Three Dimensional Geometry

Teaching learning points

- Distance between two given points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) is
$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- Direction ratio of line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$
- Let a, b, c be the direction ratio of a line whose direction cosines are l, m, n the $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$
$l^2 + m^2 + n^2 = 1$
$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}$ $m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}$ $n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$
If line makes angles α, β, γ with coordinate axes then $l = \cos\alpha$ $m = \cos\beta$ $n = \cos\gamma$
- $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
- Vector equation of a straight line passing through a fixed point with the position vector $\vec{a}$ and || to given vector $\vec{b}$
$\vec{r} = \vec{a} + \lambda\vec{b}$ where λ is parameter and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
- Cartesian equation of straight line passing through a fixed point (x_1, y_1, z_1) having direction ratio (a, b, c) is given by
$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$
- The coordinates of any point on the line $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$ are ($x_1 + a\lambda$, $y_1 + b\lambda$, $z_1 + c\lambda$) where $\lambda \in R$
- Angle between two lines whose direction rahos are a_1, b_1, c_1 and (a_2, b_2, c_2) is given by
$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$
If lines are perpendicular then $a_1a_2 + b_1b_2 + c_1c_2 = 0$
If lines are parallel then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- **Angle between tow lines :** $\vec{r} = \vec{a}_1 + \lambda\vec{m}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{m}_2$ is given as $\cos\theta = \frac{\vec{m}_1.\vec{m}_2}{|\vec{m}_1||\vec{m}_2|}$ and so, two lines are perpendicular if $\vec{m}_1.\vec{m}_2 = 0$. Lines are parallel $\vec{m}_1 = \lambda\vec{m}_2$
- **Skew lines :** Lines in space, which are neither parallel, nor intersecting are called skew lines, such pair of lines are non-coplanar.
- **Shortest distance :** $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are two skew lines, then distance 'd' between them is given by
$d = \left|\frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$
- Distance 'd' between parallel line : if $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ are two parallel lines, then distance d, between them is given by
$d = \left|\frac{(\vec{a}_2 - \vec{a}_1)\times\vec{b}}{|\vec{b}|}\right|$
- **Plane :** A plane is uniquely determined if any one of the following is known:
(*i*) The normal to the plane and its distnace from origin.
(*ii*) It passes through a given point and is perpendicular to a given direction.
(*iii*) It passes through three given non collinear points.
- Equation of a plance at a distance 'p' from origin and normal vector $\vec{n}$ is given by $\vec{r}.\hat{n} = p$ or $lx + my + nz = p$.
- General equation of plane passing through a point $\vec{a}$ and having normal vector to plane as $\vec{n}$ is $(\vec{r} - \vec{a}).\vec{n} = 0$. Corresponding Cartesian form is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where a, b, c are direction ratios of normal to plane.
- General equation of plane which cuts off intercepts a, b and c on x, y and z-axis respectively is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
- Equation of plane passing through three non-collinear points $\vec{a}, \vec{b}, \vec{c}$ is $(\vec{r} - \vec{a})\cdot\left[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\right] = 0$.

Corresponding Cartesian from is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$

where (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are co-ordinates of known point.

● Angle between two planes is $\cos\theta = \left|\frac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right|$ or

$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

where $\vec{n_1}$ & $\vec{n_2}$ are vectors normal to planes or a_1, b_1, c_1 and a_2, b_2, c_2 are dr's of normal to planes.

● **Condition of Coplanarity of two lines**

Let $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ be two lines then these lines are coplanar if $(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = 0$ and equation of plane containing them is $(\vec{r} - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = 0$ or $(\vec{r} - \vec{a}_2)\cdot(\vec{b}_1 \times \vec{b}_2) = 0$

● Distance of point $\vec{a}$ from plane $\vec{r}.\vec{n} = d$ is

Distance = $\frac{|\vec{a}\cdot\vec{n} - d|}{|\vec{n}|}$

● Angle θ between line $\vec{r} = \vec{a} + \lambda\vec{b}$ and plane $\vec{r}\cdot\vec{n} = d$ is given as

$\sin\theta = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$

A line is parallel to plane of $\vec{b}\cdot\vec{n} = 0$.

Question for Practice

Very Short Answer Type Questions (1 Mark)

Q1. Write intercept cut off by the plane $2x + y - z = 12$ on x axis.

Q2. Write the vector equation of a line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$

Q3. Write the direction cosine of line joining (1, 0, 0) and (0, 1, 1)

Q4. What are the direction cosine of a line which makes equal angles with coordinate axes.

Q5. What is the cosine of the angle which vector $\sqrt{2}\hat{i} + \hat{j} + \hat{k}$ makes with y axis

Q6. What is the distance of the following plane from origin $2x - y + 2z + 1 = 0$.

Q7. Write the distance of the point (a, b, c) from x axis.

Q8. For what value of λ the line $\frac{x-2}{9} = \frac{y-1}{\lambda} = \frac{z-3}{-6}$ its perpendicular to tbe plane $3x - y - 2z = 7$.

Q9. If a line makes α, β, γ and with the x axis, y axis and z axis respectively. Find the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

Q10. Find the coordinate of the pt where line $\frac{x-3}{2} = \frac{y-5}{-3} = \frac{z-2}{5}$ crosses the yz plane.

Q11. Write the direction ratio of the line $\frac{x-2}{2} = \frac{2y-5}{-3} = \frac{z+1}{1}$.

Q12. What is the equation of plane parallel to XOY plane and passing through (3, –4, 8)

Q13. Write the direction cosines of the perpendicular from the origin to the plane $\vec{r}\cdot(\hat{i} + \hat{j} + \hat{k}) = 10$.

Q14. Write the direction ratio of the normal of plane $2x + y + z = 7$.

Q15. Write the value of λ for which the plane $2x - 4y + 3z = 7$ and $x + 2y + \lambda z = 18$ are ⊥ to each other.

Q16. Write the vector eq. of plane $2x + y + z = 7$.

Q17. Write the equation of line || to the line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-2}{4}$ and passes through (0, 0, –1)

Q18. What is the equation of line passes through (1, 1, 1) and ⊥ to the plane $2x + y + z = 7$.

Q19. If a line makes an angle 60°, 30°, 90° with the positive direction of x, y, z axis respectively then write the direction cosines of line.

Q20. What is the equation of a plane that cut the coordinates axis $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.

Q21. Write the angle between line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{2}$ and the plane $3x + 4y + z + 5 = 0$.

Short Answer Type Questions (4 Marks)

Q1. Find the equation of the line passing through the points (1, 2, –1) and (3, –1, 2). At what point it meet yz plane.

Q2. Find the equation of the line passing through the point (–1, 3, –2) and perpendicular to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$

Q3. Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$

Q4. Show that the four points (0, –1, 0) (2, 1, –1) (1, 1, 1) and (3, 3, 0) are coplanar & also find the equation of plane containing these point.

Q5. A variable plane which remains a constant distance 3P from the origin cuts the coordinate axis A, B and C show that the locus of the centroid of ΔABC is $x^{-2} + y^{-2} + z^{-2} = P^{-2}$

Q6. Find the equation of the plane passing through pt (2, 3, 4) and || to the plane $5x - 6y + 7z = 3$.

Q7. Find the equation of the plane passing through the points (2, 2, 1) and (9, 3, 6) and perpendicular to the plane $2x + 6y + 6z = 1$.

Q8. Find the equation of plane passing through origin and perpendicular to each of the plane $x + 2y - z = 1$ and $3x - 4y + z = 5$.

Q9. Find the distance between two parallel planes

$2x - y + 3z + 4 = 0$

$6x - 3y + 8z - 3 = 0$

Q10. Find the equation of plane which contains the line of intersection of the plane $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=4$ and $\vec{r}\cdot(2\hat{i}+\hat{j}+\hat{k})=-5$ and which it ⊥ to the plane $\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})=-8$.

Q11. Find the distance of the point A(–1, –5, –10) from the point of intersection of the line $\vec{r} = (2\hat{i}-\hat{j}+2\hat{k}) + \lambda(3\hat{i}+4\hat{j}+2\hat{k})$ and the plane $\vec{r}\cdot(\hat{i}-\hat{j}+\hat{k})=5$

Q12. Find the distance of the point (1, –2, 3) from the plane $x - y + z = 5$ measured along a line parallel to $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$

Q13. Find the distance of the point A(–2, 3, –4) from the line $\frac{x+2}{3}=\frac{2y+3}{4}=\frac{3z+4}{5}$ parallel to the plane $4x + 12y - 3z + 1 = 0$.

Q14. Find a points on the line $\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}$ at a distance of 5 units from the point (1, 3, 3).

Q15. Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ at angle of $\frac{\pi}{3}$.

Q16. Show that the lines $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ and $\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$ are coplanar. Also find the equation of plane containing them.

Q17. Show that the line $\vec{r}=(2\hat{i}-2\hat{j}+3\hat{k}) + \lambda(\hat{i}-\hat{j}+4\hat{k})$ is || to the plane $\vec{r}.(\hat{i}+5\hat{j}+\hat{k})=5$. Also find the distance between them.

Q18. Find the value of λ so that the lines $\frac{1-x}{3}=\frac{7y-14}{2\lambda}=\frac{5z-10}{11}$ and $\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$ are ⊥ to each other.

Q19. Show that the plane whose vector equation is $\vec{r}\cdot(\hat{i}+2\hat{j}-\hat{k}) = 3$ contains the line whose vector equation is $\vec{r}=(\hat{i}+\hat{j})+\lambda(2\hat{i}+\hat{j}+4\hat{k})$.

Q20. If the point (1, 1, P) and (–3, 0, 1) be equidistant from plane $\vec{r}\cdot(3\hat{i}+4\hat{j}-12\hat{k})+13=0$ find value of P.

Long Answer Type Questions (6 Marks)

Q1. A line makes α, β, γ, δ with the four diagonals of a cube prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.

Q2. Show that angles between any two diagonals of cube is $\cos^{-1}\frac{1}{3}$.

Q3. If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two mutually prependicular lines. Show that direction cosines of the line ⊥ to both of them are $(m_1n_2 - m_2n_1)$ $(n_1p_2 - n_2p_1)$ $(l_1m_2 - l_2m_1)$.

Q4. Find the foot of the perpendicular from the point (0, 2, 3) on the line $\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}$.
Also find the length of perpendicular.

Q5. Find the image of the point (1, 6, 3) on the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$

Q6. Prove that the lines $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$ and $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ intersect. Also find the cordinates of their point of intersection.

Q7. Find the shortest distance between the lines

$\vec{r}=(1+\lambda)\hat{i}+(2-\lambda)\hat{j}+(1+\lambda)\hat{k}$

$\vec{r}=2(1+\mu)\hat{i}-(1-\mu)\hat{j}+(-1+2\mu)\hat{k}$

Q8. Find the length and the foot of the prependicular from the point (7, 14, 5) to the plane $2x + 4y - z = 2$.

Q9. Find the image of the point (1, 3, 4) on the plane $2x - y + z + 3 = 0$.

Q10. Find the equation of a plane passing through the points (0, 0, 0) and (3, –1, 2) and || to the line $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$.

Q11. Find the equation of the plane passing through the point (0, 7, –7) and containing the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$.

Very Short Answer (1 Mark)

1. 6 2. $\vec{r}=(5\hat{i}-4\hat{j}+6\hat{k})+\lambda(3\hat{i}+7\hat{j}+2\hat{k})$

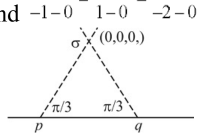
3. $\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}$ 4. $\left(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right)$

5. $\cos\theta = \frac{1}{2}$ 6. $\frac{1}{3}$
7. $\sqrt{b^2+c^2}$ 8. –3
9. 2 10. $\left(0\frac{19}{2}, \frac{-11}{2}\right)$
11. $\langle 4, -3, 2\rangle$ 12. $z = 8$
13. $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ 14. (2, 1, 1)
15. $\lambda = 2$ 16. $\vec{r}.(2\hat{i}+\hat{j}+\hat{k}) = 7$
17. $\frac{x}{2} = \frac{y}{3} = \frac{z+1}{1}$ 18. $\frac{x-1}{2} = \frac{y-1}{1} = \frac{z-1}{1}$
19. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$ 20. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
21. $\sin^{-1}\left(\frac{7}{2\sqrt{91}}\right)$

Short Answer (4 Mark)

1. $\left(0, \frac{7}{2}, \frac{-5}{2}\right)$ 2. $\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$
3. $\frac{1}{\sqrt{6}}$ 4. $4x - 3y + 2z = 3$
6. $5x - 6y + 7z = 20$ 7. $3x + 4y - 5z = 9$
8. $x + 2y + 5z = 0$ 9. $\frac{5}{\sqrt{14}}$
10. $\vec{r}.(33\hat{i}+45\hat{j}+50\hat{k}) = 41$ 11. 13
11. 1 13. $\frac{17}{2}$ units
12. (–2, –1, 3) and (4, 3, 7)
15. $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ and $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$

Hint : Give line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1} = \lambda$
Any pt on line $(2\lambda + 3, \lambda + 3, \lambda)$
DR of op $(2\lambda + 3 - 0, \lambda + 3 - 0, \lambda - 0)$
line op makes $\frac{\pi}{3}$ with line pq
$\cos\frac{\pi}{3} = \frac{|2(3+2\lambda)+1(3+\lambda)+\lambda\times 1|}{\sqrt{2^2+1^2+1^2}\sqrt{(3+2\lambda)^2+(3+\lambda)^2+\lambda^2}} \Rightarrow \frac{|6\lambda+9|}{\sqrt{6}\sqrt{6\lambda^2+18\lambda+18}}$
$\lambda = -1$, or –2 eq of requred line $\frac{x-0}{1-0} = \frac{y-0}{2-0} = \frac{z-0}{-1-0}$ and $\frac{x-0}{-1-0} = \frac{y-0}{1-0} = \frac{z-0}{-2-0}$



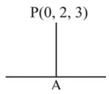
$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$, $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$
16. $x + y + z = 0$ 17. $\frac{10}{3\sqrt{3}}$ unit
18. $\lambda = 7$ 20. P = 1, $\frac{7}{3}$

Answer (6 Mark)

3. Let l, m, n be the direction cosines of the line ⊥ to each one of given line, then
$ll_1 + mm_1 + nn_1 = 0$
$ll_2 + mm_2 + nn_2 = 0$
$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{\sum(m_1n_2 - m_2n_1)^2}} = \frac{1}{\sin\theta}$
$\sqrt{\sum(m_1n_2 - m_2n_1)^2} = \sin\theta = \sin\frac{\pi}{2} = 1$
Hence direction cosines of the line are
$(m_1n_2 - m_2n_1), (n_1l_2 = n_2l_1), (l_1m_2 - l_2m_1)$
4. Gen pt on the line
$5\lambda - 3, 2\lambda + 1, 3\lambda - 4$

for some value of λ the coordinate of A
$(5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$ (1)
DR of PN $(5\lambda - 3 - 0, 2\lambda + 1 - 2, 3\lambda - 4 - 3)$
PN is $\perp$ to given line
$5(5\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 7) = 0$
$\lambda = 1$
Putting $\lambda = 1$ in (1)
We get (2, 3 – 1) as foot of $\perp$
length of $\perp$ = PN = $\sqrt{(2-0)^2 + (3-2)^2 + (-1-3)^2} = \sqrt{21}$ unit

5. Fing foot of ^ by using above method (mentioned in Q No 4) we get (1, 3, 5) since B pt is image of pt A P is the mid pt of AB

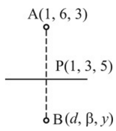


$\frac{\alpha+1}{2} = 1$ $\frac{\beta+6}{2} = 3$ $\frac{\gamma+3}{2} = 5$
$\alpha = 1$ $\beta = 0$ $\gamma = 7$

6. Give line will intersect at (5, –7, 6)

7. $\frac{3\sqrt{2}}{2}$ unit

8. Eq. of line PN (line PN is $\perp$ to plane)
$\frac{x-7}{2} = \frac{y-14}{4} = \frac{z-5}{-1} = \lambda$
Gen pt on line $2\lambda + 7, 4\lambda + 4, -\lambda + 5$

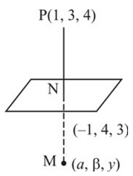


If N is the foot of $\perp$ Gen pt must satisfies the eq. of plane $2(2\lambda + 7) + 4(4\lambda + 14) - (-\lambda + 5) = 2$
$\lambda = -3$
pt N is (1, 2, 8)
Also find length of $\perp$ = PN = $\sqrt{(7-1)^2 + (14-2)5(5-8)^2} = 3\sqrt{21}$

9. (–3, 5, 2)
Hint : Find foot of $\perp$ by using above method.
(Mentioned in question No 8.
N is the mid pt of PM
$\frac{\alpha+1}{2} = 1$ $\frac{\beta+3}{2} = 4$ $\frac{\gamma+4}{2} = 3$
$\alpha = -3$ $\beta = 5$ $\gamma = 2$

10. $x - 19y - 11z = 0$

11. $x + y + z = 0$



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