4}$

Therefore, $\lim_{x\to\frac{\pi}{4}} f(x) = \lim_{x\to\frac{\pi}{4}} \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}$

$$= \lim_{x\to\frac{\pi}{4}} \frac{(\sqrt{2}\cos x - 1)\sin x}{(\cos x - \sin x)}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{(\sqrt{2}\cos x - 1)}{(\cos x - \sin x)}\cdot\frac{(\sqrt{2}\cos x + 1)}{(\sqrt{2}\cos x + 1)}\cdot\frac{(\cos x + \sin x)}{(\cos x + \sin x)}\cdot\sin x$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{2\cos^2 x - 1}{\cos^2 x - \sin^2 x}\cdot\frac{\cos x + \sin x}{(\sqrt{2}\cos x + 1)}\cdot\sin x$$

$$[\because (a-b)(a+b) = a^2 - b^2]$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\cos 2x}{\cos 2x}\cdot\frac{\cos x + \sin x}{(\sqrt{2}\cos x + 1)}\cdot\sin x$$

$$[\because \cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta]$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\cos x + \sin x}{(\sqrt{2}\cos x + 1)}\cdot\sin x$$

$$= \left(\frac{\cos\frac{\pi}{4} + \sin\frac{\pi}{4}}{\sqrt{2}\cos\frac{\pi}{4} + 1}\right)\sin\frac{\pi}{4} = \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{\sqrt{2}\times\frac{1}{\sqrt{2}} + 1}\cdot\frac{1}{\sqrt{2}}$$

$$= \frac{2/\sqrt{2}}{1+1}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}\times\frac{1}{\sqrt{2}} = \frac{1}{2}$$

So, $\lim_{x\to\frac{\pi}{4}} f(x) = \frac{1}{2}$, if we define $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$ then $f(x)$ becomes continuous at $\frac{\pi}{4}$.

Hence, $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$ for the continuity of f at $x = \frac{\pi}{4}$

4. We have, $\log(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$

TiPS

- $\dfrac{d}{dx}\{\log x\} = \dfrac{1}{x}$
- $\dfrac{d}{dx}\{\tan^{-1} x\} = \dfrac{1}{1+x^2}$

Differentiate both sides w.r.t. x, we get

$$\frac{1}{x^2+y^2}\cdot\frac{d}{dx}(x^2+y^2) = 2\,\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{1}{x^2+y^2}\left\{2x + 2y\frac{dy}{dx}\right\} = \frac{2x^2}{x^2+y^2}\cdot\left\{\frac{x\cdot\frac{dy}{dx} - y\cdot 1}{x^2}\right\}$$

$$\Rightarrow \quad \frac{2x}{x^2+y^2} + \frac{2y}{x^2+y^2}\cdot\frac{dy}{dx} = \frac{2x}{x^2+y^2}\cdot\frac{dy}{dx} - \frac{2y}{x^2+y^2}$$

$$\Rightarrow \quad \left\{\frac{2y}{x^2+y^2} - \frac{2x}{x^2+y^2}\right\}\cdot\frac{dy}{dx} = -\frac{2x}{x^2+y^2} - \frac{2y}{x^2+y^2}$$

$$\Rightarrow \quad \frac{2(x-y)}{x^2+y^2}\cdot\frac{dy}{dx} = \frac{2(x+y)}{x^2+y^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2+y^2}{2(x-y)}\times\frac{2(x+y)}{x^2+y^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{x+y}{x-y}$$

Hence proved.

5. Given, $x\sqrt{1+y} + y\sqrt{1+x} = 0 \Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides, we get

$$x^2(1+y) = y^2(1+x)$$

$$\Rightarrow \quad x^2 + x^2y = y^2 + y^2x$$

$$\Rightarrow \quad x^2 - y^2 - y^2x + x^2y = 0$$

$$\Rightarrow \quad (x-y)(x+y) + xy(x-y) = 0$$

$$\Rightarrow \quad (x-y)[x+y+xy] = 0$$

If $x - y = 0 \Rightarrow y = x$

[but $y = x$ does not satisfy the equation]

$\Rightarrow \quad x + y(1+x) = 0$

$\Rightarrow \quad y = -\dfrac{x}{1+x}$

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx} = -\frac{(1+x)(1) - x\cdot(1+0)}{(1+x)^2}$$

$$= -\frac{1+x-x}{(1+x)^2} = -\frac{1}{(1+x)^2}$$

Hence proved.

COMMON ERR(!)R

Most students differentiate directly and fail to reach the final answer.

6. Let $u = \tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$ and $v = \tan^{-1} x$

Now, put $x = \tan A$

TiP

Learn all the substitutions made in inverse trigonometric functions by heart.

$$\therefore \quad u = \tan^{-1}\left[\frac{\sqrt{1+\tan^2 A}-1}{\tan A}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{\sec^2 A}-1}{\tan A}\right] \quad [\because 1+\tan^2\theta = \sec^2\theta]$$

$$= \tan^{-1}\left[\frac{\sec A - 1}{\tan A}\right] = \tan^{-1}\left[\frac{1-\cos A}{\sin A}\right]$$

$$\left[\because \cos\theta = 1 - 2\sin^2\frac{\theta}{2} \text{ and } \sin\theta = 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\right]$$

$$= \tan^{-1}\left[\frac{2\sin^2\frac{A}{2}}{2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}\right] = \tan^{-1}\left[\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}\right]$$

$$= \tan^{-1}\left[\tan\frac{A}{2}\right] \quad [\because \tan^{-1}(\tan\theta) = \theta]$$

$$\Rightarrow \quad u = \frac{A}{2} = \frac{1}{2}\tan^{-1} x \quad [\because x = \tan A \Rightarrow A = \tan^{-1} x]$$

Differentiate both sides w.r.t. x, we get

$$\frac{du}{dx} = \frac{d}{dx}\left[\frac{1}{2}\cdot\tan^{-1} x\right] = \frac{1}{2}\times\frac{d}{dx}(\tan^{-1} x)$$

$$= \frac{1}{2}\times\frac{1}{1+x^2} = \frac{1}{2(1+x^2)}$$

and $\quad \dfrac{dv}{dx} = \dfrac{1}{1+x^2}$

$$\therefore \quad \frac{du}{dv} = \frac{du}{dx} \div \frac{dv}{dx}$$

$$= \frac{1}{2(1+x^2)} \div \frac{1}{1+x^2} = \frac{1}{2}$$

7. Let $u = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$ and $v = \cos^{-1}(2x\sqrt{1-x^2})$

Now, put $x = \sin\theta$ in $u = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$, where $\theta \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$.

$$\therefore \quad u = \tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right) = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \tan^{-1}(\cot\theta) = \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\theta\right)\right\} = \frac{\pi}{2} - \theta$$

$$[\because \tan^{-1}(\tan\theta) = \theta]$$

$$= \frac{\pi}{2} - \sin^{-1} x$$

$$\therefore \quad \frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

Now, put $x = \sin\theta$ in $v = \cos^{-1}(2x\sqrt{1-x^2})$

$$\therefore \quad v = \cos^{-1}(2x\sqrt{1-x^2})$$

$$= \cos^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$$

$$= \cos^{-1}\{2\sin\theta\cdot\cos\theta\}$$

$$= \cos^{-1}(\sin 2\theta) \quad [\because \sin 2\theta = \sin 2\theta \cos 2\theta]$$

$$= \cos^{-1}\left\{\cos\left(\frac{\pi}{2} - 2\theta\right)\right\} \quad [\because \cos^{-1}(\cos\theta) = \theta]$$

where $2\theta \in \left(\dfrac{\pi}{2}, \pi\right)$

$$= \frac{\pi}{2} - 2\theta = \frac{\pi}{2} + 2\sin^{-1} x$$

$$\therefore \quad \frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$$

Therefore, $\dfrac{dv}{du} = \dfrac{dv}{dx}\times\dfrac{dx}{du} = \dfrac{2}{\sqrt{1-x^2}}\times -\sqrt{1-x^2} = -2.$

8. Given, $x^y = e^{x-y}$

Taking logarithm on both sides,

$$\log(x^y) = \log(e^{x-y})$$

$$\Rightarrow \quad y\log x = (x-y)\log e = x - y \quad \ldots(1)$$

Differentiate both sides w.r.t. x, we get

$$\frac{d}{dx}(y\log x) = \frac{d}{dx}(x-y)$$

$$\Rightarrow \quad y\frac{d}{dx}\log x + \log x\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \quad y\cdot\frac{1}{x} + \log x\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \quad (1+\log x)\frac{dy}{dx} = 1 - \frac{y}{x} = \frac{x-y}{x}$$

$$\Rightarrow \quad (1+\log x)\frac{dy}{dx} = \frac{y\log x}{x} \quad \text{[from eq. (1)]}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log x}{(1+\log x)\cdot\left(\frac{x}{y}\right)}$$

$$\left[\because x - y = y\log x \Rightarrow x = y(1+\log x) \Rightarrow \frac{x}{y} = (1+\log x)\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

Hence proved.

9. Given, $y = (\tan x)^{\tan x \ldots \infty} \Rightarrow y = (\tan x)^y$

Taking logarithm on both sides,

$\log y = \log(\tan x)^y = y\log(\tan x)$

Differentiate both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = y\cdot\frac{1}{\tan x}\cdot\frac{d}{dx}(\tan x) + \log(\tan x)\cdot\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y} - \log \tan x\right)\frac{dy}{dx} = \frac{y}{\tan x}\cdot \sec^2 x$$

$$\Rightarrow (1 - y\log \tan x)\frac{dy}{dx} = \frac{y^2 \sec^2 x}{\tan x}$$

$$\Rightarrow (1 - y\log \tan x)\frac{dy}{dx} = y^2 \cdot \frac{1}{\cos^2 x}\cdot \frac{\cos x}{\sin x}$$

$$\Rightarrow (1 - y\log \tan x)\frac{dy}{dx} = \frac{2y^2}{2\sin x \cdot \cos x} = \frac{2y^2}{\sin 2x}$$

$[\because \sin 2\theta = 2\sin\theta\cos\theta]$

$$\Rightarrow \frac{dy}{dx} = \frac{2y^2 \operatorname{cosec} 2x}{(1 - y\log \tan x)}$$ **Hence proved.**

10. Let $u = \sqrt{3x+2}$, $v = \dfrac{1}{\sqrt{2x^2+4}}$

and $w = (\cos x)^{\tan x}$

Then, $y = u + v + w$

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} \quad ...(1)$$

$\because$ $u = \sqrt{3x+2}$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(3x+2)^{1/2} = \frac{3}{2\sqrt{3x+2}}$$

$\because$ $v = \dfrac{1}{\sqrt{2x^2+4}}$

$$\Rightarrow \frac{dv}{dx} = \frac{d}{dx}(2x^2+4)^{-1/2} = \frac{-2x}{(2x^2+4)^{3/2}}$$

and $w = (\cos x)^{\tan x}$

Taking logarithm on both sides,

$$\log w = \log(\cos x)^{\tan x} = \tan x \cdot \log \cos x$$

Differentiate both sides w.r.t. x, we get

$$\frac{1}{w}\cdot\frac{dw}{dx} = \tan x \cdot \frac{d}{dx}\log\cos x + \log\cos x \cdot \frac{d}{dx}\tan x$$

$$= \tan x \cdot \frac{1}{\cos x}\cdot(-\sin x) + \log\cos x \cdot \sec^2 x$$

$$= -\tan^2 x + \sec^2 x \cdot \log\cos x$$

$$\Rightarrow \frac{dw}{dx} = w(-\tan^2 x + \sec^2 x \cdot \log\cos x)$$

$$= (\cos x)^{\tan x}(-\tan^2 x + \sec^2 x \cdot \log\cos x)$$

Put the values of $\frac{du}{dx}, \frac{dv}{dx}$ and $\frac{dw}{dx}$ in eq. (1), we get

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3x+2}} - \frac{2x}{(2x^2+4)^{3/2}} + (\cos x)^{\tan x}(-\tan^2 x + \sec^2 x \cdot \log\cos x)$$

11. Given that, $y = x^x + (\cos x)^{\tan x} = u + v$

where, $u = x^x$ and $v = (\cos x)^{\tan x}$

TiP

Differentiation rules for different functions and forms need continuous revision and practice.

Then, $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(1)$$

Now, $u = x^x$

[taking logarithm on both sides]

$$\Rightarrow \log u = \log x^x \Rightarrow \log u = x\log x$$

Differentiate both sides w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = x\frac{d}{dx}\log x + \log x\frac{d}{dx}x$$

$$\Rightarrow \frac{du}{dx} = u\left[x\cdot\frac{1}{x} + \log x \times 1\right]$$

$$= x^x(1 + \log x)$$

Again, $v = (\cos x)^{\tan x}$

[taking logarithm on both sides]

$$\Rightarrow \log v = \log(\cos x)^{\tan x}$$

$$= \tan x \log\cos x$$

Differentiate both sides w.r.t. x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx} = \tan x\frac{d}{dx}(\log\cos x) + \log\cos x\frac{d}{dx}(\tan x)$$

$$= \tan x \cdot \frac{1}{\cos x}(-\sin x) + \log\cos x(\sec^2 x)$$

$$= \tan x\cdot(-\tan x) + \sec^2 x \cdot \log\cos x$$

$$\Rightarrow \frac{dv}{dx} = v(-\tan^2 x + \sec^2 x \cdot \log\cos x)$$

$$= (\cos x)^{\tan x}\{-\tan^2 x + \sec^2 x \cdot \log\cos x\}$$

Put the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in eq. (1), we get

$$\frac{dy}{dx} = x^x(1 + \log x) + (\cos x)^{\tan x}\{-\tan^2 x + \sec^2 x \cdot \log\cos x\}$$

12. Let $y = (\sin x)^{\cos x} + (\cos x)^{\sin x} = u + v$

TiP

Differentiation rules for different functions and forms need continuous revision and practice.

where, $u = (\sin x)^{\cos x}$ and $v = (\cos x)^{\sin x}$

Then, $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad [\because y = u + v] ...(1)$$

Now, $u = (\sin x)^{\cos x}$

[taking logarithm on both sides]

$$\Rightarrow \log u = \log(\sin x)^{\cos x}$$

$$\Rightarrow \log u = \cos x \log\sin x$$

Differentiate both sides w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = \cos x\cdot\frac{d}{dx}(\log\sin x) + \log\sin x\frac{d}{dx}(\cos x)$$

$$= \cos x \cdot \frac{1}{\sin x}\cdot\cos x + \log\sin x\cdot(-\sin x)$$

$$\therefore \frac{du}{dx} = u\left[\frac{\cos^2 x}{\sin x} - \sin x\log\sin x\right]$$

$$= (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x\log\sin x\right]$$

Again, $v = (\cos x)^{\sin x}$

[taking logarithm on both sides]

$$\Rightarrow \log v = \log(\cos x)^{\sin x}$$

$$\Rightarrow \log v = \sin x\log\cos x$$

Differentiate both sides w.r.t. x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx} = \sin x\frac{d}{dx}(\log\cos x) + \log\cos x\frac{d}{dx}(\sin x)$$

$$= \sin x\cdot\frac{1}{\cos x}\cdot(-\sin x) + \log\cos x\cdot\cos x$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left[-\frac{\sin^2 x}{\cos x} + \cos x \cdot \log \cos x\right]$$

$$= (\cos x)^{\sin x}\left[-\frac{\sin^2 x}{\cos x} + \cos x \cdot \log \cos x\right]$$

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in eq. (1), we get

$$\frac{dy}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \cdot \log \sin x\right] + (\cos x)^{\sin x}\left[\cos x \log \cos x - \frac{\sin^2 x}{\cos x}\right]$$

13. We have, $x^y - y^x = a^b$

Differentiate both sides w.r.t. x, we get

$$\frac{d}{dx}(x^y - y^x) = \frac{d}{dx}(a^b)$$

$$\Rightarrow \quad \frac{d}{dx}(x^y) - \frac{d}{dx}(y^x) = 0 \qquad ...(1)$$

TiPS

- *Derivative of constant is zero.*
- *log (a × b) = log a + log b. So, separate the function and find the derivatives.*

Let $\quad u = x^y$

Taking logarithm on both sides, we get

$$\log u = \log x^y = y \log x$$

Differentiate both sides w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = y \cdot \frac{d}{dx}\log x + \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{du}{dx} = u\left\{y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}\right\}$$

$$\therefore \quad \frac{d}{dx}(x)^y = x^y\left\{\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right\} \qquad ...(2)$$

and let $\quad v = y^x$

Taking logarithm on both sides, we get

$$\log v = \log y^x = x \log y$$

Differentiate both sides w.r.t. x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx} = x \cdot \frac{d}{dx}(\log y) + \log y \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left\{x \cdot \frac{1}{y}\cdot\frac{dy}{dx} + \log y \times 1\right\}$$

$$\therefore \quad \frac{d}{dx}(y^x) = y^x\left\{\frac{x}{y}\cdot\frac{dy}{dx} + \log y\right\} \qquad ...(3)$$

From eqs. (1), (2) and (3), we get

$$x^y\left\{\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right\} - y^x\left\{\frac{x}{y}\cdot\frac{dy}{dx} + \log y\right\} = 0$$

$$\Rightarrow y \cdot x^{y-1} + x^y \log x \cdot \frac{dy}{dx} - xy^{x-1}\frac{dy}{dx} - y^x \log y = 0$$

$$\Rightarrow \quad (x^y \log x - xy^{x-1}) \cdot \frac{dy}{dx} = y^x \log y - yx^{y-1}$$

$$\therefore \quad \frac{dy}{dx} = \frac{y^x \log y - yx^{y-1}}{x^y \log x - xy^{x-1}}$$

COMMON ERROR

Some students take logarithm directly without separating the equation, which leads to the wrong solution.

14. Given that, $\quad y = (\log x)^x + x^{\log x}$

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{(\log x)^x\} + \frac{d}{dx}\{x^{\log x}\} \qquad ...(1)$$

Let $\quad u = (\log x)^x$

TiPS

- *Differentiation rules for different functions and forms need continuous revision and practice.*
- *There is no formula to find log (a + b).*

Taking logarithm on both sides, we get

$$\log u = x \log(\log x)$$

Differentiate both sides w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = x \cdot \frac{d}{dx}\log(\log x) + \log(\log x) \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{du}{dx} = u\left\{x \cdot \frac{1}{\log x}\cdot\frac{d}{dx}(\log x) + \log(\log x) \cdot 1\right\}$$

$$\Rightarrow \frac{d}{dx}\{(\log x)^x\} = (\log x)^x\left\{\frac{x}{\log x}\cdot\frac{1}{x} + \log(\log x)\right\}$$

$$= (\log x)^x\left\{\frac{1}{\log x} + \log(\log x)\right\} \qquad ...(2)$$

and let $\quad v = x^{\log x}$

Taking logarithm on both sides, we get

$$\log v = \log x \cdot \log x$$

Differentiate both sides w.r.t. x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx} = \log x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left\{\log x \cdot \frac{1}{x} + \log x \cdot \frac{1}{x}\right\}$$

$$\Rightarrow \quad \frac{d}{dx}(x^{\log x}) = x^{\log x}\left\{\frac{2\log x}{x}\right\} \qquad ...(3)$$

From eqs. (1), (2) and (3), we get

$$\frac{dy}{dx} = (\log x)^x\left\{\log(\log x) + \frac{1}{\log x}\right\} + x^{\log x}\left\{\frac{2\log x}{x}\right\}$$

15. Given that, $y = (\cos x)^x + \tan^{-1}\sqrt{x}$

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{(\cos x)^x\} + \frac{d}{dx}(\tan^{-1}\sqrt{x}) \qquad ...(1)$$

Let $\quad u = (\cos x)^x$

Taking logarithmic both sides, we get

$$\log u = x \log(\cos x)$$

Differentiate both sides w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = x \cdot \frac{d}{dx}\log(\cos x) + \log(\cos x) \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{du}{dx} = u \cdot \left\{x \cdot \frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x) + \log(\cos x) \cdot 1\right\}$$

$$\Rightarrow \quad \frac{du}{dx} = u \cdot \left\{\frac{x}{\cos x} \times (-\sin x) + \log \cos x\right\}$$

$$\therefore \quad \frac{d}{dx}(\cos x)^x = (\cos x)^x(-x \tan x + \log \cos x) \qquad ...(2)$$

and $\quad v = \tan^{-1}\sqrt{x}$

TR!CK

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

Differentiate both sides w.r.t. x, we get

$$\frac{dv}{dx} = \frac{1}{1+(\sqrt{x})^2}\cdot\frac{d}{dx}\sqrt{x} = \frac{1}{1+x}\times\frac{1}{2\sqrt{x}}$$

$$\therefore \quad \frac{d}{dx}(\tan^{-1}\sqrt{x}) = \frac{1}{2\sqrt{x}\,(1+x)} \quad ...(3)$$

From eqs. (1), (2) and (3), we get

$$\frac{dy}{dx} = (\cos x)^x(-x\tan x + \log\cos x) + \frac{1}{2\sqrt{x}(1+x)}$$

16. Given that, $x = a\,(2\theta - \sin 2\theta)$

Differentiate both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = a\,(2 - 2\cos 2\theta) \quad ...(1)$$

$$= 2a\,\{1-(1-2\sin^2\theta)\} = 4a\sin^2\theta$$

$$[\because \cos 2\theta = 1 - 2\sin^2\theta]$$

and $\quad y = a\,(1-\cos 2\theta)$

Differentiate both sides w.r.t. θ, we get

$$\frac{dy}{d\theta} = a\,(0 + 2\sin 2\theta)$$

$$= 2a\,(2\sin\theta\cdot\cos\theta) = 4a\sin\theta\cdot\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx} = 4a\cdot\sin\theta\cdot\cos\theta\times\frac{1}{4a\cdot\sin^2\theta}$$

$$= \frac{\cos\theta}{\sin\theta} = \cot\theta$$

So, $\left[\frac{dy}{dx}\right]_{\text{at }\theta=\frac{\pi}{3}} = \cot\frac{\pi}{3} = \cot 60° = \frac{1}{\sqrt{3}}$

COMMON ERR(!)R

Some students commit error in $u \times v$ rule derivative function.

17. Given, $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$ and $y = a\sin t$

Differentiate $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$ w.r.t. t, we get

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan\frac{t}{2}}\cdot\frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{\sec^2\frac{t}{2}}{\tan\frac{t}{2}}\cdot\frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}}\cdot\sec^2\frac{t}{2}\cdot\frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{\cos t/2}{2\sin t/2}\cdot\frac{1}{\cos^2 t/2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin t/2\cos t/2}\right]$$

$$[\because \sin\theta = 2\sin\theta/2\cos\theta/2]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] = a\left[\frac{1-\sin^2 t}{\sin t}\right] = a\,\frac{\cos^2 t}{\sin t}$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

and differentiate $y = a\sin t$ w.r.t. t, we get

$$\frac{dy}{dt} = a\cos t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{a\,\frac{\cos^2 t}{\sin t}}$$

$$= \frac{\sin t\cos t}{\cos^2 t}$$

$$= \frac{\sin t}{\cos t} = \tan t$$

Hence proved.

18. Given, $e^y(x+1) = 1 \Rightarrow x + 1 = e^{-y}$

Differentiate both sides w.r.t. x, we get

$$\frac{d}{dx}(x+1) = \frac{d}{dx}\cdot e^{-y} \Rightarrow (1+0) = -e^{-y}\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{e^{-y}} = -e^y \quad ...(1)$$

Again differentiate both sides w.r.t x, we get

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\cdot e^y \Rightarrow \frac{d^2y}{dx^2} = -e^y\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{dy}{dx}\cdot\frac{dy}{dx} \quad \text{[from eq. (1)]}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

Hence proved.

19. Given, $x^m y^n = (x+y)^{m+n}$

Taking logarithm on both sides, we get

$$\log(x^m\cdot y^n) = \log(x+y)^{(m+n)}$$

$$\Rightarrow \quad \log x^m + \log y^n = (m+n)\cdot\log(x+y)$$

$$\Rightarrow \quad m\log x + n\log y = (m+n)\cdot\log(x+y)$$

Differentiate both sides w.r.t. x, we get

$$m\cdot\frac{1}{x} + n\cdot\frac{1}{y}\cdot\frac{dy}{dx} = (m+n)\cdot\frac{1}{x+y}\left\{1+\frac{dy}{dx}\right\}$$

$$\Rightarrow \quad \frac{m}{x} + \frac{n}{y}\cdot\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \left\{\frac{n}{y} - \frac{m+n}{x+y}\right\}\frac{dy}{dx} = \left\{\frac{m+n}{x+y} - \frac{m}{x}\right\}$$

$$\Rightarrow \quad \left\{\frac{nx+ny-my-ny}{y\,(x+y)}\right\}\cdot\frac{dy}{dx} = \left\{\frac{mx+nx-mx-my}{x\,(x+y)}\right\}$$

$$\Rightarrow \quad \frac{nx-my}{y\,(x+y)}\cdot\frac{dy}{dx} = \frac{nx-my}{x\,(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{nx-my}{x\,(x+y)}\times\frac{y\,(x+y)}{nx-my} = \frac{y}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = y$$

Again differentiate both sides w.r.t. x, we get

$$x\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 1 = \frac{dy}{dx}$$

$$\Rightarrow \quad x\cdot\frac{d^2y}{dx^2} = \frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{d^2y}{dx^2} = 0 \quad [\because x \neq 0]$$

Hence proved.

20. Given that, $y = \sin(\sin x)$ ...(1)

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx} = \cos(\sin x)\cdot\frac{d}{dx}\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = \cos x\cdot\cos(\sin x) \quad ...(2)$$

Again differentiate both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \cos x\cdot\left\{-\sin(\sin x)\cdot\frac{d}{dx}\sin x\right\} + \cos(\sin x)\cdot(-\sin x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \cos x\cdot\{-\sin(\sin x)\cdot\cos x\} - \sin x\cdot\cos(\sin x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\cos^2 x\{\sin(\sin x)\} - \sin x\,\{\cos(\sin x)\}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\cos^2 x\cdot y - \sin x\cdot\frac{1}{\cos x}\cdot\frac{dy}{dx}$$

[from eqs. (1) and (2)]

$$\Rightarrow \quad \frac{d^2y}{dx^2} + y\cos^2 x + \tan x\frac{dy}{dx} = 0$$

or $\dfrac{d^2y}{dx^2} + \tan x\dfrac{dy}{dx} + y\cos^2 x = 0$ **Hence proved.**

21. We have, $y = (\sin^{-1} x)^2$ ...(1)

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2\sin^{-1} x\cdot\frac{d}{dx}(\sin^{-1} x) \quad \left[\because \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = 2\sin^{-1} x\cdot\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\cdot\frac{dy}{dx} = 2\sin^{-1} x$$

Squaring on both sides, we get

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4(\sin^{-1} x)^2$$

$$\Rightarrow \quad (1-x^2)\left(\frac{dy}{dx}\right)^2 = 4y \quad \text{[from eq. (1)]}$$

Again differentiate both sides w.r.t. x, we get

$$(1-x^2)\cdot 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + (0-2x)\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0 \quad \left[\because 2\frac{dy}{dx} \neq 0\right]$$

Hence proved.

22. Given equations is $x = \cos t + \log\tan\left(\dfrac{t}{2}\right)$

Differentiate both sides w.r.t. t, we get

$$\frac{dx}{dt} = -\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)}\cdot\frac{d}{dt}\left(\tan\frac{t}{2}\right)$$

$$\left[\because \frac{d}{dx}(\tan x) = \sec^2 x\right]$$

$$= -\sin t + \frac{\sec^2\frac{t}{2}}{\tan\frac{t}{2}}\cdot\frac{d}{dt}\left(\frac{t}{2}\right) = -\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\times\frac{1}{\cos^2\frac{t}{2}}\times\frac{1}{2}$$

$$= -\sin t + \frac{1}{2\sin\frac{t}{2}\cdot\cos\frac{t}{2}} \quad \left[\because \sin\theta = 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\right]$$

$$= -\sin t + \frac{1}{\sin t} = \frac{-\sin^2 t + 1}{\sin t}$$

$$= \frac{\cos^2 t}{\sin t} \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

and $\quad y = \sin t \quad$ [given]

Differentiate both sides w.r.t. t, we get

$$\frac{dy}{dt} = \cos t$$

$$\Rightarrow \quad \frac{d^2y}{dt^2} = -\sin t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = \cos t\times\frac{\sin t}{\cos^2 t} = \frac{\sin t}{\cos t} = \tan t$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \sec^2 t\cdot\frac{dt}{dx} = \sec^2 t\times\frac{\sin t}{\cos^2 t} = \frac{\sin t}{\cos^4 t}$$

Now, $\left(\dfrac{d^2y}{dt^2}\right)_{t=\frac{\pi}{4}} = -\sin\left(\dfrac{\pi}{4}\right) = -\dfrac{1}{\sqrt{2}}$

and $\left(\dfrac{d^2y}{dx^2}\right)_{t=\frac{\pi}{4}} = \dfrac{\sin\left(\frac{\pi}{4}\right)}{\cos^4\left(\frac{\pi}{4}\right)} = \dfrac{\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^4}$

$$= \frac{\frac{1}{\sqrt{2}}}{\frac{1}{4}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

COMMON ERROR

Some students first substitute the value and then take the derivative which is wrong.

Chapter Test

Multiple Choice Questions

Q 1. The value of k which makes the function defined by

$$f(x) = \begin{cases} \sin\dfrac{1}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$ **, continuous at $x = 0$, is:**

a. 8 b. 1

c. −1 d. None of these

Q 2. The derivative of $y = (1-x)(2-x)\ldots(n-x)$ at $x = 1$ is equal to:

a. 0

b. $(-1)(n-1)!$

c. $n! - 1$

d. $(-1)^{n-1}(n-1)!$

Assertion and Reason Type Questions

Directions (Q. Nos. 3-4): *In the following questions, each question contains Assertion (A) and Reason (R). Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are:*

a. Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)
b. Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A)
c. Assertion (A) is true but Reason (R) is false
d. Assertion (A) is false but Reason (R) is true

Q 3. Assertion (A): Every differentiable function is continuous but converse is not true.
Reason (R): Function $f(x)=|x|$ is differentiable.

Q 4. Assertion (A): $\frac{d}{dx}(\sqrt{e^{\sqrt{x}}})=\frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}$

Reason (R): $\frac{d}{dx}[\log(\log(x))]=\frac{1}{x\log x}, x>1$

Case Study Based Questions

Q 5. Case Study 1

A function is continuous at $x=c$, if the function is defined at $x=c$ and if the value of the function at $x=c$ equals the limit of the function at $x=c$.

i.e., $\lim_{x\to c} f(x)=f(c)$

Based on the above information, solve the following questions:

(i) Find the relationship between a and b so that the function $f(x)=\begin{cases} ax+1; & x\le 3 \\ bx+3; & x>3 \end{cases}$ is continuous at $x=3$.

(ii) If $f(x)=\begin{cases} kx^2; & x\le 2 \\ 3; & x>2 \end{cases}$ is continuous at $x=2$, then find the value of k.

(iii) Is it true that, if $f(x)$ and $g(x)$ is continuous at $x=c$, then $f\pm g$ is continuous at $x=c$? If no then why?

Or

Discuss the continuity of the function $f(x)=\begin{cases} \frac{|x|}{x}; & x\ne 0 \\ 0; & x=0 \end{cases}$ at $x=0$.

Q 6. Case Study 2

The derivative of f at $x=c$ is defined by:

$$f(x)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

A function is said to be differentiable at a point c, if left hand derivative at $x=c$ is equal to the right hand derivative at $x=c$.

Similarly, a function is said to be differentiable in an interval (a,b), if it is differentiable at every point of (a,b).

Based on the above information, solve the following questions:

(i) If $y=1/t$ and $x=\log t^2$, then find $\frac{dy}{dx}$.

(ii) Show that $f(x)=|x|$ is differentiable at all points of $x\in R-\{0\}$.

(iii) If $y+\sin y=\cos x$, then find $\frac{dy}{dx}$.

Or

If $y=\sqrt{\sin x+y}$, then find $\frac{dy}{dx}$.

Very Short Answer Type Questions

Q 7. Show that the function $f(x)=\begin{cases} x^3+3, & \text{if } x\ne 0 \\ 1, & \text{if } x=0 \end{cases}$, is not continuous at $x=0$.

Q 8. Find the differential coefficient of the function $\frac{\log x+\log x^2}{x}$.

Short Answer Type-I Questions

Q 9. Find the differential coefficient of $\tan^{-1}x$ with respect to $\sin^{-1}x$ at $x=\frac{1}{2}$.

Q 10. Differentiate $\tan^{-1}\left[\frac{\cos x}{1+\sin x}\right]$ with respect to x.

Short Answer Type-II Questions

Q 11. If $y=\sqrt{\frac{1-x}{1+x}}$, then prove that $(1-x^2)\frac{dy}{dx}+y=0$.

Q 12. If $\sin y=x\sin(a+y)$, then prove that:

$$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}.$$

Long Answer Type Questions

Q 13. Show that the function $f(x)=|x-2|=\begin{cases} x-2, & x\ge 2 \\ 2-x, & x<2 \end{cases}$, is continuous but not differentiable at $x=2$.

Q 14. Differentiate $x^{\sin x}+(\sin x)^x$ with respect to x.