# 7. Applications of Definite Integration

**EXERCISE 7.1**

**1. Find the area of the region bounded by the following curves, the X-axis and the given lines:**

***i)*** $y = x^4, x = 1, x = 5$

**Solution:**

Required area $= \int_1^5 y\,dx$, where $y = x^4$

$= \int_1^5 x^4\,dx = \left[\frac{x^5}{5}\right]_1^5$

$= \frac{1}{5}[3125 - 1] = \frac{3124}{5}$ sq units.

***ii)*** $y = \sqrt{6x+4}, x = 0, x = 2$

**Solution:**

Required area $= \int_0^2 y\,dx$, where $y = \sqrt{6x+4}$

$= \int_0^2 \sqrt{6x+4}\,dx = \int_0^2 (6x+4)^{\frac{1}{2}}\,dx$

$= \left[\frac{(6x+4)^{\frac{3}{2}}}{3/2} \times \frac{1}{6}\right]_0^2$

$= \frac{1}{9}\left[(6x+4)^{\frac{3}{2}}\right]_0^2$

$= \frac{1}{9}[64 - 8]$

$= \frac{56}{9}$ sq units.

[**Note** : *Answer in the textbook is incorrect.* ]

***iii)*** $y = \sqrt{16 - x^2}, x = 0, x = 4$

**Solution:**

Required area $= \int_0^4 y\,dx$, where $y = \sqrt{16 - x^2}$

$= \int_0^4 \sqrt{16 - x^2}\,dx$

$= \left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4$

$\ldots \left[\because \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$

$= 0 + 8\sin^{-1}(1) - 0 - 0$ $\ldots [\because \sin^{-1}(0) = 0]$

$= 8 \times \frac{\pi}{2} = 4\pi$ sq units. $\ldots \left[\because \sin^{-1}(1) = \frac{\pi}{2}\right]$

***iv)*** $2y = 5x + 7, x = 2, x = 8$

**Solution:**

Required area $= \int_2^8 y\,dx$, where $2y = 5x + 7$

i.e. $y = \frac{5x+7}{2}$

$= \int_2^8 \left(\frac{5x+7}{2}\right)dx = \frac{1}{2}\int_2^8 (5x+7)\,dx$

$= \frac{1}{2}\left[5 \cdot \frac{x^2}{2} + 7x\right]_2^8$

$= \frac{1}{2}[160 + 56 - 10 - 14]$

$= \frac{1}{2}(192) = 96$ sq units.

***v)*** $2y + x = 8, x = 2, x = 4$

**Solution:**

Required area $= \int_2^4 y\,dx$, where $2y + x = 8$

i.e. $y = \frac{8-x}{2}$

$= \int_2^4 \left(\frac{8-x}{2}\right)dx = \frac{1}{2}\int_2^4 (8-x)\,dx$

$= \frac{1}{2}\left[8x - \frac{x^2}{2}\right]_2^4$

$= \frac{1}{2}[(32-8) - (16-2)]$

$= \frac{1}{2}(24 - 14) = 5$ sq units.

***vi)*** $\boldsymbol{y = x^2 + 1, x = 0, x = 3}$
**Solution:**

Required area $= \int_0^3 y\,dx$, where $y = x^2 + 1$

$= \int_0^3 (x^2+1)\,dx$

$= \left[\frac{x^3}{3} + x\right]_0^3$

$= 9 + 3 - 0 = 12$ sq units.

***vii)*** $\boldsymbol{y = 2 - x^2, x = -1, x = 1}$
**Solution:**

Required area $= \int_{-1}^{1} y\,dx$, where $y = 2 - x^2$

$= \int_{-1}^{1} (2 - x^2)\,dx$

$= 2\int_0^1 (2 - x^2)\,dx$

... [$\because f(x) = 2 - x^2$ is an even function]

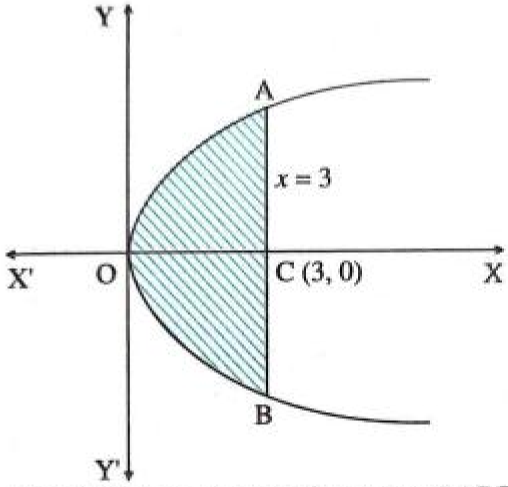
$= 2\left[2x - \frac{x^3}{3}\right]_0^1$

$= 2\left[2 - \frac{1}{3} - 0\right]$

$= 2\left(\frac{5}{3}\right)$

$= \frac{10}{3}$ sq units.

**2. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $x = 3$.**
**Solution:**




Required area = area of the region OABO

= 2 (area of the region OACO)

$= 2\int_0^3 y\,dx$, where $y^2 = 4x$, i.e. $y = 2\sqrt{x}$

$= 2\int_0^3 2\sqrt{x}\,dx$

$= 4\int_0^3 x^{\frac{1}{2}}\,dx$

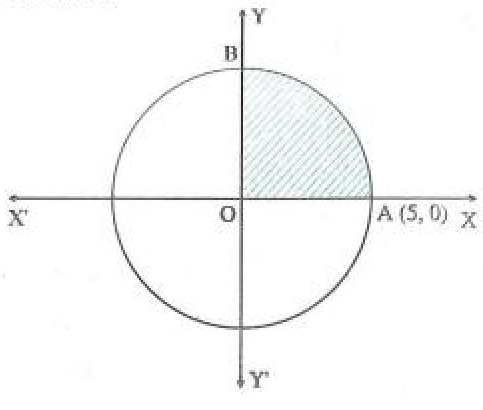
$= 4 \cdot \left[\frac{x^{\frac{3}{2}}}{3/2}\right]_0^3$

$= \frac{8}{3}\left[x^{\frac{3}{2}}\right]_0^3$

$= \frac{8}{3}(3\sqrt{3} - 0)$

$= 8\sqrt{3}$ sq units.

**3. Find the area of circle $x^2 + y^2 = 25$**
**Solution:**



By the symmetry of the circle, its area is equal to 4 times the area of the region OABO. Clearly for this region the limits of integration are 0 and 5.

From the equation of the circle, $y^2 = 25 - x^2$.

In the first quadrant $y > 0$

$\therefore\ y = \sqrt{25 - x^2}$

$\therefore$ area of the circle $= 4$(area of region OABO)

$$= 4\int_0^5 y\,dx = 4\int_0^5 \sqrt{25 - x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)\right]_0^5$$

$$\ldots\left[\because \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

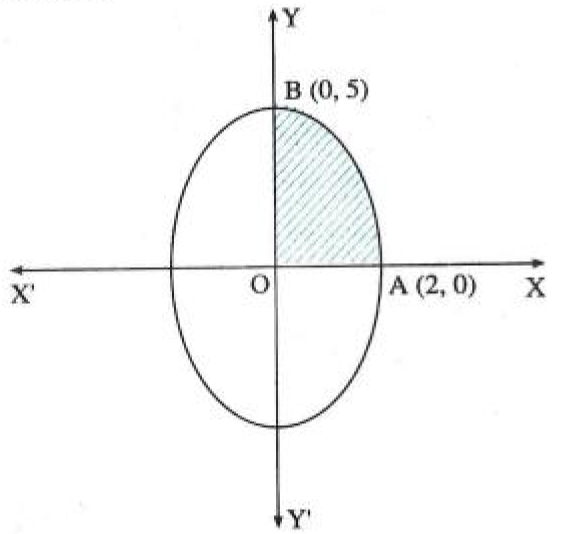
$$= 4\left[\left\{\frac{5}{2}\sqrt{25 - 25} + \frac{25}{2}\sin^{-1}(1)\right\} - \left\{\frac{0}{2}\sqrt{25 - 0} + \frac{25}{2}\sin^{-1}(0)\right\}\right]$$

$$= 4 \cdot \frac{25}{2} \cdot \frac{\pi}{2} = 25\pi \text{ sq units.}$$

$$\ldots\left[\because \sin^{-1}(1) = \frac{\pi}{2}, \sin^{-1}(0) = 0.\right]$$

**4. Find the area of ellipse $\frac{x^2}{4} + \frac{y^2}{25} = 1$**

**Solution:**




By the symmetry of the ellipse, its area is equal to 4 times the area of the region OABO. Clearly for this region, the limits of integration are 0 and 2.

From the equation of the ellipse,

$$\frac{y^2}{25} = 1 - \frac{x^2}{4} = \frac{4 - x^2}{4}$$

$$\therefore\ y^2 = \frac{25}{4}(4 - x^2)$$

In the first quadrant, $y > 0$

$$\therefore\ y = \frac{5}{2}\sqrt{4 - x^2}$$

$\therefore$ area of ellipse $= 4$(area of the region OABO)

$$= 4\int_0^2 y\,dx$$

$$= 4\int_0^2 \frac{5}{2}\sqrt{4 - x^2}\,dx$$

$$= 10\int_0^2 \sqrt{4 - x^2}\,dx$$

$$= 10\left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$\ldots\left[\because \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= 10\left[\left\{\frac{2}{2}\sqrt{4 - 4} + 2\sin^{-1}(1)\right\} - \left\{\frac{0}{2}\sqrt{4 - 0} + 2\sin^{-1}(0)\right\}\right]$$

$$= 10 \times 2 \times \frac{\pi}{2} = 10\pi \text{ sq units.}$$

$$\ldots\left[\because \sin^{-1}(1) = \frac{\pi}{2}, \sin^{-1}(0) = 0.\right]$$