2. Mathematical Methods

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1. Choose the correct option.

i) The resultant of two forces 10 N and 15 N acting along +x and -x-axes respectively, is

(A) 25 N along + x-axis
(B) 25 N along - x-axis
(C) 5 N along + x-axis
(D) 5 N along - x-axis

ii) For two vectors to be equal, they should have the

(A) same magnitude
(B) same direction
(C) same magnitude and direction
(D) same magnitude but opposite direction

iii) The magnitude of scalar product of two unit vectors perpendicular to each other is

(A) zero
(B) 1
(C) -1
(D) 2

iv) The magnitude of vector product of two unit vectors making an angle of 60° with each other is

(A) 1
(B) 2
(C) 3/2
(D) $\sqrt{3/2}$

v) If $\vec{A}, \vec{B}$ and $\vec{C}$ are three vectors,

then which of the following is not correct?
(A) $\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$
(B) $\vec{A}.\vec{B} = \vec{B}.\vec{A}$
(C) $\vec{A}\times\vec{B} = \vec{B}\times\vec{A}$
(D) $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{B}\times\vec{C}$

2. Answer the following questions.

i) Show that $\vec{a} = \frac{\hat{\imath} - \hat{\jmath}}{\sqrt{2}}$ is a unit vector.

Ans.

$$A = |\vec{A}| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

Hence, $\vec{A}$ is a unit vector

ii) If $\vec{\mathbf{v}}^1 = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{v}_2} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$,

Determine the magnitude of

$\vec{\mathbf{v}_1} + \vec{\mathbf{v}_2}$.

[Ans: 5]

Ans:

$$\begin{aligned}\vec{v}_1 + \vec{v}_2 &= (3\hat{i} + 4\hat{j} + \hat{k}) + (\hat{i} - \hat{j} - \hat{k}) \\ &= (3+1)\hat{i} + (4-1)\,\hat{j} + (1-1)\hat{k} \\ &= 4\hat{i} + 3\hat{j}\end{aligned}$$

$$\therefore\ |\vec{v}_1 + \vec{v}_2| = |4\hat{i} + 3\hat{j}| = \sqrt{(4^2 + 3^2)} = 5$$

is the required magnitude.

iii) For $\vec{\mathbf{v}_1} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$ and $\vec{\mathbf{v}}_2 = -6\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$,

Determine the magnitude and direction of

$\vec{\mathbf{v}_1} + \vec{\mathbf{v}_2}$.

$\left[\textbf{Ans: } 2\sqrt{5}, \theta = \tan^{-1}\left(-\frac{1}{2}\right) \textbf{ with x} - \textbf{axis}\right]$

Ans:

$$\vec{v} = \vec{v}_1 + \vec{v}_2 = (2\hat{i} - 3\hat{j}) + (-6\hat{i} + 5\hat{j})$$
$$= (2-6)\hat{i} + (-3+5)\hat{j}$$
$$= -4\hat{i} + 2\hat{j} = v_x\hat{i} + v_y\hat{j}$$

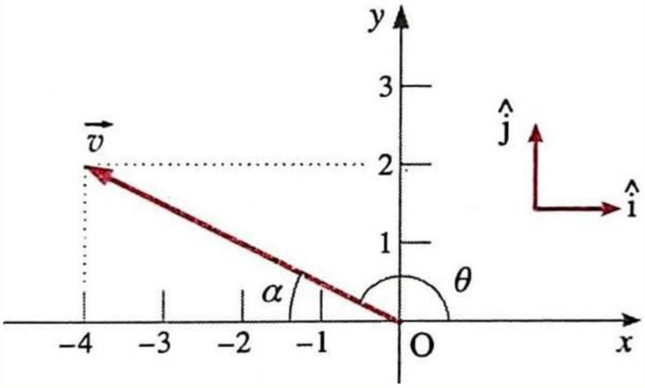
$$\therefore \ |\vec{v}| = \sqrt{(-4)^2 + (2)^2} = \sqrt{(16+4)}$$
$$= \sqrt{20} = 2\sqrt{5}$$

$$\tan\theta = \frac{v_y}{v_x} = \frac{2}{-4} = -\frac{1}{2} = \tan(180° - \alpha) = -\tan\alpha$$

$$\therefore \ \alpha = \tan^{-1}\frac{1}{2} = 26°34'$$

$$\therefore \ \theta = 180° - 26°34' = 153°26'$$

$\therefore$ $\vec{v}_1 + \vec{v}_2$ **has a magnitude of** $2\sqrt{5}$ **and makes an angle of** $\tan^{-1}\left(\frac{1}{2}\right) = 153°26'$ **with the positive** x**-axis.**



iv) Find a vector which is parallel to
$\vec{v} = \hat{i} - 2\hat{j}$
and has a magnitude 10.

Ans:

Let $\vec{u}$ be a vector of magnitude 10 and parallel to $\vec{v} = \hat{i} - 2\hat{j}$.

$\therefore \vec{u} = \lambda\vec{v}$, where the scalar multiple $\lambda = \frac{u}{v}$.

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{(1)^2 + (-2)^2} = \sqrt{5}$$

$$\therefore \lambda = \frac{10}{\sqrt{5}} = 2\sqrt{5} \qquad \text{as } u = 10$$

$$\therefore \vec{u} = 2\sqrt{5}\,(\hat{i} - 2\hat{j}) = 2\sqrt{5}\,\hat{i} - 4\sqrt{5}\,\hat{j}$$

is the required vector.

V) Show that vectors
$\vec{\mathbf{a}} = \mathbf{2\hat{i} + 5\hat{j} - 6\hat{k}}$ **and**
$\vec{\mathbf{b}} = \mathbf{\hat{i} + \frac{5}{2}\hat{j} - 3\hat{k}}$ **are parallel.**

Ans.

<u>Method 1:</u>

$\vec{a} = 2\hat{i} + 5\hat{j} - 6\hat{k} = 2\left(\hat{i} + \frac{5}{2}\hat{j} - 3\hat{k}\right) = 2\vec{b}$

Since $\vec{a}$ is a scalar multiple of $\vec{b}$,
the vectors are parallel.

<u>Method 2:</u>

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 5 & -6 \\ 1 & \frac{5}{2} & -3 \end{vmatrix}$$

$= (-15 + 15)\hat{i} + (-6 + 6)\hat{j}$
$+(5 - 5)\hat{k} = \vec{0}$

Therefore, $\vec{a}$ and $\hat{b}$ are parallel.

3. Solve the following problems.

i) Determine $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$,
given $\vec{\mathbf{a}} = \mathbf{2\hat{i} + 3\hat{j}}$
$\vec{b} = \mathbf{3\hat{\imath} + 5\hat{\jmath}}$.

<u>Solution:</u>

<u>Data:</u>

$\vec{a} = 2\hat{i} + 3\hat{j}, \vec{b} = 3\hat{i} + 5\hat{j}$
$\vec{a} \times \vec{b} = (2\hat{i} + 3\hat{j}) \times (3\hat{i} + 5\hat{j})$
$= 2\hat{i} \times 3\hat{i} + 2\hat{i} \times 5\hat{j} + 3\hat{j} \times 3\hat{i} + 3\hat{j} \times 5\hat{j}$
$= 0 + 10\hat{k} - 9\hat{k} + 0 = \hat{\mathbf{k}}$

ii) Show that vectors
$\vec{\mathbf{a}} = \mathbf{2\hat{i} + 3\hat{j} + 6\hat{k}}$,
$\vec{\mathbf{b}} = \mathbf{3\hat{i} - 6\hat{j} + 2\hat{k}}$
and $\vec{\mathbf{c}} = \mathbf{6\hat{i} + 2\hat{j} - 3\hat{k}}$
are mutually perpendicular.

Solution:

Data : $\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$, $\vec{b} = 3\hat{i} - 6\hat{j} + 2\hat{k}$,
$\vec{c} = 6\hat{i} + 2\hat{j} - 3\hat{k}$

$a \neq 0$, $b \neq 0$, $c \neq 0$

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z$$
$$= (2)(3) + (3)(-6) + (6)(2)$$
$$= 6 - 18 + 12 = 0$$

$\therefore \vec{b} \perp \vec{a}$

$$\vec{a}\cdot\vec{c} = a_x c_x + a_y c_y + a_z c_z$$
$$= (2)(6) + (3)(2) + (6)(-3)$$
$$= 12 + 6 - 18 = 0$$

$\therefore \vec{c} \perp \vec{a}$

$$\vec{b}\cdot\vec{c} = b_x c_x + b_y c_y + b_z c_z$$
$$= (3)(6) + (-6)(2) + (2)(-3)$$
$$= 18 - 12 - 6 = 0$$

$\therefore \vec{c} \perp \vec{b}$

It follows that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular.

iii) Determine the vector product of
$\vec{v_1} = 2\hat{i} + 3\hat{j} - \hat{k}$
and $\vec{v_2} = \hat{i} + 2\hat{j} - 3\hat{k}$,

Solution:

Data : $\vec{v_1} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{v_2} = \hat{i} + 2\hat{j} - 3\hat{k}$

$$\vec{v_1} \times \vec{v_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 2 & -3 \end{vmatrix}$$
$$= \hat{i}[(3)(-3) - (-1)(2)] + \hat{j}[(-1)(1) - (2)(-3)] + \hat{k}[(2)(2) - (3)(1)]$$
$$= \hat{i}(-9 + 2) + \hat{j}(-1 + 6) + \hat{k}(4 - 3)$$
$$= -7\hat{i} + 5\hat{j} + \hat{k}$$

iv) Given $\vec{v_1} = 5\hat{i} + 2\hat{j}$
and $\vec{v_2} = a\hat{i} - 6\hat{j}$
are perpendicular to each other, determine the value of a.

Solution:

Data:

$\vec{v_1} = 5\hat{i} + 2\hat{j}$ perpendicular to

$\vec{v_2} = a\hat{i} - 6\hat{j}$

$\vec{v_1}.\vec{v_2} = 5a + (2)(-6) = 0$

$\therefore 5a = 12 \therefore a = \frac{12}{5} = \mathbf{2.4}$

v) Obtain derivatives of the following functions:

(i) $x \sin x$

(ii) $x^4 + \cos x$

(iii) $x/\sin x$

Solution:

(i) $\frac{d}{dx}(x \sin x) = \left(\frac{d \sin x}{dx}\right) + \sin\ x\left(\frac{dx}{dx}\right)$

$= \mathbf{x \cos x + \sin x}$

(ii) $\frac{d}{dx}(x^4 + \cos) = \mathbf{4x^3 - \sin x}$

(iii) $\frac{d}{dx}\left(\frac{x}{\sin x}\right) = \frac{1}{\sin x}.\frac{dx}{dx} - \frac{1}{\sin^2 x}.x\frac{d(\sin x)}{dx}$

$= \frac{1}{\sin x} - \frac{x \cos x}{\sin^2 x}$

vi) Using the rule for differentiation for quotient of two functions, prove that $\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \sec^2 x$

Solution:

$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{1}{\cos x}.\frac{d \sin x}{dx} -$

$\frac{1}{\cos^2 x}.\sin x\frac{d}{dx}(\cos x)$

$= \frac{\cos^2 x}{\cos x} + \frac{\sin^2 x}{\cos^2 x} = 1 + \frac{\sin^2 x}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

vii) Evaluate the following integral:

(i) $\int_0^{\pi/2} \sin x\, dx$

(ii) $\displaystyle\int_{1}^{5} \boldsymbol{x\,dx}$.

Solution:

(i) $\displaystyle\int_{0}^{\pi/2} \sin x\,dx = -\cos x\,\Big|_{0}^{\pi/2}$

$= -\cos\frac{\pi}{2} + \cos 0 = 0 + 1 = \mathbf{1}$

(ii) $\displaystyle\int_{1}^{5} x\,dx = \frac{x^2}{2}\Big|_{1}^{5} = \frac{(5)^2}{2} - \frac{(1)^2}{2} = \frac{25-1}{2}$

$= \frac{24}{2} = \mathbf{12}$