Exercise 11.8

Q1E

A series is called a power series if it is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + ---$$

Where x is a variable, a and c_n s are constants.

We can say that the series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series centered at a.

Q2E

(A) The radius of convergence, R is a positive number such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.
We can find R generally by ratio test.

(B) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.

Since $|x-a| < R$

$$-R < x-a < R$$

$$a-R < x < a+R$$

At the end points of the interval ($x = a \pm R$) the series may converge or diverge.

Q3E

Given series $\sum_{n=1}^{\infty} (-1)^n nx^n$

Let $a_n = (-1)^n nx^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{(-1)^n n x^n}\right|$$

$$= \left|\frac{(-1)^n(-1)n\left(1+\frac{1}{n}\right)x^n x}{(-1)^n n x^n}\right|$$

$$= \left(1+\frac{1}{n}\right)|x| \to |x| \qquad \text{as n} \to \infty$$

Using the Ratio Test, we see that the series converges if
$|x| < 1$.
Thus the radius of converges is R=1.
The inequality $|x| < 1$ can be written as $-1 < x < 1$ so we test the series at the end points -1 and 1.

When x= -1, the series is $\sum_{n=1}^{\infty} n$
Which diverges by the Test for diverges.

When x=1 the series $\sum_{n=1}^{\infty} (-1)^n n$
This diverges by the Test for Divergence.
Thus the series converges only when $-1 < x < 1$
So the interval of converges in (-1, 1)

Ratio test:

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L<1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L>1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio test is inconclusive

Q4E

Given series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$

Let $a_n = \frac{(-1)^n x^n}{\sqrt[3]{n}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1} x^{n+1}}{\sqrt[3]{n+1}}}{\frac{x^n}{\sqrt[3]{n}}}\right|$$

$$= \left|\frac{\frac{(-1)^n (-1) x^n \cdot x}{\sqrt[3]{n}\sqrt[3]{1+\frac{1}{n}}}}{\frac{(-1)^n x^n}{\sqrt[3]{n}}}\right|$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^{\frac{1}{3}}}|x| \to |x| \quad \text{as } n \to \infty$$

Using the Ratio Test, we see that the series converges if $|x| < 1$ and diverges if $|x| > 1$

Thus the radius of converges is R=1.

The inequality $|x| < 1$ can be written as $-1 < x < 1$

So we test the series at the end points -1 and 1

When x= -1, the series is $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

Which is diverges since it is a p- series with $p = \frac{1}{3} < 1$

When x= 1, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$

Let $b_n = \frac{1}{\sqrt[3]{n}}$

Thus series satisfies

(1) $b_{n+1} < b_n$ because $\frac{1}{\sqrt[3]{n+1}} < \frac{1}{\sqrt[3]{n}}$

(2) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$

So the series is convergent by the Alternating series test.

Hence the interval of converges is (-1, 1]

Alternating Series Test:

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 + b_2 + \ldots \qquad b_n > 0$$

satisfies

$(i) b_{n+1} < b_n$ for all n

$(i) \lim_{n\to\infty} b_n = 0$

then the series is convergent.

Q5E

Given series $\sum_{n=1}^{\infty}\frac{x^n}{2n-1}$

Let $a_n = \frac{x^n}{2n-1}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{2n+1}}{\frac{x^n}{2n-1}}\right|$$

$$= \left|\frac{\frac{x^n \cdot x}{2n\left(1+\frac{1}{2n}\right)}}{\frac{x^n}{2n\left(1-\frac{1}{2n}\right)}}\right|$$

$$= \frac{1-\frac{1}{2n}}{\left(1+\frac{1}{2n}\right)}|x| \to |x| \quad \text{as } n \to \infty$$

Using the Ratio Test, we see that the series converges if $|x|<1$ and it diverges if $|x|>1$

Thus the radius of converges is R=1.

The inequality $|x|<1$ can be written as $-1<x<1$

So we test the series at the end points -1 and 1

When x= -1, the series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{2n-1}$

Let $b_n = \frac{1}{2n-1}$

Thus series satisfies

$(i) b_{n+1} < b_n$ *because* $\frac{1}{2n+1} < \frac{1}{2n-1}$

$(ii) \lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{2n-1} = 0$

So the series is convergent by the Alternating series test.

When x = 1, the series

$$\sum_{n=1}^{\infty} \frac{1}{2n-1}$$

Let $b_n = \frac{1}{n}$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2n-1}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}}$$

$$= \frac{1}{2} > 0$$

Since $\sum_{n=1}^{\infty} a_n$ is divergent, $\sum_{n=1}^{\infty} b_n$ is also divergent.

Thus series is divergent by the limit comparison test.

So the interval of convergence is [-1, 1)

Alternating Series Test:

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 + b_2 + \ldots \qquad b_n > 0$$

satisfies

$(i)\, b_{n+1} < b_n$ for all n

$(i)\, \lim_{n\to\infty} b_n = 0$

then the series is convergent.

Limit Comparison Test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Q6E

Given series is $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$

We have to find the radius of convergence and the interval of convergence

Let $a_n = \frac{(-1)^n x^n}{n^2}$.

Then, $a_{n+1} = \frac{(-1)^{n+1} x^{n+1}}{(n+1)^2}$

Now

$$\left|\frac{a_{n+1}}{a_n}\right| = |x|\left[\frac{n^2}{(n+1)^2}\right]$$

$$= |x|\left(1+\frac{1}{n}\right)$$

$$= \frac{|x|}{\left(1+\frac{1}{n}\right)^2}$$

We note as $n \to \infty$, $|x| \to 1$

Then, the series converges when $-1 < x < 1$ and diverges for $x < -1$ or $x > 1$.

Now, if $x = -1$, the given series becomes $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a p-series with $p = 2$ and thus convergent.

If $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. Then, by the alternating series test it is convergent.

Therefore, the radius of convergence R is 1 and the interval of convergence is $[-1, 1]$.

Q7E

We have the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

If $a_n = \frac{x^n}{n!}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \frac{|x|}{n+1}$$

And $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|\lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$ for all x

Thus, by the Ratio test, the given series converges for all values of x. Therefore the radius of convergence is $\boxed{R = \infty}$ and the interval of convergence is $\boxed{(-\infty, \infty)}$.

Q8E

We have the series $\sum_{n=1}^{\infty} n^n x^n$

If $a_n = n^n x^n$

Then $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|n^n x^n|}$

$= |x| \lim_{n\to\infty} n = \infty$

The given series is divergent. The series converges only when $|x| = 0$

Therefore the radius of convergence is $\boxed{R = 0}$ and interval of convergence is $\{0\}$.

Q9E

Consider the series, $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 x^n}{2^n}$

Let $a_n = (-1)^n \frac{n^2 x^n}{2^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}\frac{(n+1)^2 x^{n+1}}{2^{n+1}}}{(-1)^n \frac{n^2 x^n}{2^n}}\right|$$

$$= \left|(-1)^{n+1}\frac{(n+1)^2 x^{n+1}}{2^{n+1}} \times \frac{2^n}{(-1)^n n^2 x^n}\right|$$

$$= \left|(-1)\frac{(n+1)^2 x}{2n^2}\right|$$

$$= \left|(-1)\frac{n^2\left(1+\frac{1}{n}\right)^2 x}{2n^2}\right|$$

$$= \left|(-1)\frac{\left(1+\frac{1}{n}\right)^2 x}{2}\right|$$

$$= \frac{\left(1+\frac{1}{n}\right)^2}{2}|x|$$

$$\to \frac{|x|}{2} \text{ as } n \to \infty$$

By the ratio test, the above series converges if $\frac{|x|}{2} < 1$ and diverges if $\frac{|x|}{2} > 1$.

Thus it converges if $|x| < 2$ and diverges if $|x| > 2$.

Thus the radius of converges is $R = 2$

The series converges in the interval $(-2,2)$

Now test for convergence at the endpoint of this interval

If $x = -2$, the series becomes

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n \frac{n^2 x^n}{2^n} &= \sum_{n=1}^{\infty}(-1)^n \frac{n^2(-2)^n}{2^n} \\ &= \sum_{n=1}^{\infty}(-1)^{2n} \frac{n^2 2^n}{2^n} \\ &= \sum_{n=1}^{\infty}(-1)^{2n} n^2\end{aligned}$$

$$= \sum_{n=1}^{\infty} n^2$$

$$= 1 + 4 + 9 + \cdots$$

Which is divergence by the divergence test.

If $x = 2$, the series becomes

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n \frac{n^2 x^n}{2^n} &= \sum_{n=1}^{\infty}(-1)^n \frac{n^2(2)^n}{2^n} \\ &= \sum_{n=1}^{\infty}(-1)^n \frac{n^2 2^n}{2^n} \\ &= \sum_{n=1}^{\infty}(-1)^n n^2\end{aligned}$$

$$= -1 + 4 - 9 + \cdots$$

Which is divergence by the divergence test.

Therefore the above power series converges when $-2 < x < 2$. So the interval of converges is

$$\boxed{(-2,2)}$$

Q10E

Consider the series, $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$

Let $a_n = \frac{10^n x^n}{n^3}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{10^{n+1} x^{n+1}}{(n+1)^3}}{\frac{10^n x^n}{n^3}}\right|$$

$$= \left|\frac{10^{n+1} x^{n+1}}{(n+1)^3} \times \frac{n^3}{10^n x^n}\right|$$

$$= \left|\frac{10x}{n^3\left(1+\frac{1}{n}\right)^3} \times \frac{n^3}{1}\right|$$

$$= \left|\frac{10x}{\left(1+\frac{1}{n}\right)^3}\right|$$

$\rightarrow 10|x|$ as $n \rightarrow \infty$

By the ratio test, the above series converges if $10|x| < 1$ and diverges if $10|x| > 1$.

Thus it converges if $|x| < \frac{1}{10}$ and diverges if $|x| > \frac{1}{10}$.

Thus the radius of converges is $R = \frac{1}{10}$

The series converges in the interval $\left(-\frac{1}{10}, \frac{1}{10}\right)$

Now test for convergence at the endpoint of this interval

If $x = -\frac{1}{10}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3} = \sum_{n=1}^{\infty} \frac{10^n\left(-\frac{1}{10}\right)^n}{n^3}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

$$= -1 + \frac{1}{8} - \frac{1}{27} + \cdots$$

It converges by the alternating series test, as $\frac{1}{n^3}$ is a decreasing sequence converges to 0

If $x = \frac{1}{10}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3} = \sum_{n=1}^{\infty} \frac{10^n \left(\frac{1}{10}\right)^n}{n^3}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^3}$$
$$= 1 + \frac{1}{8} + \frac{1}{27} + \cdots$$

Which is converges by using integral test.

Therefore the above series observe that it is a $p-$series with $p = 3 > 1$

Therefore the above power series converges when $-\frac{1}{10} \le x \le \frac{1}{10}$ So the interval of converges is $\boxed{\left[-\frac{1}{10}, \frac{1}{10}\right]}$

Q11E

Given series is $\sum_{n=1}^{\infty} \frac{(-3)^n}{n\sqrt{n}} x^n$

We have to find the radius of convergence and interval of convergence of the given series

Let $a_n = \frac{(-3)^n x^n}{n^{\frac{3}{2}}}$ and $a_{n+1} = \frac{(-3)^{n+1} x^{n+1}}{(n+1)^{\frac{3}{2}}}$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1} x^{n+1}}{(n+1)^{\frac{3}{2}}} \cdot \frac{n^{\frac{3}{2}}}{(-3)^n x^n}\right|$$
$$= \left|\frac{(-3)^1 n^{\frac{3}{2}} x}{(n+1)^{\frac{3}{2}}}\right|$$
$$= 3|x| \frac{1}{\left(1 + \frac{1}{n}\right)^{\frac{3}{2}}}$$

We note that as $n \to \infty$, $\left|\frac{a_{n+1}}{a_n}\right| \to 3|x|$

By the Ratio test, the series is convergent if $3|x| < 1$ and diverges if $3|x| > 1$

Thus, the series converges when $|x| < \frac{1}{3}$ and diverges $|x| > \frac{1}{3}$

Now, if $x = -\frac{1}{3}$, then the given series becomes $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ which is a p-series with $p = \frac{3}{2} > 1$ and thus is convergent

If $x = \frac{1}{3}$, then the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{3}{2}}}$

Then, by the alternating series test it is convergent.

Therefore, the radius of convergence R is $\frac{1}{3}$ and the interval of convergence is $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

Q12E

Given series is $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$

We have to find the radius of convergence and interval of convergence of the given series

Let $a_n = \frac{x^n}{3^n n}$ and $a_{n+1} = \frac{x^{n+1}}{3^{n+1}(n+1)}$.

Now

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{x^n}\right|$$

$$= \left|\frac{nx}{3(n+1)}\right|$$

$$= \left|\frac{x}{3}\right|\left(\frac{n}{n+1}\right)$$

$$= \left|\frac{x}{3}\right|\left(\frac{1}{1+\frac{1}{n}}\right)$$

Then, as $n \to \infty$, $\left|\frac{a_{n+1}}{a_n}\right| \to \left|\frac{x}{3}\right|$

By the Ratio test, the series is convergent if $\left|\frac{x}{3}\right| < 1$ and diverges if $\left|\frac{x}{3}\right| > 1$

Thus, the series converges when $|x| < 3$ and diverges for $|x| > 3$.

Now, if $x = -3$, the given series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

Then, by the alternating series test, it is convergent.

If $x = 3$, then the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$ which is a harmonic series and is divergent.

Therefore, the radius of convergence R is 3 and the interval of convergence is $[-3, 3)$.

Q13E

We have been given the series.

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n}$$

Let, $a_n = (-1)^n \frac{x^n}{4^n \ln n}$

Then $a_{n+1} = (-1)^{n+1} \frac{x^{n+1}}{4^{n+1} \ln(n+1)}$

Now,
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} x^{n+1}}{4^{n+1} \ln(n+1)} \cdot \frac{4^n \ln n}{(-1)^n x^n}\right|$$
$$= \left|\frac{-x}{4} \frac{\ln n}{\ln(n+1)}\right|$$
$$= \frac{|x|}{4} \frac{\ln(n)}{\ln(n+1)}$$

Then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|}{4}\frac{\ln n}{\ln(n+1)}$

$$= \frac{|x|}{4}\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}$$

$$= \frac{|x|}{4}\lim_{n\to\infty}\frac{1/n}{1/(n+1)}$$

$\left[\text{Since by L-Hospital rule } \lim_{x\to\infty}\frac{\ln x}{\ln(x+1)} \overset{H}{=} \lim_{x\to\infty}\frac{1/x}{1/(x+1)}\right]$

$$= \frac{|x|}{4}\lim_{n\to\infty}\frac{n+1}{n}$$

$$= \frac{|x|}{4}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$$= \frac{|x|}{4} \qquad \left[\text{Since } \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1+0 = 1\right.$$

By ratio test, the series converges when $\frac{|x|}{4} < 1 \Rightarrow |x| < 4$

So, the radius of convergence is $\boxed{R = 4}$

Since the radius of convergence is R = 4, so the series converges in the interval (–4,4), we must test for convergence at the end points of this interval

If x = 4, the series is

$$\sum_{n=2}^{\infty}\frac{(-1)^n 4^n}{4^n \ln n} = \sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$$

This is an alternating series with $b_n = 1/\ln n$

Now, $\left[b_{n+1} = \frac{1}{\ln(n+1)}\right] \leq \left[b_n = \frac{1}{\ln n}\right]$ for all n

And $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\ln n} = 0$

Thus, the series converges by alternating series test.

If x = −4, the series is

$$\sum_{n=2}^{\infty}\frac{(-1)^n(-4)^n}{4^n \ln n}=\sum_{n=2}^{\infty}\frac{1}{\ln n}$$

Since, $\ln n < n$ For $n \geq 2$

$\Rightarrow \quad \frac{1}{\ln n} > \frac{1}{n}$ For, $n \geq 2$

$\Rightarrow \quad \sum_{n=2}^{\infty}\frac{1}{\ln n} > \sum_{n=2}^{\infty}\frac{1}{n}$

But the series $\sum_{n=2}^{\infty}\frac{1}{n}$ is a divergent harmonic series.

So, by comparison test, the series $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ is also divergent.

Therefore, the interval of convergence for given series is $\boxed{(-4,\ 4]}$ and the radius of convergence is $\boxed{R=4}$

Q14E

Consider the series.

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

To find the radius of convergence and interval of convergence for the given series, use the ratio test.

Ratio test states that

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Let $a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, denote the *n*th term of the series.

And

$$a_{n+1} = \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!}$$

$$= \frac{(-1)^n (-1) x^{2n+3}}{(2n+3)!}$$

$$= \frac{-(-1)^n x^{2n+3}}{(2n+3)!}$$

Now, the ratio $\frac{a_{n+1}}{a_n}$ is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{-(-1)^n x^{2n+3}}{(2n+3)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}}$$

$$= \frac{-(-1)^n x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}}$$

$$= \frac{-(-1)^n x^{2n} \cdot x^3}{(2n+3)(2n+2)(2n+1)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n} \cdot x}$$

$$= \frac{-x^2}{(2n+3)(2n+2)}$$

Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{-x^2}{(2n+3)(2n+2)}\right|$$
$$= \frac{x^2}{(2n+3)(2n+2)}$$
$$= \frac{x^2}{n^2\left(2+\frac{3}{n}\right)\left(2+\frac{2}{n}\right)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{x^2}{n^2\left(2+\frac{3}{n}\right)\left(2+\frac{2}{n}\right)}$$
$$= \lim_{n\to\infty}\frac{x^2}{n^2\left(2+\frac{3}{n}\right)\left(2+\frac{2}{n}\right)}$$
$$= \lim_{n\to\infty}\left(\frac{1}{n}\right)^2\cdot\frac{x^2}{\left(2+\frac{3}{n}\right)\left(2+\frac{2}{n}\right)}$$
$$= 0\cdot\frac{x^2}{(2+0)(2+0)}$$
$$= 0 \qquad \text{Since } \frac{1}{n}\to 0 \text{ as } n\to\infty$$

So, as $n\to\infty$, $\left|\frac{a_{n+1}}{a_n}\right| \to 0 < 1$ for all x.

Thus, by the Ratio Test, the given series converges for all values of x.

Therefore, the radius of convergence R is ∞ and the interval of convergence is $(-\infty, \infty)$.

Q15E

Consider the series,

$$\sum_{n=0}^{\infty}\frac{(x-2)^n}{n^2+1}$$

The objective is to find the radius of convergence and interval of convergence of the given series.

To find the radius of convergence, use the Ratio Test to see which values of x make the series $\sum_{n=0}^{\infty}\frac{(x-2)^n}{n^2+1}$ converge.

Let $a_n = \frac{(x-2)^n}{n^2+1}$. Then $a_{n+1} = \frac{(x-2)^{n+1}}{(n+1)^2+1}$.

Take the limit of the ratios of successive terms:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{(n+1)^2+1}}{\frac{(x-2)^n}{n^2+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2+1}\times\frac{n^2+1}{(x-2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)^n(x-2)}{(n+1)^2+1}\times\frac{n^2+1}{(x-2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)}{n^2+1+2n+1}\times\frac{n^2+1}{1}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)}{n^2\left(1+\frac{2}{n^2}+\frac{2}{n}\right)}\times\frac{n^2\left(1+\frac{1}{n^2}\right)}{1}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)}{\left(1+\frac{2}{n^2}+\frac{2}{n}\right)}\times\frac{\left(1+\frac{1}{n^2}\right)}{1}\right|$$

$$= |(x-2)| \text{ as } n\to\infty$$

By the ratio test, the above series converges if $|(x-2)| < 1$ and diverges if $|(x-2)| > 1$.

Thus the radius of converges is $\boxed{R=1}$.

Since,

$$|(x-2)|<1 \Rightarrow$$

$$-1<x-2<1$$

$$-1+2<x-2+2<1+2$$

$$1<x<3$$

So the series converges when $1<x<3$

Consider $x=1$ in the series $\sum_{n=0}^{\infty}\frac{(x-2)^n}{n^2+1}$, it becomes

$$\sum_{n=0}^{\infty}\frac{(1-2)^n}{n^2+1}=\sum_{n=0}^{\infty}\frac{1}{n^2+1}(-1)^n$$

Hence, it converges by using the alternating series test.

If we put $x=3$ in the series $\sum_{n=0}^{\infty}\frac{(x-2)^n}{n^2+1}$, it becomes

$$\sum_{n=0}^{\infty}\frac{(3-2)^n}{n^2+1}=\sum_{n=0}^{\infty}\frac{1}{n^2+1}$$

which is convergent series by comparison test.

Therefore the above power series converges when $1\le x\le 3$

So the interval of convergence is $\boxed{[1,3]}$.

Q16E

We have the series $\sum_{n=0}^{\infty}(-1)^n\frac{(x-3)^n}{2n+1}$, If $a_n=(-1)^n\frac{(x-3)^n}{2n+1}$

Then $\left|\frac{a_{n+1}}{a_n}\right|=\left|(-1)^{n+1}\frac{(x-3)^{n+1}}{2n+3}\cdot\frac{2n+1}{(-1)^n(x-3)^n}\right|=\left(\frac{2n+1}{2n+3}\right)|x-3|$

So $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left(\frac{2n+1}{2n+3}\right)|x-3|=|x-3|$

Thus by the ratio test, the given series converges when $|x-3|<1$,

Thus the radius of convergence is $\boxed{R=1}$

Since $|x-3|<1$, so $2<x<4$

When $x=4$, the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ is convergent by alternating series test

Since here $b_n=\frac{1}{2n+1}$ and $\left(b_{n+1}=\frac{1}{2n+3}\right)<\left(b_n=\frac{1}{2n+1}\right)$, $\lim_{n\to\infty}b_n=0$

When $x=2$, the series $\sum_{n=0}^{\infty}\frac{1}{2n+1}$ is divergent by integral test

Therefore, the interval of convergence is $(2,\ 4]$

Q17E

We have the series $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}}(x+4)^n$

If $a_n = \frac{3^n}{\sqrt{n}}(x+4)^n$

Then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}(x+4)^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{3^n(x+4)^n}\right|$

$$= \frac{3|x+4|}{\sqrt{1+1/n}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3|x+4|}{\sqrt{1+1/n}}$$

$$= 3|x+4|\left(\frac{1}{\sqrt{1+0}}\right) \quad \left(n\to\infty \Rightarrow \frac{1}{n}\to 0\right)$$

$$= 3|x+4|$$

Thus by the Ratio test, the given series converges when $3|x+4|<1$,

That is, $|x+4|<\frac{1}{3}$, thus the radius of convergence is $\boxed{R=1/3}$

Since $-\frac{1}{3}<x+4<\frac{1}{3} \Rightarrow -13/3<x<-11/3$

If $x=-\frac{13}{3}$, then $\sum_{n=1}^{\infty}\frac{3^n}{\sqrt{n}}\left(-\frac{1}{3}\right)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ is convergent alternating series

And if $x=-\frac{11}{3}$, then $\sum_{n=1}^{\infty}\frac{3^n}{\sqrt{n}}\frac{1}{3^n} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is divergent p-series with p <1.

Therefore the interval of convergence is $\boxed{\left[-\frac{13}{3},-\frac{11}{3}\right)}$.

Q18E

Consider the power series

$$\sum_{n=1}^{\infty}\frac{n}{4^n}(x+1)^n$$

Its need to determine the radius of convergence and interval of convergence of the given series

Observe that n^{th} term of the given series is

$$a_n = \frac{n}{4^n}(x+1)^n$$

Consider $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{4^{n+1}}(x+1)^{n+1}\cdot\frac{4^n}{n}\frac{1}{(x+1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)\cdot\frac{1}{4}(x+1)\right|$$

$$= \left|\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\cdot\frac{1}{4}(x+1)\right|$$

$$= \left|(1+0)\cdot\frac{1}{4}(x+1)\right|$$

$$= \left|\frac{x+1}{4}\right|$$

Thus $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x+1}{4}\right|$

By the Ratio Test, the series $\sum a_n x^n$ converge if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

By the above fact, the given series converges if

$$\left|\frac{x+1}{4}\right| < 1$$

$$\Leftrightarrow \quad |x+1| < 4$$

This means that given series converges if $|x+1| < 4$ and diverges if $|x+1| > 4$.

That is the radius of convergence is $R = 4$

As $|x+1| < 4 \Leftrightarrow -4 < x+1 < 4$

$$\Leftrightarrow \quad -4-1 < x+1-1 < 4-1$$

$$\Leftrightarrow \quad -5 < x < 3$$

So the series converges when $-5 < x < 3$ and diverges when $x < -5$ or $x > 3$

The Ratio test gives no information when $|x+1| = 4$

$\Leftrightarrow \quad x+1 = 4$ and $x+1 = -4$

$\Leftrightarrow \quad x = 3$ and $x = -5$

So we must consider $x = -5$ and $x = 3$ separately.

If we keep $x=-5$ in the series $\sum_{n=1}^{\infty}\frac{n}{4^n}(x+1)^n$, it becomes

$$\sum_{n=1}^{\infty}\frac{n}{4^n}(-5+1)^n=\sum_{n=1}^{\infty}\frac{n}{4^n}(-4)^n$$

$$=\sum_{n=1}^{\infty}\frac{n}{4^n}(-4)^n$$

$$=\sum_{n=1}^{\infty}\frac{n}{4^n}(-1)^n(4)^n$$

$$=\sum_{n=1}^{\infty}(-1)^n n$$

which is divergent series.

If we put $x=3$ in the series $\sum_{n=1}^{\infty}\frac{n}{4^n}(x+1)^n$, it becomes

$$\sum_{n=1}^{\infty}\frac{n}{4^n}(3+1)^n=\sum_{n=1}^{\infty}\frac{n}{4^n}(4)^n$$

$$=\sum_{n=1}^{\infty}n$$

which is divergent series.

Therefore, the given power series is converges when $-5<x<3$, so the interval of convergence is $(-5,3)$

Q19E

Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$$

Let $a_n = \frac{(x-2)^n}{n^n}$

Then, calculate $\left|\frac{a_{n+1}}{a_n}\right|$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{(x-2)^n}\right|$$

$$= \left|(x-2) \cdot \frac{n^n}{(n+1)^{n+1}}\right|$$

$$= |x-2| \left|\frac{n^n}{(n+1)^n (n+1)}\right|$$

$$= |x-2| \frac{1}{\left(1+\frac{1}{n}\right)^n (n+1)} \to 0 \text{ as } n \to \infty$$

Since $\left(1+\frac{1}{n}\right)^n \to e \quad \text{as } n \to \infty$

That is, the value of $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \text{is } 0$

Recall the Ratio Test that states that if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Hence, by the Ratio Test, the given series converges for all values of *x*.

Thus the radius of convergence is $\boxed{R = \infty}$ and interval of convergence is $\boxed{(-\infty, \infty)}$.

Q20E

Let $a_n = \frac{(2x-1)^n}{5^n\sqrt{n}}$ and

$a_{n+1} = \frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}$.

Find $\left|\frac{a_{n+1}}{a_n}\right|$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}\right)\left(\frac{5^n\sqrt{n}}{(2x-1)^n}\right)$$

$$= \frac{|2x-1|}{5}\left(\frac{1}{\sqrt{1+\frac{1}{n}}}\right)$$

Then, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|2x-1|}{5}$. This means that the series is convergent if $|2x-1| < 5$ or $-5 < 2x-1 < 5$. That is $-2 < x < 3$.

Now, if $x = -2$, then the series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ which is convergent by the alternating series test. If $x = 3$, the series becomes $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which is a p-series with $p = \frac{1}{2}$ and is divergent.

Therefore, the radius of convergence R is $\frac{5}{2}$ and the interval of convergence is $[-2, 3)$.

Q21E

Given $\sum_{n=1}^{\infty}\frac{n}{b^n}(x-a)^n, \quad b > 0$

Let $C_n = \frac{n}{b^n}(x-a)^n$

Then $C_{n+1} = \frac{(n+1)}{b^{n+1}}(x-a)^{n+1}$

Thus $$\left|\frac{C_{n+1}}{C_n}\right| = \left|\frac{(n+1)(x-a)^{n+1}}{b^{n+1}}\cdot\frac{b^n}{n(x-a)^n}\right|$$
$$= \frac{1}{b}|x-a|\left(\frac{n+1}{n}\right)$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = \frac{1}{b}|x-a|$$

Using ratio test, series converges if $\frac{|x-a|}{b} < 1$

$\Rightarrow \quad |x-a| < b$

Thus the radius of convergence $\boxed{R = b}$

The inequality $|x-a| < b$ can be written as

$a - b < x < a + b$

We test the series at the endpoints (a - b) and (a + b)

When x = a – b, the series is

$$\sum_{n=1}^{\infty} \frac{n}{b^n}(-b)^n = \sum_{n=1}^{\infty} (-1)^n . n$$

This is divergent, since $\lim_{n\to\infty} n = \infty$

When x = a + b, the series becomes as $\sum_{n=1}^{\infty} n$ which is also divergent by the test for divergence.

So the interval of convergence is $\boxed{(a-b,\ a+b)}$

Q22E

We have $\sum_{n=2}^{\infty} \frac{b^n (x-a)^n}{\ln\ n}, b > 0.$

Let $a_n = \frac{b^n (x-a)^2}{\ln\ n}$ and $|a_n|^{\frac{1}{n}} = \frac{b|x-a|}{(\ln\ n)^{\frac{1}{n}}}$.

As $n \to \infty$, $|a_n|^{\frac{1}{n}} = b|x-a|$. This means that the given series convergent if $b|x-a| < 1$ or $|x-a| < \frac{1}{b}$. That is $a - \frac{1}{b} < x < a + \frac{1}{b}$.

Also, the given series is divergent if $b|x-a| > 1$.

Now, if $x = a + \frac{1}{b}$, then the given series becomes $\sum_{n=1}^{\infty} \frac{1}{\ln\ n}$. We know that $\ln n < n$ for all n. Then $\frac{1}{\ln n} > \frac{1}{n}$ for all n. We get $\sum_{n=1}^{\infty} \frac{1}{\ln\ n} > \sum_{n=1}^{\infty} \frac{1}{n}$. The series on the right hand side is a part of the harmonic series and thus it is divergent.

Thus, if $x = a + \frac{1}{b}$, then by part **(ii)** of the comparison test, the given series also divergent.

Now, if $x = a - \frac{1}{b}$, then the given series becomes $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$. We know that $n + 1 > n$.

Then, $\ln(n + 1) > \ln n$. This means that $\frac{1}{\ln(n+1)} < \frac{1}{\ln n}$ for all n or $b_{n+1} < b_n$ for all n.

Also, $\lim_{x\to\infty} b_n = \lim_{x\to\infty} \frac{1}{\ln n} = 0$.

Thus, by alternating series test the series is convergent. The radius of convergence R is obtained as $\frac{1}{b}$ and the interval of convergence is $\left[a - \frac{1}{b}, a + \frac{1}{b}\right]$.

Q23E

The objective is to determine the radius of convergence and interval of convergence of the series.

Consider the series, $\sum_{n=1}^{\infty} n!(2x-1)^n$

Take $a_n = n!(2x-1)^n$

$$a_{n+1} = (n+1)!(2x-1)^{n+1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right|$$

$$= \left|\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right|$$

$$= |(n+1)(2x-1)|$$

$$= 2\left|(n+1)\left(x-\frac{1}{2}\right)\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 2\left|(n+1)\left(x-\frac{1}{2}\right)\right|$$

$$= 2\left|x-\frac{1}{2}\right|\lim_{n\to\infty}|(n+1)|$$

$$= \infty$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty > 1 \quad \text{provided } x \neq \frac{1}{2}$$

Thus, radius of convergence is, $R = 0$.

The power series will only converges if $x = \frac{1}{2}$

Every power series will converge for $x = a$

Take $a = \frac{1}{2}$

Power series of the form, $\left(x - \frac{1}{2}\right)^n$, the coefficient of the x must be one.

Thus, the interval of convergence is, $x = \frac{1}{2}$

Therefore, the radius of convergence is $\boxed{R = 0}$ and the interval of convergence is $\boxed{\left\{\frac{1}{2}\right\}}$

Q24E

Consider the series $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2.4.6.....(2n)}$.

Rewrite the above equation as follows:

$$\sum_{n=1}^{\infty} \frac{n^2 x^n}{2.4.6.....(2n)} = \sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n (1.2.3.......n)}$$
$$= \sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n n!} \qquad \text{......} (1)$$

Check the convergence of series (1).

Let $a_n = \frac{n^2 x^n}{2^n n!}$.

Now consider $\left|\frac{a_{n+1}}{a_n}\right|$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2 (x)^{n+1}}{2^{n+1}\,(n+1)!}\frac{2^n\ n!}{n^2\ \ x^n}\right|$$

$$= \left|\frac{(n+1)^2 (x)^n \cdot x}{2^n \cdot 2\,(n+1)\cdot n!}\cdot\frac{2^n\ n!}{n^2\ \ x^n}\right|$$

$$= \frac{(n+1)^2 |x|}{2(n+1)n^2}$$

$$= \frac{(n+1)}{2}\frac{|x|}{n^2}$$

$$= \frac{1}{2}\left(\frac{1}{n}+\frac{1}{n^2}\right)|x|$$

Take limit on both sides.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n^2}\right)|x|$$

$$= \frac{1}{2}(0+0)|x| \qquad \text{if } n\to\infty \text{ then } \frac{1}{n}\to 0$$

$$= 0\ \ (<1)$$

Ratio test:

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Every absolutely convergent series is convergent.

Therefore, $\sum_{n=1}^{\infty} a_n$ is convergent.

Thus by the ratio test, the given series converges for all values of *x*.

Therefore, the radius of converges is $\boxed{R=\infty}$.

The interval of convergence of a power series is the interval that consists of all values of *x* for which the series is converges.

Therefore, the interval of converges is $\boxed{(-\infty,\infty)}$.

Q25E

Consider the power series,

$$\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}.$$

The objective is to determine the radius of convergence and radius of convergence of the given series.

To determine the radius of convergence:

Apply the ratio test,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(5x-4)^{n+1}}{(n+1)^3}\cdot\frac{n^3}{(5x-4)^n}\right| \\
&= \lim_{n\to\infty}\left|(5x-4)\left(\frac{n}{n+1}\right)^3\right| \\
&= |5x-4|\lim_{n\to\infty}\left|\left(\frac{1}{1+\frac{1}{n}}\right)^2\right| \\
&= |5x-4|(1) \\
&= |5x-4|
\end{aligned}$$

By the ratio test, the series converges for

$$|5x-4|<1$$
$$\left|x-\frac{4}{5}\right|<\frac{1}{5}$$

Thus, the radius of convergence is $\boxed{R=\frac{1}{5}}$.

From the radius of convergence, determine the interval of convergence as follows:

$$\left|x-\frac{4}{5}\right|<\frac{1}{5}$$
$$\Rightarrow -\frac{1}{5}<x-\frac{4}{5}<\frac{1}{5}$$
$$\Rightarrow \frac{4}{5}-\frac{1}{5}<x<\frac{4}{5}+\frac{1}{5}$$
$$\Rightarrow \quad \frac{3}{5}<x<1$$

First, check the endpoint $x = \frac{3}{5}$ as below.

Substitute $x = \frac{3}{5}$ in the power series $\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$, to get:

$$\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(3-4)^n}{n^3}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

By the alternating series test, this series converges.

So, the endpoint $x = \frac{3}{5}$ can be included to the interval of convergence.

Now, check the endpoint $x = 1$ as below.

Substitute $x = 1$ in the power series $\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$, to get:

$$\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(1)^n}{n^3}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^3}$$

By the *p*-series test, this series converges.

So, the endpoint $x = 1$ can be included to the interval of convergence.

Therefore, the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$ in interval notation is

$$\boxed{I = \left[\frac{3}{5}, 1\right]}.$$

Q26E

Consider the series

$$\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$$

Need to find the radius of converges and interval of converges.

To use the ratio test first find $\frac{a_{n+1}}{a_n}$:

Let $a_n = \frac{x^{2n}}{n(\ln n)^2}$

Then

$$a_{n+1} = \frac{x^{2(n+1)}}{(n+1)(\ln(n+1))^2}$$
$$= \frac{x^{2n+2}}{(n+1)(\ln(n+1))^2}$$

So,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+2}}{(n+1)(\ln(n+1))^2} \cdot \frac{n(\ln n)^2}{x^{2n}}\right|$$
$$= \left|\frac{n}{n+1}\right|\left(\frac{\ln n}{\ln(n+1)}\right)^2 \left|\frac{x^{2n} \cdot x^2}{x^{2n}}\right|$$
$$= \frac{1}{\left(1+\frac{1}{n}\right)} \cdot \left(\frac{\ln n}{\ln(n+1)}\right)^2 \left|x^2\right|$$
$$\to \left|x^2\right| \text{ as } n \to \infty$$

Since as $n \to \infty$ $\left(\frac{\ln n}{\ln(n+1)}\right)^2 \to 1$ and $\frac{1}{\left(1+\frac{1}{n}\right)} \to 1$

By the ratio test, the original series converges if $\left|x^2\right| < 1$ and diverges if $x^2 > 1$

Thus, it converges if $x < \pm 1$ and diverges if $x > \pm 1$

This means that the radius of convergence is $\boxed{R=1}$ and the series is converges in the interval $\boxed{(-1,1)}$

Test for convergence at the end points of the interval $(-1,1)$.

If $x=1$, the series becomes

$$\sum_{n=2}^{\infty}\frac{x^{2n}}{n(\ln n)^2}=\sum_{n=2}^{\infty}\frac{1^{2n}}{n(\ln n)^2}$$

The function $\int_2^{\infty}\frac{1}{x(\ln x)^2}$ is positive and continuous for $x>2$ because logarithm function is continuous. But is it not obvious whether or not *f* is decreasing, so compute its derivative:

$$f'(x)=-\frac{2+\ln x}{x^2\ln^3(x)}$$

Thus, $f'(x)<0$

So, apply the integral test:

$$\int_2^{\infty}\frac{1}{x(\ln x)^2}=\lim_{t\to\infty}\int_2^{t}\frac{1}{x(\ln x)^2}$$
$$=\lim_{t\to\infty}\left(-\frac{1}{\ln x}\right)_2^{\infty}$$
$$=\lim_{t\to\infty}\left(-\frac{1}{\ln x}\right)-\left(-\frac{1}{\ln 2}\right)$$
$$=\frac{1}{\infty}-\frac{1}{\ln 2}$$
$$=0-\frac{1}{\ln 2}$$
$$=\frac{1}{\ln 2}$$

Since this improper integral is convergent, $\sum\frac{1}{n(\ln n)^2}$ also convergent.

If $x=-1$, the series becomes

$$\sum_{n=2}^{\infty}\frac{x^{2n}}{n(\ln n)^2}=\sum_{n=2}^{\infty}\frac{(-1)^{2n}}{n(\ln n)^2}$$
$$=\sum_{n=2}^{\infty}\frac{1^{2n}}{n(\ln n)^2}$$

By the integral test, the above series is convergent.

Therefore, the original power series converges when $-1\le x\le 1$, so the interval of convergence is $\boxed{[-1,1]}$

Q27E

Ratio test: if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, then the series $\sum_{n=1}^{\infty} a_n$ (i) converges absolutely if L < 1, (ii) diverges if L > 1 and (iii) inconclusive if L = 1.

Given series $\sum_{n=1}^{\infty} \frac{x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

$$a_n = \frac{x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{1\cdot 3\cdot 5\cdots(2n-1)\cdot(2n+1)} \times \frac{1\cdot 3\cdot 5\cdots(2n-1)}{x^n}\right|$$

$$= \frac{1}{(2n+1)}|x|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|}{2n+1}$$

$$= 0 \text{ for every real } x \quad \ldots\ldots (1)$$

$$< 1$$

So, by ratio test, the series $\sum_{n=1}^{\infty} \frac{x^n}{1\cdot 2\cdot 3\ldots(2n-1)}$ is convergent.

By (1), we confirm that the radius of convergence is $(-\infty,\infty)$.

Q28E

Ratio test: if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, then the series $\sum_{n=1}^{\infty} a_n$ (i) converges absolutely if L < 1, (ii) diverges if L > 1 and (iii) inconclusive if L = 1.

Given series $\sum_{n=1}^{\infty} \frac{n!\,x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

$$a_n = \frac{n!\,x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\,x^{n+1}}{1\cdot 3\cdot 5\cdots(2n-1)\cdot(2n+1)} \times \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!\,x^n}\right|$$

$$= \left|\frac{(n+1)}{(2n+1)}x\right|$$

$$= \left|\left(\frac{1+\frac{1}{n}}{2+\frac{1}{n}}\right)x\right|$$

Using the Ratio Test, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1+\frac{1}{n}}{2+\frac{1}{n}}\right||x|$

$$= \frac{1+0}{2+0}|x|$$

$$= \frac{|x|}{2}.$$

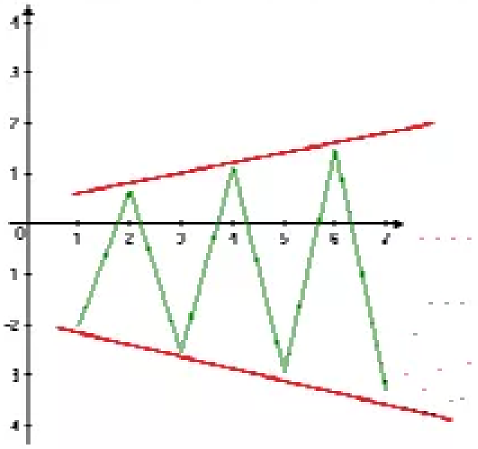
The series converges if $\frac{|x|}{2} < 1$ and diverges when $\frac{|x|}{2} > 1$

Ratio test fails when $\frac{|x|}{2} = 1$

When $\frac{|x|}{2} < 1$, we have $-2 < x < 2$, the series converges and so, the radius of convergence is $(-2, 2)$.

When $\frac{|x|}{2} = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{n!(-2)^n}{1\cdot3\cdot5\cdots(2n-1)}$

With the help of the following table, we confirm whether this series converges or not.



The graph is shown from $-u_1, -u_1+u_2, -u_1+u_2-u_3, -u_1+u_2-u_3+u_4, \ldots$

From this graph, we see that $\sum(-1)^n u_n$ tends to $-\infty$ as n proceeds along the odd natural numbers and tends to ∞ as n proceeds along the even.

Thus, the series diverges when $\frac{|x|}{2} = 1$

Q29E

Consider the series $\sum_{n=0}^{\infty} c_n (4)^n$ and it is convergent.

(a)

The objective is to tell the series $\sum_{n=0}^{\infty} c_n (-2)^n$ is convergent or not.

From the given information, if the series $\sum_{n=0}^{\infty} c_n (4)^n$ is convergence then it tells that the radius of convergence, R, for the series $\sum_{n=0}^{\infty} c_n (x)^n$ must satisfy $R \geq 4$.

The series $\sum_{n=0}^{\infty} c_n (x)^n$ converges absolutely when $|x| < 4$, it is also converges when $x = 4$ but it is possible that $x = 4$ is an end point of the interval of convergence.

In this case, the series $\sum_{n=0}^{\infty} c_n (x)^n$ might or might not converge at the other endpoint, which would be $x = -4$.

Therefore, the series $\sum_{n=0}^{\infty} c_n (-2)^n$ is also convergence because, -2 lines between -4 and 4 i.e. $-4 < -2 < 4$.

Hence, the series $\sum_{n=0}^{\infty} c_n (-2)^n$ is convergence.

(b)

The objective is to tell the series $\sum_{n=0}^{\infty} c_n (-4)^n$ is convergent or not.

It is clear that the series $\sum_{n=0}^{\infty} c_n (4)^n$ is convergence at 4 but does not tell anything that it is convergence at -4 and it is possible that -4 is an endpoint of the interval of convergence.

Hence, the series $\sum_{n=0}^{\infty} c_n (-4)^n$ may/many not convergence i.e. we cannot say whether the series is convergence or not.

Q30E

Since the series $\sum_{n=0}^{\infty} c_n x^n$ is convergent at $x = -4$, so it will be continuous for all the values of x lying between -4 and 4.

Also the series is divergent at $x = 6$ so it will be divergent for all $x \geq 6$ and $x < -6$.

(A) When $x = 1$ the given series is

$$\sum_{n=0}^{\infty} c_n (1)^n = \sum_{n=0}^{\infty} c_n$$

Since 1 lies between -4 and 4.so the series $\sum_{n=0}^{\infty} c_n$ is convergent.

(B) When $x=8$ the given series is

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n 8^n$$

Since 8 is greater than 6 and the series $\sum_{n=0}^{\infty} c_n x^n$ is divergent for all $x \geq 6$. so the series $\sum_{n=0}^{\infty} c_n 8^n$ is divergent.

(C) When $x=-3$ the series $\sum_{n=0}^{\infty} c_n x^n$ is $\sum_{n=0}^{\infty} c_n (-3)^n$ also -3 lies between -4 and 4. and series $\sum_{n=0}^{\infty} c_n x^n$ is convergent in $(-4,4)$. so the series $\sum_{n=0}^{\infty} c_n (-3)^n$ is convergent..

(D) When $x=-9$ the series $\sum_{n=0}^{\infty} c_n x^n$ is $\sum_{n=0}^{\infty} c_n (-9)^n = \sum_{n=0}^{\infty} (-1)^n c_n 9^n$.

Also -9 is less than -6 and the series $\sum_{n=0}^{\infty} c_n x^n$ is divergent for all $x < -6$

So the series $\sum_{n=0}^{\infty} (-1)^n c_n 9^n$ is divergent.

Q31E

Given series is $\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$

n^{th} Term of the given series is $a_n = \frac{(n!)^k x^n}{(kn)!}$

$(n+1)^{th}$ Term of the given series is $a_{n+1} = \frac{((n+1)!)^k x^{n+1}}{(k(n+1))!}$

Now

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left((n+1)!\right)^k \ x^{n+1}}{(kn+k)!}\frac{(kn)!}{(n!)^k x^n}\right|$$

$$= \left|\frac{\left[(n+1)(n!)\right]^k (kn)! \ x}{(n!)^k (kn+k)(kn+k-1)(kn+k-2)\ldots\ldots\ldots(kn+1)(kn!)}\right|$$

$$= \left|\frac{(n+1)^k (n!)^k \ x}{(n!)^k (kn+k)(kn+k-1)(kn+k-2)\ldots\ldots(kn+1)}\right|$$

$$= \frac{(n+1)^k}{(kn+k)(kn+k-1)(kn+k-2)\ldots\ldots\ldots(kn+1)}|x|$$

$$= \frac{(n+1)(n+1)}{(kn+k)(kn+k-1)}\frac{(n+1)}{(kn+k-2)}\ldots\ldots\frac{(n+1)}{(kn+1)}|x|$$

$$= \frac{\left(1+\frac{1}{n}\right)}{\left(k+\frac{k}{n}\right)}\frac{\left(1+\frac{1}{n}\right)}{\left(k+\frac{k-1}{n}\right)}\frac{\left(1+\frac{1}{n}\right)}{\left(k+\frac{n-2}{n}\right)}\ldots\ldots\frac{\left(1+\frac{1}{n}\right)}{\left(k+\frac{1}{n}\right)}|x|$$

Therefore

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)}{\left(k+\frac{k}{n}\right)}\frac{\left(1+\frac{1}{n}\right)}{\left(k+\frac{k-1}{n}\right)}\frac{\left(1+\frac{1}{n}\right)}{\left(k+\frac{k-2}{n}\right)}\ldots\ldots\ldots\frac{\left(1+\frac{1}{n}\right)}{\left(k+\frac{1}{n}\right)}|x|$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)}{\left(k+\frac{k}{n}\right)}.\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)}{\left(k+\frac{k-1}{n}\right)}\ldots\ldots\ldots\ldots\ldots\ldots\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)}{\left(k+\frac{1}{n}\right)}|x|$$

$$= \frac{1}{k}.\frac{1}{k}.\frac{1}{k}\ldots\ldots\frac{1}{k}|x|$$

$$= \left(\frac{1}{k}\right)^k |x|$$

$$= \frac{1}{k^k}|x|$$

Now, by ratio test the given series will be convergent if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

$$\Rightarrow \quad \frac{|x|}{k^k} < 1$$

$$\Rightarrow \quad |x| < k^k$$

Hence,

the radius of convergence is $\boxed{R = k^k}$

Q32E

(a) Given that the power series has the interval of convergence (p, q) where p < q. So, without loss of generality, we consider that in the ratio test, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|x-\frac{p+q}{2}\right|$$

For the series $\sum_{n=1}^{\infty}\frac{(x-\frac{1}{2}(p+q))^n}{\frac{1}{2}(q-p)}\cdot\frac{1}{n}$, the interval of convergence is

(b)For the series $\sum_{n=1}^{\infty}\frac{(-1)^n(x-\frac{1}{2}(p+q))^n}{\frac{1}{2}(q-p)n}$, the interval of convergence is (p, q].

(c)For the series $\sum_{n=1}^{\infty}\frac{(x-\frac{1}{2}(p+q))^n}{\frac{1}{2}(q-p)n}$, the interval of convergence is [p, q).

(d)For the series $\sum_{n=1}^{\infty}\frac{(-1)^n(x-\frac{1}{2}(p+q))^n}{\frac{1}{2}(q-p)n^2}$, the interval of convergence is [p, q].

Q33E

Is it is possible to define the power series whose interval of convergence is $[0,\infty)$?

No, it's not possible.

Explanation:

For the contrary, consider the power series

$$\sum a_n(x-c)^n$$

The radius of convergence of the power series $\sum a_n(x-c)^n$ is the number R such that the series is converge for $|x-c|<R$

That is, convergent for

$$-R<x-c<R$$

$$-R+c<x<R+c$$

For the interval of convergence to be $[0,\infty)$, we would need the following to be true.

$$R+c=\infty$$

$$-R+c=0$$

Adding these two, get

$$2c=\infty$$

$$\Rightarrow\ c=\infty$$

But there is no such thing as expanding the series about infinity.

Hence, it's not possible define the power series whose interval of convergence is $[0,\infty)$.

Q34E

Consider the function

$$f(x)=\frac{1}{1-x}$$

On rewriting the function in series notation, we have that

$$f(x)=\frac{1}{1-x}$$

$$=(1-x)^{-1}$$

$$=1+x+x^2+x^3+\cdots+x^n+\cdots$$

$$=\sum_{n=0}^{\infty}x^n$$

Thus, series together with the sum function $f(x)=1/1-x$ is $\sum_{n=0}^{\infty}x^n$

Several partial sums of the series $\sum_{n=0}^{\infty}x^n$ are given by

$$s_0=1$$

$$s_1=1+x$$

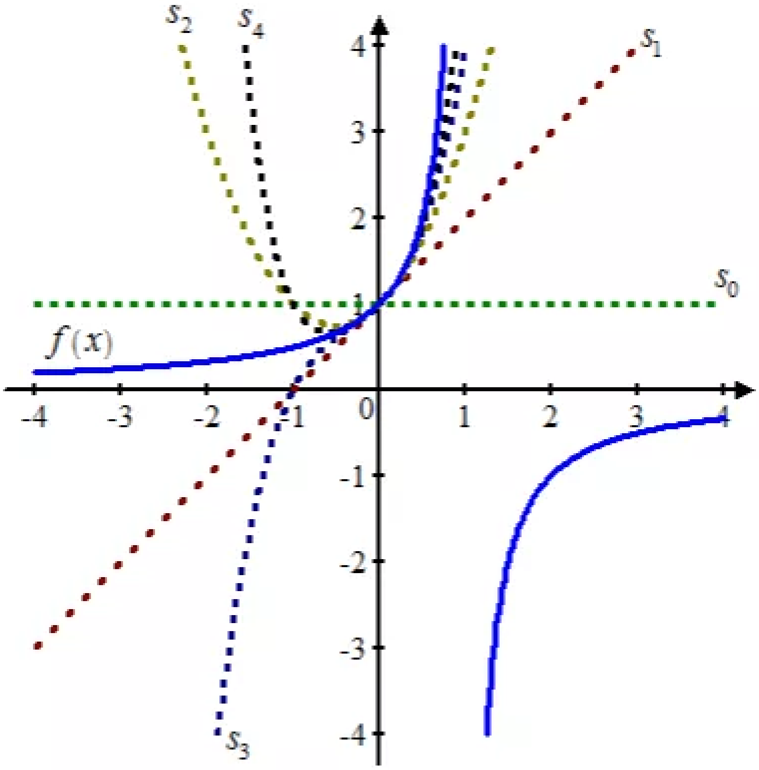
$$s_2=1+x+x^2$$

$$s_3=1+x+x^2+x^3$$

$$s_4=1+x+x^2+x^3+x^4$$

And so on.

Graph of the partial sums s_0, s_1, s_2, s_3, s_4 and the function $f(x)=1/1-x$ on the same screen is shown.



From the graph, it can be observe that graph of the partial sums s_0, s_1, s_2, s_3, s_4 approaches(converges) to the graph of the function $f(x)=1/1-x$ in the interval $(-1,1)$.

Q35E

(a)

Let's find the domain of the following function, called the Bessel function of order 1:

$$J_1(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

As usual, we can use the Ratio Test to see for which values of x the series converges. The domain of the function will be all of the values of x where the series converges.

Take the limit of ratios of successive terms of the series:

$$\lim_{x\to\infty}\left|\frac{\dfrac{(-1)^{n+1}x^{2(n+1)+1}}{(n+1)!((n+1)+1)!2^{2(n+1)+1}}}{\dfrac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}}\right|=\lim_{x\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{(n+1)!(n+2)!2^{2n+3}}\frac{n!(n+1)!2^{2n+1}}{(-1)^n x^{2n+1}}\right|$$

$$=\lim_{x\to\infty}\left|\frac{x^2}{(n+1)(n+2)2^2}\right|$$

by cancelling terms and since $|-1|=1$

$$=\left|\frac{x^2}{4}\right|\lim_{x\to\infty}\left|\frac{1}{(n+1)(n+2)}\right|$$

$$=0$$

The equivalence $\lim_{x\to\infty}\left|\frac{x^2}{(n+1)(n+2)2^2}\right| = \left|\frac{x^2}{4}\right|\lim_{x\to\infty}\left|\frac{1}{(n+1)(n+2)}\right|$ is true because we are taking the limit as n approaches infinity, and considering the limit for a fixed x. Since the value of x does not change as n increases, we treat the term x^2 as a constant and pull it out. So the limit is 0 for every x.

The Ratio Test says that $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$ converges when the limit of ratios of successive terms satisfies $\lim_{x\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2(n+1)+1}}{(n+1)!((n+1)+1)!2^{2(n+1)+1}}}{\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}}\right|<1$ and diverges when

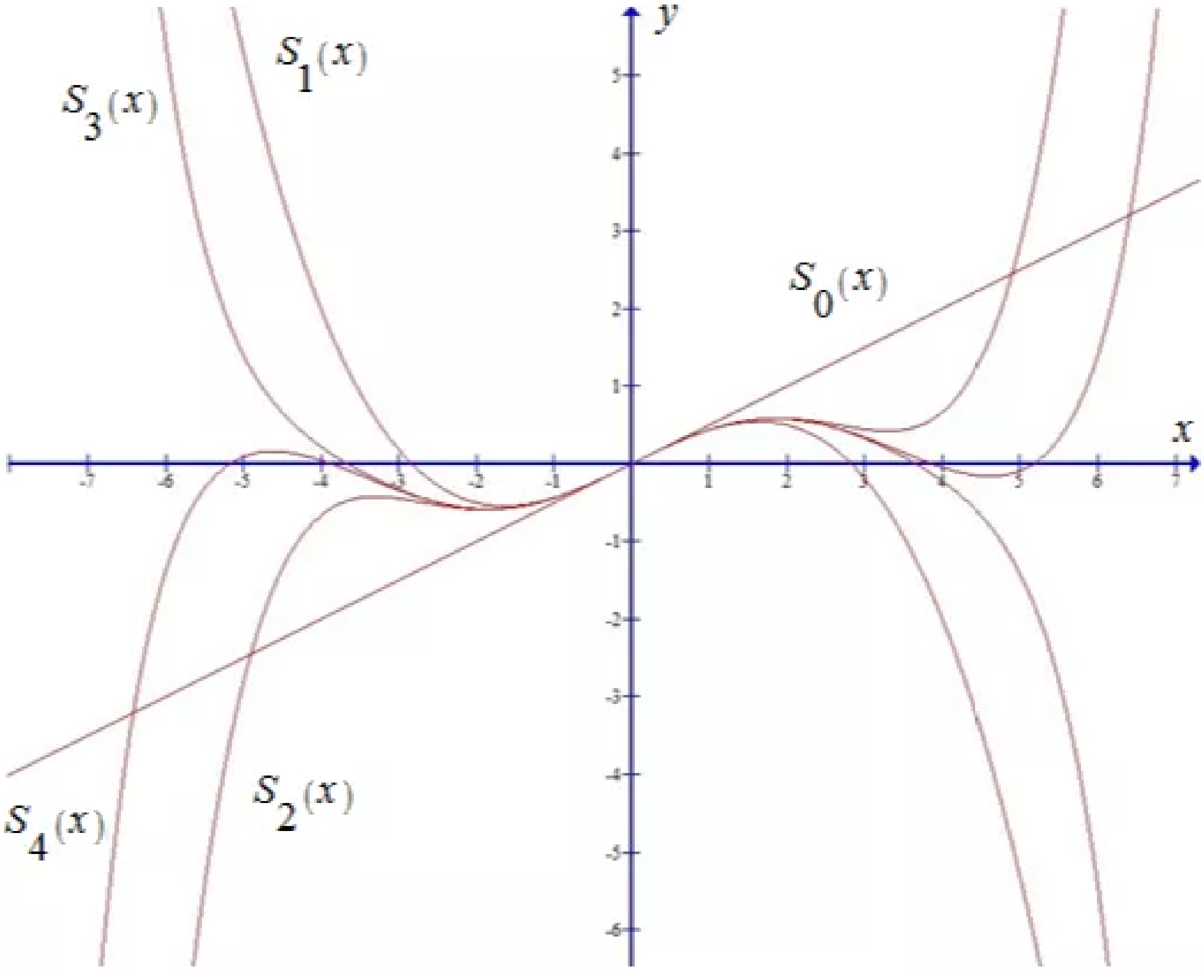
$\lim_{x\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2(n+1)+1}}{(n+1)!((n+1)+1)!2^{2(n+1)+1}}}{\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}}\right|>1$. We just showed that for every x,

$\lim_{x\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2(n+1)+1}}{(n+1)!((n+1)+1)!2^{2(n+1)+1}}}{\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}}\right|=0$, so the series converges for every x.

That is, the domain is all real numbers, $(-\infty,\infty)$.

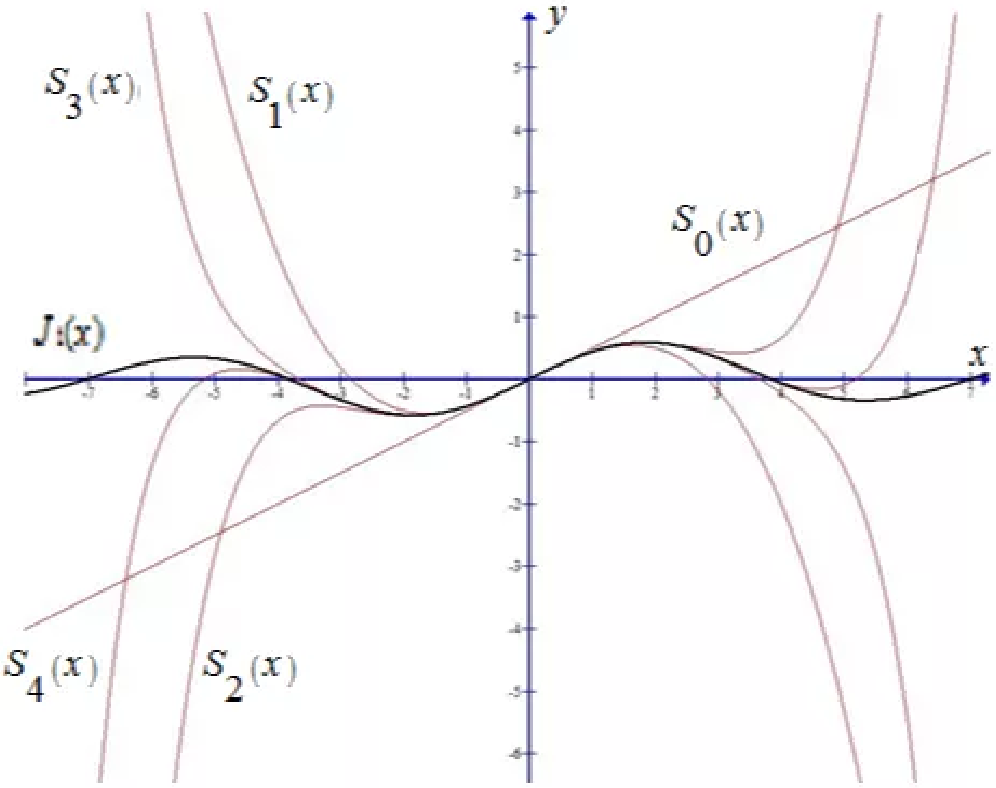
Now let's graph the first few partial sums $s_n(x)$, where $s_0(x) = \frac{(-1)^0 x^1}{0!1!2^1}$,

$s_1(x) = \frac{(-1)^0 x^1}{0!(1)!2^1} + \frac{(-1)^1 x^3}{1!2!2^3}$, $s_2(x) = \frac{(-1)^0 x^1}{0!(1)!2^1} + \frac{(-1)^1 x^3}{1!2!2^3} + \frac{(-1)^2 x^5}{2!3!2^5}$, and so on:



(c)

Here we'll graph the entire summed Bessel function along with the partial sums from part (b). The Bessel function $J_1(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$ will be in black, with the partial sums in red as before:



Q36E

(a)

Let's find the domain of the following function:

$$A(x)=1+\frac{x^3}{2\cdot 3}+\frac{x^6}{2\cdot 3\cdot 5\cdot 6}+\frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}+\ldots$$

First, find a pattern for the terms of the series. Starting with the 0th term, which is 1, we see that the nth term has a factor of x to the $3n$ power. In the denominator, we see that the last two terms of the product are $(3n-1)$, $3n$. For example, the last two terms of the first denominator are $(3\cdot 1-1)=2$, $3\cdot 1=3$. And the last two terms of the third denominator are $(3\cdot 3-1)=8$, $3\cdot 3=9$.

So the nth denominator has the form $2\cdot 3\cdot 5\cdot 6\cdot \ldots\cdot(3n-1)\cdot 3n$.

Now we can use the Ratio Test to see for which values of x the series converges. The domain of the function will be all of the values of x where the series converges.

Take the limit of ratios of successive terms of the series:

$$\lim_{x\to\infty}\left|\frac{\dfrac{x^{3(n+1)}}{2\cdot 3\cdot\ldots\cdot(3n-1)\cdot 3n\cdot(3(n+1)-1)\cdot 3(n+1)}}{\dfrac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdot\ldots\cdot(3n-1)\cdot 3n}}\right|$$

$$=\lim_{x\to\infty}\left|\frac{x^{3n+3}}{2\cdot 3\cdot\ldots\cdot(3n-1)\cdot 3n\cdot(3n+2)\cdot(3n+3)}\frac{2\cdot 3\cdot 5\cdot 6\cdot\ldots\cdot(3n-1)\cdot 3n}{x^{3n}}\right| =\lim_{x\to\infty}\left|\frac{x^3}{(3n+2)\cdot(3n+3)}\right|$$

$$=\left|x^3\right|\lim_{x\to\infty}\left|\frac{1}{(3n+2)\cdot(3n+3)}\right|$$

$$=0$$

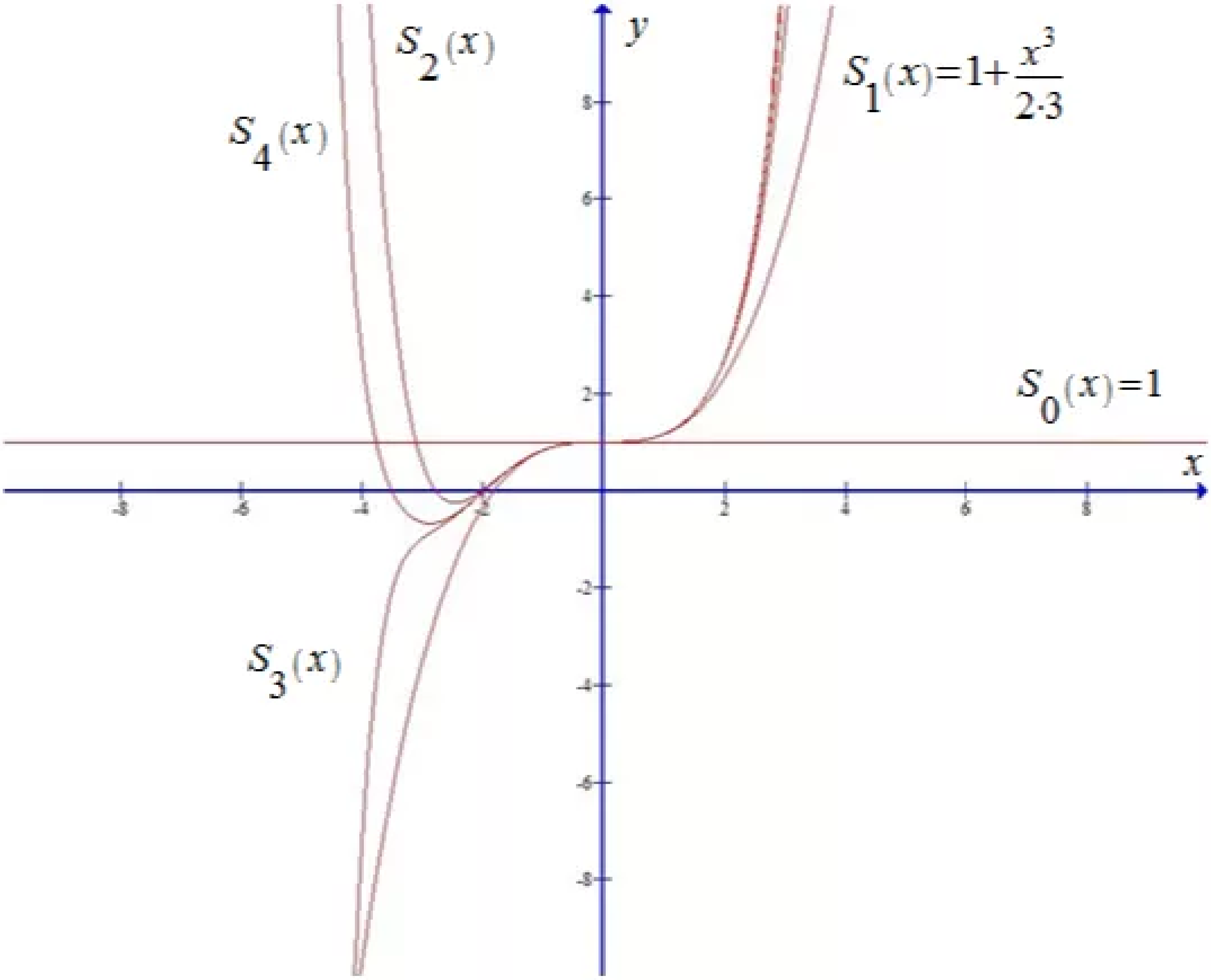
The equivalence $\lim_{x\to\infty}\left|\frac{x^3}{(3n+2)\cdot(3n+3)}\right|=\left|x^3\right|\lim_{x\to\infty}\left|\frac{1}{(3n+2)\cdot(3n+3)}\right|$ is true because we are taking the limit as n approaches infinity, and considering the limit for a fixed x. Since the value of x does not change as n increases, we treat the term x^3 as a constant and pull it out. So the limit is 0 for every x.

The Ratio Test says that $A(x)=1+\frac{x^3}{2\cdot 3}+\frac{x^6}{2\cdot 3\cdot 5\cdot 6}+\frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}+\ldots$ converges when the limit of ratios of successive terms satisfies $\lim_{x\to\infty}\left|\frac{\dfrac{x^{3(n+1)}}{2\cdot 3\cdot\ldots\cdot(3n-1)\cdot 3n\cdot(3(n+1)-1)\cdot 3(n+1)}}{\dfrac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdot\ldots\cdot(3n-1)\cdot 3n}}\right|<1$ and diverges when $\lim_{x\to\infty}\left|\frac{\dfrac{x^{3(n+1)}}{2\cdot 3\cdot\ldots\cdot(3n-1)\cdot 3n\cdot(3(n+1)-1)\cdot 3(n+1)}}{\dfrac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdot\ldots\cdot(3n-1)\cdot 3n}}\right|>1$. We just showed that for every x, $\lim_{x\to\infty}\left|\frac{\dfrac{x^{3(n+1)}}{2\cdot 3\cdot\ldots\cdot(3n-1)\cdot 3n\cdot(3(n+1)-1)\cdot 3(n+1)}}{\dfrac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdot\ldots\cdot(3n-1)\cdot 3n}}\right|=0$, so the series converges for every x. That is, the domain is all real numbers, $(-\infty,\infty)$.

(b)

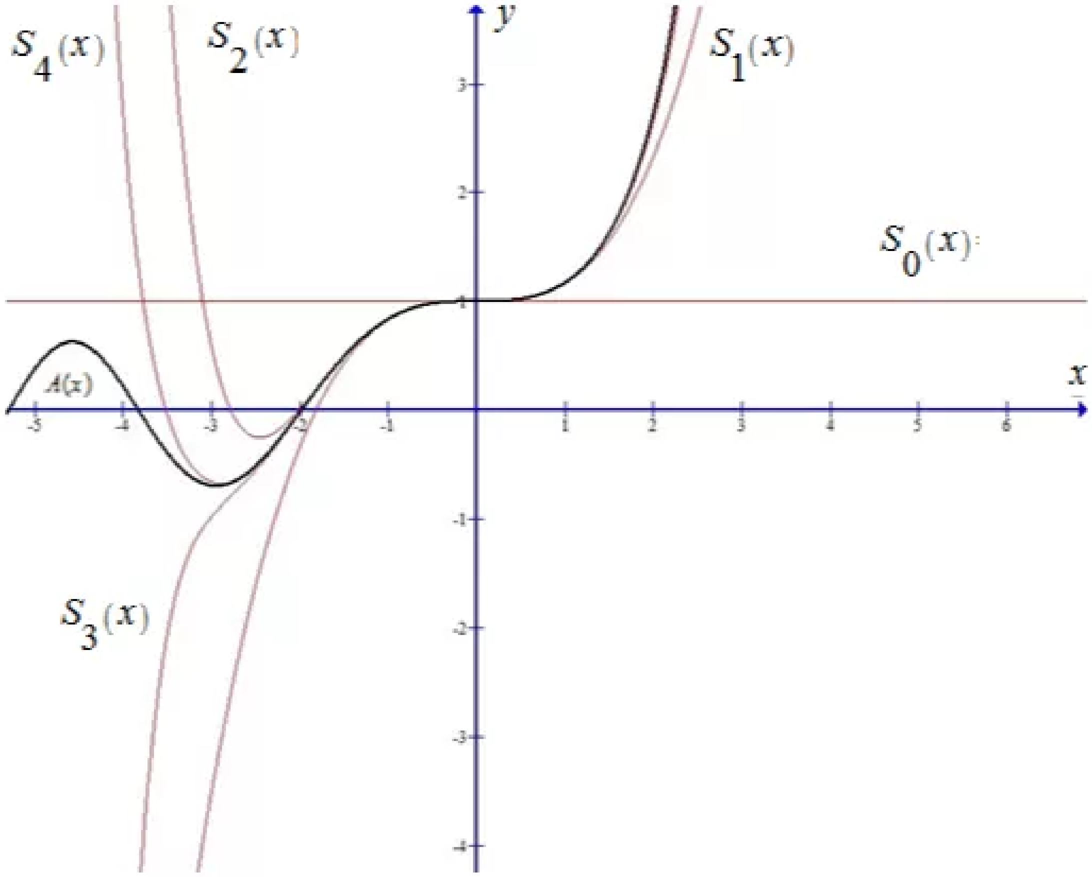
Now let's graph the first few partial sums $s_n(x)$, where $s_0(x)=1$, $s_1(x)=1+\frac{x^3}{2\cdot 3}$,

$s_2(x)=1+\frac{x^3}{2\cdot 3}+\frac{x^6}{2\cdot 3\cdot 5\cdot 6}$, and so on:



(c)

Here we'll graph the entire summed Airy function along with the partial sums from part b. The Airy function $A(x)=1+\frac{x^3}{2\cdot 3}+\frac{x^6}{2\cdot 3\cdot 5\cdot 6}+\frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}+\ldots$ will be in black, with the partial sums in red as before:



Q37E

Let S_{2n} denotes the sum of first 2n terms of the series then

$$S_{2n}=1+2x+x^2+2x^3+x^4+\ldots\ldots\ldots\ldots+x^{2n-2}+2x^{2n-1}$$
$$=\left(1+x^2+x^4+\ldots\ldots+x^{2n-2}\right)+\left(2x+2x^3+2x^5+\ldots\ldots\ldots\ldots+2x^{2n-1}\right)$$
$$=\left(1+x^2+x^4+\ldots\ldots\ldots+x^{2n-2}\right)+2x\left(1+x^2+x^4\ldots\ldots+x^{2n-2}\right)$$
$$=\left(1+x^2+x^4+\ldots\ldots+x^{2n-2}\right)(1+2x)$$

Here $1+x^2+x^4+\ldots\ldots+x^{2n-2}$ is a geometric progression containing n terms with first term 1 and common ratio x^2.

So its sum is $=\frac{1-\left(x^2\right)^n}{1-x^2}=\frac{1-x^{2n}}{1-x^2}$.

Therefore

$$S_{2n}=\frac{\left(1-x^{2n}\right)(1+2x)}{\left(1-x^2\right)}$$

If $x^2<1$ or $|x|<1$ then0020 $\lim\limits_{n\to\infty} x^{2n}\to 0$ and thus $\lim\limits_{n\to\infty} S_{2n}=\lim\limits_{n\to\infty}\frac{\left(1-x^{2n}\right)(1+2x)}{\left(1-x^2\right)}$

$$=\frac{1+2x}{1-x^2}$$

Q38E

Given $f(x)=\sum_{n=0}^{\infty} c_n x^n$

Given $c_{n+4}=c_n$ for all $n\ge 0$. Therefore, the given series can be written as,

$$c_0+c_1x+c_2x^2+c_3x^3+c_0x^4+c_1x^5+c_2x^6+c_3x^7+c_0x^8+c_1x^9+......\infty.$$

Let s_{4n} denotes the sum of first 4n terms of the series. Therefore,

$$s_{4n}=c_0+c_1x+c_2x^2+c_3x^3+c_0x^4+c_1x^5+c_2x^6+c_3x^7+c_0x^8+........c_3x^{4n-1}$$

$$=\left[c_0+c_0x^4+c_0x^8+......c_0x^{4n-4}\right]+\left[c_1x+c_1x^5+c_1x^9+......+c_1x^{4n-3}\right]+$$
$$\left[c_2x^2+c_2x^6+c_2x^{10}+......+c_2x^{4n-2}\right]+\left[c_3x^3+c_3x^7+c_3x^{11}+......+c_3x^{4n-1}\right]$$
$$=c_0\left[1+x^4+x^8+....+x^{4n-4}\right]+c_1x\left[1+x^4+x^8+.....+x^{4n-4}\right]+$$
$$c_2x^2\left[1+x^4+x^8+......+x^{4n-4}\right]+c_3x^3\left[1+x^4+x^8+.........+x^{4n-4}\right]$$
$$=\left[1+x^4+x^8+......+x^{4n-4}\right]\left[c_0+c_1x+c_2x^2+c_3x^3\right]$$

Here $1+x^4+x^8+.......+x^{4n-4}$ is a geometric progression containing n terms with first term 1 and common ratio x^4. So its sum is $\dfrac{1-\left(x^4\right)^n}{1-x^4}=\dfrac{1-x^{4n}}{1-x^4}$

Therefore

$$s_{4n}=\frac{\left(1-x^{4n}\right)\left(c_0+c_1x+c_2x^2+c_3x^3\right)}{\left(1-x^4\right)}$$

If $x^4<1$ or $|x|<1$. Then $\lim_{n\to\infty} x^{4n}=0$ and thus

$$\lim_{n\to\infty} S_{4n}=\lim_{n\to\infty}\frac{\left(1-x^{4n}\right)}{\left(1-x^4\right)}\left(c_0+c_1x+c_2x^2+c_3x^3\right)$$

$$=\frac{c_0+c_1x+c_2x^2+c_3x^3}{\left(1-x^4\right)}$$

Also, sum of first $(4n+1)$ terms of the series

$$S_{4n+1}=S_{4n}+(4n+1)^{th}\text{ Term}$$
$$=S_{4n}+c_0x^{4n}$$

And $\lim_{n\to\infty} S_{4n+1}=\lim_{n\to\infty} S_{4n}+\lim_{n\to\infty} c_0x^{4n}$

$$=\frac{\left(c_0+c_1x+c_2x^2+c_3x^3\right)}{\left(1-x^4\right)}+c_0\times(0)\qquad \text{Since } \lim_{n\to\infty} x^{4n}=0$$

$$=\frac{\left(c_0+c_1x+c_2x^2+c_3x^3\right)}{\left(1-x^4\right)}$$

Also sum of first $(4n+2)$ terms of the series,

$$S_{4n+2}=S_{4n+1}+(4n+2)^{th}\text{ Term}$$
$$=S_{4n+1}+c_1x^{4n+1}$$

And

$$\lim_{n\to\infty} S_{4n+2}=\lim_{n\to\infty} S_{4n+1}+\lim_{n\to\infty} c_1x^{4n+1}$$

$$=\frac{\left(c_0+c_1x+c_2x^2+c_3x^3\right)}{\left(1-x^4\right)}+0\qquad \text{Since } \lim_{n\to\infty} x^{4n+1}=0$$

Q39E

The given series is $\sum c_n x^n$

Here n[th] term of the series $a_n = c_n x^n$

Therefore

$$\sqrt[n]{|a_n|} = \sqrt[n]{|c_n x^n|}$$

$$= \sqrt[n]{|c_n||x^n|}$$

$$= |x|\sqrt[n]{|c_n|}$$

And $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |x|\sqrt[n]{|c_n|}$

$= c|x|$ Since $\lim_{n\to\infty} \sqrt[n]{|c_n|} = c$

Thus, by the root test, the given series will be convergent if $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$

So $c|x| < 1$

Or $|x| < \frac{1}{c}$

Hence, the radius of convergence of the given series is $R = \frac{1}{c}$.

Q40E

Given series is $\sum c_n (x-a)^n$

Here n[th] term of the series is $a_n = c_n (x-a)^n$ where $c_n \neq 0$

Then $a_{n+1} = c_{n+1}(x-a)^{n+1}$

Now

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n}\right|$$

$$= \left|\frac{c_{n+1}}{c_n}\right||x-a|$$

$$= \frac{|x-a|}{|c_n/c_{n+1}|} \qquad \text{Since} \left|\frac{c_{n+1}}{c_n}\right| = \frac{1}{\left|\frac{c_n}{c_{n+1}}\right|}$$

Therefore $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-a|}{|c_n/c_{n+1}|}$

Since $\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$ exists let $\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = L \neq 0$

Thus $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-a|}{L}$

By ratio test, series converges when $\frac{|x-a|}{L} < 1$

$\Rightarrow \quad |x-a| < L$

$\Rightarrow \quad |x-a| < \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$

Hence, the radius of convergence is $R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$

If $\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = 0$ then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$$

And so series diverges.

Therefore the radius of convergence is $R = 0 = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$

Hence the radius of convergence is $R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$ for all cases.

Q41E

The series $\sum c_n x^n$ has radius of convergence 2. Therefore, the series will be convergent for $|x| < 2$ i. e for $-2 < x < 2$

The series $\sum d_n x^n$ has radius of convergence 3. Therefore this series will be convergent for $|x| < 3$ or for $-3 < x < 3$.

We know that if $\sum a_n$ is convergent and $\sum b_n$ is divergent. Then the series $\sum (a_n + b_n)$ is divergent.

Now for the interval $2 < x < 3$ and $-3 < x < -2$, the series $\sum d_n x^n$ is convergent and series $\sum c_n x^n$ divergent.

Therefore the series $\sum (c_n + d_n) x^n$ is divergent.

Also for $-2 < x < 2$, both the series $\sum c_n x^n$ and $\sum d_n x^n$ are convergent, therefore the series $\sum (c_n + d_n) x^n$ is convergent and the radius of convergence of the series $\sum (c_n + d_n) x^n$ is 2.

Q42E

For the power series $\sum c_n x^n$

n^{th} Term $a_n = c_n x^n$

$(n+1)^{th}$ Term $a_{n+1} = c_{n+1} x^{n+1}$

Now

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{c_{n+1} x^{n+1}}{c_n x^n}\right|$$

$$= \left|\frac{c_{n+1}}{c_n} x\right|$$

$$= \left|\frac{c_{n+1}}{c_n}\right| |x|$$

And $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| |x|$

For the convergence of the series $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right||x| < 1$

$$\Rightarrow \quad |x| < \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

Therefore, radius of convergence $= \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$.

Given that the radius of convergence of the series is R.

So $\quad \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = R$

Now, n^{th} term of the series $\sum c_n x^{2n}$ is $b_n = c_n x^{2n}$

$(n+1)^{th}$ term $b_{n+1} = c_{n+1} x^{2n+2}$

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{c_{n+1} x^{2n+2}}{c_n x^{2n}}\right| = \left|\frac{c_{n+1}}{c_n}\right| x^2$$

And

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| x^2$$

For the convergence of the series, by ratio test

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| x^2 < 1$$

$$\Rightarrow \quad x^2 < \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

$$\Rightarrow \quad x^2 < R$$

$$\Rightarrow \quad |x| < \sqrt{R}$$

Therefore radius of convergence of series $\sum c_n x^{2n} = \sqrt{R}$

Hence

$$\boxed{\text{Radius of convergence} = \sqrt{R}}$$