# STRAIGHT LINES

## 1. DISTANCE FORMULA

The distance between the points $A(x_1,y_1)$ and $B(x_2,y_2)$ is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ .

## 2. SECTION FORMULA

The P(x,y) divided the line joining $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio m : n, then ;

$$x=\frac{mx_2+nx_1}{+n};\ y=\frac{my_2+ny_1}{+n}$$

*Note...*

(i) If m/n is positive, the division is internal, but if m/n is negative, the division is external.

(ii) If P divides AB internally in the ratio m:n & Q divides AB externally in the ratio m:n then

P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Mathematically, $\frac{2}{AB}=\frac{1}{AP}+\frac{1}{AQ}$ i.e. AP, AB & AQ are in H.P.

## 3. CENTROID, INCENTRE & EXCENTRE

If $A(x_1, y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ are the vertices of triangle ABC, whose sides BC, CA, AB are of lengths a, b, c respectively, then the co-ordinates of the special points of triangle ABC are as follows :

Centroid $G \equiv \left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$

Incentre $I \equiv \left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right)$ and

Excentre (to A) $I_1$

$\equiv \left(\frac{-ax_1+bx_2+cx_3}{-a+b+c},\frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$ and so on.

*Note...*

(i) Incentre divides the angle bisectors in the ratio, (b+c): a; (c+a) : b & (a+b) : c.

(ii) Incentre and excentre are harmonic conjugate of each other w.r.t. the angle bisector on which they lie.

(iii) Orthocentre, Centroid & Circumcentre are always collinear & centroid divides the line joining orthocentre & circumcentre in the ratio 2:1.

(iv) In an isosceles triangle G, O, I & C lie on the same line and in an equilateral traingle, all these four points coincide.

## 4. AREA OF TRIANGLE

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of triangle ABC, then its area is equal to

$\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$, provided the vertices are considered in the counter clockwise sense.

The above formula will give a (–)ve area if the vertices $(x_i, y_i)$, i = 1, 2, 3 are placed in the clockwise sense.

*Note...*

Area of n-sided polygon formed by points $(x_1, y_1); (x_2, y_2); ............(x_n, y_n)$ is given by :

$$\frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + ........... \begin{vmatrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{vmatrix}\right)$$

## 5. SLOPE FORMULA

If θ is the angle at which a straight line is inclined to the positive direction of x-axis, and $0° \le \theta < 180°$, $\theta \ne 90°$, then the slope of the line, denoted by m, is defined by $m = \tan\theta$. If θ is 90°, m does n't exist, but the line is parallel to the y-axis, If θ = 0, then m = 0 and the line is parallel to the x-axis.

If $A(x_1, y_1)$ & $B(x_2, y_2)$, $x_1 \ne x_2$, are points on a straight line, then the slope m of the line is given by :

$$m = \left(\frac{y_1 - y_2}{x_1 - x_2}\right)$$

## 6. CONDITION OF COLLINEARITY OF THREE POINTS

Points A $(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ are collinear if :

(i) $m_{AB} = m_{BC} = m_{CA}$ i.e. $\left(\frac{y_1 - y_2}{x_1 - x_2}\right) = \left(\frac{y_2 - y_3}{x_2 - x_3}\right)$

(ii) $\Delta ABC = 0$ i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(iii) AC = AB + BC or AB ~ BC

(iv) A divides the line segment BC in some ratio.

## 7. EQUATION OF A STRAIGHT LINE IN VARIOUS FORMS

**(i) Point-Slope form :** $y - y_1 = m(x - x_1)$ is the equation of a straight line whose slope is m and which passes through the point $(x_1,y_1)$.

**(ii) Slope-Intercept form :** $y = mx + c$ is the equation of a straight line whose slope is m and which makes an intercept c on the y-axis.

**(iii) Two point form :** $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of a straight line which passes through the point $(x_1,y_1)$ & $(x_2,y_2)$

**(iv) Determinant form :** Equation of line passing through $(x_1,y_1)$ and $(x_2,y_2)$ is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

**(v) Intercept form :** $\frac{x}{a} + \frac{y}{b} = 1$ is the equation of a straight line which makes intercepts a & b on OX & OY respectively.

**(vi) Perpendicular/Normal form :** $x\cos\alpha + y\sin\alpha = p$ (where $p > 0$, $0 \le \alpha < 2\pi$) is the equation of the straight line where the length of the perpendicular from the origin O on the line is p and this perpendicular makes an angle α with positive x-axis.

**(vii) Parametric form :** $P(r) = (x,y) = (x_1 + r\cos\theta, y_1 + r\sin\theta)$ or $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$ is the equation of the line is parametric form, where 'r' is the parameter whose absolute value is the distance of any point (x,y) on the line from fixed point $(x_1,y_1)$ on the line.

**(viii) General Form :** $ax + by + c = 0$ is the equation of a straight line in the general form. In this case, slope of line $= -\frac{a}{b}$.

## 8. POSITION OF THE POINT $(x_1,y_1$ RELATIVE OF THE LINE ax + by + c = 0

If $ax_1 + by_1 + c$ is of the same sign as c, then the point $(x_1, y_1)$ lie on the origin side of $ax + by + c = 0$. But if the sign of $ax_1 + by_1 + c$ is opposite to that of c, the point $(x_1,y_1)$ will lie on the non-origin side of $ax + by + c = 0$.

In general two points $(x_1,y_1)$ and $(x_2,y_2)$ will lie on same side or opposite side of $ax+by+c=0$ according as $ax_1+ by_1+ c$ and $ax_2 + by_2 + c$ are of same or opposite sign respectively.

## 9. THE RATIO IN WHICH A GIVEN LINE DIVIDES THE LINE SEGMENT JOINING TWO POINTS

Let the given line $ax + by + c = 0$ divides the line segment joining $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio m:n, then $\frac{m}{n} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$. If A and B are on the same side of the given line then m/n is negative but if A and B are on opposite sides of the given line, then m/n is positive.

## 10. LENGTH OF PERPENDICULAR FROM A POINT ON A LINE

The length of perpendicular from $P(x_1,y_1)$ on $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

## 11. REFLECTION OF A POINT ABOUT A LINE

(i) The image of a point $(x_1,y_1)$ about the line $ax + by + c = 0$ is :

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -2\,\frac{ax_1 + by_1 + c}{a^2 + b^2}$$

(ii) Similarly foot of the perpendicular from a point on the line is :

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$$

## 12. ANGLE BETWEEN TWO STRAIGHT LINES IN TERMS OF THEIR SLOPES

If $m_1$ and $m_2$ are the slopes of two intersecting straight lines ($m_1m_2 \neq -1$) and θ is the acute angle between them, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$.

*Note...*

Let $m_1$, $m_2$, $m_3$ are the slopes of three line $L_1=0$; $L_2=0$; $L_3=0$ where $m_1 > m_2 > m_3$ then the interior angles of the ΔABC found by these lines are given by,

$$\tan A = \frac{m_1 - m_2}{1 + m_1m_2};\ \tan B\frac{m_2 - m_3}{1 + m_2m_3};\ \text{and } \tan C = \frac{m_3 - m_1}{1 + m_3m_1}$$

## 13. PARALLEL LINES

(i) When two straight lines are parallel their slopes are equal. Thus any line parallel to $y = mx + c$ is of the type $y = mx + d$, where d is parameter.

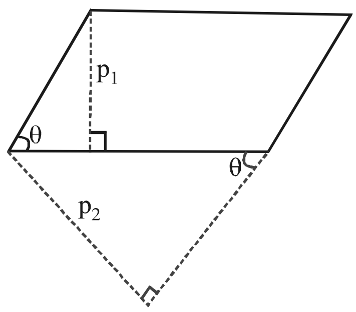
(ii) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are parallel if : $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$

Thus any line parallel to $ax + by + c = 0$ is of the type $ax + by + k = 0$, where k is a parameter.

(iii) The distance between two parallel lines with equations $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$

Coefficient of x & y in both the equations must be same.

(iv) The area of the parallelogram $= \frac{p_1p_2}{\text{Sin}\theta}$, where $p_1$ and $p_2$ are




distance between two pairs of opposite sides and θ is the angle between any two adjacent sides. Note that area of the parallogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$. and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$

## 14. PERPENDICULAR LINES

(i) When two lines of slopes $m_1$ & $m_2$ are at right angles, the product of their slope is –1 i.e., $m_1m_2 = -1$. Thus any line perpendicular to $y = mx + c$ is of the form.

$$y = -\frac{1}{m}x + d, \text{ where d is any parameter.}$$

(ii) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are perpendicular if $aa' + bb' = 0$. Thus any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$, where k is any parameter.

## 15. STRAIGHT LINES MAKING ANGLE a WITH GIVEN LINE

The equation of lines passing through point $(x_1,y_1)$ and making angle $\alpha$ with the line $y = mx + c$ are given by $(y - y_1) = \tan(\theta - \alpha)(x - x_1)$ & $(y - y_1) = \tan(\theta + \alpha)(x - x_1)$, where $\tan\theta = m$.

## 16. BISECTOR OF THE ANGLES BETWEEN TWO LINES

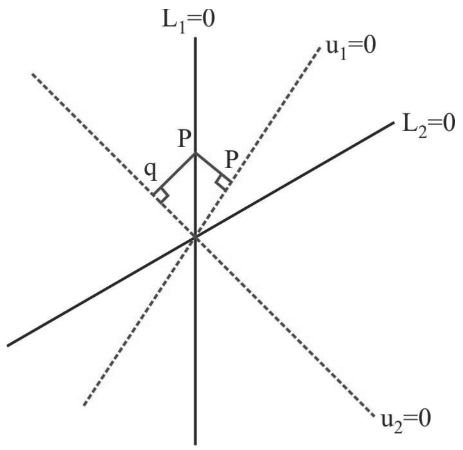
Equations of the bisectors of angles between the lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ $(ab' \neq a'b)$ are :

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = \pm \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$

*Note...*

Equation of straight lines through $P(x_1,y_1)$ & equally inclined with the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisector between these two lines & passing throught the point P.

## 17. METHODS TO DISCRIMINATE BETWEEN THE ACUTE BISECTOR AND THE OBTUSE ANGLE BISECTOR




(i) If $\theta$ be the angle between one of the lines & one of the bisectors, find $\tan\theta$. if $|\tan\theta| < 1$, then $2\theta < 90°$ so that this bisector is the acute angle bisector. if $|\tan\theta| > 1$, then we get the bisector to be the obtuse angle bisector

(ii) Let $L_1=0$ & $L_2=0$ are the given lines & $u_1=0$ and $u_2=0$ are bisectors between $L_1=0$ and $L_2=0$. Take a point P on any one of the lines $L_1=0$ or $L_2=0$ and drop perpendicular on $u_1=0$ and $u_2=0$ as shown. If.

$|p| < |q| \Rightarrow u_1$ is the acute angle bisector.

$|p| > |q| \Rightarrow u_1$ is the obtuse angle bisector.

$|p| = |q| \Rightarrow$ the lines $L_1$ and $L_2$ are perpendicular.

(iii) if $aa' + bb' < 0$, while c & c′ are postive, then the angle between the lines is acute and the equation of the bisector of this acute angle is $\frac{ax + by + c}{\sqrt{a^2 + b^2}} = +\frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$

If, however, $aa' + bb' > 0$, while c and c′ are postive, then the angle between the lines is obtuse & the equation of the bisector of this obtuse angle is :

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = +\frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$

The other equation represents the obtuse angle bisector in both cases.

## 18. TO DISCRIMINATE BETWEEN THE BISECTOR OF THE ANGLE CONTAINING A POINT

To discriminate between the bisector of the angle containing the origin & that of the angle not containing the origin. Rewrite the equation, $ax + by + c = 0$ & $a'x + b'y + c' = 0$ such that the constant term c, c' are positive.

Then ; $\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = +\dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle containing origin and $\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = -\dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle not containing the origin. In general equation of the bisector which contains the point $(\alpha, \beta)$ is.

$$\frac{ax+by+c}{\sqrt{a^2+b^2}} = \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}} \text{ or } \frac{ax+by+c}{\sqrt{a^2+b^2}} = -\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$$

according as $a\alpha + b\beta + c$ and $a'\alpha + b'\beta + c'$ having same sign or otherwise.

## 19. CONDITION OF CONCURRENCY

Three lines $a_1x + b_1y + c_1=0$, $a_2x + b_2y + c_2=0$ and $a_3x + b_3y + c_3=0$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

**Alternatively :** If three constants A, B and C (not all zero) can be found such that $A(a_1x + b_1y + c_1) + B(a_2x + b_2y + c_2) + C(a_3x + b_3y + c_3) \equiv 0$, then the three straight lines are concurrent.

## 20. FAMILY OF STRAIGHT LINES

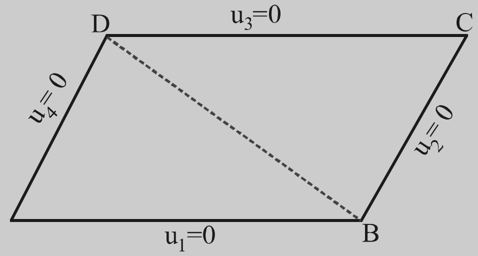
The equation of a family of straight lines passing through the points of intersection of the lines,

$L_1 \equiv a_1x + b_1y + c_1=0$ & $L_2 \equiv a_2x + b_2y + c_2=0$ is given by $L_1 + kL_2 = 0$ i.e.

$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$, where k is an arbitray real number.

*Note...*

(i) If $u_1= ax+by+c$, $u_2= a'x+b'y+d$, $u_3= ax+by+c'$, $u_4= a'x+b'y+d'$, then $u_1= 0$; $u_2= 0$; $u_3= 0$; $u_4= 0$; form a parallegoram




The diagonal BD can be given by $u_2u_3 - u_1u_4=0$

**Proof :** Since it is the first degree equation in x & y, it is a straight line. Secondly point B satisfies $u_2 = 0$ and $u_1 = 0$ while point D satisfies $u_3 = 0$ and $u_4 = 0$. Hence the result. Similarly, the diagonal AC can be given by $u_1u_2 - u_3u_4 = 0$

(ii) The diagonal AC is also given by $u_1 + \lambda u_4=0$ and $u_2+\mu u_3 = 0$, if the two equation are identical for some real $\lambda$ and $\mu$.

[For getting the values of $\lambda$ and $\mu$ compare the coefficients of x, y & the constant terms.]

## 21. A PAIR OF STRAIGHT LINES THROUGH ORIGIN

(i) A homogeneous equation of degree two, "$ax^2 + 2hxy + by^2 = 0$" always represents a pair of straight lines passing through the origin if :

(a) $h^2 > ab \Rightarrow$ lines are real and distinct.

(b) $h^2 = ab \Rightarrow$ lines are coincident.

(c) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. (0,0)

(ii) If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then;

$$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

(iii) If θ is the acute angle between the pair of straight lines represented by,

$ax^2 + 2hxy + by^2 = 0$, then; $\tan\theta = \left|\dfrac{2\sqrt{h^2 - ab}}{a + b}\right|$

(iv) The condition that these lines are :

(a) At right angles to each other is a + b = 0 i.e. co-efficient of $x^2$ + co-efficient of $y^2 = 0$

(b) Coincident is $h^2 = ab$.

(c) Equally inclined to the axis of x is h =0 i.e. coeff. of xy=0.

*Note...*

A homogeneous equation of degree n represents n straight lines passing through origin.

(v) The equation to the pair of straight lines bisecting the angle between the straight lines,

$ax^2 + 2hxy + by^2 = 0$, is $\dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}$

## 22. GENERAL EQUATION OF SECOND DEGREE REPRESENTING A PAIR OF STRAIGHT LINES

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ = represents a pair of straight lines if :

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, i.e. if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(ii) The angle θ between the two lines representing by a general equation is the same as that between two lines represented by its homogenous part only.

## 23. HOMOGENIZATION

The equation of a pair of straight lines joining origin to the points of intersection of the line

$L \equiv lx + my + n = 0$ and a second degree curve,

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx + my}{-n}\right) +$$

$$2fy\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0$$

The equal is obtained by homogenizing the equation of curve with the help of equation of line.

*Note...*

Equation of any cure passing through the points of intersection of two curves $C_1 = 0$ and $C_2 = 0$ is given by $\lambda C_1 + \mu C_2 = 0$ where λ and μ are parameters.