Exercise 15.10

Chapter 15 Multiple Integrals 15.10 1E

Given equations are x = 5u-v, y = u + 3v

Therefore Jacobian is $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix}$$

=5.3-1.(-1)

=15+1

=16

Chapter 15 Multiple Integrals 15.10 2E

Given equation is x = u.v, y = u/v

Therefore Jacobian is $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix}$$

$$= v.\frac{(-u)}{v^2} - u.\frac{1}{v}$$

$$= -\frac{2u}{v}$$

Chapter 15 Multiple Integrals 15.10 3E

Given equation is $x = e^{-r}\sin\theta,\ y = e^{r}\cos\theta$

Therefore Jacobian is $\dfrac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$

$$= \begin{vmatrix} -e^{-r}\sin\theta & e^{-r}\cos\theta \\ e^{r}\cos\theta & -e^{r}\sin\theta \end{vmatrix}$$

$$= \sin^2\theta - \cos^2\theta$$

Chapter 15 Multiple Integrals 15.10 4E

Given that $x = e^{s+t},\ y = e^{s-t}$

Therefore Jacobian is $\dfrac{\partial(x, y)}{\partial(s, t)} = \begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{vmatrix}$

$$= \begin{vmatrix} e^{s+t} & e^{s+t} \\ e^{s-t} & -e^{s-t} \end{vmatrix}$$

$$= -e^{s+t}.e^{s-t} - e^{s+t}.e^{s-t}$$

$$= -e^{s+t+s-t} - e^{s+t+s-t}$$

$$= -e^{2s} - e^{2s}$$

$$= -2e^{2s}$$

Chapter 15 Multiple Integrals 15.10 5E

Consider the transformation,

$$x=\frac{u}{v}, y=\frac{v}{w}, z=\frac{w}{u}.$$

The object is to find the Jacobean of the transformations.

Recall that, the Jacobean of the transformation

$x=f(u,v,w), y=g(u,v,w), z=h(u,v,w)$ is defined as,

$$J\left(\frac{\partial(x,y,z)}{\partial(u,v,w)}\right)=\begin{vmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w}\\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w}\\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w}\end{vmatrix}.$$

The partial derivatives of the transformation are

$$\begin{aligned}\frac{\partial x}{\partial u}&=\frac{1}{v}\frac{\partial}{\partial u}(u)\\ &=\frac{1}{v}(1)\\ &=\frac{1}{v}\end{aligned}$$

$$\begin{aligned}\frac{\partial x}{\partial v}&=u\frac{\partial}{\partial u}\left(\frac{1}{v}\right)\\ &=u\left(\frac{-1}{v^2}\right)\\ &=\frac{-u}{v^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial x}{\partial w}&=\frac{\partial}{\partial w}\left(\frac{u}{v}\right)\\ &=0\end{aligned}$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left(\frac{v}{w}\right)$$

$$= 0$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\left(\frac{v}{w}\right)$$

$$= \frac{1}{w}(1)$$

$$= \frac{1}{w}$$

$$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}\left(\frac{w}{u}\right)$$

$$= w\frac{\partial}{\partial u}\left(\frac{1}{u}\right)$$

$$= w\left(\frac{-1}{u^2}\right)$$

$$= \frac{-w}{u^2}$$

$$\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}\left(\frac{w}{u}\right)$$

$$= 0$$

$$\frac{\partial z}{\partial w} = \frac{\partial}{\partial w}\left(\frac{w}{u}\right)$$

$$= \frac{1}{u}(1)$$

$$= \frac{1}{u}$$

Substitute all the values into $J\left(\frac{\partial(x,y,z)}{\partial(u,v,w)}\right)=\begin{vmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w}\\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w}\\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w}\end{vmatrix}$, and then the Jacobean is

$$J\left(\frac{\partial(x,y,z)}{\partial(u,v,w)}\right)=\begin{vmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w}\\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w}\\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w}\end{vmatrix}$$
$$=\begin{vmatrix}\frac{1}{v} & \frac{-u}{v^2} & 0\\ 0 & \frac{1}{w} & \frac{-v}{w^2}\\ \frac{-w}{u^2} & 0 & \frac{1}{u}\end{vmatrix}$$

The determinant of the matrix is calculated as,

$$J\left(\frac{\partial(x,y,z)}{\partial(u,v,w)}\right)=\begin{vmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w}\\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w}\\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w}\end{vmatrix}$$
$$=\begin{vmatrix}\frac{1}{v} & \frac{-u}{v^2} & 0\\ 0 & \frac{1}{w} & \frac{-v}{w^2}\\ \frac{-w}{u^2} & 0 & \frac{1}{u}\end{vmatrix}$$
$$=\frac{1}{v}\begin{vmatrix}\frac{1}{w} & \frac{-v}{w^2}\\ 0 & \frac{1}{u}\end{vmatrix}-\left(\frac{-u}{v^2}\right)\begin{vmatrix}0 & \frac{-v}{w^2}\\ \frac{-w}{u^2} & \frac{1}{u}\end{vmatrix}+0\begin{vmatrix}0 & \frac{1}{w}\\ \frac{-w}{u^2} & 0\end{vmatrix}$$
$$=\frac{1}{v}\left(\frac{1}{uw}\right)+\frac{u}{v^2}\left(0-\frac{vw}{u^2w^2}\right)+0$$
$$=\frac{1}{uvw}-\frac{1}{uvw}$$
$$=0$$

Hence, the Jacobean of the transformation is

$$J=\boxed{0}.$$

Chapter 15 Multiple Integrals 15.10 6E

Givne that $x = v + w^2, y = w + u^2, z = u + v^2$

Jacobian is $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2w \\ 2u & 0 & 1 \\ 1 & 2v & 0 \end{vmatrix}$

$$= 0 - 1(0 - 1) + 2w(2u.2v - 0)$$

$$= 1 + 8\text{uvw}$$

Chapter 15 Multiple Integrals 15.10 7E

$S = \{(u,v)/0 \le u \le 3,\ 0 \le v \le 2\}$
$x = 2u + 3v,\ y = u - v$

The first side S_1 is given by $v = 0$, we have
$x = 2u$, $y = u$, we get $x = 2y$ which is equation of straight line with end points (0, 0) and (6, 3)

The second side S_2 is given by $u = 3$, we get
$x = 6 + 3v$, $y = 3 - v$. Thus $x = 15 - 3y$ is equation of straight line with end points $(6,3)$ and $(12,1)$

The S_3 is given by $v = 2$, we get $x = 2u + 6$ and $y = u - 2$, so $x = 2y + 10$ is equation of straight line with end points $(6,-2)$ and $(12,1)$

The S_4 is given by $u = 0$, we get $x = 3v, y = -v$. Thus $x = -3y$ is the equation of straight line with end points (0, 0) and (6, -2).
Therefore the image of S is the parallelogram with vertices (0, 0), (6, 3), (12, 1), (6, -2)

Chapter 15 Multiple Integrals 15.10 8E

Consider the following transformation S is square bounded by the lines,

$$u = 0, u = 1, v = 0, v = 1, x = v, y = u\left(1+v^2\right)$$

The objective is to find image of the set S.

Write the transformation as set notation as follows:

$$S = \{(u,v) \mid 0 \le u \le 1, 0 \le v \le 1\}$$
$$x = v, y = u\left(1+v^2\right)$$

Let S_1 be the first transformation.

Substitute $v = 0$ in $x = v, y = u\left(1+v^2\right)$.

$$x = 0, y = u\left(1+0^2\right)$$
$$x = 0, y = u$$

Since $u = 0$ and $u = 1$. So $y = 0$ and $y = 1$.

Thus, S_1 is mapped in to the line segment from $(0,0)$ to $(0,1)$.

Let S_2 be the second transformation.

Substitute $v = 1$ in $x = v, y = u\left(1+v^2\right)$.

$$x = 1, y = u\left(1+1^2\right)$$
$$x = 1, y = 2u$$

Since $u = 0$ and $u = 1$. So $y = 0$ and $y = 2$.

Thus, S_2 is mapped in to the line segment from $(1,0)$ to $(1,2)$.

Let S_3 be the third transformation.

Substitute $u = 0$ in $x = v, y = u\left(1+v^2\right)$.

$$x = v, y = u\left(1+1^2\right)$$
$$x = v, y = 0\left(1+v^2\right)$$
$$x = v, y = 0$$

Since $v = 0$ and $v = 1$. So $x = 0$ and $x = 1$.

Thus, S_3 is mapped in to the line segment from $(0,0)$ to $(1,0)$.

Therefore, the image of S is the region $x-$axis, $y-$axis, line segment from $(1,0)$ to $(1,2)$ and the parabola given by equation $x^2 = y-1$.

Let S_4 be the fourth transformation.

Substitute $u=1$ in $x=v, y=u\left(1+v^2\right)$.

$$x=v, y=u\left(1+1^2\right)$$
$$x=v, y=1\left(1+v^2\right)$$
$$x=v, y=1+v^2$$

Substitute $x=v$ in $y=1+v^2$.

$$y=1+x^2$$
$$x^2=y-1$$

Thus, the transformation S_4 is a parabola.

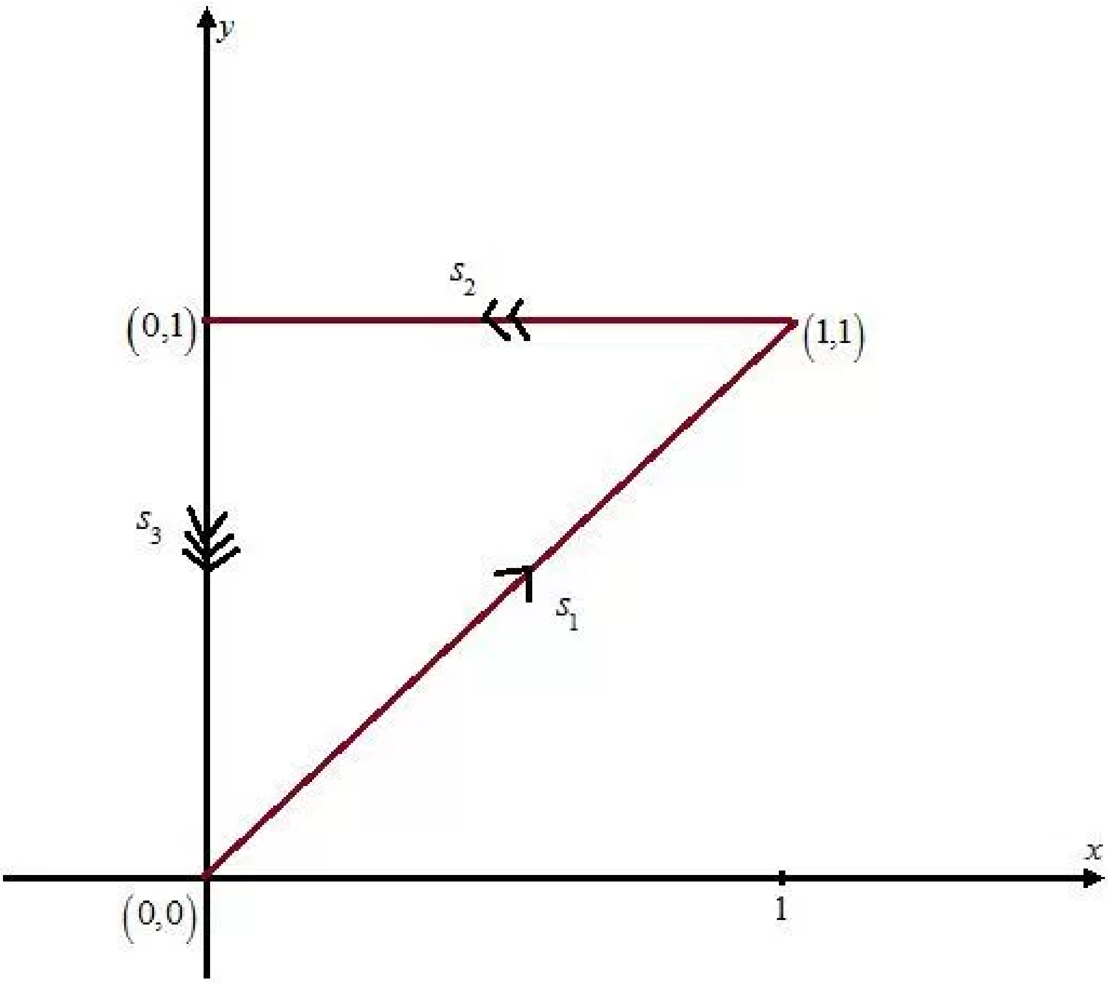
Chapter 15 Multiple Integrals 15.10 9E

Consider the triangular region *S* with vertices

$$(0,1),(1,1),(0,1); x=u^2, y=v$$

Find the image of the set S under the transformation:

Sketch the triangle $(0,1),(1,1),(0,1)$ is as follows:



Let S_1 be the line segment $u = v, 0 \le u \le 1$

So, $y = v = u$ and $x = u^2 = y^2$.

Since $0 \le u \le 1$, the image is the portion of the parabola $x = y^2, 0 \le y \le 1$.

Let S_2 be the line segment $v = 1$, $0 \le u \le 1$

Thus, $y = v = 1$ and $x = u^2$

The image is line segment $y = 1, \ 0 \le u \le 1$

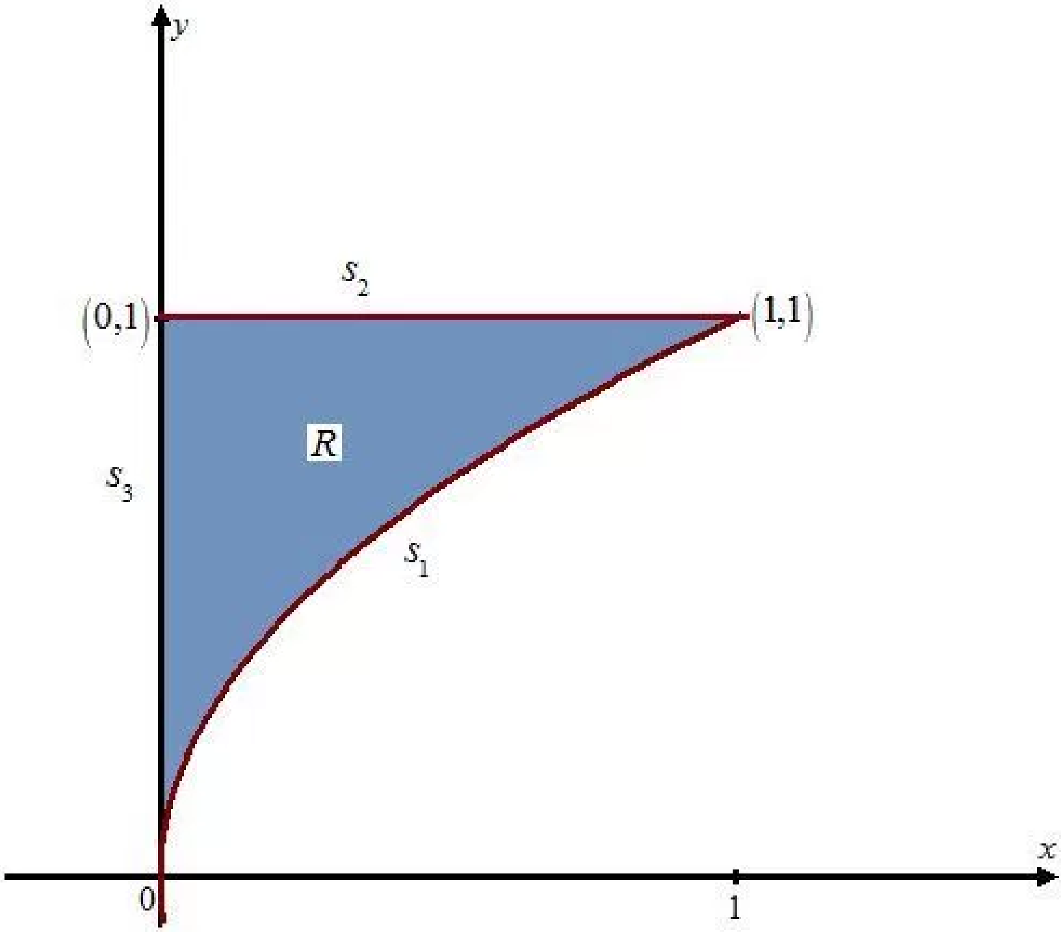
Let S_3 be the segment $u = 0, 0 \le v \le 1$

So $x = u^2 = 0$ and $y = v$

The image is the segment $x = 0$, $0 \le y \le 1$.

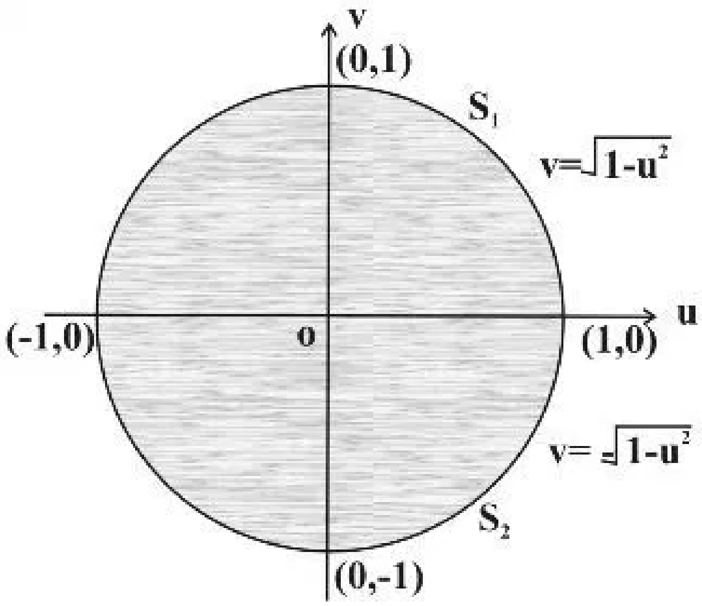
Therefore, the image of S is the region R in the first quadrant bounded by the parabola $\sqrt{x} = y$, the $y-\text{axis}$, and the line $y = 1$.

Sketch the image of S is the region R. $y = \sqrt{x}, y = 1, x = 0$



Chapter 15 Multiple Integrals 15.10 10E

S is the disk given by $u^2 + v^2 \le 1$



We divide the boundary of S into two parts S_1 and S_2 where S_1 is given by $-1 \le u \le 1,\ 0 \le v \le \sqrt{1-u^2}$ and S_2 is given by $-1 \le u \le 1,\ -\sqrt{1-u^2} \le v \le 0$

Now it is given that $x = au$ and $y = bv$

On S_1, $x = au,\ y = b\sqrt{1-u^2}$

Eliminating u from these equations, we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

That is the image of S_1 is an elliptic arc $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Similarly on S_2, $x = au,\ y = -b\sqrt{1-u^2}$

On eliminating u we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

That is image of S_2 is an elliptic arc $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Hence the image of S is the region R bounded by an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Chapter 15 Multiple Integrals 15.10 11E

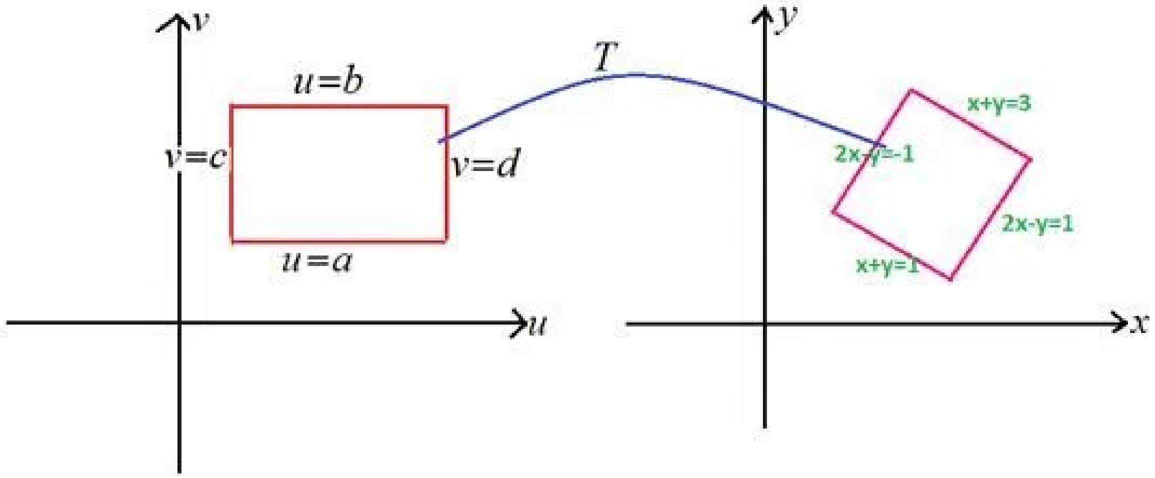
Consider the region R in the $xy-\text{plane}$ is

$y = 2x - 1, y = 2x + 1, y = 1 - x, y = 3 - x$

Now need to find transformation T from uv plane to xy plane such that the rectangular region S maps to R

Where the sides of S are parallel to the $u - and\ v - axis$

Consider the rectangular region $u = a, u = b, v = c, v = d$



The lines are $u = a, u = b$ parallel in $uv-\text{plane}$ and $x + y = 1, x + y = 3$ are parallel in $xy-\text{plane}$

So one of the possibilities is $u = a$ maps to $x + y = 1, u = b$ maps to $x + y = 3$

$1 \le u \le 3$

Let $u = x + y$

The lines are $v = c, v = d$ parallel in $uv-\text{plane}$ and $2x - y = -1, 2x - y = 1$ are parallel in $xy-\text{plane}$

So one of the possibilities is $v = c$ maps to $2x - y = -1, v = d$ maps to $2x - y = 1$

$-1 \le v \le 1$

Let $v = 2x - y$

Add the equations for *u* and *v* and solve for *x*.

$u + v = x + y + 2x - y$

$u + v = 3x$

$$x = \frac{u + v}{3}$$

Replace x with $\frac{u+v}{3}$ in $v = 2x - y$.

$$v = 2\left(\frac{u+v}{3}\right) - y$$

$$v = \frac{2u+2v}{3} - y$$

$$y = \frac{2u+2v-3v}{3}$$

$$y = \frac{2u-v}{3}$$

Thus, we get $x = \frac{u+v}{3}$ and $y = \frac{2u-v}{3}, 1 \le u \le 3, -1 \le v \le 1$

That is $S = \{(u,v) / 1 \le u \le 3, -1 \le v \le 1\}$

Chapter 15 Multiple Integrals 15.10 12E

From the given vertices, we get the sides of the parallelogram as $x + 2y = 0$, $x + 2y = 10$, $4y - 3x = 0$, and $4y - 3x = 10$.

Let $u = x + 2y$ and $v = 4y - 3x$. Then, we have $u = 0$, $u = 10$, $v = 0$ and $v = 10$.

Multiply both sides of $u = x + 2y$ by 3 and add to $v = 4y - 3x$.

$$3u + v = 3x + 6y + 4y - 3x$$

$$3u + v = 10y$$

$$y = \frac{3u+v}{10}$$

Replace y with $\frac{3u+v}{10}$ in $v = 4y - 3x$.

$$v = 4\left(\frac{3u+v}{10}\right) - 3x$$

$$v = \frac{6u}{5} + \frac{2v}{5} - 3x$$

$$3x = \frac{6u}{5} + \frac{2v}{5} - \frac{5v}{5}$$

$$x = \frac{1}{5}(2u - v)$$

Thus, we get $\boxed{x = \frac{1}{5}(2u - v) \text{ and } y = \frac{3u+v}{10}}$, where $0 \le u \le 10$ and $0 \le v \le 10$.

Chapter 15 Multiple Integrals 15.10 13E

Let $u^2 = x^2 + y^2$ and $v = \tan^{-1}\left(\frac{y}{x}\right)$.

Then, obtain that $u = 1$, $u = \sqrt{2}$, $v = 0$ and $v = \pi/2$.

Take the square root on both sides of the equation.

$$\sqrt{u^2} = \sqrt{x^2 + y^2}$$
$$u = \sqrt{x^2 + y^2}$$

Since

$$v = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\tan v = \frac{y}{x}$$

$$y = x \tan v$$

Replace y with $x \tan v$ in $u = \sqrt{x^2 + y^2}$.

$$u = \sqrt{x^2 + (x \tan v)^2}$$
$$u^2 = x^2 + (x \tan v)^2$$
$$= x^2 + x^2 \tan^2 v$$
$$= x^2 (1 + \tan^2 v)$$
$$u^2 = x^2 \sec^2 v$$
$$\frac{u^2}{\sec^2 v} = x^2$$
$$x^2 = u^2 \cos^2 v$$
$$x = u \cos v$$

Substitute $u \cos v$ for x in $u = \sqrt{x^2 + y^2}$.

$$u = \sqrt{(u \cos v)^2 + y^2}$$
$$u^2 = u^2 \cos^2 v + y^2$$
$$u^2 - u^2 \cos^2 v = y^2$$
$$u^2 (1 - \cos^2 v) = y^2$$
$$u^2 \sin^2 v = y^2$$
$$y = u \sin v$$

Thus, $x = u \cos v$ and $y = u \sin v$, where $1 \le u \le \sqrt{2}$ and $0 \le v \le \pi/2$.

Hence the required equations are $\boxed{x = u \cos v \text{ and } y = u \sin v}$.

Chapter 15 Multiple Integrals 15.10 14E

Let $u = xy$ and $v = \dfrac{y}{x}$.

Then, we have $u = 1$, $u = 4$, $v = 1$ and $v = 4$.

Multiply both the equations for u and v.

$$uv = (xy)\frac{y}{x}$$
$$uv = y^2$$
$$y = \sqrt{uv}$$

We get $y = \sqrt{uv}$.

Divide the equation $u = xy$ by $v = \dfrac{y}{x}$.

$$\frac{u}{v} = \frac{xy}{\frac{y}{x}}$$
$$\frac{u}{v} = x^2$$
$$x = \sqrt{\frac{u}{v}}$$

Thus, we get $\boxed{x = \sqrt{\frac{u}{v}} \text{ and } y = \sqrt{uv}}$, where $1 \le u \le 4$ and $1 \le v \le 4$.

Chapter 15 Multiple Integrals 15.10 15E

Determine the equations of the lines of the triangle.

The line R_1 passes through the points $A(0,0)$ and $B(2,1)$.

So, the equation of the line R_1 is:

$$\frac{x-0}{2-0} = \frac{y-0}{1-0}$$
$$y = \frac{x}{2}$$

The line R_2 passes through the points $B(2,1)$ and $C(1,2)$.

So, the equation of the line R_2 is:

$$\frac{x-2}{1-2} = \frac{y-1}{2-1}$$
$$y = 3 - x$$

The line R_3 passes through the points $C(1,2)$ and $A(0,0)$.

So, the equation of the line R_3 is:

$$\frac{x-1}{0-1} = \frac{y-2}{0-2}$$
$$y = 2x$$

The transformation $x = 2u + v$, $y = u + 2v$ on R_1 gives:

$$y = \frac{x}{2}$$

$$u + 2v = \frac{1}{2}(2u + v)$$

$$v = 0 \quad \text{......} \ (1)$$

On R_2:

$$x + y = 3$$

$$(2u + v) + (u + 2v) = 3$$

$$u + v = 1 \quad \text{......} \ (2)$$

And the transformation $x = 2u + v$, $y = u + 2v$ on R_3 gives:

$$y = 2x$$

$$u + 2v = 2(2u + v)$$

$$u = 0 \quad \text{......} \ (3)$$

From (1), (2) and (3) observe that the transformed region R is given by:

$$S = \{(u, v),\ 0 \le u \le 1,\ 0 \le v \le 1 - u\}$$

The jacobian of the transformation is:

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 4 - 1$$

$$= 3$$

The integral is evaluated as follows:

$$\begin{aligned}\iint_R (x-3y)\,dA &= \iint_S [2u+v-3u-6v]\frac{\partial(x,y)}{\partial(u,v)}\,dudv \\ &= -\iint_S (u+5v)3\,dudv \\ &= -3\int_0^1\int_0^{1-u}(u+5v)\,dvdu \\ &= -3\int_0^1\left[uv+\frac{5}{2}u^2\right]_{u=0}^{u=1-v} du\end{aligned}$$

Further evaluate:

$$\begin{aligned}\iint_R (x-3y)\,dA &= -3\int_0^1\left(u-u^2+\frac{5}{2}(1-u)^2\right)du \\ &= -3\left[\frac{1}{2}u^2-\frac{1}{3}u^3-\frac{5}{6}(1-u)^3\right]_0^1 \\ &= -3\left(\frac{1}{2}-\frac{1}{3}+\frac{5}{6}\right) \\ &= -3\end{aligned}$$

Therefore, the value of the integral is:

$$\boxed{\iint_R (x-3y)\,dA = -3}$$

Chapter 15 Multiple Integrals 15.10 16E

Consider the integral,

$$\iint_R (4x+8y)\,dA$$

Here, R is the parallelogram with vertices $A(-1,3), B(1,-3), C(3,-1), D(1,5)$, and the transformation $x=\frac{1}{4}(u+v)$ and $y=\frac{1}{4}(v-3u)$.

The objective is to evaluate the above integral.

Use the change of variables in a double integral to evaluate the integral as follows:

Change of variables in a double integral:

Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the $uv-$ plane onto a region R in the $xy-$ plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x,y)\,dA = \iint_S f\left(x(u,v), y(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \quad \text{.....} \quad (\mathrm{I})$$

Find the Jacobean of T as follows:

$$T = \frac{\partial(x,y)}{\partial(u,v)}$$
$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Find the partials of x, y with respect to u and v as follows:

$$x = \frac{1}{4}(u+v)$$
$$\frac{\partial x}{\partial u} = \frac{1}{4}$$
$$\frac{\partial x}{\partial v} = \frac{1}{4}$$
$$y = \frac{1}{4}(v-3u)$$
$$\frac{\partial y}{\partial u} = \frac{-3}{4}$$
$$\frac{\partial y}{\partial v} = \frac{1}{4}$$

Substitute these values in the Jacobean T.

$$\text{jac}(T) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{-3}{4} & \frac{1}{4} \end{vmatrix}$$

$$= \frac{1}{16} + \frac{3}{16}$$

$$= \frac{4}{16}$$

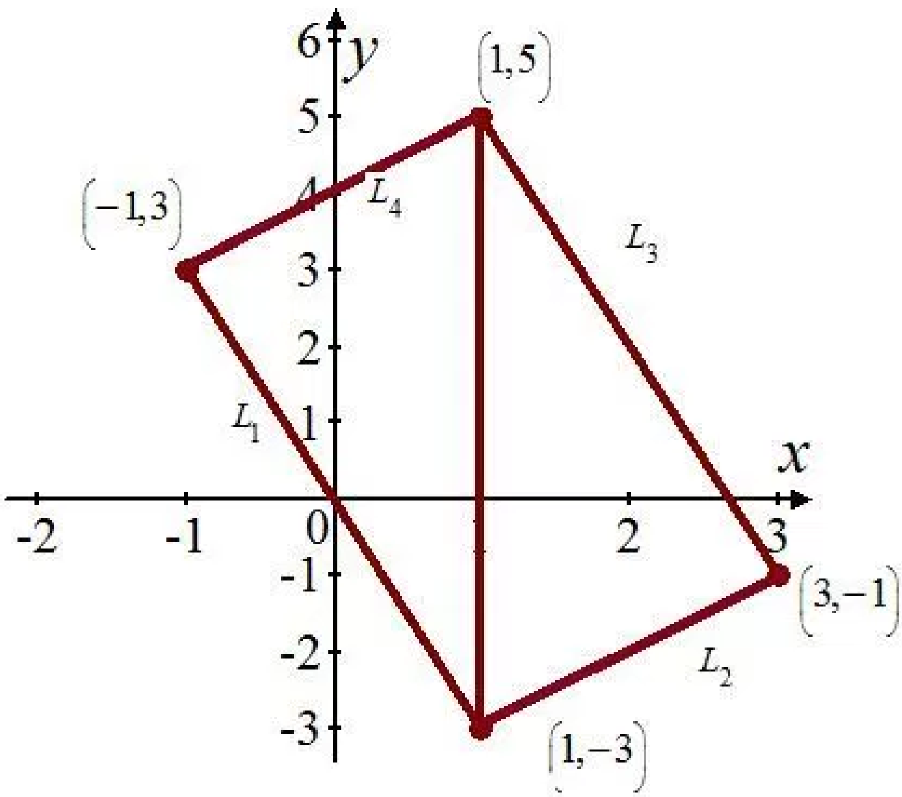
$$= \frac{1}{4}$$

Thus, the required value is $\text{jac}(T) = \frac{1}{4}$.

Consider the region R is the parallelogram with vertices

$A(-1,3), B(1,-3), C(3,-1) \text{ and } D(1,5)$

Sketch the rough graph of the parallelogram using above vertices as follows:



The equation of the straight line passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

The equation of the straight line L_1 passing through $A(-1,3), B(1,-3)$ is,

$$\frac{x+1}{1+1} = \frac{y-3}{-3-3}$$
$$\frac{x+1}{2} = \frac{y-3}{-6}$$
$$-3(x+1) = y-3$$
$$-3x-3 = y-3$$

$$3x + y = 0$$

The equation of the straight line L_2 passing through $B(1,-3), C(3,-1)$ is

$$\frac{x-1}{3-1} = \frac{y+3}{-1+3}$$
$$\frac{x-1}{2} = \frac{y+3}{2}$$
$$x-1 = y+3$$
$$x = y+4$$

$$x - y = 4$$

The equation of the straight line L_3 passing through $C(3,-1)$ and $D(1,5)$ is,

$$\frac{x-1}{3-1} = \frac{y-5}{-1-5}$$
$$\frac{x-1}{2} = \frac{y-5}{-6}$$
$$-3(x-1) = y-5$$
$$-3x+3 = y-5$$

$$3x + y = 8$$

The equation of the straight line L_4 passing through $A(-1,3)$ and $D(1,5)$ is,

$$\frac{x+1}{1+1} = \frac{y-3}{5-3}$$
$$\frac{x+1}{2} = \frac{y-3}{2}$$
$$x+1 = y-3$$

$$x - y = -4$$

Now calculating u, v from x, y as follows:

$$x = \frac{1}{4}(u+v)$$
$$4x = u + v \qquad \text{......}(1)$$
$$y = \frac{1}{4}(v-3u)$$
$$4y = v - 3u \qquad \text{......}(2)$$

Subtracting equation (2) from equation (1),

$$\begin{array}{r} 4x = u + v \\ 4y = -3u + v \\ \hline 4x - 4y = 4u \end{array}$$
$$x - y = u \qquad \text{.....}(3)$$

Now, $(1)\times 3 + (2)$

$$\begin{array}{r} 12x = 3u + 3v \\ 4y = -3u + v \\ \hline 12x + 4y = 4v \end{array}$$
$$3x + y = v \qquad \text{......}(4)$$

Substitute the values of $u = x - y$ and $v = 3x + y$ in the equations $3x + y = 0, x - y = 4$, $3x + y = 8,$ and $x - y = -4.$

That implies,

$v = 0, u = 4, v = 8,$ and $u = -4$

Thus, the region is,

$R = \{(u,v) \mid -4 \le u \le 4, 0 \le v \le 8\}$

Convert the function $f(x,y) = 4x + 8y$ in terms of u and v as follows:

$$\begin{aligned} 4x + 8y &= 4\left(\frac{1}{4}(u+v)\right) + 8\left(\frac{1}{4}(v-3u)\right) \\ &= u + v + 2(v - 3u) \\ &= u + v + 2v - 6u \\ &= 3v - 5u \end{aligned}$$

Substitute these values in the equation (I).

$$\iint_R (4x+8y)\,dA = \iint_S (3v-5u)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_0^8\int_{-4}^4 \frac{1}{4}(3v-5u)\,du\,dv$$

$$= \int_0^8 \frac{1}{4}\left(3vu-5\left[\frac{u^2}{2}\right]\right)_{-4}^4 dv$$

$$= \frac{1}{4}\int_0^8\left[(12v+12v)-\frac{5}{2}(16-16)\right]dv$$

$$= \frac{24}{4}\int_0^8 (v)\,dv$$

$$= 6\left(\frac{v^2}{2}\right)_0^8$$

$$= 3(8^2-0)$$

$$= 3(64)$$

$$= 192$$

Thus, the required value is, $\iint_R (4x+8y)\,dA = \boxed{192}$.

Chapter 15 Multiple Integrals 15.10 17E

$x = 2u$, $y = 3v$

$\frac{\partial x}{\partial u} = 2$, $\frac{\partial x}{\partial v} = 0$, $\frac{\partial y}{\partial u} = 0$, $\frac{\partial y}{\partial v} = 3$

So $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}$$
$$= 6-0$$
$$= 6$$

Now R is region bounded by the ellipse

$$9x^2+4y^2 = 36,\text{ thus } 36u^2+36v^2 = 36$$
$$\Rightarrow u^2+v^2 = 1$$

And $S = \left\{(u,v)/-1 \le v \le 1,\ -\sqrt{1-v^2} \le u \le \sqrt{1-v^2}\right\}$

$$\iint_R x^2\, dA = \int_{-1}^{1}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} 4u^2.6.du\,dv$$

$Let\, u = r\cos\theta, and\ v = r\sin\theta, then\, dudv = r\, dr d\theta.$

$$So, \iint_R x^2\, dA = 24\int_{-1}^{1}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} u^2 du\, dv = 24\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\cos^2\theta . r dr\, d\theta$$

$$= 24\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^3\cos^2\theta\, dr\, d\theta = 24\int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 \cos^2\theta\, d\theta$$

$$= 6\int_0^{2\pi}\cos^2\theta\, d\theta = 3\int_0^{2\pi}(1+\cos 2\theta)\, d\theta$$

$$= 3\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 3(2\pi - 0) = 6\pi$$

Chapter 15 Multiple Integrals 15.10 18E

The region R is bounded by ellipse $x^2 - xy + y^2 = 2$

The given transformation is

$$x = \sqrt{2}u - \sqrt{2/3}\,v$$
$$y = \sqrt{2}u + \sqrt{2/3}\,v$$

On using this transformation we find that the image of the region R is

$$\left(\sqrt{2}u - \sqrt{2/3}\,v\right)^2 - \left(\sqrt{2}u - \sqrt{2/3}\,v\right)\left(\sqrt{2}u + \sqrt{2/3}\,v\right) + \left(\sqrt{2}u + \sqrt{2/3}\,v\right)^2 = 2$$

i.e. $2u^2 + \frac{2}{3}v^2 - \frac{u}{\sqrt{3}}uv - 2u^2 + \frac{2}{3}v^2 + 2u^2 + \frac{2}{3}u^2 + \frac{u}{3}uv = 2$

i.e. $2u^2 + 2v^2 = 2$

i.e. $u^2 + v^2 = 1$

Which is a circle with center at (0, 0) and radius 1.

Now $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix}$$

$$= \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}}$$

$$= \frac{4}{\sqrt{3}}$$

Then by theorem of change of variables in double integrals:

$$\iint_R (x^2 - xy + y^2)\, dA = \iint_3 2(u^2 + v^2) \frac{\partial(x,y)}{\partial(u,v)} dS$$

Where $S = \{(u,v):\ u^2 + v^2 \le 1\}$

On using polar co – ordinates we have

$$2\iint_S (u^2 + v^2) . \frac{4}{\sqrt{3}} dS$$

$$= \frac{8}{\sqrt{3}} \int_0^{2\pi} \int_0^1 r^2 . r\, dr\, d\theta$$

$$= \frac{8}{\sqrt{3}} \left[\frac{r^4}{4} \right]_0^1 (\theta)_0^{2\pi}$$

$$= \frac{8}{\sqrt{3}} \left[\frac{r^4}{4} \right]_0^1 (\theta)_0^{2\pi}$$

$$= \frac{8}{\sqrt{3}} \left(\frac{1}{4} \right) (2\pi)$$

$$= \frac{4\pi}{\sqrt{3}}$$

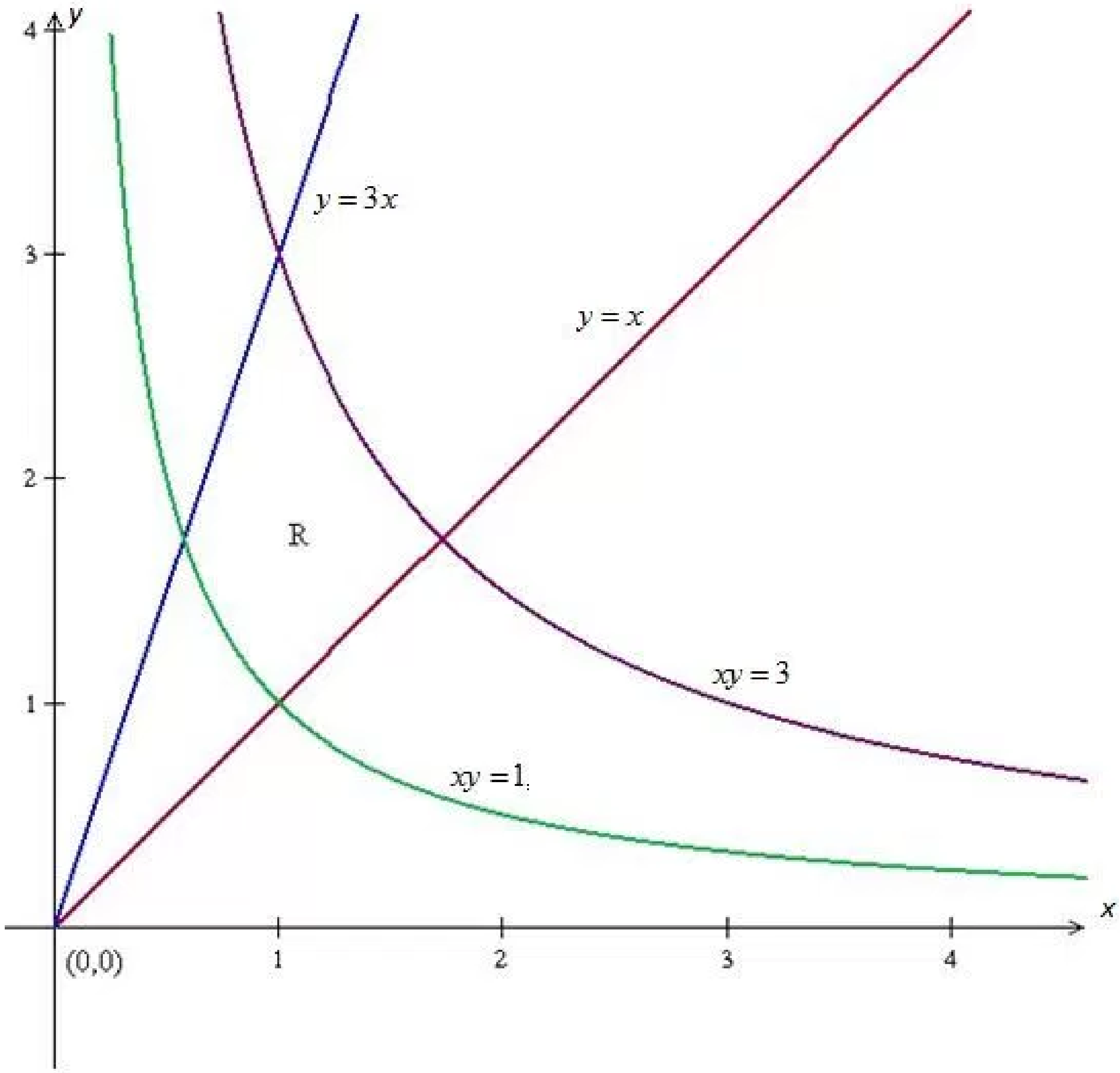
Hence $\iint_R (x^2 - xy + y^2)\, dA = \boxed{\frac{4\pi}{\sqrt{3}}}$

Chapter 15 Multiple Integrals 15.10 19E

Consider the double integral $\iint_R xy\, dA$ where the region in the first quadrant bounded by the lines $y = x, y = 3x$, and the hyperbolas $xy = 1, xy = 3$.

The objective is to find the double integral $\iint_R xy\, dA$ in the given region by using transformations $x = \frac{u}{v}$ and $y = v$.

The graph of four equations in the first quadrant is as shown below.



To transform these four equations in terms of u and v, replace each x and y with their transformation equations $x = \frac{u}{v}$ and $y = v$.

First, transform equation $y = x$,

$$y = x$$
$$v = \frac{u}{v}$$
$$v^2 = u$$
$$v = \pm\sqrt{u}$$

Next transform equation $y = 3x$,

$$y = 3x$$
$$v = 3\frac{u}{v}$$
$$v^2 = 3u$$
$$v = \pm\sqrt{3u}$$

Now transform the third equation $xy = 1$

$$xy = 1$$
$$\left(\frac{u}{v}\right)v = 1$$
$$u = 1$$

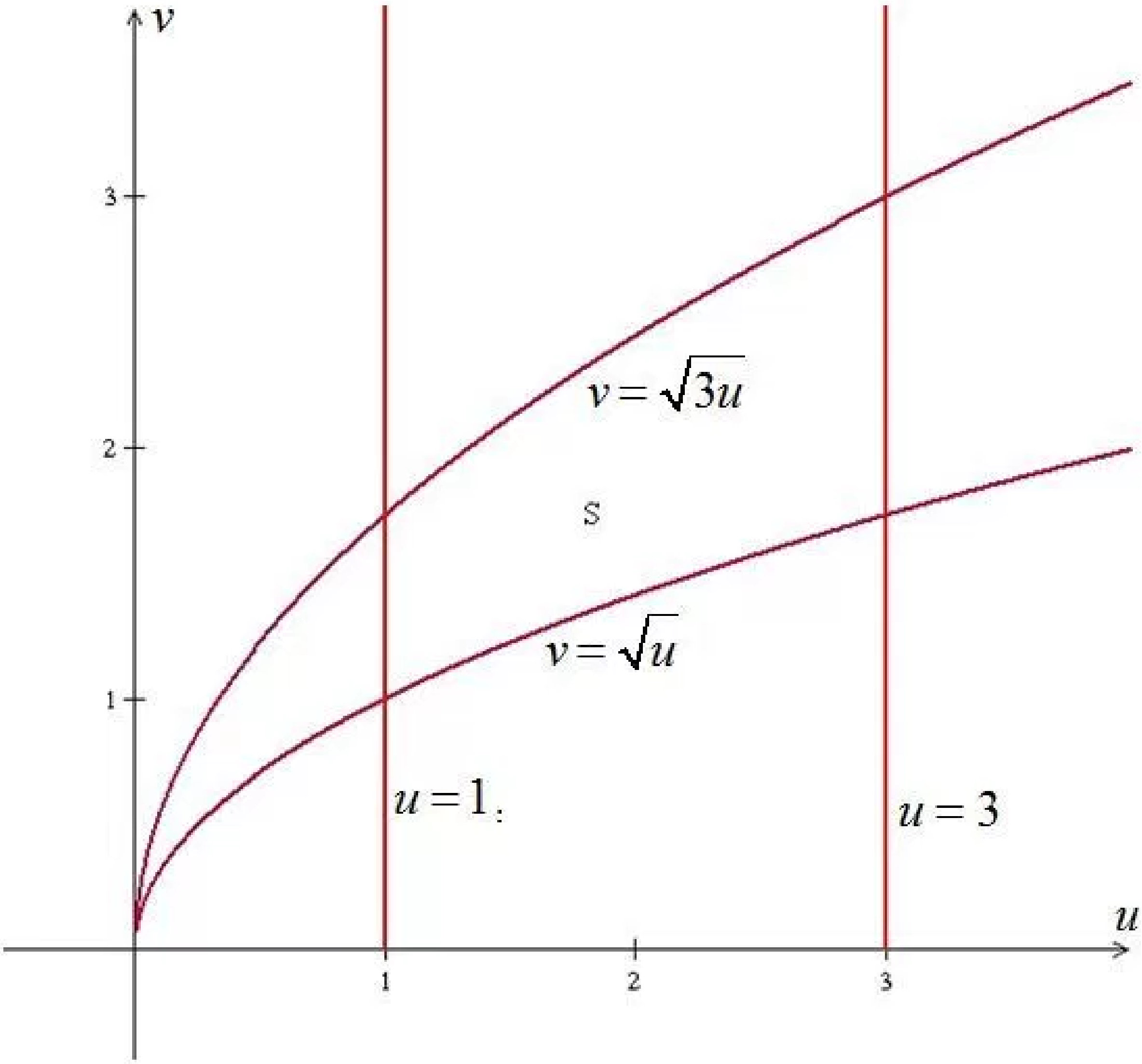
Now transform the third equation $xy = 3$

$$xy = 3$$
$$\left(\frac{u}{v}\right)v = 3$$
$$u = 3$$

The region S in the uv – plane is bound by the equations $u = 1$, $u = 3$, $v = \sqrt{u}$ and $v = \sqrt{3u}$

i.e. $S = \left\{(u,v) \mid 1 \le u \le 3,\ \sqrt{u} \le v \le \sqrt{3u}\right\}$.

The region is as shown below.



The value of dA using Jacobian is,

$$dA=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}du\,dv$$

$$=\begin{vmatrix}\frac{1}{v} & -\frac{u}{v^2}\\ 0 & 1\end{vmatrix}du\,dv$$

$$=\frac{1}{v}du\,dv$$

Hence, the value of dA is $dA=\frac{1}{v}du\,dv$.

The value of the integral $\iint_R xy\,dA$ is,

$$\iint_R xy dA = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \left(\frac{u}{v}\right)(v)\left(\frac{1}{v}\right) dv\,du$$

$$= \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\,dv\,du$$

$$= \int_1^3 u\left[\ln(v)\right]_{v=\sqrt{u}}^{v=\sqrt{3u}} du$$

$$= \int_1^3 u\left[\ln\left(\sqrt{3u}\right) - \ln\left(\sqrt{u}\right)\right] du$$

$$= \frac{1}{2}\left[\int_1^3 u \cdot \ln(3u)\,du - \int_1^3 (u)\cdot \ln(u)\,du\right] \qquad \left(\text{Since, } \ln\left(\sqrt{x}\right) = \frac{1}{2}\ln(x)\right)$$

$$= \frac{1}{2}\left[\left[\frac{u^2}{2}\ln(3u) - \int \frac{u^{\not 2}}{2}\cdot\frac{1}{\not u}\,du\right]_{u=1}^{u=3} - \left[\frac{u^2}{2}\ln(u) - \int \frac{u^{\not 2}}{2}\frac{1}{\not u}\,du\right]_{u=1}^{u=3}\right]$$

$$= \frac{1}{2}\left[\frac{u^2}{2}\ln(3u) - \frac{u^2}{4} - \frac{u^2}{2}\ln(u) + \frac{u^2}{4}\right]_{u=1}^{u=3}$$

$$= \frac{1}{2}\left[\frac{u^2}{2}\ln(3u) - \frac{u^2}{2}\ln(u)\right]_{u=1}^{u=3}$$

$$= \frac{1}{4}\left[u^2\left(\ln(3u) - \ln(u)\right)\right]_{u=1}^{u=3}$$

$$= \frac{1}{4}\left[u^2 \ln(3)\right]_{u=1}^{u=3} \qquad \left(\begin{aligned}\text{Since, } \ln(3u) - \ln(u) &= \ln\left(\frac{3u}{u}\right)\\ &= \ln(3)\end{aligned}\right)$$

$$= \frac{1}{4}\left(3^2 - 1^2\right)\ln(3)$$

$$= \frac{1}{4}(8)\ln(3)$$

$$= 2\ln(3)$$

Hence, the value of the integral is $\boxed{\iint_R xy\,dA = 2\ln(3)}$.

Chapter 15 Multiple Integrals 15.10 20E

Consider the following region:

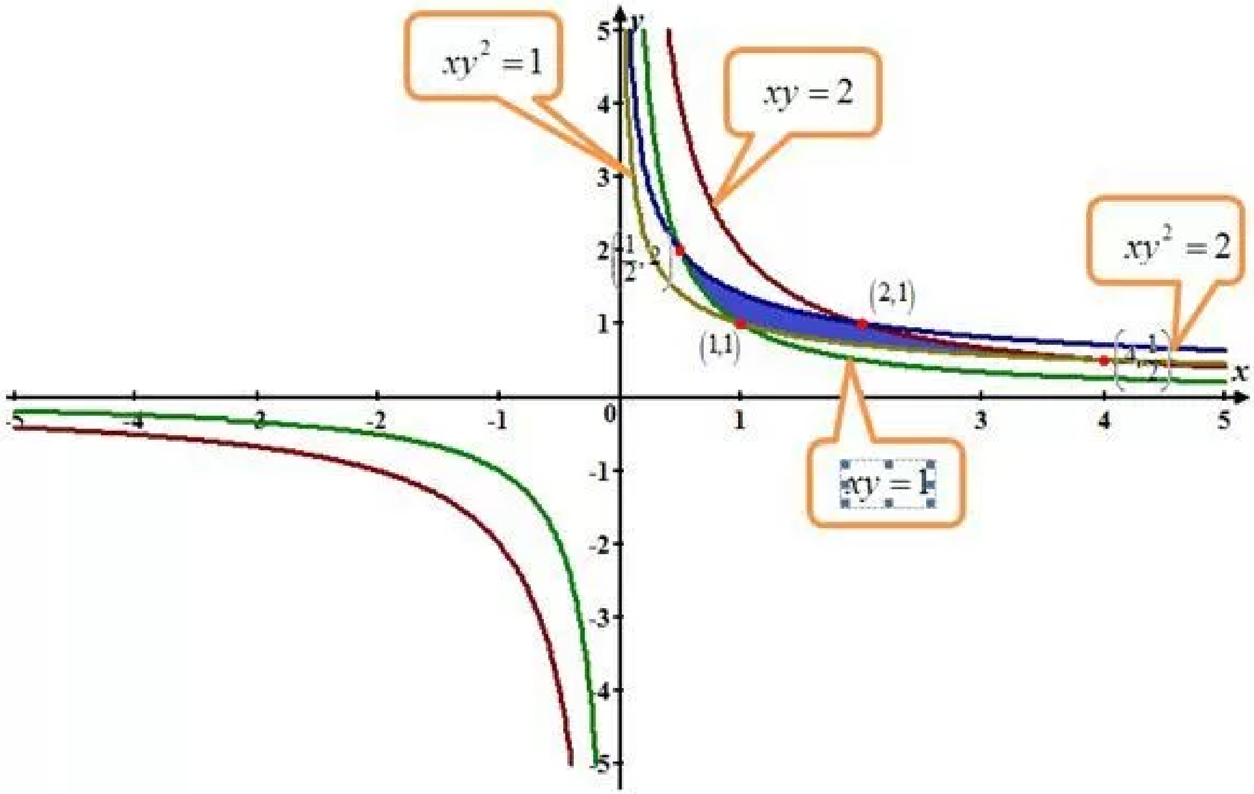
$$\iint_R y^2 dA$$

Here, R is the region bounded by the curves.

$xy = 1, xy = 2, xy^2 = 1, xy^2 = 2; u = xy,$ and $v = xy^2$

Evaluate the region bounded by the given curves using a graphing calculator.

Sketch the graph of the given curves as shown below:



Observe the above figure; the required region is blue in colour.

Since $u = xy, \Rightarrow y = \frac{u}{x} \Rightarrow y^2 = \frac{u^2}{x^2}$.

Substitute $y^2 = \frac{u^2}{x^2}$ in $v = xy^2$ to get $x = \frac{u^2}{v}$.

Now substitute $x = \frac{u^2}{v}$ in $y = \frac{u}{x}$, to get $y = \frac{v}{u}$.

Therefore, $y = \frac{v}{u}, x = \frac{u^2}{v}$.

Differentiate y with respect to u and v, to get the following result:

$$\frac{\partial y}{\partial u} = -\frac{v}{u^2}, \text{ and } \frac{\partial y}{\partial v} = \frac{1}{u}$$

Differentiate x with respect to u and v, to get the following result:

$$\frac{\partial x}{\partial u} = \frac{2u}{v}, \text{ and } \frac{\partial x}{\partial v} = -\frac{u^2}{v^2}$$

Observe the given curves x and y functions in u and v.

Compute the Jacobian as follows:

$$\begin{aligned}\frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} \\ &= \left(\frac{2u}{v}\right)\left(\frac{1}{u}\right) - \left(-\frac{u^2}{v^2}\right)\left(-\frac{v}{u^2}\right)\end{aligned}$$

$$= \frac{2}{v} - \frac{1}{v}$$

$$= \frac{1}{v}$$

The region R is the region with vertices $(1,1), (2,1), \left(4, \frac{1}{2}\right)$, and $\left(\frac{1}{2}, 2\right)$.

Therefore, the region of R is calculated as follows:

$$\iint_R y^2 dA = \int_1^2\int_1^2 \frac{v^2}{u^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$= \int_1^2\int_1^2 \frac{v^2}{u^2}\left(\frac{1}{v}\right) dudv$$

$$= \int_1^2\int_1^2 \frac{v}{u^2} dudv$$

$$= \int_1^2 v\left[-\frac{1}{u}\right]_1^2 dv$$

$$= \int_1^2 v\left[-\frac{1}{2}+\frac{1}{1}\right] dv$$

$$= \frac{1}{2}\int_1^2 vdv$$

$$= \frac{1}{2}\left[\frac{v^2}{2}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{4}{2}-\frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{3}{2}\right]$$

$$= \boxed{\frac{3}{4}}$$

Chapter 15 Multiple Integrals 15.10 21E

(a)

Consider the solid E which is enclosed by ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

The volume of this solid is given by the following triple integral

$$\iiint_E dV$$

Use the transformation

$$x = au, y = bv, z = cw$$

The ellipsoid E_{xyz} in xyz space transform to E_{uvw} in uvw space

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\frac{(au)^2}{a^2}+\frac{(bv)^2}{b^2}+\frac{(cw)^2}{c^2}=1$$

$$u^2+v^2+w^2=1$$

So E_{uvw} is a solid enclosed by sphere with radius is 1.

$$\iiint_{E_{xyz}} dV = \iiint_{E_{xyz}} dxdydz$$
$$= \iiint_{E_{uvw}} \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$
$$= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$
$$= abc$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = abc \qquad \text{...........}(1)$$

Using this Jacobean value to find the integral

$$\iiint_{E_{xyz}} dV = \iiint_{E_{uvw}} \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$
$$= \iiint_{E_{uvw}} abc\,dudvdw$$
$$= abc\iiint_{E_{uvw}} dudvdw$$
$$= (abc)\times(\text{volume of the sphere } \mathrm{u}^2 + \mathrm{v}^2 + w^2 = 1)$$
$$= (abc)\times\left(\frac{4}{3}\pi(1)^3\right)$$
$$= \frac{4}{3}\pi abc$$

(b)

The earth is approximately ellipsoid shape with

$a = b = 6378km$ and $c = 6356km$

The volume of earth is approximately from result of part (a) is

$$V = \frac{4\pi abc}{3}$$
$$\approx \frac{4\pi(6378)(6378)(6356)}{3}$$
$$\approx \frac{4\pi(258554986704)}{3}$$
$$\approx \boxed{1.0830\times 10^{12}}$$

Thus, we get the volume of the earth as $1.0830\times 10^{12}\,\mathrm{km}^3$.

(c)

It is known that the moment of inertia of a solid about the z axis is given by

$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\, dV$$

But it is given that the solid has constant density k

$$I_z = \iiint_{E_{xyz}} (x^2 + y^2)\rho(x, y, z)\, dV$$

$$= \iiint_{E_{xyz}} (x^2 + y^2)k\ dxdydz$$

$$= k\iiint_{E_{uvw}} \left((au)^2 + (bv)^2\right)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

$$= k\iiint_{E_{uvw}} (a^2u^2 + b^2v^2)(abc)\ dudvdw \qquad (\text{from equation (1)})$$

Since boundary of E_{uvw} is a sphere we use spherical coordinates

$$E_{uvw} = \{(\rho,\theta,\phi) \mid 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$$

$$u = \rho\sin\phi\cos\theta, v = \rho\sin\phi\sin\theta, w = \rho\cos\phi$$

Now calculate above triple integral using spherical coordinates, that is coordinates (u,v,w) convert into (ρ,θ,ϕ) so in this case use the Jacobean $\left|\frac{\partial(u,v,w)}{\partial(\rho,\theta,\phi)}\right|$

$$k\iiint_{E_{uvw}} (a^2u^2 + b^2v^2)(abc)\ dudvdw$$

$$k(abc)\int_0^1\int_0^{2\pi}\int_0^{\pi}\left(a^2(\rho\sin\phi\cos\theta)^2 + b^2(\rho\sin\phi\sin\theta)^2\right)\left|\frac{\partial(u,v,w)}{\partial(\rho,\theta,\phi)}\right| d\phi d\theta d\rho$$

$$k(abc)\int_0^1\int_0^{2\pi}\int_0^{\pi}(\rho^2\sin^2\phi)(a^2\cos^2\theta + b^2\sin^2\theta)(\rho^2\sin\phi)\, d\phi d\theta d\rho$$

$$k(abc)\int_0^1\int_0^{2\pi}\int_0^{\pi}(\rho^4\sin^3\phi)(a^2\cos^2\theta + b^2\sin^2\theta)\, d\phi d\theta d\rho$$

$$k(abc)\int_0^1 \rho^4 d\rho\int_0^{2\pi}(a^2\cos^2\theta + b^2\sin^2\theta)\, d\theta\int_0^{\pi}\sin^3\phi d\phi$$

These three limits are independent so calculate separate integrals and then product all the values

$$\int_0^1 \rho^4 d\rho = \left[\frac{\rho^5}{5}\right]_0^1 = \frac{1}{5} \qquad \text{............}(2)$$

$$\int_0^{2\pi} \left(a^2\cos^2\theta + b^2\sin^2\theta\right)d\theta = \int_0^{2\pi}\left(a^2\left(\frac{1+\cos 2\theta}{2}\right)+b^2\left(\frac{1-\cos 2\theta}{2}\right)\right)d\theta$$

$$= \int_0^{2\pi} a^2\left[\frac{\theta}{2}+\frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)\right]+b^2\left[\frac{\theta}{2}-\frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)\right]$$

$$= a^2\left[\frac{2\pi}{2}+\frac{1}{2}\left(\frac{\sin 2(2\pi)}{2}\right)-\frac{0}{2}-\frac{1}{2}\left(\frac{\sin 2(0)}{2}\right)\right]$$

$$+b^2\left[\frac{2\pi}{2}-\frac{1}{2}\left(\frac{\sin 2(2\pi)}{2}\right)-\frac{0}{2}+\frac{1}{2}\left(\frac{\sin 2(0)}{2}\right)\right]$$

$$= \pi\left(a^2+b^2\right)$$

$$\int_0^{2\pi} \left(a^2\cos^2\theta + b^2\sin^2\theta\right)d\theta = \pi\left(a^2+b^2\right) \qquad \text{...........}(3)$$

Now consider the integral on that variable ϕ

$$\int_0^{\pi} \sin^3\phi d\phi = \int_0^{\pi}\left(\frac{3\sin\phi - \sin 3\phi}{4}\right)d\phi$$

$$= \frac{3}{4}\left[-\cos\phi\right]_0^{\pi} - \frac{1}{4}\left[\frac{-\cos 3\phi}{3}\right]_0^{\pi}$$

$$= \frac{3}{4}\left[-\cos\pi + \cos 0\right] - \frac{1}{4}\left[\frac{-\cos 3\pi}{3}+\frac{\cos 3(0)}{3}\right]$$

$$= \frac{3}{2}-\frac{1}{6}$$

$$= \frac{4}{3}$$

$$\int_0^{\pi} \sin^3\phi d\phi = \frac{4}{3} \qquad \text{.........}(4)$$

From the equations (2),(3) and (4)

$$k(abc)\int_0^1 \rho^4 d\rho \int_0^{2\pi} \left(a^2\cos^2\theta + b^2\sin^2\theta\right)d\theta \int_0^{\pi} \sin^3\phi d\phi$$

$$\left(k(abc)\right)\left(\frac{1}{5}\right)\left(\pi\left(a^2+b^2\right)\right)\left(\frac{4}{3}\right)$$

$$\frac{4}{15}\cdot(\pi kabc)\cdot\left(a^2+b^2\right)$$

Therefore, the moment of inertia about the z axis is $\frac{4}{15}\pi\left(a^2+b^2\right)abck$.

Chapter 15 Multiple Integrals 15.10 22E

Use a change of variables to find an easy integral for calculating the area.
The change of variables can be used to calculate a double integral as follows:

$$\iint_R f(x,y)dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv \quad \ldots\ldots (1)$$

Where $\frac{\partial(x,y)}{\partial(u,v)}$ is the Jacobin of the transformation, calculable by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

The equations the problem provides suggest an easy transformation to u and v, as follows:

$u = xy$

$v = xy^{1.4}$

This is a transformation from the xy-plane to the uv-plane, however. We want the inverse transformation: the transformation from the xy-plane to the uv-plane, which we hope will make the region easier to integrate. To find the inverse transformation, solve these equations to get them in terms of x and y.

First solve for x in the first equation:

$$x = \frac{u}{y} \quad \ldots\ldots (2)$$

Plug this into the equation for v:

$$v = \frac{u}{y}y^{1.4}$$

$$v = uy^{0.4}$$

The decimal .4 is the same as the fraction 2/5; we rewrite the exponent this way so we can solve for y by taking both sides to the power of 5/2:

$$v = uy^{2/5}$$

$$y^{2/5} = \frac{v}{u}$$

$$y = \left(\frac{v}{u}\right)^{5/2}$$

Now plug this back into (2):

$$x = \frac{u}{\left(\frac{v}{u}\right)^{5/2}}$$

$$= u\left(\frac{u}{v}\right)^{5/2}$$

$$= \frac{u^{7/2}}{v^{5/2}}$$

We now have the desired transformation equations,

$$x(u,v) = \frac{u^{7/2}}{v^{5/2}}$$

$$y(u,v) = \frac{v^{5/2}}{u^{5/2}}$$

We'll need the partial derivatives of $x(u,v)$ and $y(u,v)$ to plug into (1). Hold v constant and take the derivatives in terms of u:

$$\frac{\partial x}{\partial u} = \frac{7u^{5/2}}{2v^{5/2}}$$

$$\frac{\partial y}{\partial u} = -\frac{5v^{5/2}}{2u^{7/2}}$$

Now hold u constant and take the derivative in terms of v:

$$\frac{\partial x}{\partial v} = -\frac{5u^{7/2}}{2v^{7/2}}$$

$$\frac{\partial y}{\partial v} = \frac{5v^{3/2}}{2u^{5/2}}$$

Use the partial derivatives to calculate the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$= \left(\frac{7u^{5/2}}{2v^{5/2}}\right)\left(\frac{5v^{3/2}}{2u^{5/2}}\right) - \left(-\frac{5u^{7/2}}{2v^{7/2}}\right)\left(-\frac{5v^{5/2}}{2u^{7/2}}\right)$$

$$= \frac{35}{4v} - \frac{25}{4v}$$

$$= \frac{5}{2v}$$

Since we are trying to find the area, the function f we are double-integrating equals 1. The limits of integration are given in the problem; since xy ranges from a to b, now that we have $u = xy$, the limits of u are a and b. Similarly, since $xy^{1.4}$ is given as ranging from c to d, the limits of v are c and d.

We now plug into the transformation integral given in (1):

$$\iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \int_c^d \int_a^b (1)\left(\frac{5}{2v}\right) dudv$$

$$= \int_c^d \int_a^b \left(\frac{5}{2v}\right) dudv$$

Integrate in terms of u:

$$\int_c^d \int_a^b \left(\frac{5}{2v}\right) dudv = \int_c^d \left(\frac{5u}{2v}\right)\Bigg|_a^b dv$$

$$= \int_c^d \left(\frac{5b-5a}{2v}\right) dv$$

$$= \frac{5b-5a}{2}\int_c^d \left(\frac{1}{v}\right) dv$$

Integrate in terms of v:

$$\frac{5b-5a}{2}\int_c^d \left(\frac{1}{v}\right) dv = \frac{5b-5a}{2}(\ln v)\Big|_c^d$$

$$= \boxed{\frac{5b-5a}{2}(\ln d - \ln c)}$$

Chapter 15 Multiple Integrals 15.10 23E

Consider the integral,

$$\iint_R \frac{x-2y}{3x-y}\,dA$$

As the integrand $\frac{x-2y}{3x-y}$ is not easily integrable, make a change of variables.

Let us use the transformations,

$u = x-2y,\ \ v = 3x-y$

These equations define a transformation T^{-1} from $xy-$plane to the $uv-$plane.

Recall the theorem change of variables in a double integral,

If a transformation is defined from *uv*-plane onto *xy*- plane and the Jacobian is non-zero then

$$\iint_R f(x,y)\,dA = \iint f\left(x(u,v), y(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du dv \quad \ldots\ldots (1)$$

So, the transformation from *uv*-plane to *xy*-plane is

$$x = \frac{2v-u}{5},\quad y = \frac{3u-v}{-5} = \frac{v-3u}{5}$$

Differentiate partially *x, y* with respect to *u, v*.

$$\frac{\partial x}{\partial u} = \frac{-1}{5},\ \frac{\partial x}{\partial v} = \frac{2}{5},\ \frac{\partial y}{\partial u} = \frac{-3}{5},\ \frac{\partial y}{\partial v} = \frac{1}{5}$$

Find the Jacobian.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{-1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix}$$

$$= \frac{-1}{25} + \frac{6}{25}$$

$$= \frac{5}{25}$$

$$= \frac{1}{5}$$

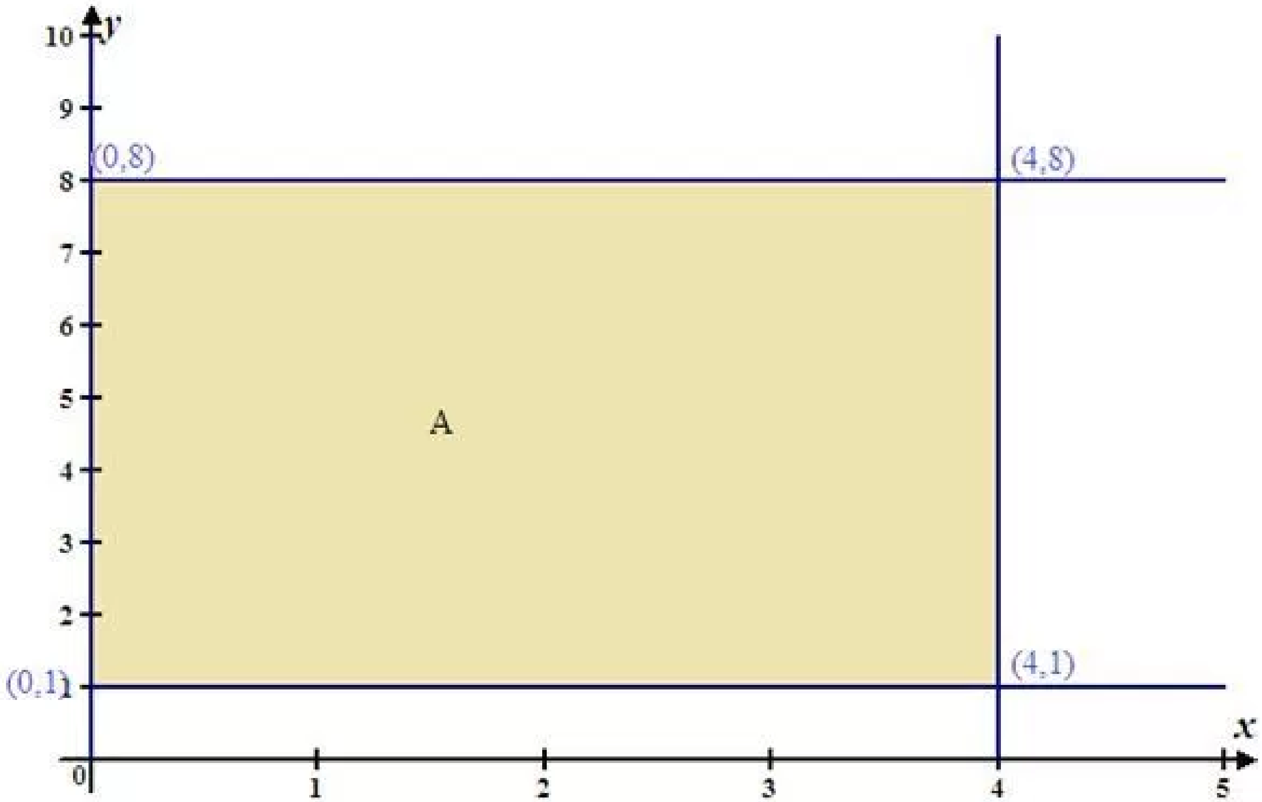
Change the limits of integration.

The region R is given by $x-2y=0, x-2y=4, 3x-y=1$ and $3x-y=8$

After applying the transformation, the region is

$u=0, u=4, v=1, v=8$

Sketch the region bounded by this lines.



Use (1) to evaluate the integral.

$$\iint_R \frac{x-2y}{3x-y}\,dA = \int_1^8\int_0^4 \frac{u}{v}.\frac{1}{5}\,du\,dv$$

$$=\frac{1}{5}\int_1^8 \frac{1}{v}dv\int_0^4 u\,du$$

$$=\frac{1}{5}.\ \ln|v|\Big]_1^8 \frac{u^2}{2}\Big]_0^4$$

$$=\frac{1}{5}[\ln 8-\ln 1].\frac{16}{2}$$

$$=\frac{8}{5}(\ln 8-0)$$

$$=\boxed{\frac{8}{5}\ln 8}$$

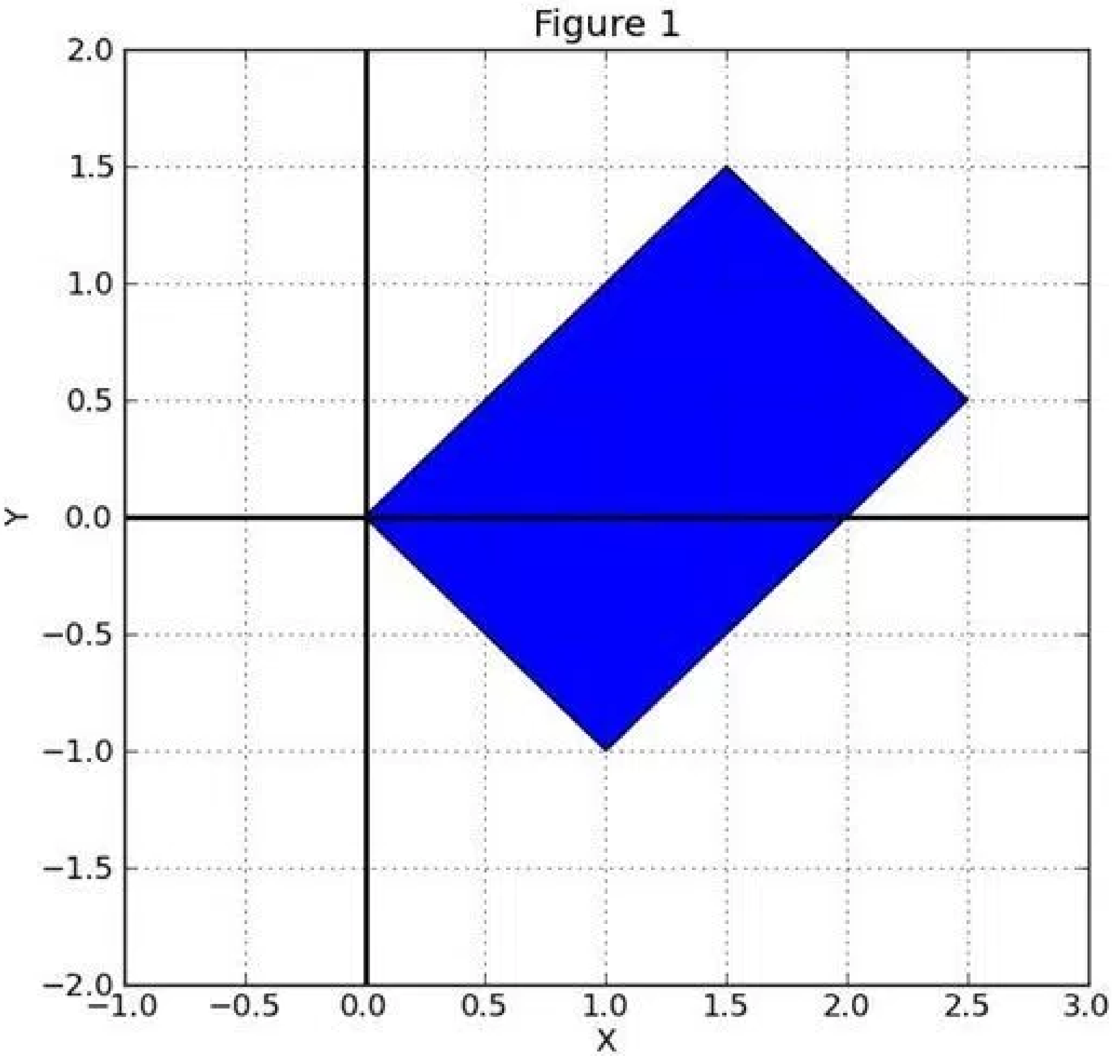
Chapter 15 Multiple Integrals 15.10 24E

Consider the following integral:

$$\iint_R (x+y)e^{x^2-y^2}\,dA$$

Here, *R* is the region bounded by equations $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$.

The region *R* is as shown below:



Let $u=x-y$ and $v=x+y$

$$x(u,v)=\frac{u+v}{2} \text{ and } y(u,v)=\frac{v-u}{2}$$

The limits of new region is,

$$T=\{(u,v)/0\le u\le 2, 0\le v\le 3\}$$

To find the value of the integral, use the following formula:

$$\iint_R f(x,y)\,dA=\iint_T f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv \qquad ......(1)$$

Find the partial derivatives as follows:

$$x_u = \frac{\partial}{\partial u}\left(\frac{u+v}{2}\right)$$
$$= \frac{1}{2}$$
$$x_v = \frac{\partial}{\partial v}\left(\frac{u+v}{2}\right)$$
$$= \frac{1}{2}$$
$$y_u = \frac{\partial}{\partial u}\left[\frac{v-u}{2}\right]$$
$$= -\frac{1}{2}$$
$$y_v = \frac{1}{2}$$

The Jacobian of T:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$
$$= x_u y_v - x_v y_u$$
$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)$$
$$= \frac{1}{4} + \frac{1}{4}$$
$$= \frac{1}{2}$$

Evaluate the integral is as follows:

$$\iint_R (x+y)e^{x^2-y^2}\,dA = \iint_T \left[\frac{u+v}{2} + \frac{v-u}{2}\right]e^{uv}\left(\frac{1}{2}\right)dudv \qquad \text{use } (1)$$
$$= \frac{1}{2}\int_0^3\int_0^2 ve^{uv}\,dudv$$
$$= \frac{1}{2}\int_0^3 v\left(\frac{e^{uv}}{v}\right)_0^2 dv$$
$$= \frac{1}{2}\int_0^3 \left(e^{2v}-1\right)dv$$
$$= \frac{1}{2}\left(\frac{e^{2v}}{2} - v\right)_0^3$$
$$= \frac{1}{2}\left(\frac{e^6}{2} - 3 - \frac{1}{2}\right)$$
$$= \frac{1}{4}\left(e^6 - 7\right)$$

Therefore, the value of the integral is $\boxed{\frac{1}{4}\left(e^6-7\right)}$.

Chapter 15 Multiple Integrals 15.10 25E

Evaluate $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$

Since it is not easy to integrate $\cos\left(\frac{y-x}{y+x}\right)$, we make a change of variables suggested by the form of this function

$u = x + y$, $v = y - x$ ...... (I)

These equations define a transformation T^{-1} from the xy-plane to the uv-plane.

Change of variable in a double integral: Suppose that *T* is a C^1 transformation whose jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that *f* is continuous on R and that R and S are type I and type II regions. Suppose also that T is one-to-one, except on the boundary of *S*

Then $\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

It is obtained by solving (I) for *x* and *y*

$u + v = 2y, u - v = 2x$

Then $y = \frac{1}{2}(u+v)$, $x = \frac{1}{2}(u-v)$

And then the Jacobian of T is:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

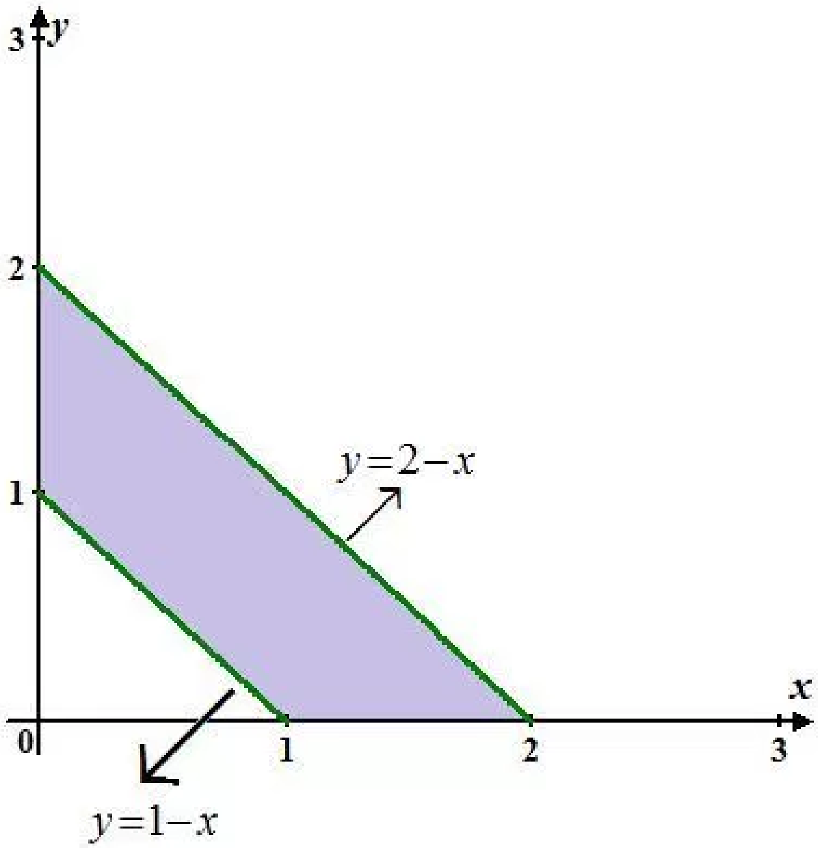
$$= \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix}$$

$$= \frac{1}{4} + \frac{1}{4}$$

$= \frac{1}{2}$

The given region *R* is a trapezoidal with vertices (1, 0), (2, 0), (0, 2), (0, 1)

The region *R* is a trapezoidal with vertices (1, 0), (2, 0), (0, 2), (0, 1) shown in the below



To find the region *S* (the image of *R*) in *uv* – plane corresponding is *R*, we see that the sides of R lie on the lines: $x = 0$, $y = 0$, $y = 1 - x$, $y = 2 - x$

Then on using (I) we find that the image lines in uv – plane are:

$u = v,\ \ u = -v,\ \ u = 1,\ \ u = 2$

Thus $S = \{(u,v):\ 1 \le u \le 2,\ -u \le v \le u\}$

Then using the theorem of change of variables in double integral,

$$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA = \iint_S \cos\left(\frac{v}{u}\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_1^2 \int_{-u}^{u} \cos\left(\frac{v}{u}\right).\frac{1}{2} dv\,du$$

$$= \frac{1}{2}\int_1^2 \left[u\sin\left(\frac{v}{u}\right)\right]_{v=-u}^{v=u} du$$

$$= \frac{1}{2}\int_1^2 \left[u\sin(1) - u\sin(-1)\right] dv$$

$$= \frac{1}{2}\int_1^2 \left[u\sin(1) + u\sin(1)\right] du$$

$$= \sin 1\int_1^2 u\,du$$

$$= \sin 1\left(\frac{u^2}{2}\right)_1^2$$

$$= \sin(1)\left[2 - \frac{1}{2}\right]$$

$$= \frac{3}{2}\sin 1$$

Hence $\boxed{\iint_R \cos\left(\frac{y-x}{y+x}\right) dA = \frac{3}{2}\sin 1}$

Chapter 15 Multiple Integrals 15.10 26E

We need to integrate $\sin\left(9x^2 + 4y^2\right)$, then we make transformation of the form $x = \frac{u}{3}$ and $y = \frac{v}{2}$

Then $u = 3x$ and $v = 2y$

And then $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{6}$$

The given region R is the region in first quadrant bounded by ellipse $9x^2+4y^2=1$
To find the region S (the image of R) in uv – plane corresponding to R, we see that S is the region of circle $u^2+v^2=1$ in the first quadrant.

In polar co – ordinates the region S is

$$S=\left\{(r,\theta):\ 0\le r\le 1,\ 0\le\theta\le\frac{\pi}{2}\right\}$$

Then using the theorem of change of variables in double integral,

$$\iint_R \sin\left(9x^2+4y^2\right)dA=\iint_S \sin\left(u^2+v^2\right)\frac{\partial(x,y)}{\partial(u,v)}du\,dv$$

$$=\frac{1}{6}\int_0^{\pi/2}\int_0^1 \sin\left(r^2\right)r\,dr\,d\theta$$

$$=\frac{1}{6}\int_0^{\pi/2}d\theta.\int_0^1 r\sin\left(r^2\right)dr$$

$$=\frac{1}{6}(\theta)_0^{\pi/2}\left[\frac{-1}{2}\cos r^2\right]_0^1$$

$$=\frac{1}{6}\left(\frac{\pi}{2}\right)\left[\frac{-1}{2}\cos 1+\frac{1}{2}\cos 0\right]$$

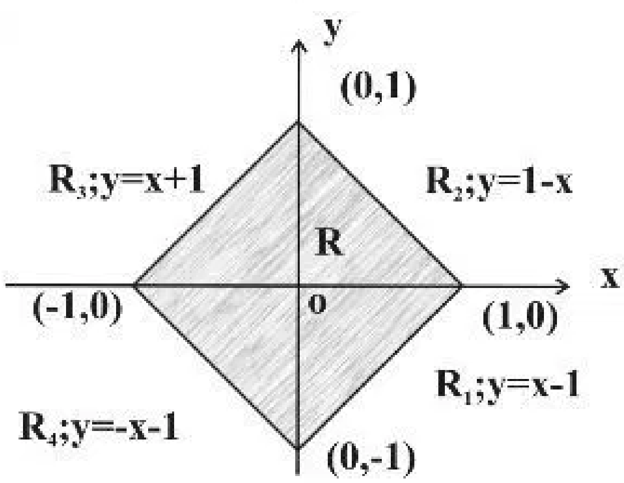
$$=\frac{\pi}{12}\left(-\frac{1}{2}\cos 1+\frac{1}{2}\right)$$

$$=\frac{\pi}{24}(1-\cos 1)$$

Hence $\boxed{\iint_R \sin\left(9x^2+4y^2\right)dA=\frac{\pi}{24}(1-\cos 1)}$

Chapter 15 Multiple Integrals 15.10 27E

We need to integrate e^{x+y} over the region R given by $|x|+|y|\le 1$



We make transformation $x+y=u$ and $x-y=v$

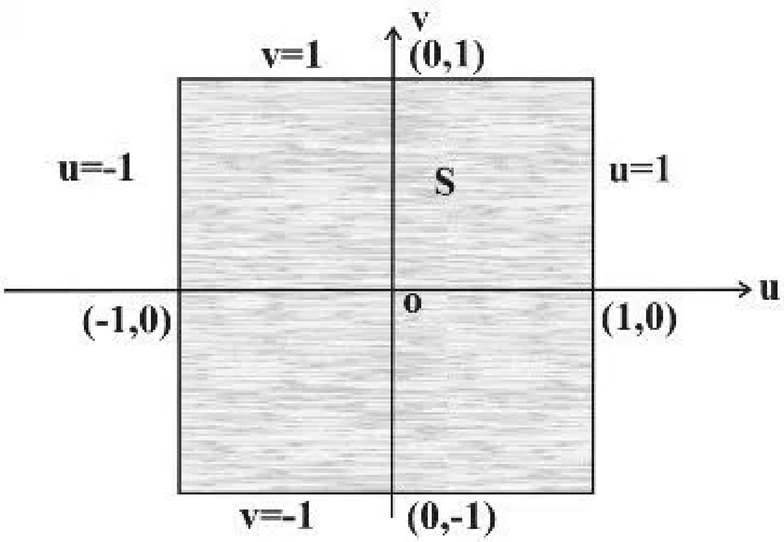
Or $\quad x=\frac{1}{2}(u+v),\ y=\frac{1}{2}(u-v)$ ---------- (I)

Then region R is bounded by lines $R_1:\ y=x-1,\ R_2:\ y=1-x,\ R_3:\ y=x+1$ and $R_4:\ y=-x-1$, then to find the image of R in uv – plane, on using (I) we find that the image lines are given by

$$u=1,\ u=-1,\ v=1,\ v=-1 \qquad (\text{Since } x+y=u \text{ and } x-y=v)$$

Then the image of R is given by

$$S=\{(u,v):\ -1\le u\le 1,\ -1\le v\le 1\}$$



Also $\quad \dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}$

$$=\begin{vmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{vmatrix}$$

$$=-\frac{1}{4}-\frac{1}{4}$$

$$=-\frac{1}{2}$$

Then by the theorem of change of variables in double integrals we have

$$\iint_R e^{x+y}\,dA=\iint_S e^u\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

$$=\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}e^u\,du\,dv$$

$$=\frac{1}{2}\int_{-1}^{1}dv\int_{-1}^{1}e^u\,du$$

$$=\frac{1}{2}(v)_{-1}^{1}(e^u)_{-1}^{1}$$

$$=\frac{1}{2}(2)(e^1-e^{-1})$$

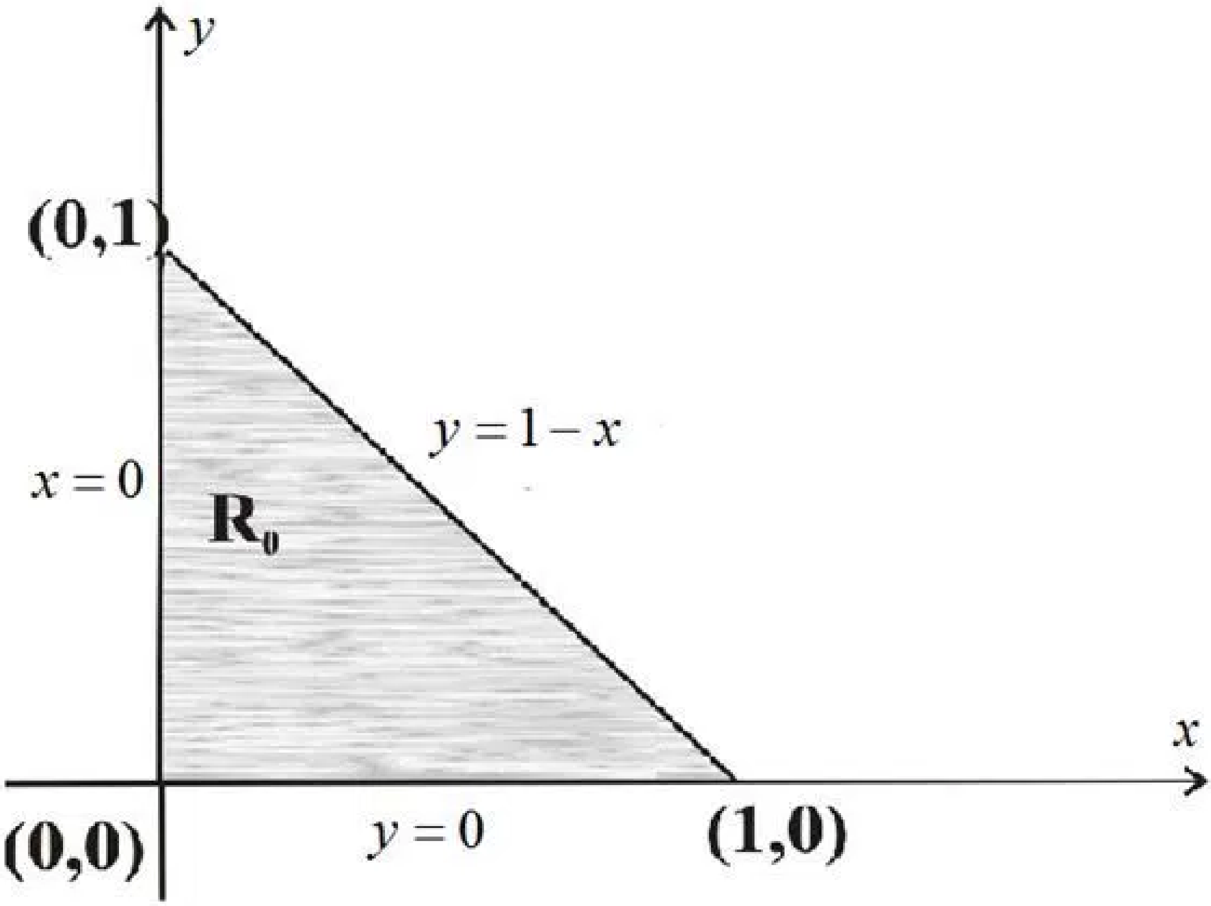
$$=e-e^{-1}$$

Hence $\boxed{\iint_R e^{x+y}\,dA=e-e^{-1}}$

Chapter 15 Multiple Integrals 15.10 28E

Consider

Triangular region with vertices $(0,0),(1,0)$ and $(0,1)$



Take transformation: $x+y=u$ and $x-y=v$

Adding these two equations

$x+y=u$

$x-y=v$

$2x=u+v$

$x=\frac{1}{2}(u+v)$

Now, subtracting the equations

$x+y=u$

$x-y=v$

$2y=u-v$

$y=\frac{1}{2}(u-v)$

Now,

$$x = \frac{1}{2}(u+v)$$

Taking partial derivative with respect to u

$$\frac{\partial x}{\partial u} = \frac{1}{2}$$

Taking partial derivative with respect to v

$$\frac{\partial x}{\partial v} = \frac{1}{2}$$

And $y = \frac{1}{2}(u-v)$

Taking partial derivative with respect to u

$$\frac{\partial y}{\partial u} = \frac{1}{2}$$

Taking partial derivative with respect to v

$$\frac{\partial y}{\partial v} = -\frac{1}{2}$$

The region R is bounded by lines $y=0$, $x=0$ and $y=1-x$

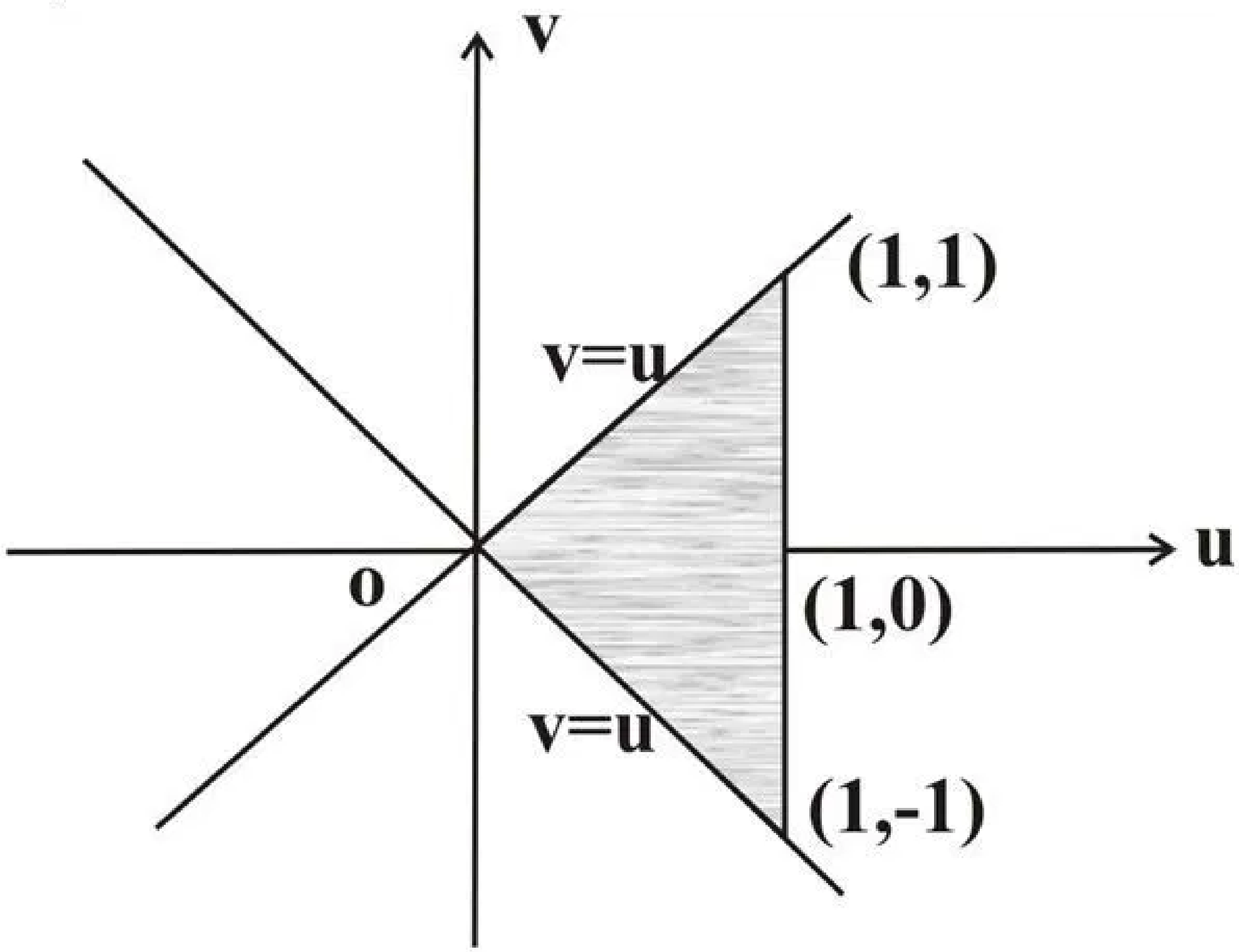
Then on using $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$, the image lines in uv – plane are:

$$u=0, u=1, v=u \text{ and } v=-u$$

Then the set S (the image of R in uv-plane) is:

$$S = \{(u,v):\ 0 \le u \le 1,\ -u \le v \le u\}$$

Graph is



Also $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$

$= -\frac{1}{4} - \frac{1}{4}$

$= -\frac{1}{2}$

Change of variables in a double integral:

Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the $uv-$ plane onto a region R in the $xy-$ plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x,y)\,dA = \iint_S f\left(x(u,v), y(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du dv$$

Then by the theorem of change of variables in double integrals we have

$$\iint_R f(x,y)\,dA = \iint_S f(u)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \frac{1}{2}\int_0^1\int_{-u}^{u} f(u)\,dv\,du \quad \left(\text{Since } \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|-\frac{1}{2}\right|\right)$$

$$= \frac{1}{2}\int_0^1 f(u).(v)_{-u}^{u}\,du$$

$$= \frac{1}{2}\int_0^1 f(u)(2u)\,du$$

$$= \int_0^1 (u)\,f(u)\,du$$

Hence

$$\iint_R f(x+y)\,dA = \int_0^1 u\,f(u)\,du$$