

Ex 15.1

Mean Value Theorems Ex 15.1 Q1(i)

$$f(x) = 3 + (x - 2)^{\frac{2}{3}} \text{ on } [1, 3]$$

Differentiating it with respect to x ,

$$f'(x) = \frac{2}{3} \times \frac{1}{(x - 2)^{\frac{1}{3}}}$$

$$\text{Clearly, } \lim_{x \rightarrow 2} = \frac{2}{3} \times \frac{1}{(x - 2)^{\frac{1}{3}}}$$

Thus, $f(x)$ is not differentiable at $x = 2 \in (1, 3)$

Hence, Rolle's theorem is not applicable for $f(x)$ in $x \in [1, 3]$.

Mean Value Theorems Ex 15.1 Q1(ii)

Here, $f(x) = [x]$ and $x \in [-1, 1]$, at $n = 1$

$$\begin{aligned} \text{LHL} &= \lim_{x \rightarrow (1-h)} [x] \\ &= \lim_{h \rightarrow 0^+} [1 - h] \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{RHL} &= \lim_{x \rightarrow (1+h)} [x] \\ &= \lim_{h \rightarrow 0^+} [1 + h] \\ &= 1 \end{aligned}$$

$$\text{LHL} \neq \text{RHL}$$

So, $f(x)$ is not continuous at $1 \in [-1, 1]$

Hence, Rolle's theorem is not applicable on $f(x)$ in $[-1, 1]$.

Mean Value Theorems Ex 15.1 Q1(iii)

Here, $f(x) = \sin\left(\frac{1}{x}\right)$, $x \in [-1, 1]$, at $n = 0$

$$\begin{aligned}
 \text{LHS} &= \lim_{x \rightarrow (0-h)} \sin\left(\frac{1}{x}\right) \\
 &= \lim_{h \rightarrow 0} \sin\left(\frac{1}{0-h}\right) \\
 &= \lim_{h \rightarrow 0} \sin\left(\frac{-1}{h}\right) \\
 &= -\lim_{h \rightarrow 0} \sin\left(\frac{1}{h}\right) \\
 &= -k \quad \left[\text{Let } \lim_{h \rightarrow 0} \sin\left(\frac{1}{h}\right) = k \text{ as } k \in [-1, 1] \right]
 \end{aligned}$$

$$\begin{aligned}
 \text{RHS} &= \lim_{x \rightarrow (0+h)} \sin\left(\frac{1}{x}\right) \\
 &= \lim_{h \rightarrow 0} \sin\left(\frac{1}{h}\right) \\
 &= k
 \end{aligned}$$

$\Rightarrow$ LHS $\neq$ RHS
 $\Rightarrow$ $f(x)$ is not continuous at $n = 0$

So, rolle's theorem is not applicable on $f(x)$ in $[-1, 1]$

Mean Value Theorems Ex 15.1 Q1(iv)

Here, $f(x) = 2x^2 - 5x + 3$ on $[1, 3]$

$f(x)$ is continuous in $[1, 3]$ and $f(x)$ is differentiable in $(1, 3)$ since it is a polynomial function.

Now,

$$\begin{aligned}
 f(x) &= 2x^2 - 5x + 3 \\
 f(1) &= 2(1)^2 - 5(1) + 3 \\
 &= 2 - 5 + 3 \\
 f(1) &= 0 \quad \text{---(i)} \\
 f(3) &= 2(3)^2 - 5(3) + 3 \\
 &= 18 - 15 + 3 \\
 f(3) &= 6 \quad \text{---(ii)}
 \end{aligned}$$

From equation (i) and (ii),
 $f(1) \neq f(3)$

So, rolle's theorem is not applicable on $f(x)$ in $[1, 3]$.

Mean Value Theorems Ex 15.1 Q1(v)

Here, $f(x) = x^{\frac{2}{3}}$ on $[-1, 1]$

$$\begin{aligned}
 f'(x) &= \frac{2}{3x^{\frac{1}{3}}} \\
 f'(0) &= \frac{2}{3(0)^{\frac{1}{3}}} \\
 f'(0) &= \infty
 \end{aligned}$$

So, $f'(x)$ does not exist at $x = 0 \in (-1, 1)$
 $\Rightarrow$ $f(x)$ is not differentiable in $x \in (-1, 1)$

So, rolle's theorem is not applicable on $f(x)$ in $[-1, 1]$.

Mean Value Theorems Ex 15.1 Q1(vi)

$$\text{Here, } f(x) = \begin{cases} -4x + 5, & 0 \leq x \leq 1 \\ 2x - 3, & 1 < x \leq 2 \end{cases}$$

For $n = 1$

$$\begin{aligned} \text{LHS} &= \lim_{x \rightarrow (1-h)} (-4x + 5) \\ &= \lim_{h \rightarrow 0} [-4(1-h) + 5] \\ &= -4 + 5 \\ \text{LHS} &= 1 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \lim_{x \rightarrow (1+h)} (2x - 3) \\ &= \lim_{h \rightarrow 0} [2(1+h) - 3] \\ &= 2 - 3 \\ \text{RHS} &= -1 \end{aligned}$$

So, $\text{LHS} \neq \text{RHS}$
 $\Rightarrow f(x)$ is not continuous at $x = 1 \in [0, 2]$
 $\Rightarrow$ Rolle's theorem is not applicable on $f(x)$ in $[0, 2]$.

Mean Value Theorems Ex 15.1 Q2(i)

Here,

$$f(x) = x^2 - 8x + 12 \text{ on } [2, 6]$$

$f(x)$ is continuous in $[2, 6]$ and differentiable in $(2, 6)$ as it is a polynomial function

$$\begin{aligned} \text{And } f(2) &= (2)^2 - 8(2) + 12 = 0 \\ f(6) &= (6)^2 - 8(6) + 12 = 0 \\ \Rightarrow f(2) &= f(6) \end{aligned}$$

So, Rolle's theorem is applicable, therefore we show have $f'(c) = 0$ such that $c \in (2, 6)$

$$\begin{aligned} \text{So, } f(x) &= x^2 - 8x + 12 \\ \Rightarrow f'(x) &= 2x - 8 \end{aligned}$$

$$\begin{aligned} \text{So, } f'(c) &= 0 \\ 2c - 8 &= 0 \\ c &= 4 \in (2, 6) \end{aligned}$$

Therefore, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(ii)

The given function is $f(x) = x^2 - 4x + 3$

f , being a polynomial function, is continuous in $[1, 4]$ and is differentiable in $(1, 4)$ whose derivative is $2x - 4$.

$$\begin{aligned} f(1) &= 1^2 - 4 \times 1 + 3 = 0, f(4) = 4^2 - 4 \times 4 + 3 = 3 \\ \therefore \frac{f(b) - f(a)}{b - a} &= \frac{f(4) - f(1)}{4 - 1} = \frac{3 - (0)}{3} = \frac{3}{3} = 1 \end{aligned}$$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c) = 1$

$$\begin{aligned} f'(c) &= 1 \\ \Rightarrow 2c - 4 &= 1 \\ \Rightarrow c &= \frac{5}{2}, \text{ where } c = \frac{5}{2} \in (1, 4) \end{aligned}$$

Hence, Mean Value Theorem is verified for the given function

Mean Value Theorems Ex 15.1 Q2(iii)

Here,

$$f(x) = (x-1)(x-2)^2 \text{ on } (1,2)$$

$f(x)$ is continuous on $[1,2]$ and differentiable on $(1,2)$ since it is a polynomial function.

$$\text{And } f(1) = (1-1)(1-2)^2 = 0$$

$$f(2) = (2-1)(2-2)^2 = 0$$

$$\Rightarrow f(1) = f(2)$$

So, Rolle's theorem is applicable on $f(x)$ in $[1,2]$, therefore, there exist a $c \in (1,2)$ such that $f'(c) = 0$

Now,

$$f(x) = (x-1)(x-2)^2$$

$$f'(x) = (x-1) \times 2(x-2) + (x-2)^2$$

$$f'(x) = (x-2)(3x-4)$$

$$\text{So, } f'(c) = 0$$

$$(c-2)(3c-4) = 0$$

$$\Rightarrow c = 2 \text{ or } c = \frac{4}{3} \in (1,2)$$

Thus, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(iv)

Here,

$$f(x) = x(x-1)^2 \text{ on } [0,1]$$

$f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$ as it is a polynomial function.

Now,

$$f(0) = 0(0-1)^2 = 0$$

$$f(1) = 1(1-1)^2 = 0$$

$$\Rightarrow f(0) = f(1)$$

So, Rolle's theorem is applicable on $f(x)$ in $[0,1]$ therefore, we should show that there exist a $c \in (0,1)$ such that $f'(c) = 0$

Now,

$$f(x) = x(x-1)^2$$

$$f'(x) = (x-1)^2 + x \times 2(x-1)$$

$$= (x-1)(x-1+2x)$$

$$f'(x) = (x-1)(3x-1)$$

$$\text{So, } f'(c) = 0$$

$$(c-1)(3c-1) = 0$$

$$\Rightarrow c = 1 \text{ or } c = \frac{1}{3} \in (0,1)$$

Thus, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(v)

Here,

$$f(x) = (x^2 - 1)(x - 2) \text{ on } [-1, 2]$$

$f(x)$ is continuous in $[-1, 2]$ and differentiable in $(-1, 2)$ as it is a polynomial function.

Now,

$$f(-1) = (1 - 1)(-1 - 2) = 0$$

$$f(2) = (4 - 1)(2 - 2) = 0$$

$$\Rightarrow f(-1) = f(2)$$

So, Rolle's theorem is applicable on $f(x)$ is $[-1, 2]$ therefore, we have to show that there exist a $c \in (-1, 2)$ such that $f'(c) = 0$

Now,

$$f(x) = (x^2 - 1)(x - 2)$$

$$f'(x) = 2x(x - 2) + (x^2 - 1)$$

$$= 2x^2 - 4 + x^2 - 1$$

$$f'(x) = 3x^2 - 5$$

Now,

$$f'(c) = 0$$

$$\Rightarrow 3x^2 - 5 = 0$$

$$\Rightarrow x = -\sqrt{\frac{5}{3}} \text{ or } x = \sqrt{\frac{5}{3}} \in (-1, 2)$$

Thus, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(vi)

Here, $f(x) = x(x - 4)^2$ on $[0, 4]$

$f(x)$ is continuous in $[0, 4]$ and differentiable in $(0, 4)$ since $f(x)$ is a polynomial function.

Now,

$$f(x) = x(x - 4)^2$$

$$f(0) = 0(0 - 4)^2$$

$$f(0) = 0 \quad \text{---(i)}$$

$$f(4) = 4(4 - 4)^2$$

$$f(4) = 0 \quad \text{---(ii)}$$

From equation (i) and (ii),

$$f(0) = f(4)$$

So, Rolle's theorem is applicable, therefore, we have to show that $f'(c) = 0$ for $c \in (0, 4)$

$$\begin{aligned} f'(x) &= x \times 2(x - 4) + (x - 4)^2 \\ &= 2x^2 - 8x + x^2 + 16 - 8x \end{aligned}$$

$$\text{So, } f'(c) = 3c^2 - 16c + 16$$

$$0 = 3c^2 - 16c + 16$$

$$0 = 3c(c - 4) - 4(c - 4)$$

$$0 = (c - 4)(3c - 4)$$

$$\Rightarrow c = 4 \text{ or } c = \frac{4}{3} \in (0, 4)$$

So, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(vii)

Here, $f(x) = x(x-2)^2$ on $[0, 2]$
 $f(x)$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$
as it is a polynomial function.

$$\begin{aligned}\text{And } f(0) &= 0(0-2)^2 = 0 \\ f(2) &= 2(2-2)^2 = 0 \\ \Rightarrow f(0) &= f(2)\end{aligned}$$

So, Rolle's theorem is applicable on $f(x)$ is $[0, 2]$, therefore,
we have to show that $f'(c) = 0$ as $c \in (0, 2)$

$$\begin{aligned}f(x) &= x(x-2)^2 \\ f'(x) &= x \times 2(x-2) + (x-2) \\ f'(x) &= 2x(x-2) + (x-2) \\ \Rightarrow f'(c) &= 0 \\ 2c(c-2) + (c-2) &= 0 \\ (c-2)(2c+1) &= 0 \\ c = 2 \text{ or } c &= -\frac{1}{2} \\ \Rightarrow c = 2 &\in (0, 2)\end{aligned}$$

So, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(viii)

Here, $f(x) = x^2 + 5x + 6$ on $[-3, -2]$
 $f(x)$ is continuous on $[-3, -2]$ and $f(x)$ is differentiable on $(-3, -2)$
since it is a polynomial function.

Now,

$$\begin{aligned}f(x) &= x^2 + 5x + 6 \\ f(-3) &= (-3)^2 + 5(-3) + 6 \\ &= 9 - 15 + 6 \\ f(-3) &= 0 \quad \text{--- (i)} \\ f(-2) &= (-2)^2 + 5(-2) + 6 \\ &= 4 - 10 + 6 \\ f(-2) &= 0 \quad \text{--- (ii)}\end{aligned}$$

From equation (i) and (ii),
 $f(-3) = f(-2)$

So, Rolle's theorem is applicable on $[-3, -2]$, we have to show that
 $f'(c) = 0$ as $c \in (-3, -2)$.

Now,

$$\begin{aligned}f(x) &= x^2 + 5x + 6 \\ f'(x) &= 2x + 5 \\ \Rightarrow f'(c) &= 0 \\ 2c + 5 &= 0 \\ c &= -\frac{5}{2} \in (-3, -2)\end{aligned}$$

So, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(i)

Here,

$$f(x) = \cos 2\left(x - \frac{\pi}{4}\right) \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that cosine function is continuous and differentiable

every where, so $f(x)$ is continuous is $\left[0, \frac{\pi}{2}\right]$ and differentiable is $\left[0, \frac{\pi}{2}\right]$.

Now,

$$f(0) = \cos 2\left(0 - \frac{\pi}{4}\right) = 0$$

$$f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = 0$$

$$\Rightarrow f(0) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable.

Hence, there must exists a $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now,

$$f'(x) = -\sin 2\left(x - \frac{\pi}{4}\right) \times 2$$

$$f'(x) = -2 \sin\left(2x - \frac{\pi}{2}\right)$$

$$\Rightarrow -2 \sin\left(2c - \frac{\pi}{2}\right) = 0$$

$$\Rightarrow \sin\left(2c - \frac{\pi}{2}\right) = \sin 0$$

$$\Rightarrow 2c - \frac{\pi}{2} = 0$$

$$c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(ii)

Here,

$$f(x) = \sin 2x \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that $\sin x$ is a continuous and differentiable every where. So,

$f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$ and differentiable is $\left(0, \frac{\pi}{2}\right)$.

Now,

$$f(0) = \sin 0 = 0$$

$$f\left(\frac{\pi}{2}\right) = \sin \pi = 0$$

$$\Rightarrow f(0) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable, so, there must exist a $c \in \left(0, \frac{\pi}{2}\right)$

such that $f'(c) = 0$

Now,

$$f'(x) = 2 \cos 2x$$

$$f'(c) = 2 \cos 2c = 0$$

$$\Rightarrow \cos 2c = 0$$

$$\Rightarrow 2c = \frac{\pi}{2}$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Thus, Rolle's theorem verified.

Mean Value Theorems Ex 15.1 Q3(iii)

Here,

$$f(x) = \cos 2x \text{ on } \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$$

We know that $\cos x$ is a continuous and differentiable every where. So,

$f(x)$ is continuous in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ and differentiable is $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

$$\text{Now, } f\left(-\frac{\pi}{4}\right) = \cos 2\left(-\frac{\pi}{4}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$$

$$f\left(\frac{\pi}{4}\right) = \cos 2\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$\Rightarrow f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right)$$

So, Rolle's theorem is applicable, so, there must exist a $c \in \left(0, \frac{\pi}{2}\right)$

such that $f'(c) = 0$

Now,

$$f'(x) = 2 \sin 2x$$

$$f'(c) = 2 \sin 2c = 0$$

$$\Rightarrow \sin 2c = 0$$

$$\Rightarrow 2c = 0$$

$$\Rightarrow c = 0 \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

Thus, Rolle's theorem verified.

Mean Value Theorems Ex 15.1 Q3(iv)

Here,

$$f(x) = e^x \times \sin x \text{ on } [0, \pi]$$

We know that since exponential function are continuous and differentiable every where so, $f(x)$ is continuous is $[0, \pi]$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = e^0 \sin 0 = 0$$

$$f(\pi) = e^\pi \sin \pi = 0$$

$$\Rightarrow f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (0, \pi)$

such that $f'(c) = 0$.

Now,

$$f(x) = e^x \sin x$$

$$f'(x) = e^x \cos x + e^x \sin x$$

$$\text{Now, } f'(c) = 0$$

$$e^c (\cos c + \sin c) = 0$$

$$\Rightarrow e^c = 0 \text{ or } \cos c = -\sin c$$

$$\Rightarrow e^c = 0 \text{ gives no value of } c \text{ or } \tan c = -1$$

$$\Rightarrow \tan c = \tan\left(\pi - \frac{\pi}{4}\right)$$

$$c = \frac{3\pi}{4} \in (0, \pi)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(v)

Here,

$$f(x) = e^x \cos x \text{ on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

We know that exponential and cosine function are continuous and differentiable every where so, $f(x)$ is continuous is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and differentiable is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now,

$$f\left(-\frac{\pi}{2}\right) = e^{\frac{\pi}{2}} \cos\left(-\frac{\pi}{2}\right) = 0$$

$$f\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}} \cos\left(\frac{\pi}{2}\right) = 0$$

$$\Rightarrow f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now,

$$f(x) = e^x \cos x$$

$$f'(x) = -e^x \sin x + e^x \cos x$$

$$\text{So, } f'(c) = 0$$

$$e^c (-\sin c + \cos c) = 0$$

$$\Rightarrow e^c = 0 \text{ gives no value of } c$$

$$\Rightarrow -\sin c + \cos c = 0$$

$$\Rightarrow \tan c = 1$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(vi)

Here,

$$f(x) = \cos 2x \text{ on } [0, \pi]$$

We know that, cosine function is continuous and differentiable every where, so $f(x)$ is continuous is $[0, \pi]$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = \cos 0 = 1$$

$$f(\pi) = \cos(2\pi) = 1$$

$$\Rightarrow f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$$f(x) = \cos 2x$$

$$f'(x) = -2 \sin 2x$$

$$\text{So, } f'(c) = 0$$

$$\Rightarrow -2 \sin 2c = 0$$

$$\Rightarrow \sin 2c = 0$$

$$\Rightarrow 2c = 0 \quad \text{or} \quad 2c = \pi$$

$$\Rightarrow c = 0 \quad \text{or} \quad c = \frac{\pi}{2} \in (0, \pi)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(vii)

$$f(x) = \frac{\sin x}{e^x} \text{ on } x \in [0, \pi]$$

We know that, exponential and sine both functions are continuous and differentiable every where, so $f(x)$ is continuous is $[0, \pi]$ and differentiable is $[0, \pi]$

Now,

$$f(0) = \frac{\sin 0}{e^0} = 0$$

$$f(\pi) = \frac{\sin \pi}{e^\pi} = 0$$

$$\Rightarrow f(0) = f(\pi)$$

Since Rolle's theorem applicable, therefore there must exist a point $c \in [0, \pi]$ such that $f'(c) = 0$

Now,

$$f(x) = \frac{\sin x}{e^x}$$

$$\Rightarrow f'(x) = \frac{e^x(\cos x) - e^x(\sin x)}{(e^x)^2}$$

Now,

$$f'(c) = 0$$

$$\Rightarrow e^c(\cos c - \sin c) = 0$$

$$\Rightarrow e^c \neq 0 \text{ and } \cos c - \sin c = 0$$

$$\Rightarrow \tan c = 1$$

$$c = \frac{\pi}{4} \in [0, \pi]$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(viii)

Here,

$$f(x) = \sin 3x \text{ on } [0, \pi]$$

We know that, sine function is continuous and differentiable every where. So, $f(x)$ is continuous is $(0, \pi)$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = \sin 0 = 0$$

$$f(\pi) = \sin 3\pi = 0$$

$$\Rightarrow f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exists a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$$f(x) = \sin 3x$$

$$f'(x) = 3 \cos 3x$$

Now,

$$f'(c) = 0$$

$$\Rightarrow 3 \cos 3x = 0$$

$$\Rightarrow \cos 3x = 0$$

$$\Rightarrow 3x = \frac{\pi}{2}$$

$$\Rightarrow x = \frac{\pi}{6} \in (0, \pi)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(ix)

Here,

$$f(x) = e^{1-x^2} \text{ on } [-1, 1]$$

We know that, exponential function is continuous and differentiable every where. So, $f(x)$ is continuous is $[-1, 1]$ and differentiable is $(-1, 1)$.

Now,

$$f(-1) = e^{1-1} = 1$$

$$f(1) = e^{1-1} = 1$$

$$\Rightarrow f(-1) = f(1)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (-1, 1)$ such that $f'(c) = 0$.

Now,

$$f(x) = e^{1-x^2}$$

$$f'(x) = e^{1-x^2} (-2x)$$

Now,

$$f'(c) = 0$$

$$-2ce^{1-c^2} = 0$$

$$\Rightarrow c = 0 \quad \text{or} \quad e^{1-c^2} = 0$$

$$\Rightarrow c = 0 \in (-1, 1)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(x)

Here,

$$f(x) = \log(x^2 + 2) - \log 3 \text{ on } [-1, 1]$$

We know that, logarithmic function is continuous and differentiable is its domain, so $f(x)$ is continuous is $[-1, 1]$ and differentiable is $(-1, 1)$.

Now,

$$f(-1) = \log(1+2) - \log 3 = 0$$

$$f(1) = \log(1+2) - \log 3 = 0$$

$$\Rightarrow f(-1) = f(1)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (-1, 1)$ such that $f'(c) = 0$.

Now,

$$f(x) = \log(x^2 + 2) - \log 3$$

$$f'(x) = \frac{(2x)}{x^2 + 2}$$

Now,

$$f'(c) = 0$$

$$\frac{2c}{c^2 + 2} = 0$$

$$\Rightarrow c = 0 \in (-1, 1)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xi)

Here,

$$f(x) = \sin x + \cos x \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that $\sin x$ and $\cos x$ are continuous and differentiable every where, so

$f(x)$ is continuous is $\left[0, \frac{\pi}{2}\right]$ and differentiable is $\left(0, \frac{\pi}{2}\right)$.

Now,

$$f(0) = \sin 0 + \cos 0 = 1$$

$$f\left(\frac{\pi}{2}\right) = \frac{\sin \pi}{2} + \frac{\cos \pi}{2} = 1$$

$$\Rightarrow f(0) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now,

$$f(x) = \sin x + \cos x$$

$$f'(x) = \cos x - \sin x$$

Now,

$$f'(c) = 0$$

$$\cos c - \sin c = 0$$

$$\Rightarrow \tan c = 1$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xii)

Here,

$$f(x) = 2 \sin x + \sin 2x \text{ on } [0, \pi]$$

We know that sine function is continuous and differentiable every where, so

$f(x)$ is continuous is $[0, \pi]$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = 2 \sin 0 + \sin 0 = 0$$

$$f(\pi) = 2 \sin \pi + \sin 2\pi = 0$$

$$\Rightarrow f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$$f(x) = 2 \sin x + \sin 2x$$

$$f'(x) = 2 \cos x + 2 \cos 2x$$

Now,

$$f'(c) = 0$$

$$2 \cos c + 2 \cos 2c = 0$$

$$\Rightarrow 2(\cos c + 2 \cos^2 c - 1) = 0$$

$$\Rightarrow (2 \cos^2 c + 2 \cos c - \cos c - 1) = 0$$

$$\Rightarrow (2 \cos c - 1)(\cos c + 1) = 0$$

$$\Rightarrow \cos c = \frac{1}{2}, \cos c = -1$$

$$\Rightarrow \tan c = 1$$

$$c = \frac{\pi}{3} \in (0, \pi), c = \pi$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xiii)

Here,

$$f(x) = \frac{x}{2} - \sin \frac{\pi x}{6} \text{ on } [-1, 0]$$

We know that sine function is continuous and differentiable every where, so $f(x)$ is continuous is $[-1, 0]$ and differentiable is $(-1, 0)$.

Now,

$$\begin{aligned} f(-1) &= \frac{-1}{2} - \sin\left(-\frac{\pi}{6}\right) \\ &= -\frac{1}{2} + \sin \frac{\pi}{6} \\ &= -\frac{1}{2} + \frac{1}{2} \\ f(-1) &= 0 \end{aligned} \quad \text{---(i)}$$

$$\begin{aligned} \text{And } f(0) &= 0 - \sin 0 \\ f(0) &= 0 \end{aligned} \quad \text{---(ii)}$$

From equation (i) and (ii),

$$f(-1) = f(0)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (-1, 0)$ such that $f'(c) = 0$.

Now,

$$\begin{aligned} f(x) &= \frac{x}{2} - \sin\left(\frac{\pi x}{6}\right) \\ f'(x) &= \frac{1}{2} - \frac{\pi}{6} \cos\left(\frac{\pi x}{6}\right) \end{aligned}$$

Now,

$$\begin{aligned} f'(c) &= 0 \\ \frac{1}{2} - \frac{\pi}{6} \cos\left(\frac{\pi c}{6}\right) &= 0 \\ \Rightarrow -\frac{\pi}{6} \cos\left(\frac{\pi c}{6}\right) &= -\frac{1}{2} \\ \Rightarrow \cos\left(\frac{\pi c}{6}\right) &= \frac{3}{\pi} \\ \Rightarrow \frac{\pi c}{6} &= \cos^{-1}\left(\frac{3}{\pi}\right) \\ \Rightarrow c &= \frac{6}{\pi} \cos^{-1}\left(\frac{3}{\pi}\right) \\ \Rightarrow c &= \frac{21}{11} \cos^{-1}\left(\frac{66}{7}\right) \\ \Rightarrow c &\in \left(-\frac{21}{11}, \frac{21}{11}\right) && \left[\text{since, } \cos^{-1} x \in [-1, 1]\right] \\ \Rightarrow c &\in (-1.9, 1.9) \\ \Rightarrow c &\in (-1, 0) \end{aligned}$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xiv)

Here,

$$f(x) = \frac{6x}{\pi} - 4\sin^2 x \text{ on } \left[0, \frac{\pi}{6}\right]$$

We know that sine and its square function is continuous and differentiable every where, so

$f(x)$ is continuous is $\left[0, \frac{\pi}{6}\right]$ and differentiable is $\left(0, \frac{\pi}{6}\right)$.

Now,

$$f(0) = 0 - 0 = 0$$

$$f\left(\frac{\pi}{6}\right) = 1 - 1 = 0$$

$$\Rightarrow f(0) = f\left(\frac{\pi}{6}\right)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in \left(0, \frac{\pi}{6}\right)$ such

that $f'(c) = 0$.

Now,

$$f(x) = \frac{6x}{\pi} - 4\sin^2 x$$

$$f'(x) = \frac{6}{\pi} - 8\sin x \cos x$$

$$f'(x) = \frac{6}{\pi} - 4\sin 2x$$

Now,

$$f'(c) = 0$$

$$\frac{6}{\pi} - 4\sin 2c = 0$$

$$\Rightarrow 4\sin 2c = \frac{6}{\pi}$$

$$\Rightarrow \sin 2c = \frac{3}{2\pi}$$

$$\Rightarrow 2c = \sin^{-1}\left(\frac{21}{44}\right)$$

$$\Rightarrow c = \frac{1}{2}\sin^{-1}\left(\frac{21}{44}\right)$$

$$\Rightarrow c \in \left(-\frac{1}{2}, \frac{1}{2}\right) \quad \left[\text{since, } \sin^{-1} x \in [-1, 1]\right]$$

$$\Rightarrow c \in \left(0, \frac{11}{21}\right)$$

$$\Rightarrow c \in \left(0, \frac{\pi}{6}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xv)

Here,

$$f(x) = 4^{\sin x} \text{ on } [0, \pi]$$

We know that exponential and $\sin x$ both are continuous and differentiable, so $f(x)$ is continuous is $[0, \pi]$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = 4^{\sin 0} = 4^0 = 1$$

$$f(\pi) = 4^{\sin \pi} = 4^0 = 1$$

$$\Rightarrow f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$$f(x) = 4^{\sin x}$$

$$f'(x) = 4^{\sin x} \log 4 \times \cos x$$

Now,

$$f'(c) = 0$$

$$4^{\sin c} \times \cos c \log 4 = 0$$

$$\Rightarrow \cos c = 0$$

$$\Rightarrow c = \frac{\pi}{2} \in (0, \pi)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xvi)

Here,

$$f(x) = x^2 - 5x + 4 \text{ on } [1, 4]$$

$f(x)$ is continuous and differentiable as it is a polynomial function.

Now,

$$f(1) = (1)^2 - 5(1) + 4 = 0$$

$$f(4) = (4)^2 - 5(4) + 4 = 0$$

$$\Rightarrow f(1) = f(4)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (1, 4)$ such that $f'(c) = 0$.

Now,

$$f(x) = x^2 - 5x + 4$$

$$f'(x) = 2x - 5$$

So,

$$f'(c) = 0$$

$$\Rightarrow 2c - 5 = 0$$

$$\Rightarrow c = \frac{5}{2} \in (1, 4)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xvii)

Here,

$$f(x) = \sin^4 x + \cos^4 x \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that sine and cosine function are differentiable and continuous.

So, $f(x)$ is continuous is $\left[0, \frac{\pi}{2}\right]$ and it is differentiable is $\left(0, \frac{\pi}{2}\right)$.

Now,

$$f(0) = \sin^4(0) + \cos^4(0) = 1$$

$$f\left(\frac{\pi}{2}\right) = \sin^4\left(\frac{\pi}{2}\right) + \cos^4\left(\frac{\pi}{2}\right) = 1$$

$$\Rightarrow f(0) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now,

$$f(x) = \sin^4 x + \cos^4 x$$

$$\begin{aligned} f'(x) &= 4\sin^3 x \cos x - 4\cos^3 x \sin x \\ &= -2(2\sin x \cos x)(\cos^2 x - \sin^2 x) \\ &= -2\sin 2x \cos 2x \\ f'(x) &= -\sin 4x \end{aligned}$$

Now,

$$f'(c) = 0$$

$$-\sin 4x = 0$$

$$\sin 4x = 0$$

$$\Rightarrow 4x = 0 \quad \text{or} \quad 4x = \pi$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xvii)

Since trigonometric functions are differentiable and continuous, the given function, $f(x) = \sin x - \sin 2x$ is also continuous and differentiable.

$$\text{Now } f(0) = \sin 0 - \sin 2 \times 0 = 0$$

and

$$f(\pi) = \sin \pi - \sin 2 \times \pi = 0$$

$$\Rightarrow f(0) = f(\pi)$$

Thus, $f(x)$ satisfies conditions of the Rolle's Theorem on $[0, \pi]$.

Therefore, there exists $c \in [0, \pi]$ such that $f'(c) = 0$

$$\text{Now } f(x) = \sin x - \sin 2x$$

$$\Rightarrow f'(x) = \cos x - 2\cos 2x = 0$$

$$\Rightarrow \cos x = 2\cos 2x$$

$$\Rightarrow \cos x = 2(2\cos^2 x - 1)$$

$$\Rightarrow \cos x = 4\cos^2 x - 2$$

$$\Rightarrow 4\cos^2 x - \cos x - 2 = 0$$

$$\Rightarrow \cos x = \frac{1 \pm \sqrt{33}}{8} = 0.8431 \text{ or } -0.5931$$

$$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } \cos^{-1}(-0.5931)$$

$$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } 180^\circ - \cos^{-1}(0.5931) \quad [\because \cos^{-1}(-x) = \pi - \cos^{-1}(x)]$$

$$\Rightarrow x = 32^\circ 32' \text{ or } x = 126^\circ 23'$$

Both $32^\circ 32'$ and $126^\circ 23' \in [0, \pi]$ such that $f'(c) = 0$.

Hence Rolle's Theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xviii)

Since trigonometric functions are differentiable and continuous,
the given function, $f(x) = \sin x - \sin 2x$ is also continuous and differentiable.

$$\text{Now } f(0) = \sin 0 - \sin 2 \times 0 = 0$$

and

$$f(\pi) = \sin \pi - \sin 2 \times \pi = 0$$

$$\Rightarrow f(0) = f(\pi)$$

Thus, $f(x)$ satisfies conditions of the Rolle's Theorem on $[0, \pi]$.

Therefore, there exists $c \in [0, \pi]$ such that $f'(c) = 0$

$$\text{Now } f(x) = \sin x - \sin 2x$$

$$\Rightarrow f'(x) = \cos x - 2\cos 2x = 0$$

$$\Rightarrow \cos x = 2\cos 2x$$

$$\Rightarrow \cos x = 2(2\cos^2 x - 1)$$

$$\Rightarrow \cos x = 4\cos^2 x - 2$$

$$\Rightarrow 4\cos^2 x - \cos x - 2 = 0$$

$$\Rightarrow \cos x = \frac{1 \pm \sqrt{33}}{8} = 0.8431 \text{ or } -0.5931$$

$$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } \cos^{-1}(-0.5931)$$

$$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } 180^\circ - \cos^{-1}(0.5931) \quad [\because \cos^{-1}(-x) = \pi - \cos^{-1}(x)]$$

$$\Rightarrow x = 32^\circ 32' \text{ or } x = 126^\circ 23'$$

Both $32^\circ 32'$ and $126^\circ 23' \in [0, \pi]$ such that $f'(c) = 0$.

Hence Rolle's Theorem is verified.

Mean Value Theorems Ex 15.1 Q7

Let $f(x) = 16 - x^2$, then $f'(x) = -2x$

$f(x)$ is continuous on $[-1, 1]$ because it is a polynomial function.

$$\text{Also } f(-1) = 16 - (-1)^2 = 15$$

$$f(1) = 16 - (1)^2 = 15$$

$$f(-1) = f(1)$$

There exists a $c \in [-1, 1]$ such that $f'(c) = 0$

$$\Rightarrow -2c = 0$$

$$\Rightarrow c = 0$$

Thus, at $0 \in [-1, 1]$ the tangent is parallel to the x -axis.

Mean Value Theorems Ex 15.1 Q8(i)

Let $f(x) = x^2$, then $f'(x) = 2x$

$f(x)$ is continuous on $[-2, 2]$ because it is a polynomial function.

$f(x)$ is differentiable on $(-2, 2)$ as it is a polynomial function.

$$\text{Also } f(-2) = (-2)^2 = 4$$

$$f(2) = 2^2 = 4$$

$$\Rightarrow f(-2) = f(2)$$

$\therefore$ There exists $c \in (-2, 2)$ such that $f'(c) = 0$

$$\Rightarrow 2c = 0$$

$$\Rightarrow c = 0$$

Thus, at $0 \in [-2, 2]$ the tangent is parallel to the x -axis.

$x = 0$, then $y = 0$

Therefore, the point is $(0, 0)$

Mean Value Theorems Ex 15.1 Q8(ii)

Let $f(x) = e^{1-x^2}$ on $[-1, 1]$

Since, $f(x)$ is a composition of two continuous functions, it is continuous on $[-1, 1]$

Also $f(x) = -2xe^{1-x^2}$

$$f(2) = 2^2 = 4$$

$\therefore f'(x)$ exists for every value of x in $(-1, 1)$

$\Rightarrow f(x)$ is differentiable on $(-1, 1)$

By Rolle's theorem, there exists $c \in (-1, 1)$ such that $f'(c) = 0$

$$\Rightarrow -2ce^{1-c^2} = 0$$

$$\Rightarrow c = 0$$

Thus, at $c = 0 \in [-1, 1]$ the tangent is parallel to the x-axis.

$x = 0$, then $y = e$

Therefore, the point is $(0, e)$

Mean Value Theorems Ex 15.1 Q8(iii)

Let $f(x) = 12(x+1)(x-2)$

Since, $f(x)$ is a polynomial function, it is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$

Also $f'(x) = 12[(x-2) + (x+1)] = 12[2x-1]$

By Rolle's theorem, there exists $c \in (-1, 2)$ such that $f'(c) = 0$

$$\Rightarrow 12(2c-1) = 0$$

$$\Rightarrow c = \frac{1}{2}$$

Thus, at $c = \frac{1}{2} \in (-1, 2)$ the tangent to $y = 12(x+1)(x-2)$ is parallel to x-axis

Mean Value Theorems Ex 15.1 Q9

It is given that $f: [-5, 5] \rightarrow \mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) f is continuous on $[-5, 5]$.

(b) f is differentiable on $(-5, 5)$.

Therefore, by the Mean Value Theorem, there exists $c \in (-5, 5)$ such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

$$\Rightarrow 10f'(c) = f(5) - f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\therefore f'(c) \neq 0$$

$$\Rightarrow 10f'(c) \neq 0$$

$$\Rightarrow f(5) - f(-5) \neq 0$$

$$\Rightarrow f(5) \neq f(-5)$$

Hence, proved.

Mean Value Theorems Ex 15.1 Q10

By Rolle's Theorem, for a function $f:[a, b] \rightarrow \mathbf{R}$, if

(a) f is continuous on $[a, b]$

(b) f is differentiable on (a, b)

(c) $f(a) = f(b)$

then, there exists some $c \in (a, b)$ such that $f'(c) = 0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x) = [x]$ for $x \in [5, 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$f(x)$ is not continuous in $[5, 9]$.

Also, $f(5) = [5] = 5$ and $f(9) = [9] = 9$

$\therefore f(5) \neq f(9)$

The differentiability of f in $(5, 9)$ is checked as follows.

Let n be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h \rightarrow 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^-} \frac{n-1-n}{h} = \lim_{h \rightarrow 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \rightarrow 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^+} \frac{n-n}{h} = \lim_{h \rightarrow 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

f is not differentiable in $(5, 9)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$.

$$(ii) f(x) = [x] \text{ for } x \in [-2, 2]$$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$f(x)$ is not continuous in $[-2, 2]$.

$$\text{Also, } f(-2) = [-2] = -2 \text{ and } f(2) = [2] = 2$$

$$\therefore f(-2) \neq f(2)$$

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h \rightarrow 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^-} \frac{n-1-n}{h} = \lim_{h \rightarrow 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \rightarrow 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^+} \frac{n-n}{h} = \lim_{h \rightarrow 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

f is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

Mean Value Theorems Ex 15.1 Q11

It is given that the Rolle's Theorem holds for the function $f(x) = x^3 + bx^2 + cx$, $x \in [1, 2]$

at the point $x = \frac{4}{3}$.

We need to find the values of b and c .

$$f(x) = x^3 + bx^2 + cx$$

Since it satisfies the Rolle's theorem, we have,

$$f(1) = f(2)$$

$$\Rightarrow 1^3 + b \times 1^2 + c \times 1 = 2^3 + b \times 2^2 + c \times 2$$

$$\Rightarrow 1 + b + c = 8 + 4b + 2c$$

$$\Rightarrow 3b + c = -7 \dots (1)$$

Differentiating the given function, we have,

$$f'(x) = 3x^2 + 2bx + c$$

$$f'\left(\frac{4}{3}\right) = 3 \times \left(\frac{4}{3}\right)^2 + 2b \times \left(\frac{4}{3}\right) + c$$

$$\Rightarrow 0 = \frac{16}{3} + \frac{8b}{3} + c \dots (2)$$

Solving the equations (1) and (2), we have,

$$b = -5 \text{ and } c = 8$$

Ex 15.2

Mean Value Theorems Ex 15.2 Q1(i)

Here,

$$f(x) = x^2 - 1 \text{ on } [2, 3]$$

It is a polynomial function so it is continuous in $[2, 3]$ and differentiable in $(2, 3)$. So, both conditions of Lagrange's mean value theorem are satisfied.

Therefore, there exist a point $c \in (2, 3)$ such that

$$\begin{aligned} f'(c) &= \frac{f(3) - f(2)}{3 - 2} \\ 2c &= \frac{(3)^2 - 1 - ((2)^2 - 1)}{1} \\ 2c &= (8 - 3) \\ c &= \frac{5}{2} \in (2, 3) \end{aligned}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(ii)

Here,

$$f(x) = x^3 - 2x^2 - x + 3 \text{ on } [0, 1]$$

Since, $f(x)$ is a polynomial function. So, $f(x)$ is continuous in $[0, 1]$ and differentiable in $(0, 1)$.
So, Lagrange's mean value theorem is applicable. Thus, there exists a point $c \in (0, 1)$ such that

$$\begin{aligned} f'(c) &= \frac{f(1) - f(0)}{1 - 0} \\ \Rightarrow 3c^2 - 4c - 1 &= \frac{[(1)^3 - 2(1)^2 - (1) + 3] - 3}{1} \\ \Rightarrow 3c^2 - 4c - 1 &= 1 - 3 \\ \Rightarrow 3c^2 - 4c + 1 &= 0 \\ \Rightarrow 3c^2 - 3c - c + 1 &= 0 \\ \Rightarrow 3c(c - 1) - 1(c - 1) &= 0 \\ \Rightarrow (3c - 1)(c - 1) &= 0 \\ \Rightarrow c = \frac{1}{3} \in (0, 1) \end{aligned}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(iii)

Here,

$$\begin{aligned} f(x) &= x(x - 1) \\ f(x) &= x^2 - x \text{ on } [1, 2] \end{aligned}$$

We know that, polynomial function is continuous and differentiable. So,
 $f(x)$ is continuous in $[1, 2]$ and $f(x)$ is differentiable in $(1, 2)$. So, Lagrange's
mean value theorem is applicable. Thus, there exists a point $c \in (1, 2)$ such that

$$\begin{aligned} f'(c) &= \frac{f(2) - f(1)}{2 - 1} \\ \Rightarrow 2c - 1 &= \frac{(4 - 2) - (1 - 1)}{1} \\ \Rightarrow 2c - 1 &= \frac{2 - 0}{1} \\ \Rightarrow 2c &= 3 \\ \Rightarrow c &= \frac{3}{2} \in (1, 2) \end{aligned}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(iv)

Here,

$$f(x) = x^2 - 3x + 2 \text{ on } [-1, 2]$$

We know that, polynomial function is continuous and differentiable. So,
 $f(x)$ is continuous in $[-1, 2]$ and differentiable in $(-1, 2)$. So, Lagrange's
mean value theorem is applicable, so there exist a point $c \in (-1, 2)$ such that

$$\begin{aligned} f'(c) &= \frac{f(2) - f(-1)}{2 - (-1)} \\ \Rightarrow 2c - 3 &= \frac{(4 - 6 + 2) - (1 + 3 + 2)}{3} \\ \Rightarrow 2c - 3 &= -\frac{6}{3} \\ \Rightarrow 2c &= 1 \\ \Rightarrow c &= \frac{1}{2} \in (-1, 2) \end{aligned}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(v)

Here,

$$f(x) = 2x^2 - 3x + 1 \text{ on } [1, 3]$$

We know that, polynomial function is continuous and differentiable. So, $f(x)$ is continuous in $[1, 3]$ and $f(x)$ is differentiable in $(1, 3)$. So, Lagrange's mean value theorem is applicable, so there exist a point $c \in (1, 3)$ such that

$$\begin{aligned} f'(c) &= \frac{f(3) - f(1)}{3 - 1} \\ \Rightarrow 4c - 3 &= \frac{(2(3)^2 - 3(3) + 1) - (2 - 3 + 1)}{3 - 1} \\ \Rightarrow 4c - 3 &= \frac{10}{2} \\ \Rightarrow 4c &= 5 + 3 \\ \Rightarrow 4c &= 8 \\ \Rightarrow c &= 2 \in (1, 3) \end{aligned}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(vi)

Here,

$$f(x) = x^2 - 2x + 4 \text{ on } [1, 5]$$

We know that, polynomial is always continuous and differentiable. So, $f(x)$ is continuous in $[1, 5]$ and it is differentiable in $(1, 5)$. So, Lagrange's mean value theorem is applicable. Thus, there exists a point $c \in (1, 5)$ such that

$$\begin{aligned} f'(c) &= \frac{f(5) - f(1)}{5 - 1} \\ \Rightarrow 2c - 2 &= \frac{((5)^2 - 2(5) + 4) - (1 - 2 + 4)}{4} \\ \Rightarrow 2c - 2 &= \frac{19 - 3}{4} \\ \Rightarrow 2c - 2 &= 4 \\ \Rightarrow 2c &= 6 \\ \Rightarrow c &= 3 \in (1, 5) \end{aligned}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(vii)

Here,

$$f(x) = 2x - x^2 \text{ on } [0, 1]$$

We know that, polynomial is continuous and differentiable. So, $f(x)$ is continuous in $[0, 1]$ and differentiable in $(0, 1)$. So, Lagrange's mean value theorem is applicable. Thus, there exists a point $c \in (0, 1)$ such that

$$\begin{aligned} f'(c) &= \frac{f(1) - f(0)}{1 - 0} \\ \Rightarrow 2 - 2c &= \frac{(2(1) - (1)^2) - (0)}{1} \\ \Rightarrow 2 - 2c &= 1 \\ \Rightarrow 1 &= 2c \\ \Rightarrow c &= \frac{1}{2} \in (0, 1) \end{aligned}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(viii)

$$f(x) = (x-1)(x-2)(x-3) \text{ on } [0, 4]$$

We know that, polynomial is continuous and differentiable every where. So, $f(x)$ is continuous in $[0, 4]$ and differentiable in $(0, 4)$. So, Lagrange's mean value theorem is applicable. Thus, there exists a point $c \in (0, 4)$ such that

$$\begin{aligned} f'(c) &= \frac{f(4) - f(0)}{4 - 0} \\ \Rightarrow (c-1)(c-2) + (c-2)(c-3) + (c-1)(c-3) &= \frac{(3)(2)(1) - (-1)(-2)(-3)}{4 - 0} \\ \Rightarrow c^2 - 3c + 2 + c^2 + 5c + 6 + c^2 - 4c + 3 &= \frac{6 + 6}{4} \\ \Rightarrow 3c^2 - 12c + 11 &= 3 \\ \Rightarrow 3c^2 - 12c + 8 &= 0 \\ \Rightarrow c &= \frac{-(-12) \pm \sqrt{144 - 4 \times 3 \times 8}}{6} \\ \Rightarrow c &= \frac{12 \pm \sqrt{48}}{6} \\ \Rightarrow c &= 2 \pm \frac{2\sqrt{3}}{3} \in (0, 4) \\ \Rightarrow c &= 2 \pm \frac{2}{\sqrt{3}} \in (0, 4) \end{aligned}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(ix)

Here,

$$f(x) = \sqrt{25 - x^2} \text{ on } [-3, 4]$$

Given function is continuous as it has unique value for each $x \in [-3, 4]$ and

$$\begin{aligned} f'(x) &= \frac{-2x}{2\sqrt{25 - x^2}} \\ f'(x) &= \frac{-x}{\sqrt{25 - x^2}} \end{aligned}$$

So, $f'(x)$ exists for all values for $x \in (-3, 4)$ so, $f(x)$ is differentiable in $(-3, 4)$. So, Lagrange's mean value theorem is applicable. Thus, there exists a point $c \in (-3, 4)$ such that

$$\begin{aligned} f'(c) &= \frac{f(4) - f(-3)}{4 - (-3)} \\ \Rightarrow \frac{-2c}{2\sqrt{25 - c^2}} &= \frac{\sqrt{9} - \sqrt{16}}{7} \\ \Rightarrow -7c &= -\sqrt{25 - c^2} \end{aligned}$$

Squaring both the sides,

$$\begin{aligned} 49c^2 &= 25 - c^2 \\ c^2 &= \frac{1}{2} \\ c &= \pm \frac{1}{\sqrt{2}} \in (-3, 4) \end{aligned}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(x)

Here,

$$f(x) = \tan^{-1} x \text{ on } [0, 1]$$

We know that, $\tan^{-1} x$ has unique value in $[0, 1]$ so, it is continuous in $[0, 1]$

$$f'(x) = \frac{1}{1+x^2}$$

So, $f'(x)$ exists for each $x \in (0, 1)$

So, $f'(x)$ is differentiable in $(0, 1)$, thus Lagrange's mean value theorem is applicable, so there exist a point $c \in (0, 1)$ such that

$$\begin{aligned} f'(c) &= \frac{f(1) - f(0)}{1 - 0} \\ \Rightarrow \frac{1}{1+c^2} &= \frac{\tan^{-1}(1) - \tan^{-1}(0)}{1} \\ \Rightarrow \frac{1}{1+c^2} &= \frac{\frac{\pi}{4} - 0}{1} \\ \Rightarrow \frac{4}{\pi} &= 1+c^2 \\ \Rightarrow c &= \sqrt{\frac{4}{\pi} - 1} \end{aligned}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(xi)

Here,

$$f(x) = x + \frac{1}{x} \text{ on } [1, 3]$$

$f(x)$ attains a unique value for each $x \in [1, 3]$, so it is continuous

$f'(x) = 1 - \frac{1}{x^2}$ is defined for each $x \in (1, 3)$

$\Rightarrow f(x)$ is differentiable in $(1, 3)$, so Lagrange's mean value theorem is applicable, so there exist a point $c \in (1, 3)$ such that

$$\begin{aligned} f'(c) &= \frac{f(3) - f(1)}{3 - 1} \\ \Rightarrow 1 - \frac{1}{c^2} &= \frac{\left(3 + \frac{1}{3} - (1 + 1)\right)}{2} \\ \Rightarrow 1 - \frac{1}{c^2} &= \frac{\frac{10}{3} - 2}{2} \\ \Rightarrow 1 - \frac{1}{c^2} &= \frac{4}{3 \times 2} \\ \Rightarrow 1 - \frac{2}{3} &= \frac{1}{c^2} \\ \Rightarrow c^2 &= 3 \\ \Rightarrow c &= \sqrt{3} \in (1, 3) \end{aligned}$$

So, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(xii)

Here,

$$f(x) = x(x+4)^2 \text{ on } [0, 4]$$

We know that every polynomial function is continuous and differentiable everywhere, so, $f(x)$ is continuous in $[0, 4]$ and differentiable in $(0, 4)$, so, Lagrange's mean value theorem is applicable, thus there exist a point $c \in (0, 4)$ such that

$$\begin{aligned} f'(c) &= \frac{f(4) - f(0)}{4 - 0} \\ \Rightarrow 3c^2 + 16c + 16 &= \frac{4 \times (8)^2 - 0}{4} \\ \Rightarrow 3c^2 + 16c + 16 &= 64 \\ \Rightarrow 3c^2 + 16c - 48 &= 0 \\ \Rightarrow c &= \frac{-16 \pm \sqrt{256 + 576}}{6} \\ \Rightarrow &= \frac{-16 \pm \sqrt{832}}{6} \\ \Rightarrow &= \frac{-16 \pm 8\sqrt{13}}{6} \\ \Rightarrow c &= \frac{-8 \pm 4\sqrt{13}}{3} \\ c &= \frac{-8 + 4\sqrt{13}}{3} \in (0, 4) \end{aligned}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(xiii)

Here,

$$f(x) = x\sqrt{x^2 - 4} \text{ on } [2, 4]$$

$f(x)$ is continuous as it attains a unique value for each $x \in [2, 4]$ and

$$\begin{aligned} f'(x) &= \frac{2x}{2\sqrt{x^2 - 4}} \\ f'(x) &= \frac{x}{\sqrt{x^2 - 4}} \\ \Rightarrow f'(x) &\text{ exists for each } x \in (2, 4) \\ \Rightarrow f(x) &\text{ is differentiable in } (2, 4), \text{ so} \end{aligned}$$

Lagrange's mean value theorem is applicable, so there exist a $c \in (2, 4)$ such that

$$\begin{aligned} f'(c) &= \frac{f(4) - f(2)}{4 - 2} \\ \Rightarrow \frac{c}{\sqrt{c^2 - 4}} &= \frac{\sqrt{12} - 0}{2} \end{aligned}$$

Squaring both the sides,

$$\begin{aligned} \Rightarrow \frac{c^2}{c^2 - 4} &= \frac{12}{4} \\ \Rightarrow 4c^2 &= 12c^2 - 48 \\ \Rightarrow 8c^2 &= 48 \\ \Rightarrow c^2 &= 6 \\ \Rightarrow c &= \sqrt{6} \in (2, 4) \end{aligned}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(xiv)

Here,

$$f(x) = x^2 + x - 1 \text{ on } [0, 4]$$

$f(x)$ is polynomial, so it is continuous in $[0, 4]$ and differentiable in $(0, 4)$

as every polynomial is continuous and differentiable every where. So,

Lagrange's mean value theorem is applicable, so there exists a point $c \in [0, 4]$ such that

$$\begin{aligned} f'(c) &= \frac{f(4) - f(0)}{4 - 0} \\ \Rightarrow 2c + 1 &= \frac{((4)^2 + 4 - 1) - (0 - 1)}{4} \\ \Rightarrow 2c + 1 &= \frac{19 + 1}{4} \\ \Rightarrow 2c + 1 &= 5 \\ \Rightarrow c &= 2 \in (0, 4) \end{aligned}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(xv)

Here,

$$f(x) = \sin x - \sin 2x - x \text{ on } [0, \pi]$$

We know that $\sin x$ and polynomial is continuous and differentiable every where so,

$f(x)$ is continuous in $[0, \pi]$ and differentiable in $(0, \pi)$. So, Lagrange's mean value theorem is applicable. So, there exist a point $c \in (0, \pi)$ such that

$$\begin{aligned} f'(c) &= \frac{f(\pi) - f(0)}{\pi - 0} \\ \Rightarrow \cos c - 2 \cos 2c - 1 &= \frac{(\sin \pi - \sin 2\pi - \pi) - (0)}{\pi} \\ \Rightarrow \cos c - 2 \cos 2c &= -1 + 1 \\ \Rightarrow \cos c - 2(2 \cos^2 c - 1) &= 0 \\ \Rightarrow 4 \cos^2 c - \cos c - 2 &= 0 \\ \Rightarrow \cos c &= \frac{-(-1) \pm \sqrt{1 - 4 \times 4 \times (-2)}}{8} \\ \Rightarrow \cos c &= \frac{1 \pm \sqrt{33}}{8} \\ \Rightarrow c &= \cos^{-1} \left(\frac{1 \pm \sqrt{33}}{8} \right) \in (0, \pi) \end{aligned}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(xvi)

The given function is $f(x) = x^3 - 5x^2 - 3x$, f being a polynomial function, is continuous in $[1, 3]$ and is differentiable in $(1, 3)$ whose derivative is $3x^2 - 10x - 3$.

$$f(1) = 1^3 - 5(1)^2 - 3(1) = -7$$

$$f(3) = 3^3 - 5(3)^2 - 3(3) = 27 - 45 - 9 = -27$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{-27 + 7}{2} = -10$$

Mean value theorem states that there is a point $c \in (1, 3)$ such that $f'(c) = 3c^2 - 10c - 3$

$$\begin{aligned} f'(c) &= -10 \\ 3c^2 - 10c - 3 &= -10 \\ 3c^2 - 10c + 7 &= 0 \\ 3c^2 - 3c - 7c + 7 &= 0 \\ c &= \frac{7}{3}, \text{ where } c = \frac{7}{3} \in (1, 3) \end{aligned}$$

Hence, Mean value theorem is verified for the given function.

Mean Value Theorems Ex 15.2 Q2

Here,

$$f(x) = |x| \text{ on } [-1, 1]$$

$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

For differentiability at $x = 0$

$$\begin{aligned} \text{LHD} &= \lim_{x \rightarrow 0^-} \frac{f(0-h) - f(0)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{-(0-h) - 0}{-h} \\ &= \lim_{h \rightarrow 0} \frac{h}{-h} \\ \text{LHD} &= -1 \end{aligned}$$

$$\begin{aligned} \text{RHD} &= \lim_{x \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(0+h) - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} \\ &= 1 \end{aligned}$$

$\therefore$ LHD $\neq$ RHD

$\Rightarrow f(x)$ is not differentiable at $x = 0 \in (-1, 1)$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q3

Here,

$$f(x) = \frac{1}{x} \text{ on } [-1, 1]$$

$$f'(x) = -\frac{1}{x^2}$$

$\Rightarrow f'(x)$ does not exist at $x = 0 \in (-1, 1)$

$\Rightarrow f(x)$ is not differentiable in $(-1, 1)$

Hence, LMVT is verified

Mean Value Theorems Ex 15.2 Q4

Here,

$$f(x) = \frac{1}{4x-1}, x \in [1, 4]$$

$f(x)$ attain unique value for each $x \in [1, 4]$, so $f(x)$ is continuous in $[1, 4]$.

$$f'(x) = -\frac{4}{(4x-1)^2}$$

$\Rightarrow f'(x)$ exists for each $x \in (1, 4)$

$\Rightarrow f'(x)$ is differentiable in $(1, 4)$

So, Lagrange's mean value theorem is applicable.

So, there exist a point $c \in (1, 4)$ such that,

$$\begin{aligned} f'(c) &= \frac{f(4) - f(1)}{4 - 1} \\ \Rightarrow -\frac{4}{(4x-1)^2} &= \frac{\frac{1}{15} - \frac{1}{3}}{3} \\ \Rightarrow -\frac{4}{(4x-1)^2} &= -\frac{4}{45} \\ \Rightarrow (4x-1)^2 &= 45 \\ \Rightarrow 4x-1 &= \pm 3\sqrt{5} \\ \Rightarrow x &= \frac{3\sqrt{5}+1}{4} \in [1, 4] \end{aligned}$$

Mean Value Theorems Ex 15.2 Q5

Here,

$$\text{curve is } y = (x - 4)^2$$

Since, it a polynomial function so it is differentiable and continuous. So, it Lagrange's mean value theorem is applicable, so, there exist a point c such that,

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ \Rightarrow 2(c - 4) &= \frac{f(5) - f(4)}{5 - 4} \\ \Rightarrow 2c - 8 &= \frac{1 - 0}{1} \\ \Rightarrow 2c &= 9 \\ \Rightarrow c &= \frac{9}{2} \\ \Rightarrow y &= \left(\frac{9}{2} - 4\right)^2 \\ y &= \frac{1}{4} \end{aligned}$$

Thus, $(c, y) = \left(\frac{9}{2}, \frac{1}{4}\right)$ is required point.

Mean Value Theorems Ex 15.2 Q6

Here,

$$y = x^2 + x$$

Since, y is a polynomial function, so it continuous differentiable,
 $\Rightarrow$ Lagrange's mean value theorem is applicable, so, there exist a point c such that,

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ \Rightarrow 2c + 1 &= \frac{f(1) - f(0)}{1 - 0} \\ \Rightarrow 2c + 1 &= 2 \\ \Rightarrow c &= \frac{1}{2} \\ \Rightarrow y &= \left(\frac{1}{2}\right)^2 + \frac{1}{2} \\ \Rightarrow y &= \frac{3}{4} \end{aligned}$$

So, $(c, y) = \left(\frac{1}{2}, \frac{3}{4}\right)$ is the required point.

Mean Value Theorems Ex 15.2 Q7

Here,

$$y = (x - 3)^2$$

Since, y is a polynomial function, so it continuous differentiable,

$\Rightarrow$ Lagrange's mean value theorem is applicable
 $\Rightarrow$ There exist a point c such that,

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ \Rightarrow 2(c - 3) &= \frac{f(4) - f(3)}{4 - 3} \\ \Rightarrow 2c - 6 &= \frac{1 - 0}{1} \\ \Rightarrow 2c &= 7 \\ \Rightarrow c &= \frac{7}{2} \\ \Rightarrow y &= \left(\frac{7}{2} - 3\right)^2 \\ \Rightarrow y &= \frac{1}{4} \end{aligned}$$

So, $(c, y) = \left(\frac{7}{2}, \frac{1}{4}\right)$ is the required point.

Mean Value Theorems Ex 15.2 Q8

Here,

$$y = x^3 - 3x$$

y is a polynomial function, so it is continuous differentiable, so

Lagrange's mean value theorem is applicable thus there exists a point c such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow 3c^2 - 3 = \frac{f(2) - f(1)}{2 - 1}$$

$$\Rightarrow 3c^2 - 3 = \frac{2 + 2}{1}$$

$$\Rightarrow 3c^2 = 7$$

$$\Rightarrow c = \pm \sqrt{\frac{7}{3}}$$

$$\Rightarrow y = \mp \frac{2}{3} \sqrt{\frac{7}{3}}$$

So, $(c, y) = \left(\pm \sqrt{\frac{7}{3}}, \mp \frac{2}{3} \sqrt{\frac{7}{3}} \right)$ is the required point.

Mean Value Theorems Ex 15.2 Q9

Here,

$$y = x^3 + 1$$

It is a polynomial function, so it is continuous differentiable.

$\Rightarrow$ Lagrange's mean value theorem is applicable, so there exists a point c such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow 3c^2 = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow 3c^2 = \frac{28 - 2}{2}$$

$$\Rightarrow c^2 = \frac{13}{3}$$

$$\Rightarrow c = \sqrt{\frac{13}{3}}$$

$$\Rightarrow y = \left(\frac{13}{3} \right)^{\frac{3}{2}} + 1$$

So, $(c, y) = \left(\sqrt{\frac{13}{3}}, \left(\frac{13}{3} \right)^{\frac{3}{2}} + 1 \right)$ is the required point.

Mean Value Theorems Ex 15.2 Q10

Trigonometric functions are continuous and differentiable.

Thus, the curve C is continuous between the points $(a,0)$ and $(0,a)$ and is differentiable on $[a,a]$

Therefore, by Lagrange's Mean Value Theorem, there exists a real number $c \in (a,a)$ such that

$$f'(c) = \frac{a-0}{0-a} = -1$$

Now consider the parametric functions of the given function

$$x = a \cos^3 \theta$$

and

$$y = a \sin^3 \theta$$

$$\Rightarrow \frac{dx}{d\theta} = 3a \cos^2 \theta (-\sin \theta)$$

and

$$\Rightarrow \frac{dy}{d\theta} = 3a \sin^2 \theta (\cos \theta)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3a \sin^2 \theta (\cos \theta)}{3a \cos^2 \theta (-\sin \theta)}$$

$$\Rightarrow \frac{dy}{dx} = -\tan \theta$$

Slope of the chord joining the points $(a,0)$ and $(0,a)$

= Slope of the tangent at $(c, f(c))$, where c lies on the curve

$$\Rightarrow \frac{a-0}{0-a} = -\tan \theta$$

$$\Rightarrow -1 = -\tan \theta$$

$$\Rightarrow \tan \theta = 1$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Now substituting $\theta = \frac{\pi}{4}$, in the

parametric representations, we have,

$$x = a \cos^3 \theta, y = a \sin^3 \theta$$

$$\Rightarrow x = a \cos^3 \left(\frac{\pi}{4} \right), y = a \sin^3 \left(\frac{\pi}{4} \right)$$

$$\Rightarrow x = \frac{a}{2\sqrt{2}}, y = \frac{a}{2\sqrt{2}}$$

Thus, $P\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$ is a point on C, where the tangent

is parallel to the chord joining the points $(a,0)$ and $(0,a)$.

Mean Value Theorems Ex 15.2 Q11

Consider the function as

$$f(x) = \tan x, \quad \left\{ x \in [a, b] \text{ such that } 0 < a < b < \frac{\pi}{2} \right\}$$

We know that $\tan x$ is continuous and differentiable in $\left(0, \frac{\pi}{2}\right)$, so, Lagrange's mean value theorem is applicable on (a, b) , so there exists a point c such that,

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ \Rightarrow \sec^2 c &= \frac{\tan b - \tan a}{b - a} \quad \text{---(i)} \end{aligned}$$

Now,

$$\begin{aligned} c &\in (a, b) \\ \Rightarrow a &< c < b \\ \Rightarrow \sec^2 a &< \sec^2 c < \sec^2 b \\ \Rightarrow \sec^2 a &< \left(\frac{\tan b - \tan a}{b - a} \right) < \sec^2 b \end{aligned}$$

Using equation (i),

$$\Rightarrow (b - a) \sec^2 a < (\tan b - \tan a) < (b - a) \sec^2 b$$