Chapter 3

Friction, Centre of Gravity, Moment of Inertia

CHAPTER HIGHLIGHTS

- *Friction*
- *Laws of Friction*
- *Cone of Friction*
- *determination of the Center of Gravity of a Thin Irregular Lamina*
- *Integration Method for Centroid Determination in a Thin Lamina or Solid*
- *Area Moment of Inertia*
- *Polar Moment of Inertia*
- *Perpendicular Axis Theorem*
- *Centroid of Solids*
- *Mass Moment of Inertia*
- *Mass Moment of Inertia and Radius of Gyration*

Friction

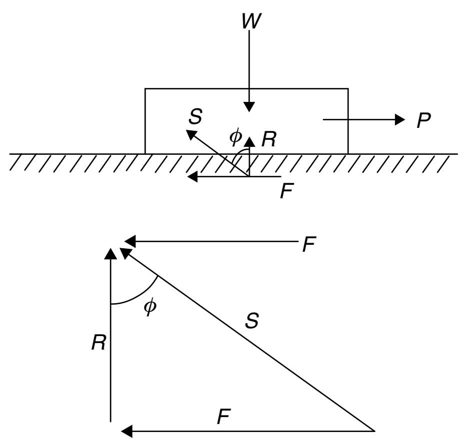
Definitions

1. **Static friction:** It is the friction between two bodies which is a tangential force and which opposes the sliding of one body relative to the other.
2. **Limiting friction:** It is the maximum value of the static friction that occurs when motion is impending.
3. **Kinetic friction:** It is the tangential force between two bodies after motion begins. Its value is less than the corresponding static friction.
4. **Angle of friction:** It is the angle between the action line of the total reaction of one body on another and the normal to the common tangent between the bodies when motion is impending.

   It is also defined as the angle made by the resultant (S) of the normal reaction (R) and the limiting force of friction (F) with the normal reaction R (see the figure given below). It is denoted by f. From the figure, we have:

$$\tan \lambda = \frac{F}{R} = \frac{\mu R}{R} = \mu = \text{coefficient of friction}$$

5. **Coefficient of static friction**

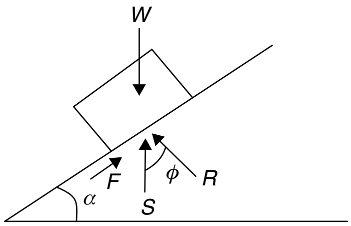


It is defined as the ratio of the limiting force of friction (F) to the normal reaction (R) between two bodies (see above figure, where a solid body rests on a horizontal plane). It is denoted by μ.

$$\mu = \frac{\text{Limiting force of friction}}{\text{Normal reaction}} = \frac{F}{R}$$

$$\therefore F = \mu R$$

7. **Angle of repose**



The above figure shows a block of weight W on a rough and plane inclined at an angle α with the horizontal. Let R be the normal reaction and F be the force of friction. From applying the condition of equilibrium, algebraic sum of the forces resolved along the plane:

$$= W \sin\alpha = F \tag{1}$$

Algebraic sum of the forces resolved perpendicular to the plane:

$$= W \cos\alpha = R \tag{2}$$

From equations (1) and (2) $\tan\alpha = \dfrac{F}{R}$

But $\tan\phi = \dfrac{F}{R}$

∴ Angle of plane = Angle of friction

Suppose the angle of the plane α is increased to a value ϕ, so that the block is at the point of sliding or the state of impending motion occurs, then at this angle,

$$\mu = \tan\lambda = \tan\alpha \qquad \therefore \lambda = \alpha$$

Hence, the angle of repose is defined as the angle to which an inclined plane may be raised before an object resting on it will move under the action of the force of gravity and the reaction of the plane.

Hence, angle of repose = angle of plane

Laws of Friction

First law: Friction always opposes motion and comes into play only when a body is urged to move. Frictional force will always act in a direction opposite to that in which the body tends to move.

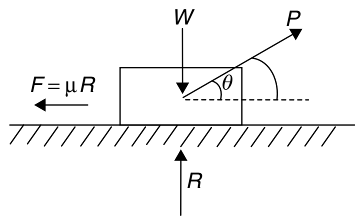
Second law: The magnitude of the frictional force is just sufficient to prevent the body from moving. That is, only as much resistance as required to prevent motion will be offered as friction.

Third law: The limiting frictional force or resistance bears a constant ratio with the normal reaction. This ratio depends on the nature of the surfaces in contact. The limiting frictional resistance is independent of the area of contact.

Fourth law: When motion takes place as one body slides over the other, the magnitude of the frictional force or resistance will be slightly less than the offered force at that condition of limiting equilibrium. The magnitude of the frictional force will depend only on the nature of the sliding surfaces and independent of the shape or extent of the contact surfaces.

Force Determinations for Different Scenarios

Least force required to drag a body on a rough horizontal plane:

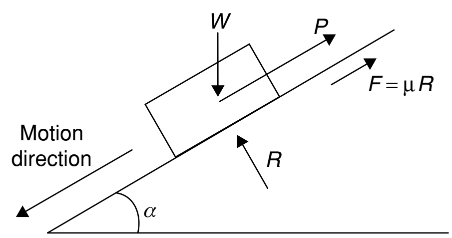


Force P is applied, at an angle θ to the horizontal, on a block of weight W such that the motion impends or the block tends to move.

$$\text{Force, } P = \frac{W \sin\alpha}{\cos(\theta - \phi)}$$

Least force required, $P_{\text{least}} = W \sin\phi$

Force acting on a block (weight =) along a rough inclined plane:



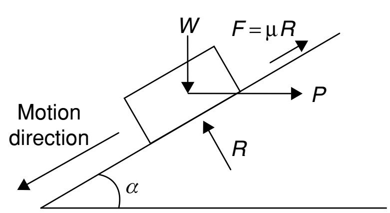
$$\text{For motion down the plane, } P = \frac{W \sin(\alpha - \phi)}{\cos\phi}$$

$$\text{For motion up the plane, } P = \frac{W \sin(\alpha + \phi)}{\cos\phi}$$

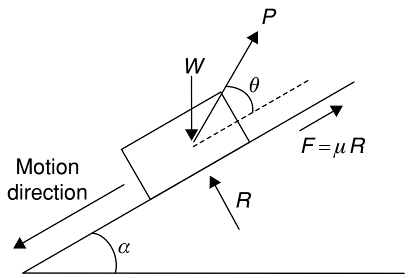
Force acting horizontally on a block (weight =) resting on a rough inclined plane:

For motion down the plane, $P = W \tan(\alpha - \phi)$

For motion up the plane, $P = W \tan(\alpha + \phi)$



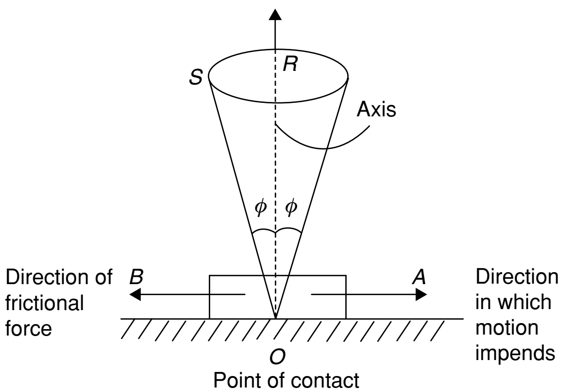
Force acting, at an angle θ to the plane, on a block (weight =) resting on a rough inclined plane:



$$\text{For motion down the plane, } P = \frac{W\sin(\alpha - \phi)}{\cos(\theta + \phi)}$$

$$\text{For motion up the plane, } P = \frac{W\sin(\alpha + \phi)}{\cos(\theta - \phi)}$$

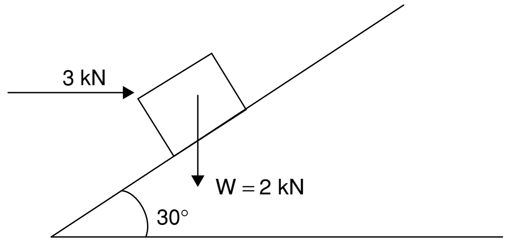
Cone of Friction



Let OR represent the normal reaction offered by a surface on a body and let the direction of impending motion be along OA while the direction in which the frictional force acts is in the opposite direction, i.e., along *OB*. Assuming that the body is in a state of limiting equilibrium, the resultant reaction S makes an angle of ϕ with the normal *OR*. If the body slides in any other direction, the resultant reaction *S* will still make the same angle ϕ with the normal. It is, therefore, seen that when limiting equilibrium is maintained, then the line of action of the resultant reaction should always lie on the surface of an inverted right circular cone; whose semi-vertical angle is ϕ. This cone is known as the cone of friction.

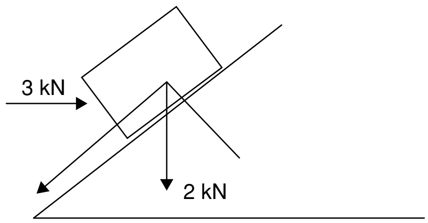
Solved Examples

Example 1: Determine whether the 2 kN block, shown in the figure, will be held in equilibrium by a horizontal force of 3 kN? The coefficient of static friction is 0.3.



(A) 0.96 kN (B) 0.86 kN
(C) 0.75 kN (D) 0.65 kN

Solution:



Applying the conditions of equilibrium and summing the force parallel and perpendicular to the plane, we have:

$$\Sigma F(\text{parallel to the plane}) = 0$$
$$= -F - 2\sin 30° + 3\cos 30° = 0$$
$$F = -2 \times \frac{1}{2} + 3 \times 0.866$$
$$= -1 + 2.598 = 1.598 \text{ kN}$$
$$\Sigma F(\text{perpendicular to the plane}) = 0$$
$$R - 2\cos 30° - 3\sin 30° = 0$$
$$R = 2 \times 0.866 + 3 \times 0.5 = 1.732 + 1.5$$
$$= 3.232 \text{ kN}$$

This indicates that the value of *F* necessary to hold the block from moving up the plane is 1.598 kN. However, the maximum value obtainable as the frictional force,

$$F = \mu R = 0.3 \times 3.232 = 0.9696 \text{ kN}$$

This means that the block will move up the plane.

Example 2: An effort of 2 kN is required just to move a certain body up an inclined plane of angle 15°, the force acting parallel to the plane. If the angle of inclination of the plane is made 20°, the effort required, again applied parallel to the plane, is found to be 2.3 kN. Find the weight of the body and the coefficient of friction.

(A) 3.9 kN, 0.258 (B) 4.5 kN, 0.26
(C) 3.8 kN, 0.24 (D) 3.8 kN, 0.268

Solution:

Let *W* be the weight of the body, μ be the coefficient of friction and *P* be the effort when the inclination of the plane is α.

Applying the conditions of equilibrium and summing the forces parallel and perpendicular to the plane, we have,

$\Sigma F(\text{parallel to the plane}) = 0$

$$P - \mu R - W\sin\alpha = 0 \qquad (1)$$

$\Sigma F(\text{perpendicular to the plane}) = 0$

$$R - W\cos\alpha = 0 \qquad (2)$$

Eliminating *R* from equations (1) and (2) we have,

$$P = \mu W\cos\alpha + W\sin\alpha \text{ or}$$

$$P = W(\mu \cos \alpha + \sin \alpha) \quad (3)$$

When $\alpha = 15°$, $P = 2$ kN and when $\alpha = 20°$, $P = 2.3$ kN. Substituting in equation (3) we have,

$$2 = W(\mu \cos \alpha + \sin \alpha)$$
$$2 = W(\mu \cos 15° + \sin 15°) \quad (4)$$
$$2.3 = W(\mu \cos 20° + \sin 20°) \quad (5)$$

Dividing equation (5) by (4), we have,

$$\frac{2}{2.3} = \frac{\mu \cos 15° + \sin 15°}{\mu \cos 20° + \sin 20°}$$

$$\frac{2}{2.3} = \frac{\mu \times 0.966 + 0.258}{\mu \times 0.939 + 0.342} \text{ or}$$

$$\mu[(2.3 \times 0.969) - (2 \times 0.939)]$$
$$= [(2 \times 0.342) - (2.3 \times 0.258)]$$

or $0.3507\,\mu = 0.0906$

$$\mu = \frac{0.0906}{0.3507} = 0.258$$

From equation (5),

$$2.3 = W[0.258 \times 0.939 + 0.342]$$
$$= W(0.242 + 0.342) = 0.584W$$
$$W = \frac{2.3}{0.584} = 3.938 \text{ kN}$$

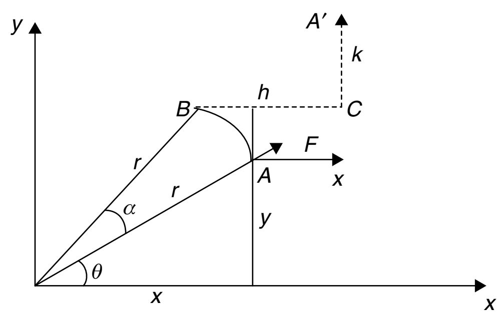
Virtual Work

Virtual displacement: Virtual displacement is defined as an infinitesimal (exceedingly small) and displacement, given hypothetically to a particle or to a body or a system of bodies in equilibrium consistent with the constraints. The displacement is only imagined and it does not have to take place for which it is called virtual displacement.

Virtual work: Virtual work is defined as the work done, by a force on a body due to a small virtual (i.e., imaginary) displacement of the body.

Principle of Virtual Work

If a system of forces acting on a body or a system of bodies be in equilibrium and if the system be assumed to undergo a small displacement consistent with the geometrical conditions, then the algebraic sum of the virtual work done by the forces of the system is zero.



To illustrate the principle of work, let us consider a body at equilibrium at a point A. A force F acts on the body and displaces it to the point A', where the displacement consists of the following:

1. A very small rotation through the angle α about the origin of the rectangular 2-D coordinate system, say origin O in the xy plane.
2. A very small displacement h along the x-axis, and,
3. A very small displacement k along the y-axis.

If the components of the force F along the x-axis and y-axis are F_x and F_y respectively, then work done by the force F when its point of application is displaced from point A to A'

$$= hF_x + kF_y + \alpha(xF_y - yF_x)$$

If a system of forces act on the body where h, k and α are the same for every force, then work done by all the forces:

$$= h\sum F_x + k\sum F_y + \alpha\sum(xF_y - yF_x),$$

Where, $\sum F_X$ and $\sum F_Y$ are the sums of the resolved parts of the forces along the x-axis and y-axis respectively, and $\sum(xF_y - yF_x)$ is the moments of the forces about the origin O.

Since the system is in equilibrium, all the three terms in the above expression, for the work done by all the forces, is zero. Hence, the sum of the virtual works done by the forces is zero.

Lifting Machine

Lifting machines are defined as those appliances or machines which are used for lifting heavy loads. They are also called simple machines. Some commonly used machines are:

1. Lever
2. Inclined plane
3. Wedge
4. Wheel and axle
5. Winch crab
6. A pulley and system of pulleys
7. Screw jack

Screw jack is the most important among all the above simple machines.

Load or resistance: A machine has to either lift a load or overcome a resistance. It is usually denoted by W and its unit is N. Examples: A lifting device lifts a load or heavy weight whereas a bicycle overcomes the frictional resistance between the wheels and the road.

Efforts: It is the force which is applied to a machine to lift a load or to overcome the resistance against a movement. It is usually denoted by P and its unit is N. Examples: Force applied on the pedals of a bicycle or on the handle of a screw jack.

Input of a machine: It is defined as the amount of total work done on the machine. This is measured by the product of the effort and the distance through which it moves.

Input = Effort × Distance moved by the effort = $P \times y$

It has the unit of Nm.

Output of a machine: It is defined as the amount of work got out of a machine or the actual work done by the machine

Output of the machine:

= Load × Distance through which load is lifted = $W \times x$

It has the unit of Nm.

Velocity Ratio (VR): It is defined as the ratio of the distance moved by the effort and to the distance moved by the load during the same interval of time.

$$VR = \frac{\text{Distance moved by the effort}}{\text{Distance moved by the load}} = \frac{y}{x}$$

NOTE

In all machines $y > x$.

Mechanical advantage (): It is defined as the ratio of the load or weight lifted to the effort applied.

$$MA = \frac{\text{Weight lifted}}{\text{Effort applied}} = \frac{W}{P}$$

NOTE

In all machines $W > P$.

Ideal machine: It is defined as the machine which is absolutely free from frictional resistances. In such a machine, input = output.

For an ideal machine, $VR = MA$

Efficiency of a machine: It is the ratio of output of the machine to the input of the machine.

$$\eta = \frac{\text{output of the machine} \times 100}{\text{input of the machine}}$$

$$= \frac{\text{useful work done by the machine} \times 100}{\text{energy supplied to the machine}} = \frac{W \times x}{P \times y} \times 100$$

For an ideal machine, $\eta = 100\%$. For an actual machine,

$$\eta = \frac{\text{Ideal effort}}{\text{Actual effort}} = \frac{\text{Actual load}}{\text{Ideal load}}.$$

Relation between *MA*, *VR* and η

$$\eta = \frac{W \times x}{P \times y} = \frac{\frac{W}{P}}{\frac{y}{x}} = \frac{MA}{VR}$$

Frictional losses

Output = Input – Losses due to friction

$$\text{Effort lost in friction} = P - \frac{W}{VR}$$

Loss in load lifted due to friction = $P \times VR - W$.

Here P is the actual effort required to overcome resistance W or lift load W.

Reversible and Irreversible Machine

A machine is said to be reversible when the load W gets lowered on the removal of the effort. In such a case, work is done by the machine in reverse direction.

A machine is said to be irreversible when the load W does not fall down on the removal of the effort. In such a case, work is not done in the reverse direction.

The condition of irreversibility or self locking of a machine is that its efficiency should be less than 50%.

Compound Efficiency

It is defined as the overall efficiency of the combination of machines and it is the product of the efficiencies of the individual machines.

The overall efficiency η of n machines coupled together is $\eta = \prod_{i=1}^{n} \eta_i$, where, η_i is the efficiency of the ith machine.

Law of a Machine

It is defined as the relationship which exists between the effort applied and the load lifted.

$$P = mW + C$$

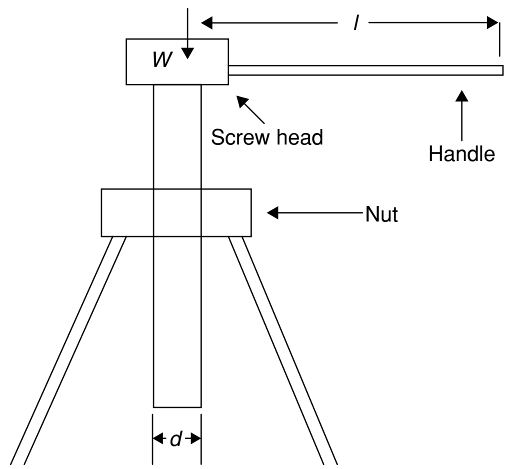
P is the effort applied, W is the corresponding load, m and C are coefficients which are determined in any machine after conducting a series of tests and plotting the W versus P graph.

The expression for maximum mechanical advantage is given by $(MA)_{max} = \frac{1}{m}$.

The expression for maximum efficiency is given by $\eta_{max} = \frac{1}{m \times (VR)}$.

Screw Jack

It is a device for lifting heavy loads by applying comparatively a smaller effort at the end of the handle. The screw jack works on the principle of inclined plane.



It mainly consists of a nut which forms the body of the jack and a screw is fitted into it. The threads are generally square. The load W is placed on the head of the screw. By rotating the screw with a handle the load is lifted or lowered. Let W be the load lifted, α be the angle of helix of the screw and ϕ be the angle of friction.

Here, efficiency $= \dfrac{\tan\alpha}{\tan(\alpha+\varphi)}$, which shows that efficiency is independent of the load lifted or lowered.

Assuming that the effort is applied at the end of the handle, let us consider the following two cases.

Let the weight W be lifted: Let P_E be the effort applied at the end of the handle. Let l be the length of the handle and let d be the mean diameter of the screw.

Σm about, the axis is zero.

Let p be the pitch and μ be the coefficient of friction, then:

$$\tan\alpha = \frac{p}{\pi d}$$

$$\tan\phi = \mu$$

$$\boxed{P_E \frac{Wd}{2l} \cdot \frac{p+\mu\pi d}{\pi d - \mu p}}$$

Let the weight W be lowered: Let Q be the effort applied at the circumference of the screw and let Q_E be the actual effort applied at the end of the handle.

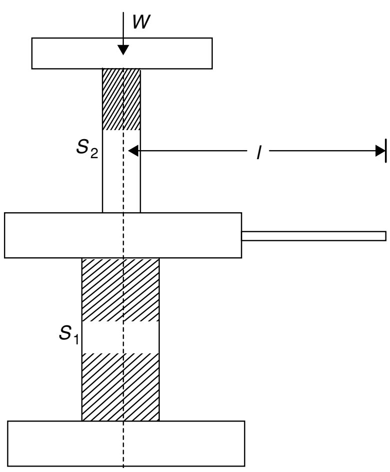
$$Q = W\tan(\phi - \alpha)$$

$$Q_E = \frac{Wd}{2l} \cdot \frac{\mu\pi d - p}{\pi d + \mu p}$$

For an n-threaded screw, $\tan\alpha = np/\pi d$.

Differential Screw Jack

Instead of only one threaded spindle as in the case of a simple screw jack it has two threaded spindles S_1 and S_2. The spindle S_1 is screwed to the base which is fixed.



This spindle carries both internal as well as external threads. The spindle S_2 is engaged to spindle S_1 by means of an internal thread. When spindle S_1 ascends, the spindle S_2 descends. This is also known as 'Differential Screw' jack.

The principle of working of this jack is similar to the one as described in the above figure.

Let p_{s_1} = pitch of the threads on S_1

p_{s_2} = pitch of the threads on S_2

Let the lever length be l and the effort be applied at the end of this lever.

When the lever is moved by one revolution, the distance covered by the effort P is 2πl and correspondingly the load distance is equal to $p_{s_1} - p_{s_2}$.

Then, velocity ratio $(VR) = \dfrac{2\pi l}{p_{S_1} - p_{S_2}}$.

NOTE

p_{s_1} is always greater than p_{s_2}. Due to this difference, the mechanical advantage as well as the velocity ratio will be more.

Direction for questions 3 and 4: A screw jack has a pitch of 12 mm with a mean radius of thread equal to 25 mm a lever 500 mm long is used to raise a load of 1500 kg. The coefficient of friction is 0.10.

Example 3: Find the helix angle α and θ (i.e., friction angle)

(A) 6.2°, 4.5° (B) 4.85°, 5.7°
(C) 4.85°, 5.7° (D) 4.36°, 5.7°

Solution: Given $P = 12$ mm, $d = 2r = 25 \times 2 = 50$ mm,

l = 500 mm

$W = 1500$ kg, $\mu = 0.10$, $\tan\phi = \mu = 0.10$,

$\phi = 5.71°$

$$\tan\alpha = \frac{P}{\pi d} = \frac{12}{\pi \times 50} = 0.076$$

$\alpha = 4.36°$

Example 4: What force is necessary when applied normal to the lever at its free end

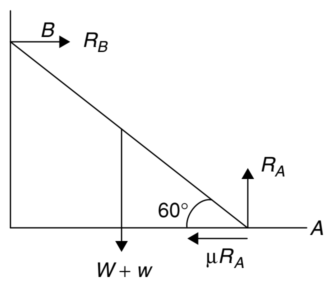
(A) 13.319 kg (B) 12.8 kg.
(C) 14.5 kg (D) 18.3 kg.

Solution: $P = \dfrac{wd}{2l}\tan(\alpha+\varphi) = \dfrac{1500 \times 50}{2 \times 500} \times \tan(4.36 + 5.71)$

$P = 13.319$ kg.

Direction for questions 5, 6 and 7: A uniform ladder of weight 500 N and the length 8 m rests on a horizontal ground and leans against a smooth vertical wall. The angle made by the ladder with the horizontal is 60°. When a man of weight 500 N, stands on the ladder at a distance of 4 metre from the top of the ladder, the ladder is at the point of sliding.

Example 5: Find the coefficient of friction in terms of R_B.



(A) $\mu = \frac{R_B}{1000}$ (B) $\mu = 1400\, R_B$

(C) $\mu = 500\, R_B$ (D) $\mu = \frac{R_B}{500}$

Solution: Resolving all the forces $R_B = \mu R_A$

$$R_A = W + w = 500 + 500 = 1000$$
$$R_B = \mu \times R_A = \mu \times 1000$$
$$\mu = \frac{R_B}{1000}.$$

Example 6: Find the reaction at B (i.e., R_B)

(A) 289 (B) 300

(C) 350 (D) 400

Solution: Taking moment at A, $M_A = 0$

$$R_B \times 8 \times \frac{\sqrt{3}}{2} = 500 \times \frac{8}{2} \times \frac{1}{2} + 500 \times 4 \times \frac{1}{2}$$

$$R_B = \frac{500 \times 2 + 1000}{6.92} = 289$$

Example 7: Find the value of coefficient of friction

(A) 0.370 (B) 0.486

(C) 0.289 (D) 0.355

Solution: From equation $\mu = \frac{R_B}{1000} = \frac{289}{1000} = 0.289$

Centre of Gravity

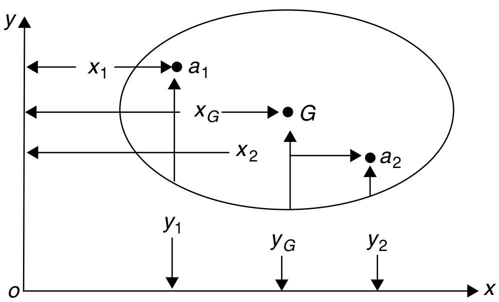
The centre of gravity of a body is the point, through which the whole weight of the body acts, irrespective of the position in which body is placed. This can also be defined as the centre of the gravitational forces acting on the body. It is denoted by G or c.g.

Centroid: It is defined as that point at which the total area of a plane figure (like rectangle, square, triangle, quadrilateral, circle etc.) is assumed to be concentrated. The centroid and the centre of gravity are one and the same point. It is also denoted by G or c.g..

Centroidal axis: It is defined as that axis which passes through the centre of gravity of a body or through the centroid of an area.

Lamina: A very thin plate or sheet of any cross-section is known as lamina. Its thickness is so small that it can be considered as a plane figure or area having no mass.

Determination of the Centre of Gravity of a Thin Irregular Lamina



The above figure shows an irregular lamina of total area A whose center of gravity is to be determined. Let the lamina be composed of small areas $a_1, a_2 \ldots$ etc. Such that:

$$A = a_1 + a_2 + \cdots = \Sigma a_i$$

Let the distances of the centroids of the areas $a_1, a_2, \ldots$ etc. from the x-axis be $y_1, y_2, \ldots$ etc. respectively and from the y-axis be $x_1, x_2, \ldots$ etc. The sum of moments of all the small areas about the y-axis

$$= a_1 x_1 + a_2 x_2 + \cdots = \Sigma\, a_i x_i$$

Let x_G and y_G be the coordinates of the centre of gravity G from the y-axis and x-axis respectively. From the principle of moments, it can be written that:

$$A x_G = \Sigma a_i x_i$$

or

$$x_G = \frac{\sum a_i x_i}{A} = \frac{\sum a_i x_i}{\sum a_i}$$

Similarly, it can be shown that:

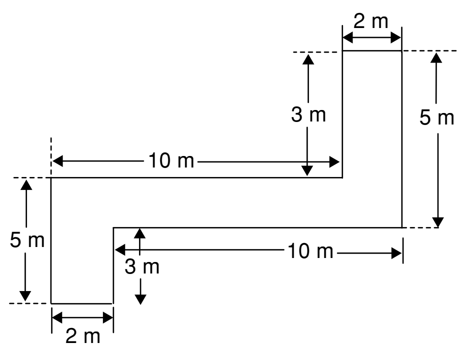
$$y_G = \frac{\sum a_i\, y_i}{\sum a_i}$$

NOTES

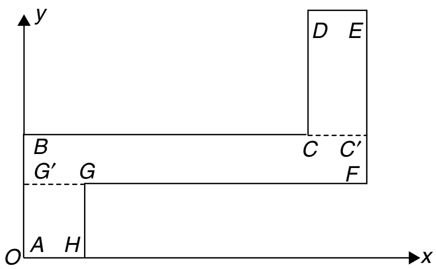
1. The axis of reference of a plane figure is generally taken as the bottom most line of the figure for determining y_G and the left most line of the figure for calculating x_G.
2. If the figure is symmetrical about the x-axis or y-axis, then the centre of gravity will lie on the axis of symmetry.
3. For solid bodies, elementary masses m_1, m_2, etc., are considered instead of the areas a_1, a_2, etc., and the coordination of centre of gravity are given as follows:

$$x_G = \frac{\sum m_i x_i}{\sum m_i}, \quad y_G = \frac{\sum m_i y_i}{\sum m_i}$$

Example 8: Determine the position of the center of gravity for the following figure.



Solution:



The x-axis and y-axis of reference are chosen as shown in the above figure such that the origin O coincides with the point A of the figure and the axes coincide with the left most and bottom most lines of the figure respectively. The position of the centre of gravity is determined with respect to the origin O.

The figure is broken down into the three areas $AHGG'$, $G'FC'B$ and $CC'ED$

For rectangle $AHGG'$,

Area $A_1 = 3 \times 2 = 6\ m^2$

c.g. coordinates, $x_1 = \frac{2}{2} = 1$ m

$$y_1 = \frac{3}{2} = 1.5 \text{ m}$$

For rectangle $G'FC'B$,

Area $A_2 = (2 + 10) \times (5 - 3) = 24$ m^2

c.g. coordinates, $x_2 = \frac{(2+10)}{2} = 6$ m

$$y_2 = 3 + \frac{(5-3)}{2} = 4 \text{ m}$$

For rectangle $CC'ED$,

Area $A_3 = 3 \times 2 = 6$ m^2

c.g. coordinates, $x_3 = 10 + \frac{2}{2} = 11$ m

$$y_3 = 5 + \frac{3}{2} = 6.5 \text{ m}$$

c.g. of the figure coordinates,

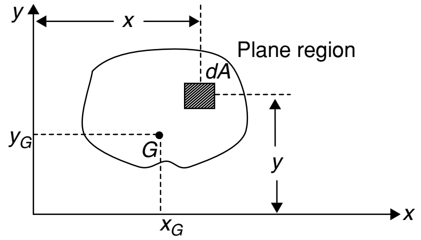
$$x_G = \frac{A_1x_1 + A_2x_2 + A_3x_3}{A_1 + A_2 + A_3} = 6 \text{ m}$$

$$y_G = \frac{A_1y_1 + A_2y_2 + A_3y_3}{A_1 + A_2 + A_3} = 4 \text{ m}$$

Integration Method for Centroid Determination in a Thin Lamina or Solid

In this method, the given figure is not split into shapes of figures of known centroid as done in the previous section. The centroid is directly found out by determining $\Sigma a_i y_i$ or $\Sigma a_i x_i$ and Σa_i by direct integration.

First moment of area: Consider a plane region of area A as shown in the following figure.



Let dA be a differential (i.e., infinitesimal) area located at the point (x, y) in the plane region area A.

Here, $A = \int_A dA$

First moments of the area about the x-axis and y-axis are respectively:

$$M_X = \int_A y\, dA$$

$$M_Y = \int_A x\, dA$$

The coordinates (x_G, y_G) of the centre of gravity of the plane region is given by:

$$x_G = \frac{M_y}{A} = \frac{\int_A x dA}{\int_A dA}$$

$$y_G = \frac{M_X}{A} = \frac{\int_A y dA}{\int_A dA}$$

NOTES

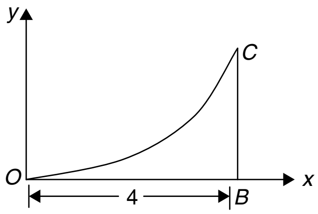
1. If the x-axis passes through the centre of gravity, then $M_x = 0$. Similarly, $M_y = 0$, when the y-axis passes through the centre of gravity.
2. If the plane region is symmetric about the y-axis, then $M_y = 0$ and $x_G = 0$, i.e., the centre of gravity would lie somewhere on the y-axis. Similarly, $M_x = 0$ and $y_G = 0$, if the plane region is symmetric about the x-axis, i.e., the centre of gravity would lie somewhere on the x-axis.

If instead of a plane region, we have a plane curve of length L and on which a differential length dL is considered which is located at the point (x, y) on the curve, then the coordinates of the centre of gravity for the planar curve is given as follows:

$$x_G = \frac{M_y}{L} = \frac{\int_L x dL}{\int_L dL}$$

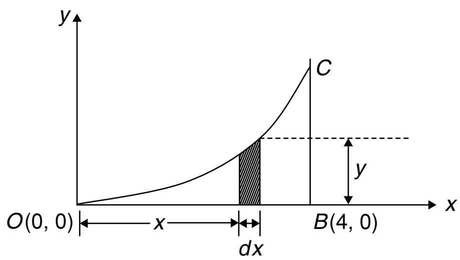
$$y_G = \frac{M_X}{A} = \frac{\int_L y dL}{\int_L dL}$$

Example 9: The centre of gravity of the following shown area *OBC*, where the curve *OC* is given by the equation $y = 0.625x^2$, with respect to the point *O* (0, 0) is:



(A) (6, 5)
(B) (6, 3)
(C) (3, 5)
(D) (3, 3)

Solution:



Let us consider an elementary rectangular area of height *y* and width *dx* as shown in the above figure.

Area of the elementary rectangle, $dA = ydx = 0.625x^2\,dx$

Area of *OBC*, $A = \int_0^4 dA = \int_0^4 0.625x^2 dx$

$$= 0.625 \times \frac{4^3}{3}$$

Moment of area about *x*-axis,

$$M_X = \int_0^4 dA\frac{y}{2} = \int_0^4 0.625x^2\,dx\,\frac{0.625x^2}{2}$$

$$= \frac{0.625^2}{2} \times \frac{4^5}{5}$$

Moment of area about *y*-axis,

$$Mx = \int_0^4 dAx = \int_0^4 0.625x^2 dxx = 0.625 \times \frac{4^4}{4}$$

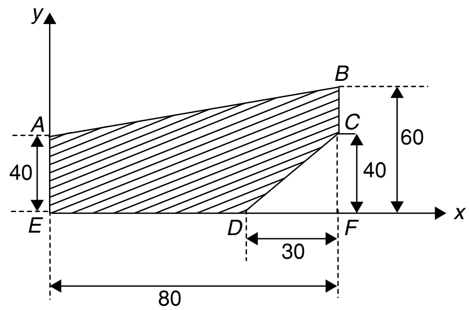
Let x_G and y_G be the *x* and *y* coordinates of the centre of gravity of *OBC* with respect to the point *O*.

Then, $M_x = Ay_G$ and $M_y = Ax_G$

$$y_G \frac{0.625^2}{2} \times \frac{4^5}{5} \times \frac{3}{0.625 \times 4^3} = 3$$

$$x_G = 0.625 \times \frac{4^4}{4} \times \frac{3}{0.625 \times 4^3} = 3$$

Example 10: The centre of gravity of the following hatched figure with respect to the point *E* is:



(A) (20, 30)
(B) (37.84, 27.45)
(C) (20, 27.45)
(D) (37.84, 30)

Solution:

For Δ *ABC*, area $A_1 = \frac{1}{2} 80 \times (60 - 40) = 800$

c.g. coordinates, $x_1 = \frac{2}{3} \times 80 = \frac{160}{3}$

$$y_1 = 40 + \frac{1}{3} \times (60 - 40) = \frac{140}{3}$$

For □*ACFE*, area $A_2 = 40 \times 80 = 3200$

c.g. coordinates, $x_2 = \frac{80}{2} = 40$

$$y_2 = \frac{40}{2} = 20$$

For Δ *CFD*, area $A_3 = \frac{1}{2} 30 \times 40 = 600$

c.g. coordinates, $x_3 = 50 + \frac{2}{3} \times 30 = 70$

$$y_3 = \frac{1}{3} \times 40 = \frac{40}{3}$$

Since Δ *CFD* is cut out from the figure *ABFE* to obtain the hatched figure, the area of Δ *CFD* is assigned a negative sign.

$\therefore A_3 = -600$

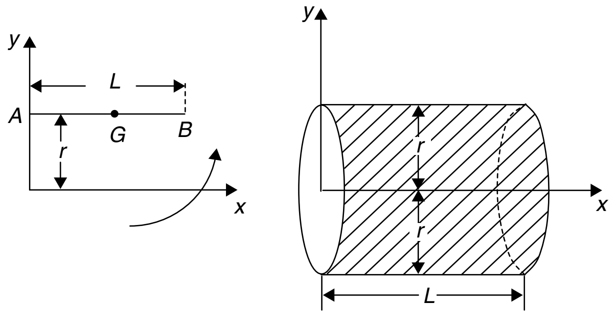
Let x_G and y_G be the *x* and *y* coordinates of the centre of gravity of the hatched figure with respect to the point *E*, then:

$$x_G = \frac{A_1x_1 + A_2x_2 + A_3x_3}{A_1 + A_2 + A_3} = 37.84$$

$$y_G = \frac{A_1y_1 + A_2y_2 + A_3y_3}{A_1 + A_2 + A_3} = 27.45$$

Theorems of Pappus–Guldinus

A surface of revolution is a surface which can be generated by rotating a plane curve about a fixed axis.



For example, in the above figure, the curved surface of a cylinder is obtained by rotating the line AB about the x-axis.

Theorem I

The area of a surface of revolution is equal to the product of the length of the generating curve and the distance traveled by the centroid of the curve while the surface is being generated.

NOTE

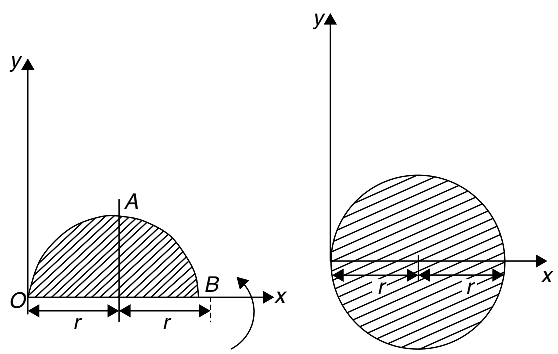
The generating curve must not cross the axis about which it is rotated.

In the above figure, length of the generating curve $= L$.

Distance traveled by the centroid while the surface is being generated $= 2\pi r$ (circumference of a circle of radius r)

$\therefore$ Area of the surface of the cylinder generated $= L \times 2\pi r = 2\pi rL$

A body of revolution is a body which can be generated by rotating a plane area about a fixed axis.



(a) (b)

For example, in the above figure, the volume of a sphere is obtained by rotating the semi-circle OAB about the x-axis.

Theorem II

The volume of a body of revolution is equal to the product of the generating area and the distance traveled by the centroid of the area while the body is being generated.

NOTE

The theorem does not apply if the axis of rotation intersects the generating area.

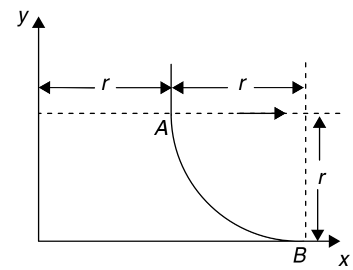
In the figure, generating area $= \frac{1}{2}\pi r^2$.

Distance traveled by the centroid of the area while the body is being generated $= 2\pi \times \frac{4r}{3\pi}$ (circumference of a circle of radius $\frac{4r}{3\pi}$)

$\therefore$ Volume of the sphere generated

$$= \frac{1}{2}\pi r^2 \times 2\pi \times \frac{4r}{3\pi} = \frac{4\pi r^3}{3}$$

Example 11: A quartered circular arc AB when rotated about the y-axis generates a surface of area A_y. The same



arc when rotated about the x-axis generates a surface of area A_x. If the ratio $A_y : A_x$ is related to the length r by the equation $\frac{Ay}{Ax} = kr^n$, where k, n are constants, then the value of k and n respectively are

(A) 0.27 and 0 (B) 0.27 and 1
(C) 3.75 and 0 (D) 3.75 and 1

Solution:

Length of the arc $= \frac{1}{2}\pi r$;

x coordinate of the centroid of the arc $= 2r - \frac{2r}{\pi}$.

Distance travelled by the centroid when the arc is rotated about the y-axis $= \frac{2\pi \times 2r(\pi - 1)}{\pi}$

Using Pappus–Guldinus theorem I,

$$A_y = \left(2r - \frac{2r}{\pi}\right) \times \frac{2\pi \times 2r(\pi - 1)}{\pi} = 2r^2\pi(\pi - 1)$$

y coordinate of the centroid of the arc $= r - \frac{2r}{\pi}$.

Distance travelled by the centroid when the arc is rotated about the x-axis $= \frac{2\pi \times r(\pi - 2)}{\pi}$.

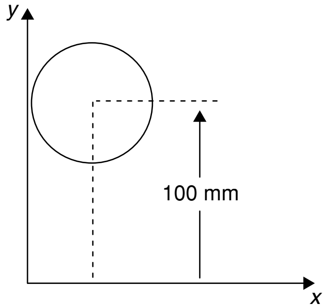
Using Pappus–Guldinus theorem I,

$$A_x = \left(r - \frac{2r}{\pi}\right) \times \frac{2\pi \times r(\pi - 2)}{\pi} = r^2\pi(\pi - 2)$$

$$\therefore \frac{Ay}{Ax} = kr^n = \frac{2(\pi - 1)}{\pi - 2}$$

$$\Rightarrow \quad n = 0 \text{ and } k = \frac{2(\pi - 1)}{\pi - 2}.$$

Example 12: A solid ring (torus) of circular cross-section is obtained by rotating a circle of radius 25 mm about the x-axis as shown in the following figure.



If the density of the material making up the circular cross-section is 7800 kg/m^3, the weight of the ring generated is:

(A) 82.6 N (B) 94.4 N
(C) 123.4 N (D) 90.6 N

Solution:

y coordinate of the centroid of the circle = 100 mm = 0.1 m

Area of the circle = $\pi \times (0.025)^2$

Distance traveled by the centroid of the circle while generating the ring = $2\pi \times (0.1)$ (circumference of a circle of radius 0.1 m)

Using Pappus–Guldinus theorem II,

Volume of the ring generated

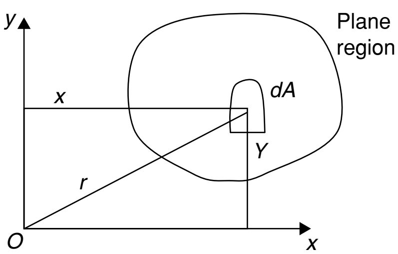
$= \pi \times (0.025)^2 \times 2\pi \times (0.1) = 0.001233$ m^3

Weight of the generated ring

$= 7800 \times 0.001233 \times 9.81 = 94.4$ N.

Area Moment of Inertia

In a plane region of area A, a differential area dA located at the point (x, y) is considered as shown in the below figure.



The moment of inertia of the area about the x-axis and y-axis respectively are:

$$I_x = \int_A y^2 dA \quad \text{and} \quad I_y = \int_A x^2 dA$$

I_x and I_y are also called as the second moments of the area.

Polar Moment of Inertia

In the above figure, the polar moment of inertia of the area about the point O (actually, about an axis through the point O, perpendicular to the plane of the area) is

$$J_0 = \int_A r^2 dA$$

$$J_0 = I_x + I_y$$

The above equation states that the polar moment of inertia of an area about a point O is the sum of the moments of inertia of the area about two perpendicular axes that intersect at O.

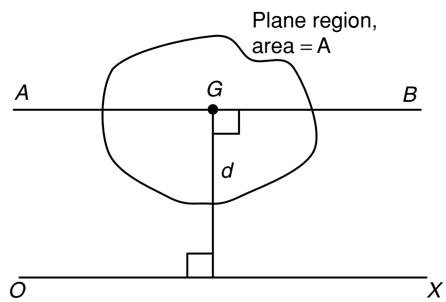
Radius of Gyration

In the above figure, the radii of gyration of an area about the x-axis, y-axis and the origin O are:

$$k_x = \sqrt{\frac{I_x}{A}}, \quad k_y = \sqrt{\frac{I_y}{A}} \quad and \quad k_o = \sqrt{\frac{J_o}{A}}$$

Parallel Axis Theorem

The moment of inertia of a plane region area about an axis, say AB, in the plane of area through the centre of gravity of the plane region area be represented by I_G, then the moment of inertia of the given plane region area about a parallel axis, say OX, in the plane of the area at a distance d from the centre of gravity of the area is $I_X = I_G + Ad^2$,



Where,

I_X = moment of inertia of the given area about the OX axis

I_G = moment of inertia of the given area about AB axis

A = area of the plane region

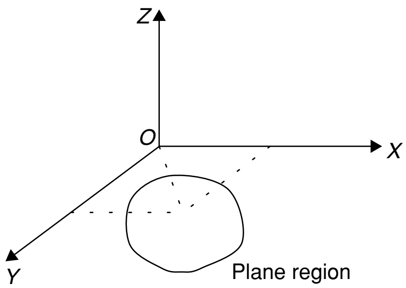
d = perpendicular distance between the parallel axes AB and OX

G = centre of gravity of the plane region

Perpendicular Axis Theorem

If I_{OX} and I_{OY} are the moments of inertia of a plane region area about two mutually perpendicular axes OX and OY in the plane of the area, then the moment of inertia of the plane region area I_{OZ} about the axis OZ, perpendicular to the plane and passing through the intersection of the axes OX and OY is:

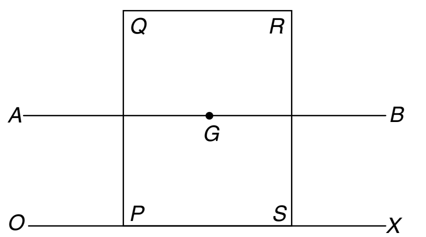
$$\boxed{I_{OZ} = I_{OX} + I_{OY}}$$



NOTE

I_{OZ} is also called as the polar moment of inertia and the axis OZ is called as the polar axis.

Example 13: In the below figure, the axes AB and OX are parallel to each other. If the moments of inertia of the rectangle $PQRS$ along the axis AB, which passes through the centroid of the rectangle, and the axis OX are I_G and I_X respectively, then the value of I_X/I_G is



(A) 4 (B) 12 (C) 3 (D) 0.25

Solution:

From parallel axis theorem, we have $I_X = I_G + A$ (perpendicular distance between axes)^2,

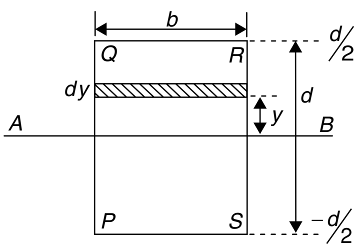
Let $PQ = d$ and $QR = b$, then the perpendicular distance between the axes $= \frac{d}{2}$

$$\therefore \; I_X = I_G + A\frac{d^2}{4} = I_G + bd\frac{d^2}{4}$$

So,

$$\boxed{\frac{I_X}{I_G} = 1 + \frac{bd^3}{4I_G}}$$

To determine I_G, let us consider a rectangular strip of thickness dy at a distance y from the axis AB as shown below:



Area of the rectangular strip = bdy

Moment of inertia of the strip about the axis

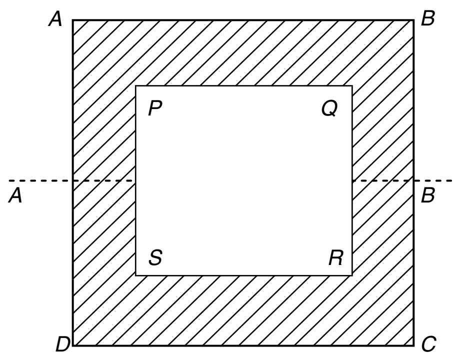
$$AB = (bdy)\,y^2$$

Moment of inertia of the rectangle $PQRS$ about the axis

$$AB,\; I_G = \int_{-d/2}^{d/2} by^2 dy = \frac{bd^3}{12}$$

$$\therefore \; \frac{I_X}{I_G} = 4.$$

Example 14: The moment of inertia for the following hatched figure about the axis AB (which passes through the centroid of the figure), where $AB = DC = 30$ m, $PQ = SR = 20$ m, $BC = AD = 20$ m and $QR = PS = 10$ m, is:



(A) $6.78 \times 10^4\ m^4$ (B) $5.41 \times 10^3\ m^4$
(C) $1.83 \times 10^4\ m^4$ (D) $2.6 \times 10^5\ m^4$

Solution:

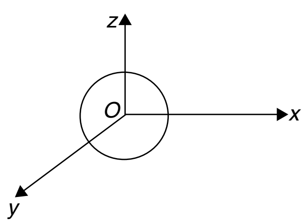
Moment of inertia of the hatched figure = moment of inertia of □ $ABCD$ – Moment of inertia of □ $PQRS$

$$= \frac{1}{12} \times (DC \times AD^3 - SR \times QR^3$$

$$= \frac{1}{12} \times (30 \times 20^3 - 20 \times 10^3)$$

$$= 18333.33\,\text{m}^4.$$

Example 15: A circular section of diameter d is lying on the xy–plane where the centre of the circular section coincides with the origin O as shown in the following figure.

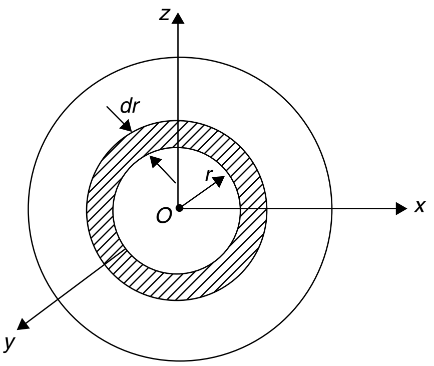


If the moments of inertia of the circular section along the x, y and z axes are I_X, I_Y and I_Z respectively, then which of the following statements is NOT correct?

(A) $I_X = \frac{\pi d^4}{32}$ (B) $I_X = I_Y$

(C) $I_Z = \frac{\pi d^4}{32}$ (D) $I_Y = \frac{\pi d^4}{64}$

Solution:



Let us consider an elementary ring of thickness dr and located at a distance r from the origin O.

Area of the elementary ring $= 2\pi r dr$.

Moment of inertia of the elementary ring about the z-axis $= 2\pi r dr \times r^2 = 2\pi r^3 dr$.

Moment of inertia of the whole circular section about the z-axis $= \int_0^{D/2} 2\pi r^3 dr = \frac{\pi d^4}{32}$

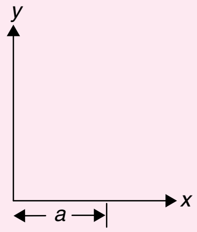
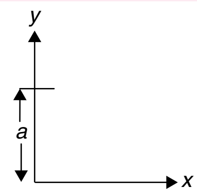
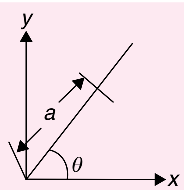
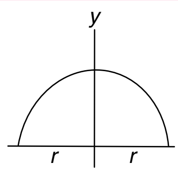
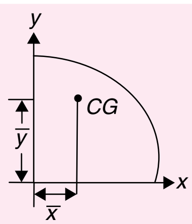
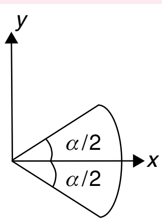
From the symmetry of the circular section, it can be written that $I_X = I_Y$.

From the perpendicular axis theorem, we have,

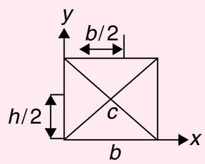
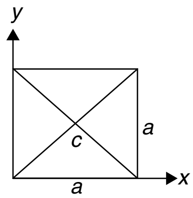
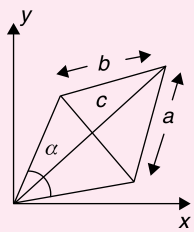
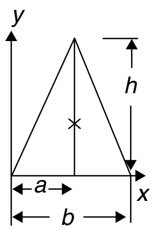
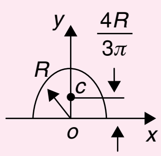
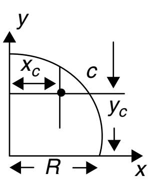
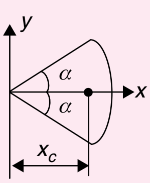
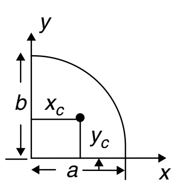
$$I_Z = I_X + I_Y$$

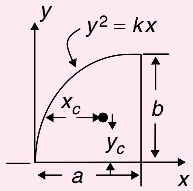
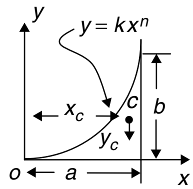
i.e., $I_Z = 2I_X$

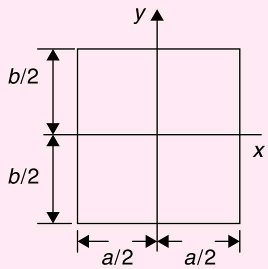
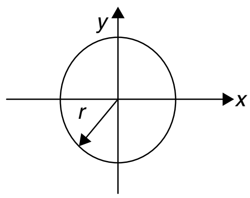
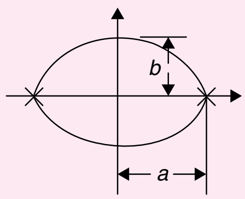
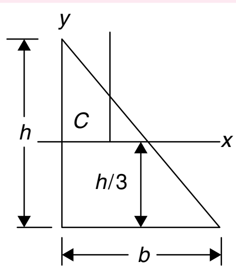
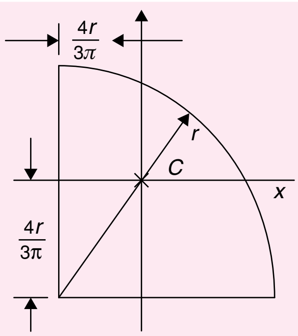
$$\therefore I_X \frac{\pi d^4}{64} = I_Y$$

Description	Shape	L	$\overline{x_c}$	$\overline{y_c}$
Horizontal line		a	$\frac{a}{2}$	0
Vertical line		a	0	$\frac{a}{2}$
Inclined line with θ		a	$\left(\frac{a}{2}\right)\cos\theta$	$\left(\frac{a}{2}\right)\sin\theta$
Semicircular arc		πr	0	$\frac{2r}{\pi}$
Quarter circular arc		$\frac{\pi r}{2}$	$\frac{2r}{\pi}$	$\frac{2r}{\pi}$
Circular arc		αr	$\frac{2r\sin\alpha/2}{\alpha}$	0

(Continued)

Description	Shape	L	$\overline{x_c}$	$\overline{y_c}$
Rectangle		bh	$\frac{b}{2}$	$\frac{h}{2}$
Square		a^2	$\frac{a}{2}$	$\frac{a}{2}$
Parallelogram		$ab \sin\alpha$	$\frac{b + a\cos\alpha}{2}$	$\frac{a\sin\alpha}{2}$
Triangle		$\frac{bh}{2}$	$\frac{a+b}{3}$	$\frac{h}{3}$
Semi circle		$\frac{\pi R^2}{2}$	0	$\frac{4R}{3\pi}$
Quarter circle		$\frac{\pi r^2}{2}$	$\frac{4R}{3\pi}$	$\frac{4R}{3\pi}$
Sector of a circle		$R^2\alpha$	$\frac{2}{3}\frac{R\sin\alpha}{\alpha}$	0
Quarter ellipse		$\frac{\pi ab}{3}$	$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$

Quarter parabola		$\frac{\pi ab}{3}$	$\frac{3a}{5}$	$\frac{3b}{5}$
General spandrel		$\frac{ab}{3}$	$\frac{3a}{4}$	$\frac{3b}{4}$

Description	Figure	$\bar{I}_X$	$\bar{I}_Y$
Rectangle		$\frac{ab^3}{12}$	$\frac{ba^3}{12}$
Circle		$\frac{\pi r^4}{4}$	$\frac{\pi r^4}{4}$
Ellipse		$\frac{\pi ab^3}{4}$	$\frac{\pi ba^3}{4}$
Triangle		$\frac{bh^3}{36}$	$\frac{hb^3}{36}$
Quadrant Circle		$0.0549\ r^4$	$0.0549\ r^4$

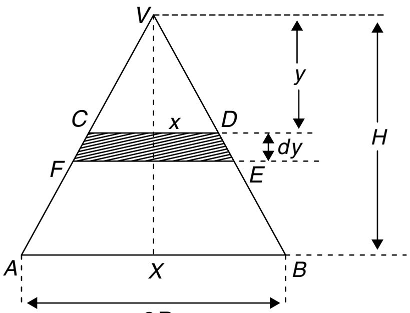
Centroid of Solids

If dm is an elemental mass in a body of mass M and x_G, y_G are the coordinates of the center of gravity of the body from the reference axes y-axis and x-axis respectively, then:

$$X_G = \frac{\int xdm}{\int dm} = \frac{\int xdm}{M}, \quad y_G = \frac{\int ydm}{\int dm} = \frac{\int ydm}{M}$$

Let us consider a right circular solid cone whose centre of gravity is to be determined. Let the diameter of the base of the right circular solid cone be $2R$ and its height H as shown in the following figure.

Since the cone is symmetric about the VX axis, its centre of gravity will lie on this axis. The cone can be imagined to be consisting of an infinite number of circular discs with different radii, parallel to the base.



Consider one such disc of radius x, thickness dy and at a depth y from the vertex of the cone, i.e., from V.

From the geometry of the above figure,

$$\frac{x}{R} = \frac{y}{H} \quad or \quad X = \frac{yR}{H}$$

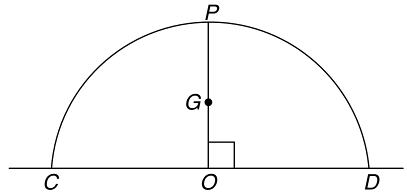
Volume of disc $= \pi X^2 dy = \pi \frac{y^2 R^2}{H^2} dy$

If ρ is the density of the material making up the cone, then dm $= \rho \frac{\pi y^2 R^2}{H^2} dy$

$$\therefore y_G \frac{\int ydm}{dm} = \frac{\int_0^H \rho \frac{\pi y^3 R^2}{H^2} dy}{\int_0^H \rho \frac{\pi y^2 R^2}{H^2} dy} = \frac{3}{4}[y]_0^H = \frac{3}{4}H$$

$\therefore$ Centroid or centre of gravity of a right circular cone is situated at a distance of $\frac{3}{4}$ H from its vertex V and lies on its axis VX.

Example 16: In the homogenous hollow hemisphere, shown in the following figure, OP = 10 cm = the radius of the hemisphere. The points P, G and O lie on a straight line that is perpendicular to the base CD. If G is the centroid of the hollow hemisphere, then which one of the following statements is not correct?



(A) $OG = 5$ cm (B) $OG = \frac{3}{8}OP$

(C) $CO = 10$ cm (D) $OD = 2 \times OG$

Solution:

The centre of gravity of a hollow hemisphere with respect to the x-axis would lie on an perpendicular axis along which the homogeneous hemisphere is symmetrical.

Since G is the centre of gravity, then the hemisphere should be symmetrical along OP, i.e., $CO = OD$.

It can also be deciphered that $CO = OD$ = radius of the hemisphere = OP = 10 cm.

Now OG will be equal to $R/2$, where R is the radius of the hollow hemisphere.

$\therefore OG = 0.5\ OP = 5$ cm

It can be written $OP = CO = OD = 2\ OG$, hence the option (B) is NOT correct.

NOTE

Option B would be right if the hemisphere had been a homogeneous solid hemisphere.

Mass Moment of Inertia

The Moment of Inertia of an element of mass is the product of the mass of the element and the square of the distance of the element from the axis.

The mass moment of inertia of the body with respect to Cartesian frame xyz is given by:

$$I_{xx} = \int (y^2 + z^2)\,dm = \int_v (y^2 + z^2)\,\rho dv$$

$$I_{YY} = \int (x^2 + z^2)\,dm = \int_v (x^2 + z^2)\,\rho dv$$

$$I_{zz} = \int (x + y^2)\,dm = \int_v (x^2 + y^2)\,\rho dv,$$ where, I_{XX}, I_{YY} and I_{ZZ} are the axial moments of inertia of mass with respect to the x-, y- and z-axes respectively.

For thin plates essentially in the x-y plane, the following relations hold.

$$I_{xx} = \int y^2 dm$$

$$I_{YY} = \int x^2 dm$$

$$I_{zz} = \int z^2 dm = \int (x^2 + y^2)\,dm$$

$$I_{zz} = I_{XX} + I_{YY}$$

I_{zz} is also called the polar moment of inertia.

Mass Moment of Inertia and Radius of Gyration

$$I_{xx} = K_x^2 m$$
$$I_{yy} = K_y^2 m$$
$$I_{zz} = K_z^2 m$$
$$K_x = \sqrt{\frac{I_{xx}}{m}}$$
$$K_y = \sqrt{\frac{I_{YY}}{m}}$$
$$K_z = \sqrt{\frac{I_{zz}}{m}}$$

The parallel-axis theorem for the mass moment of inertia states that the mass moment of inertia with respect to any axis is equal to the moment of inertia of the mass with respect to a parallel axis through the centre of mass plus the product of the mass and the square of the perpendicular distance between the axes.

Mathematically $I_{AB} = I_G + md^2$

For a thin plate,

$$I_{xx(\text{mass})} = \rho t\, I_{xx(\text{area})}$$
$$I_{YY(\text{mass})} = \rho t\, I_{YY(\text{area})}$$
$$I_{zz(\text{mass})} = \rho t\, I_{zz(\text{area})}$$

Where t is the uniform thickness and ρ is the mass of the thin plate.

$$I_{zz} = I_{xx} + I_{YY}$$

The mass moment of inertia about a centroidal axis perpendicular to a uniform thin rod of length ℓ, mass m and small cross section is given by

$$I_{YY} = \frac{1}{12} m\ell^2$$

Radius of gyration about a centroidal axis perpendicular to a uniform thin rod of the length ℓ, mass m and a small cross section is given by

$$K_y = \frac{\ell}{\sqrt{12}}$$

The mass moment of inertia about the longitudinal and transverse axes passing through the centre of mass of a rectangular prism (block) of cross section (axb), uniform density ρ and length ℓ is given by $I_{xx} = \frac{1}{12} m(\ell^2 + b^2)$.

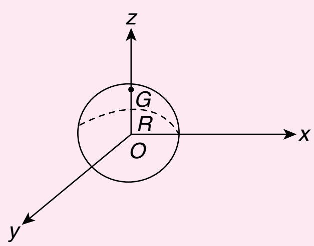
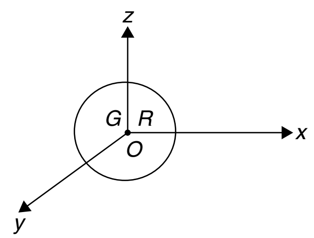
$$I_{YY} = \frac{1}{12} m(a^2 + b^2)$$
$$I_{zz} = \frac{1}{12} m(a^2 + \ell^2)$$

In the above case, if the three axes were chosen through a corner instead of centre of mass, the results are:

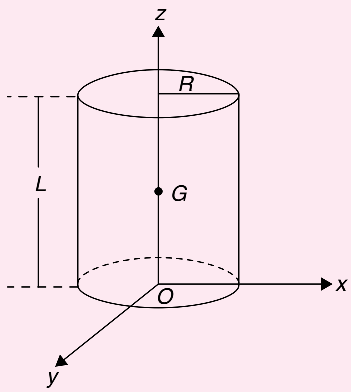
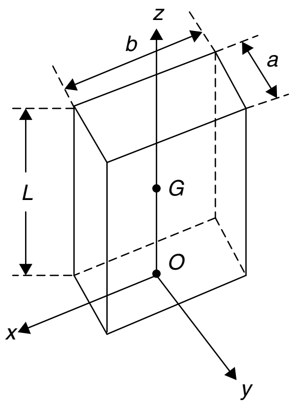
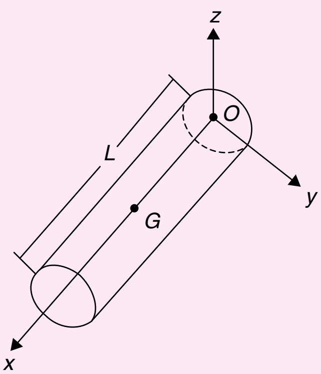
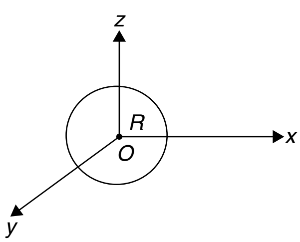
$$I_{xx} = \frac{1}{3} m(\ell^2 + b^2)$$
$$I_{YY} = \frac{1}{3} m(a^2 + b^2)$$
$$I_{zz} = \frac{1}{3} m(a^2 + \ell^2)$$

For a right circular cylinder of radius R, length or height ℓ and mass m, the mass moment of inertia about the centroidal x-axis is given by

$$I_{xx} = m\left[\frac{R^4}{4} + \frac{\ell^2}{12}\right]$$

Solid Body	Centroid	Mass moment of inertia
Solid hemisphere 	$x_G = y_G = 0$, $Z_G = \frac{3}{8}R$	$I_{XX} = I_{YY} = I_{ZZ} = \frac{2}{5}mR^2$
Solid sphere 	$x_G = y_G = z_G = 0$	$I_{XX} = I_{YY} = I_{ZZ} = \frac{2}{5}mR^2$, $K_y = \sqrt{\frac{2}{5}}R$

(Continued)

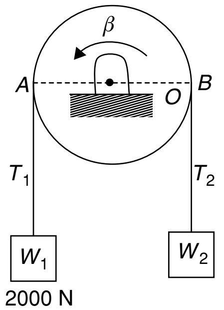
Solid Body	Centroid	Mass moment of inertia
Solid cylinder 	$x_G = y_G = 0$ $z_G = \frac{L}{2}$	$I_{XX} = I_{YY} = \frac{1}{4}mR^2 + \frac{1}{3}mL^2$ $I_{ZZ} = \frac{1}{2}mR^2$
Rectangular block (cuboid) 	$x_G = y_G = 0$ $z_G = \frac{L}{2}$	$I_{xx} = \frac{1}{12}ma^2 + \frac{1}{3}mL^2$ $I_{yy} = \frac{1}{12}mb^2 + \frac{1}{3}mL^2$ $I_{zz} = \frac{1}{12}m(a^2 + b^2)$
Slender rod (thin cylinder) 	$z_G = \frac{L}{2}$ $y_G = z_G = 0$	$I_{xx} = 0$ $I_{YY} = I_{ZZ} = \frac{mL^2}{3}$
Solid disk 	$x_G = y_G = z_G = 0$	$I_{XX} = I_{YY} = \frac{mR^2}{4}$ $I_{zz} = \frac{mR^2}{2}$ $K_z = \frac{r}{\sqrt{2}}$

Exercises

Practice Problems I

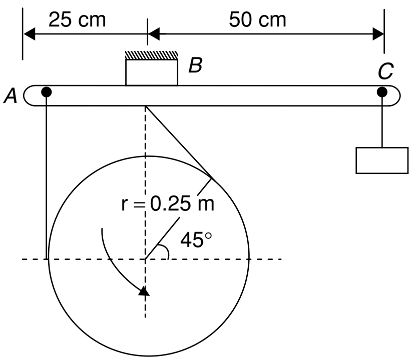
Select the correct alternative from the given choices.

1. A belt supports two weights W_1 and W_2 over a pulley as shown in the figure. If $W_1 = 2000$ N, the minimum weight W_2 to keep W_1 in equilibrium (assume that the pulley is locked and $\mu = 0.25$) is:



(A) 911.9 N (B) 812.8 N
(C) 913 N (D) 715.5 N

2.



A rotating wheel is braked by a belt *AB* attached to the lever *ABC* hinged at *B*. The coefficient of friction between the belt and the wheel is 0.5. The braking moment exerted by the vertical weight W = 200 N is:

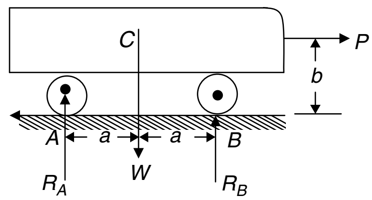
(A) 98.23 Nm (B) 95.96 Nm
(C) 95.00 Nm (D) 93.24 Nm

3. A screw jack has square threaded screw of 5 cm diameter and 1 cm pitch. The coefficient of friction at the screw thread is 0.15. The force required at the end of a 70 cm long handle to raise a load of 1000 N and the force required, at the end of the same handle to raise the same load, if the screw jack is considered to be an ideal machine, respectively, are:

(A) 7.702 N and 2.123 N
(B) 7.702 N and 2.273 N
(C) 8.162 N and 1.850 N
(D) 8.162 N and 1.798 N

Direction for question 4: A locomotive of weight *W* is at rest.

4. The reactions at *A* and *B* are



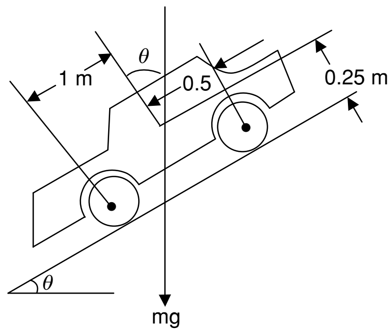
(A) $\frac{W}{2}N$ (B) $2WN$
(C) $\frac{2}{3}WN$ (D) $\sqrt{3}WN$

Direction for question 5: When it is pulling a wagon, the draw bar pull *P* is just equal to the total friction at the points of contact, *A* and *B*.

5. The new magnitudes of the vertical reactions at *A* and B respectively are:

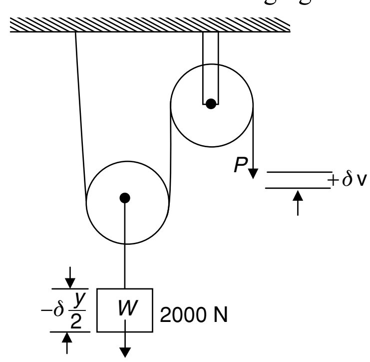
(A) $\frac{Wa-Pb}{2a}, \frac{Wa+Pb}{2a}$ (B) $\frac{W}{2}, \frac{W}{2a}$
(C) $\frac{W}{2}, \frac{W}{3}$ (D) $\frac{W}{2}, \frac{2}{3}W$

6. A four wheel vehicle with passengers has a mass of 2000 kg passengers. The road, on which the vehicle is moving, is inclined at an angle θ with the horizontal. If the coefficient of static friction between tyres and the road is 0.3, the maximum inclination θ at which the vehicle can still climb is:



(A) 18° (B) 16.7° (C) 15° (D) 17.2°

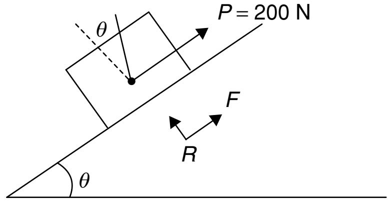
7. A weight *W* of 2000 N is to be raised by a system of pulleys as shown in the following figure.



The value of the force P which can hold the system in equilibrium is:

(A) 5000 N (B) 1000 N
(C) 2000 N (D) 1500 N

Direction for questions 8, 9 and 10: A weight of 600 N just starts moving down a rough inclined plane supported by a force of 200 N acting parallel to the plane and it is at the point of moving up the plane when pulled by a force of 300 N parallel to the plane.



8. The values of the normal reaction R and the limiting friction F, respectively are:

(A) $500 \cos\theta$ and $500\mu \cos\theta$
(B) $400 \cos\theta$ and $400\mu \cos\theta$
(C) $600 \cos\theta$ and $600\mu \cos\theta$
(D) $600 \cos\theta$ and $500\mu \cos\theta$

9. The inclination of the plane θ is:

(A) 30° (B) 25.6°
(C) 24.6° (D) 32.1°

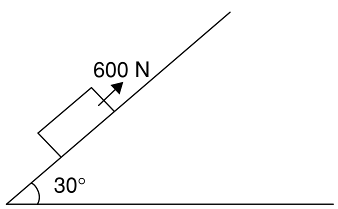
10. The coefficient of friction is:

(A) 0.092 (B) 0.1124
(C) 0.1510 (D) 0.2130

Practice Problems 2

Direction for questions 1 to 10: Select the correct alternative from the given choices.

1. The block shown in figure below is kept in equilibrium and prevented from sliding down by applying a force of 600 N. The co-efficient of friction is $\frac{\sqrt{3}}{5}$. The weight of the block would be:



(A) 4000 N (B) 2500 N
(C) 3000 N (D) 5000 N

2. Mention the statements which are governing the laws of friction between dry surfaces.

(i) The friction force is independent on the velocity of sliding.
(ii) The friction force is proportional to the normal force across surface of contact
(iii) The friction force is dependent on the materials of the contacting surfaces.
(iv) The friction force is independent of the area of contact.

(A) 2, 3, 4 (B) 1 and 3
(C) 2 and 4 (D) 1, 2, 3 and 4

3. The limiting friction between two bodies in contact is independent of

(A) Nature of surfaces in contact;
(B) The area of surfaces in contact;
(C) Normal reaction between the surfaces.
(D) All of the above.

4. A body of weight 50 N is kept on a plane inclined at an angle of 30° to the horizontal. It is in limiting equilibrium. The co-efficient friction is the equal to:

(A) $\frac{1}{\sqrt{3}}$ (B) $\sqrt{3}$
(C) $\frac{1}{50\sqrt{3}}$ (D) $\frac{\sqrt{3}}{5}$

5. A man of weight 60 N stands on the middle rung of a ladder of weight 15 N. The co-efficient of friction between contacting surfaces is 0.25. The reaction at the floor is:

(A) 80 N (B) 73.25 N
(C) 85.6 N (D) 72.75 N

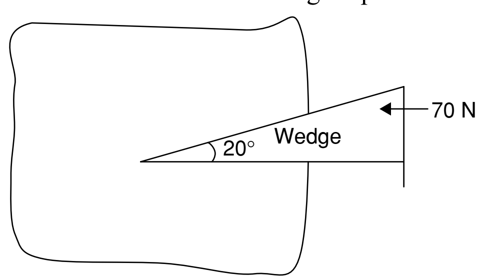
6. Determine the effort required at the end of an arm 50 cm long to lift a load of 5 kN by means of a simple screw jack with screw threads of pitch 1 cm if the efficiency at this load is 45%.

(A) 40.8 N (B) 43.6 N
(C) 44.8 N (D) 35.36 N

7. Determine the effort needed if the jack in above question is converted into a differential screw jack with internal threads of pitch 7 mm and efficiency of operation is 30%.

(A) 15.9 N (B) 19.8 N
(C) 17.2 N (D) 18 N

8. A wooden block is being split by a 20° wedge with a force of 70 N applied horizontally as shown. Taking the co-efficient of friction between wood and the wedge as 0.4 estimate the vertical force tending to split the wood apart.



(A) −54.24 N, 54 N (B) −65 N, 64 N
(C) −48 N, 46 N (D) −56 N, 54 N

9. A screw thread of screw jack has a mean diameter of 10 cm and a pitch of 1.25 cm. The co-efficient of friction between the screw and its nut housing is 0.25. The force *F* that must be applied at the end of a 50 cm lever arm to raise a mass 6000 kg, is:

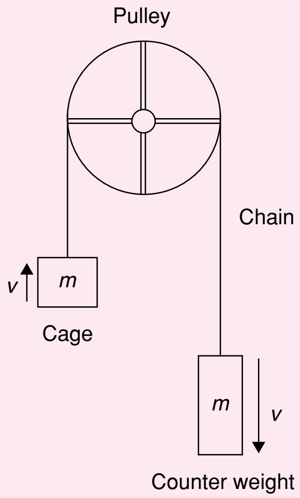
(A) 1985 N (B) 1723 N
(C) 1630 N (D) 1874 N

10. Efficiency of the screw jack in problem above is:

(A) 12% (B) 13.7%
(C) 15% (D) 16.4%

PREVIOUS YEARS' QUESTIONS

1. An elevator (lift) consists of the elevator cage and a counter weight, of mass m each. The cage and the counterweight are connected by a chain that passes over a pulley. The pulley is coupled to a motor. It is desired that the elevator should have a maximum stopping time of t seconds from a peak speed v. If the inertia of the pulley and the chain are neglected, the minimum power that the motor must have is: **[2005]**



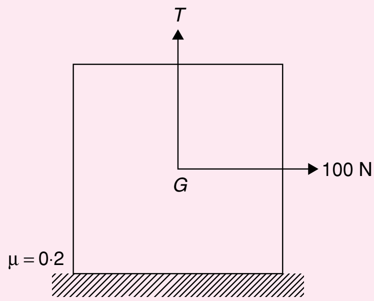
(A) $\frac{1}{2}mv^2$ (B) $\frac{mv^2}{2t}$

(C) $\frac{mv^2}{t}$ (D) $\frac{2mv^2}{t}$

2. If a system is in equilibrium and the position of the system depends upon many independent variables, the principle of virtual work states that the partial derivatives of its potential energy with respect to each of the independent variable must be: **[2006]**

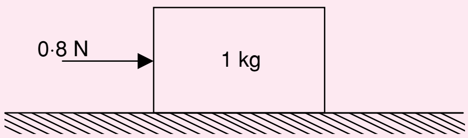
(A) −1.0 (B) 0
(C) 1.0 (D) ∞

3. A block weighing 981 N is resting on a horizontal surface. The coefficient of friction between the block and the horizontal surface is $\mu = 0.2$. A vertical cable attached to the block provides partial support as shown. A man can pull horizontally with a force of 100 N. What will be the tension, *T* (in *N*) in the cable if the man is just able to move the block to the right? **[2009]**



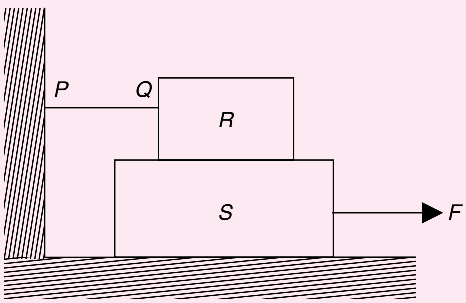
(A) 176.2 (B) 196.0
(C) 481.0 (D) 981.0

4. A 1 kg block is resting on a surface with coefficient of friction $\mu = 0.1$. A force of 0.8 N is applied to the block as shown in the figure. The friction force is: **[2011]**



(A) 0 (B) 0.8 N
(C) 0.98 N (D) 1.2 N

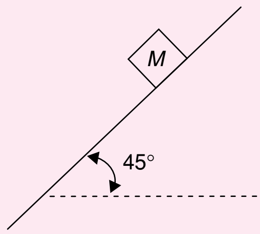
5. A block *R* of mass 100 kg is placed on a block *S* of mass 150 kg as shown in the figure. Block *R* is tied to the wall by a massless and inextensible string *PQ*. If the coefficient of static friction for all surfaces is 0.4, the minimum force *F* (in kN) needed to move the block *S* is: **[2014]**



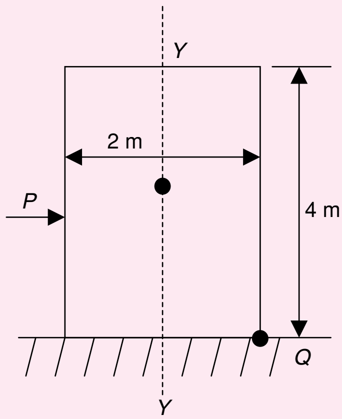
(A) 0.69 (B) 0.88
(C) 0.98 (D) 1.37

6. A block weighing 200 N is in contact with a level plane whose coefficients of static and kinetic friction are 0.4 and 0.2 respectively. The block is acted upon by a horizontal force (in newton) $P = 10t$, where t denotes the time in seconds. The velocity (in m/s) of the block attained after 10 seconds is: ____ **[2014]**

7. A body of mass (M) 10 kg is initially stationary on a 45° inclined plane as shown in figure. The coefficient of dynamic friction between the body and the plane is 0.5.? The body slides down the plane and attains a velocity of 20 m/s. The distance travelled (in meter) by the body along the plane is ________ **[2014]**

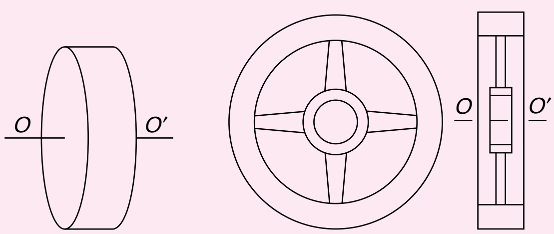


8. A wardrobe (mass 100 kg, height 4 m, width 2 m, depth 1 m), symmetric about the Y-Y axis, stands on a rough level floor as shown in the figure. A force P is applied at mid-height on the wardrobe so as to tip it about point Q without slipping. What are the minimum values of the force (in newton) and the static coefficient of friction μ between the floor and the wardrobe, respectively? **[2014]**

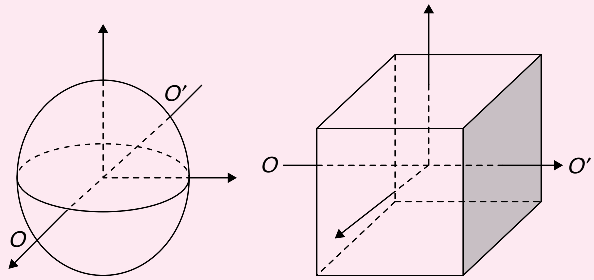


(A) 490.5 and 0.5
(B) 981 and 0.5
(C) 1000.5 and 0.15
(D) 1000.5 and 0.25

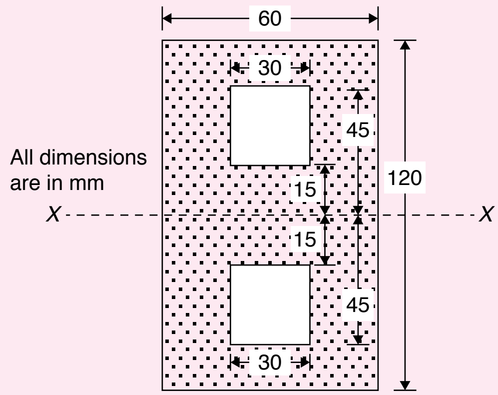
9. For the same material and the mass, which of the following configurations of flywheel will have maximum mass moment of inertia about the axis of rotation OO' passing through the center of gravity. **[2015]**



(A) Solid Cylinder (B) Rimmed wheel
(C) Solid sphere (D) Solid cube



10. The value of moment of inertia of the section shown in the figure about the axis-XX is: **[2015]**

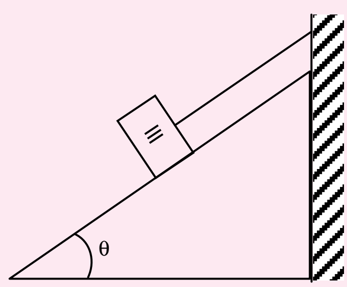


(A) 8.5050×10^6 mm^4 (B) 6.8850×10^6 mm^4
(C) 7.7625×10^6 mm^4 (D) 8.5725×10^6 mm^4

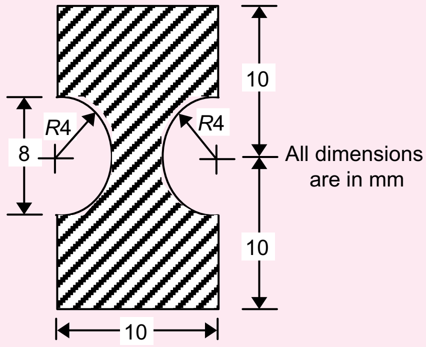
11. A block of mass m rests on an inclined plane and is attached by a string to the wall as shown in the figure. The coefficient of static friction between the plane and the block is 0.25. The string can withstand a maximum force of 20 N. The maximum value of the mass (m) for which the string will not break and the block will be in static equilibrium is ________ kg. **[2016]**

Take cos θ = 0.8 and sin θ = 0.6

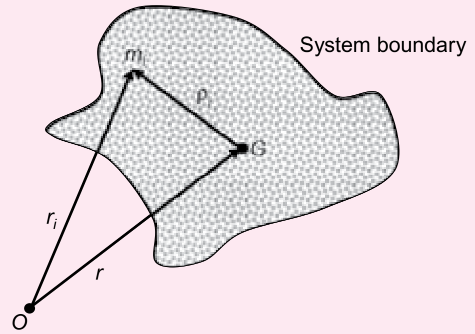
Acceleration due to gravity $g = 10$ m/s^2



12. The figure shows cross-section of a beam subjected to bending. The area moment of inertia (in mm^4) of this cross-section about its base is ________. **[2016]**



13. A system of particles in motion has mass center G as shown in the figure. The particle i has mass m_i and its position with respect to a fixed point O is given by the position vector r_i. The position of the particle with respect to G is given by the vector ρ_i. The time rate of change of the angular momentum of the system of particles about G is (The quantity $\ddot{\rho}_i$ indicates second derivative of ρ_i with respect to time and likewise for r_i). **[2016]**



(A) $\Sigma_i r_i \times m_i \ddot{\rho}_i$ (B) $\Sigma_i \rho_i \times m_i \ddot{r}_i$
(C) $\Sigma_i r_i \times m_i \ddot{r}_i$ (D) $\Sigma_i \rho_i \times m_i \ddot{\rho}_i$

Answer Keys

Exercises

Practice Problems 1

1. A **2.** B **3.** B **4.** A **5.** A **6.** B **7.** B **8.** C **9.** C **10.** A

Practice Problems 2

1. C **2.** A **3.** B **4.** A **5.** D **6.** D **7.** A **8.** A **9.** B **10.** B

Previous Years' Questions

1. C **2.** B **3.** C **4.** B **5.** D **6.** 4.8 to 5 **7.** 56 to 59 **8.** A **9.** B **10.** B
11. 5 **12.** 1873 to 1879 **13.** B