2 Integral Calculus – I

Bernhard Riemann
(17th September 1826 - 20th July 1866)

Introduction

Georg Friedrich Bernhard Riemann (Nineteenth century) was an inspiring German mathematician. He was very much recognised for his contribution in calculus.

Calculus divides naturally into two parts, namely (i) differential calculus and (ii) integral calculus. Differential calculus deals with the derivatives of a function whereas, integral calculus deals with the anti derivative of the derived function that is, finding a function when its rate of change / marginal function is known. So integration is the technique to find the original function from the derived function, the function obtained is called the indefinite integral. Definite integral is the evaluation of the indefinite integral between the specified limits, and is equal to the area bounded by the graph of the function (curve) between the specified limits and the axis. The area under the curve is approximately equal to the area obtained by summing the area of the number of inscribed rectangles and the approximation becomes exact in the limit that the number of rectangles approaches infinity. Therefore both differential and integral calculus are based on the theory of limits.

The word 'integrate' literally means that 'to find the sum'. So, we believe that the name "Integral Calculus" has its origin from this process of summation. Calculus is the mathematical tool used to test theories about the origins of the universe, the development of tornadoes and hurricanes. It is also used to find the surplus of consumer and producer, identifying the probability density function of a continuous random variable, obtain an original function from its marginal function and etc., in business applications.

In this chapter, we will study about the concept of integral and some types of method of indefinite and definite integrals.

Learning Objectives

After studying this chapter, the students will be able to understand

- the indefinite integral.
- how to find the indefinite integral of a function involving sum, difference and constant multiples.
- how to use and where to apply the substitution technique in indefinite integrals.
- the techniques involved in integration by parts and some special type of integrals.
- the fundamental theorems of integral calculus.
- the properties of definite integral and its applications.
- the application of a particular case of gamma integral.
- the properties of gamma function.
- the evaluation of the definite integral as the limit of a sum.

2.1 Indefinite Integrals

2.1.1 Concept of Indefinite Integral

In differential calculus, we have learned how to calculate the differential coefficient $f'(x)$ of a given function $f(x)$ with respect to x. In this chapter, we have to find out the primitive function $f(x)$ (i.e. original function) whenever its derived function $f'(x)$ (i.e. derivative of a function) is given, such process is called integration or anti differentiation.

$\therefore$ Integration is the reverse process of differentiation

We know that $\frac{d}{dx}(\sin x) = \cos x$. Here $\cos x$ is known as **Derived function**, and $\sin x$ is known as **Primitive function** [also called as Anti derivative function (or) Integral function].

Definition 2.1

A function $F(x)$ is said to be a primitive function of the derived function $f(x)$, if $\frac{d}{dx}[F(x)] = f(x)$

Now, consider the following examples which are already known to us.

$\frac{d}{dx}(x^3) = 3x^2$, $\quad \frac{d}{dx}(x^3+5) = 3x^2$,

$\frac{d}{dx}(x^3 - \frac{3}{2}) = 3x^2$, $\quad \frac{d}{dx}(x^3+e) = 3x^2$,

$\frac{d}{dx}(x^3-\pi) = 3x^2$, $\quad$ ...

From the above examples, we observe that $3x^2$ is the derived function of the primitive functions x^3, x^3+5, $x^3-\frac{3}{2}$, x^3+e, $x^3-\pi$, ... and which indicates that the primitive functions are need not be unique, even though the derived function is unique. So we come to a conclusion that x^3+c is the primitive function of the derived function $3x^2$.

$\therefore$ For every derived function, there are infinitely many primitives by choosing c arbitrarily from the set of real numbers $\mathbb{R}$. So we called these integrals as indefinite integrals.

In general,

$$\frac{d}{dx}[F(x)] = f(x) \Rightarrow \int f(x)\,dx = F(x)+c,$$

where c is called the constant of integration.

Remarks

- If two different primitive functions $F(x)$ and $G(x)$ have the same derived function $f(x)$, then they differ only by a constant term.
- $\int f(x)\,dx$ is called as indefinite integral.

- The symbol looks like an elongated S [$\int$], which stands for "summation" is the sign of integration.
- $\int f(x)\,dx$ is read as integral of $f(x)$ with respect to x.
- $f(x)$ in $\int f(x)\,dx$ [i.e. the function to be integrated] is called as integrand.
- x in $\int f(x)\,dx$ is the variable of integration.
- The term integration means the process of finding the integral.
- The term constant of integration means any real number c, considered as a constant function.

Definition 2.2

The process of determining an integral of a given function is defined as integration of a function.

2.1.2 Two important properties of Integral Calculus

(i) if k is any constant, then $\int k\,f(x)\,dx = k\int f(x)\,dx$

(ii) if $f(x)$ and $g(x)$ are any two functions, then

$$\int\left[f(x)\pm g(x)\right]dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Methods of Integration

The following are the four principal methods of integration.

(i) Integration by decomposition.
(ii) Integration by parts.
(iii) Integration by substitution.
(iv) Integration by successive reduction.

Remember !

Integrate a function $f(x)$ means, finding a function $F(x)$ such that $\frac{d}{dx}\left[F(x)+c\right]=f(x)$

Note

Here, we discuss only the first three methods of integration, because the method of integration by successive reduction is beyond the scope of the syllabus.

2.1.3 Integration by decomposition

Apply this method, whenever the integrand is impossible to integrate directly, but it can be integrated after decomposing into a sum or difference of integrands whose individual integrals are already known.

Note

The integrals which are directly obtained from their corresponding derivatives are known as the standard results of integration.

Some standard results of integration

Type: I

(i) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1}+c,\ \ n\neq -1$

(ii) $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)}+c,\ n\neq -1$

DO YOU KNOW?

$y = \int f(x)\,dx = F(x)+c$ denotes family of curves having parallel tangents at $x=k$

Example 2.1

Evaluate $\int \dfrac{ax^2+bx+c}{\sqrt{x}}\,dx$

Solution:

$$\int \frac{ax^2+bx+c}{\sqrt{x}}\,dx$$

$$= \int\left(ax^{\frac{3}{2}}+bx^{\frac{1}{2}}+cx^{-\frac{1}{2}}\right)dx$$

$$= a\int x^{\frac{3}{2}}dx + b\int x^{\frac{1}{2}}dx + c\int x^{-\frac{1}{2}}dx$$

$$= \frac{2ax^{\frac{5}{2}}}{5}+\frac{2bx^{\frac{3}{2}}}{3}+2c\,x^{\frac{1}{2}}+k$$

DO YOU KNOW?

$\int dx = x+c$

Example 2.2

Evaluate $\int \sqrt{2x+1}\,dx$

Solution:

$$\int \sqrt{2x+1}\,dx = \int (2x+1)^{\frac{1}{2}}dx$$

$$= \frac{(2x+1)^{\frac{3}{2}}}{3} + c$$

DO YOU KNOW?

$$\int a(ax+b)^n\,dx = \int (ax+b)^n\,d(ax+b)$$

$$= \frac{(ax+b)^{n+1}}{n+1} + c$$

Example 2.3

Evaluate $\int \frac{dx}{(2x+3)^2}$

Solution:

$$\int \frac{dx}{(2x+3)^2} = \int (2x+3)^{-2}dx$$

$$= -\frac{1}{2(2x+3)} + c$$

DO YOU KNOW?

(i) $\int \frac{1}{x^n}dx = -\frac{1}{(n-1)x^{n-1}} + c\ ,\ n \neq 1$

(ii) $\int \frac{a}{(ax+b)^n}dx = \int \frac{d(ax+b)}{(ax+b)^n}$

$$= -\frac{1}{(n-1)(ax+b)^{n-1}} + c\ ,\ n \neq 1$$

Example 2.4

Evaluate $\int \left(x+\frac{1}{x}\right)^2 dx$

Solution:

$$\int \left(x+\frac{1}{x}\right)^2 dx = \int \left(x^2+2+\frac{1}{x^2}\right) dx$$

$$= \int x^2dx + 2\int dx + \int \frac{1}{x^2}dx$$

$$= \frac{x^3}{3} + 2x - \frac{1}{x} + c$$

Example 2.5

Evaluate $\int (x^3+7)(x\ -4)\,dx$

Solution:

$$\int (x^3+7)(x\ -4)\,dx$$

$$= \int (x^4 - 4x^3 + 7x - 28)\,dx$$

$$= \frac{x^5}{5} - x^4 + \frac{7x^2}{2} - 28x + c$$

Example 2.6

Evaluate $\int \frac{2x^2-14x+24}{x-3}\,dx$

Solution:

By factorisation,

$$2x^2 - 14x + 24 = (x-3)(2x-8)$$

$$\int \frac{2x^2-14x+24}{x-3}\,dx = \int \frac{(x-3)(2x-8)}{x-3}\,dx$$

$$= \int (2x-8)\,dx$$

$$= x^2 - 8x + c$$

Example 2.7

Evaluate $\int \frac{x+2}{\sqrt{2x+3}}\,dx$

Solution:

Split into simple integrands

$$\frac{x+2}{\sqrt{2x+3}} = \frac{\frac{1}{2}(2x+4)}{(2x+3)^{\frac{1}{2}}}$$

$$= \frac{1}{2}\left\{\frac{(2x+3)+1}{(2x+3)^{\frac{1}{2}}}\right\}$$

$$= \frac{1}{2}\left\{(2x+3)^{\frac{1}{2}} + (2x+3)^{-\frac{1}{2}}\right\}$$

$$\int \frac{x+2}{\sqrt{2x+3}}dx = \int \frac{1}{2}\left\{(2x+3)^{\frac{1}{2}} + (2x+3)^{-\frac{1}{2}}\right\}dx$$

$$= \frac{1}{2}\left\{\frac{(2x+3)^{\frac{3}{2}}}{3} + (2x+3)^{\frac{1}{2}}\right\} + c$$

Example 2.8

Evaluate $\int \frac{1}{\sqrt{x+2}-\sqrt{x-2}}dx$

Solution:

> By rationalisation,
>
> $$\frac{1}{\sqrt{x+2}-\sqrt{x-2}} = \frac{1}{\sqrt{x+2}-\sqrt{x-2}} \times \frac{\sqrt{x+2}+\sqrt{x-2}}{\sqrt{x+2}+\sqrt{x-2}}$$
>
> $$= \frac{\sqrt{x+2}+\sqrt{x-2}}{4}$$

$$\int \frac{1}{\sqrt{x+2}-\sqrt{x-2}}dx$$

$$= \int \frac{\sqrt{x+2}+\sqrt{x-2}}{4}dx$$

$$= \frac{1}{6}\left\{(x+2)^{\frac{3}{2}} + (x-2)^{\frac{3}{2}}\right\} + c$$

Exercise 2.1

Integrate the following with respect to x.

1. $\sqrt{3x+5}$
2. $\left(9x^2 - \frac{4}{x^2}\right)^2$
3. $(3+x)(2-5x)$
4. $\sqrt{x}\,(x^3 - 2x + 3)$
5. $\frac{8x+13}{\sqrt{4x+7}}$
6. $\frac{1}{\sqrt{x+1}+\sqrt{x-1}}$
7. If $f'(x) = x+b$, $f(1) = 5$ and $f(2) = 13$, then find $f(x)$
8. If $f'(x) = 8x^3 - 2x$ and $f(2) = 8$, then find $f(x)$

Type: II

(i) $\int \frac{1}{x}dx = \log|x| + c$

(ii) $\int \frac{1}{ax+b}dx = \frac{1}{a}\log|ax+b| + c$

Example 2.9

Evaluate $\int \frac{3x^2+2x+1}{x}dx$

Solution:

$$\int \frac{3x^2+2x+1}{x}dx = \int\left(3x+2+\frac{1}{x}\right)dx$$

$$= \frac{3x^2}{2} + 2x + \log|x| + c$$

Example 2.10

Evaluate $\int \frac{2}{3x+5}dx$

Solution:

$$\int \frac{2}{3x+5}dx = 2\int \frac{1}{3x+5}dx$$

$$= \frac{2}{3}\log|3x+5| + c$$

> **DO YOU KNOW?**
>
> $$\int \frac{a}{ax+b}dx = \int \frac{d(ax+b)}{ax+b}$$
>
> $$= \log|ax+b| + c$$

Example 2.11

Evaluate $\int \frac{x^2+2x+3}{x+1}dx$

Solution:

> Split into simple integrands
>
> $$\frac{x^2+2x+3}{x+1} = \frac{(x^2+2x+1)+2}{x+1}$$
>
> $$= (x+1) + \frac{2}{x+1}$$

$$\int \frac{x^2+2x+3}{x+1}dx = \int\left\{(x+1) + \frac{2}{x+1}\right\}dx$$

$$= \frac{x^2}{2} + x + 2\log|x+1| + c$$

Example 2.12

Evaluate $\int \frac{x^3+5x^2-9}{x+2}dx$

Solution:

> By simple division,
>
> $$\frac{x^3+5x^2-9}{x+2} = x^2 + 3x - 6 + \frac{3}{x+2}$$

$$\int \frac{x^3+5x^2-9}{x+2}dx = \int\left[x^2+3x-6+\frac{3}{x+2}\right]dx$$

$$= \frac{x^3}{3}+\frac{3x^2}{2}-6x+3\log|x+2|+c$$

Example 2.13

Evaluate $\int \frac{7x-1}{x^2-5x+6}dx$

Solution:

By partial fractions,

$$\frac{7x-1}{x^2-5x+6} = \frac{A}{x-3}+\frac{B}{x-2}$$

$$\Rightarrow \frac{7x-1}{x^2-5x+6} = \frac{20}{x-3}-\frac{13}{x-2}$$

$$\int \frac{7x-1}{x^2-5x+6}dx = \int\left[\frac{20}{x-3}-\frac{13}{x-2}\right]dx$$

$$= 20\int\frac{dx}{x-3}-13\int\frac{dx}{x-2}$$

$$= 20\log|x-3|-13\log|x-2|+c$$

Example 2.14

Evaluate $\int \frac{3x+2}{(x-2)^2(x-3)}dx$

Solution:

By partial fractions,

$$\frac{3x+2}{(x-2)^2(x-3)} = \frac{A}{(x-2)}+\frac{B}{(x-2)^2}+\frac{C}{(x-3)}$$

$$\Rightarrow \frac{3x+2}{(x-2)^2(x-3)} = -\frac{11}{(x-2)}-\frac{8}{(x-2)^2}+\frac{11}{(x-3)}$$

$$\int \frac{3x+2}{(x-2)^2(x-3)}dx$$

$$= \int\left[-\frac{11}{(x-2)}-\frac{8}{(x-2)^2}+\frac{11}{(x-3)}\right]dx$$

$$= -11\int\frac{dx}{(x-2)}-8\int\frac{dx}{(x-2)^2}+11\int\frac{dx}{(x-3)}$$

$$= -11\log|x-2|+\frac{8}{x-2}+11\log|x-3|+c$$

$$= 11\log\left|\frac{x-3}{x-2}\right|+\frac{8}{x-2}+c$$

Example 2.15

Evaluate $\int \frac{3x^2+6x+1}{(x+3)(x^2+1)}dx$

Solution:

By partial fractions,

$$\frac{3x^2+6x+1}{(x+3)(x^2+1)} = \frac{A}{(x+3)}+\frac{Bx+C}{(x^2+1)}$$

$$\Rightarrow \frac{3x^2+6x+1}{(x+3)(x^2+1)} = \frac{1}{(x+3)}+\frac{2x}{(x^2+1)}$$

$$\int \frac{3x^2+6x+1}{(x+3)(x^2+1)}dx$$

$$= \int\left[\frac{1}{(x+3)}+\frac{2x}{(x^2+1)}\right]dx$$

$$= \int\frac{dx}{(x+3)}+\int\frac{2x}{(x^2+1)}dx$$

$$= \log|x+3|+\log|x^2+1|+c$$

$$= \log|(x+3)(x^2+1)|+c$$

$$= \log|x^3+3x^2+x+3|+c$$

Exercise 2.2

Integrate the following with respect to x.

1. $\left(\sqrt{2x}-\frac{1}{\sqrt{2x}}\right)^2$
2. $\frac{x^4-x^2+2}{x-1}$
3. $\frac{x^3}{x+2}$
4. $\frac{x^3+3x^2-7x+11}{x+5}$
5. $\frac{3x+2}{(x-2)(x-3)}$
6. $\frac{4x^2+2x+6}{(x+1)^2(x-3)}$
7. $\frac{3x^2-2x+5}{(x-1)(x^2+5)}$
8. If $f'(x)=\frac{1}{x}$ and $f(1)=\frac{\pi}{4}$, then find $f(x)$

Type: III

(i) $\int e^x \, dx = e^x + c$

(ii) $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$

(iii) $\int a^x \, dx = \frac{1}{\log a} a^x + c, \ a > 0$ and $a \neq 1$

(iv) $\int a^{mx+n} dx = \frac{1}{m \log a} a^{mx+n} + c, \ a > 0$ and $a \neq 1$

Example 2.16

Evaluate $\int 3^{2x+3} \, dx$

Solution:

$$\int 3^{2x+3} \, dx = \int 3^{2x} \cdot 3^3 \, dx$$
$$= 3^3 \int 3^{2x} \, dx$$
$$= 27 \frac{3^{2x}}{2 \log 3} + c$$

DO YOU KNOW?

$$\int m a^{mx+n} \, dx = \int a^{mx+n} \, d(mx+n)$$
$$= \frac{1}{\log a} a^{mx+n} + c, \quad a > 0 \text{ and } a \neq 1$$

Example 2.17

Evaluate $\int \frac{e^x + 7}{e^x} dx$

Solution:

$$\int \frac{e^x + 7}{e^x} dx = \int \left(1 + 7e^{-x}\right) dx$$
$$= x - 7e^{-x} + c$$

DO YOU KNOW?

$$\int (2ax + b) e^{ax^2+bx+c} \, dx$$
$$= \int e^{ax^2+bx+c} \, d\left(ax^2 + bx + c\right)$$
$$= e^{ax^2+bx+c} + k$$

Example 2.18

Evaluate $\int \frac{5 + 5e^{2x}}{e^x + e^{-x}} dx$

Solution:

$$\int \frac{5 + 5e^{2x}}{e^x + e^{-x}} dx = 5 \int \frac{e^x \left(e^{-x} + e^x\right)}{e^x + e^{-x}} dx$$
$$= 5 \int e^x dx$$
$$= 5e^x + c$$

Example 2.19

Evaluate $\int \left(e^x + \frac{1}{e^x}\right)^2 dx$

Solution:

$$\int \left(e^x + \frac{1}{e^x}\right)^2 dx = \int \left(e^{2x} + \frac{1}{e^{2x}} + 2\right) dx$$
$$= \int \left(e^{2x} + e^{-2x} + 2\right) dx$$
$$= \frac{e^{2x}}{2} - \frac{e^{-2x}}{2} + 2x + c$$

Exercise 2.3

Integrate the following with respect to x.

1. $e^{x \log a} + e^{a \log a} - e^{n \log x}$
2. $\frac{a^x - e^{x \log b}}{e^{x \log a} \ b^x}$
3. $\left(e^x + 1\right)^2 e^x$
4. $\frac{e^{3x} - e^{-3x}}{e^x}$
5. $\frac{e^{3x} + e^{5x}}{e^x + e^{-x}}$
6. $\left[1 - \frac{1}{x^2}\right] e^{\left(x + \frac{1}{x}\right)}$
7. $\frac{1}{x (\log x)^2}$
8. If $f'(x) = e^x$ and $f(0) = 2$, then find $f(x)$

Type: IV

(i) $\int \sin x \, dx = -\cos x + c$

(ii) $\int \sin(ax+b) \, dx = -\frac{1}{a} \cos\left(ax+b\right) + c$

(iii) $\int \cos x \, dx = \sin x + c$

(iv) $\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + c$

(v) $\int \sec^2 x \, dx = \tan x + c$

(vi) $\int \sec^2(ax+b) \, dx = \frac{1}{a} \tan(ax+b) + c$

(vi) $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

(viii) $\int \operatorname{cosec}^2 \left(ax+b\right) dx = -\frac{1}{a} \cot\left(ax+b\right) + c$

Example 2.20

Evaluate $\int \left(2 \sin x - 5 \cos x\right) dx$

Solution:

$$\int \left(2 \sin x - 5 \cos x\right) dx = 2 \int \sin x \, dx - 5 \int \cos x \, dx$$
$$= -2 \cos x - 5 \sin x + c$$

Example 2.21

Evaluate $\int \sin^2 x\,dx$

Solution:

> Change into simple integrands
>
> $\cos 2x = 1 - 2\sin^2 x$
>
> $\therefore \sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx$$

$$= \frac{1}{2}\left[\int dx - \int \cos 2x\,dx\right]$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$$

Example 2.22

Evaluate $\int \frac{\cos 2x}{\sin^2 x \cos^2 x}\,dx$

Solution:

> Change into simple integrands
>
> $\frac{\cos 2x}{\sin^2 x\cos^2 x} = \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cos^2 x}$
>
> $= \frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}$
>
> $= \operatorname{cosec}^2 x - \sec^2 x$

$$\int \frac{\cos 2x}{\sin^2 x\cos^2 x}\,dx = \int \left(\operatorname{cosec}^2 x - \sec^2 x\right)dx$$

$$= -\cot x - \tan x + c$$

Example 2.23

Evaluate $\int \sqrt{1+\sin 2x}\,dx$

Solution:

> Change into simple integrands
>
> $1+\sin 2x = \sin^2 x + \cos^2 x + 2\sin x\cos x$
>
> $= \left(\sin x + \cos x\right)^2$

$$\int \sqrt{1+\sin 2x}\,dx = \int \sqrt{\left(\sin x + \cos x\right)^2}\,dx$$

$$= \int \left(\sin x + \cos x\right)dx$$

$$= -\cos x + \sin x + c$$

Example 2.24

Evaluate $\int \cos^3 x\,dx$

Solution:

> Change into simple integrands
>
> $\cos 3x = 4\cos^3 x - 3\cos x$
>
> $\cos^3 x = \frac{1}{4}\left[\cos 3x + 3\cos x\right]$
>
> $= \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$

$$\int \cos^3 x\,dx$$

$$= \frac{1}{4}\int \cos 3x\,dx + \frac{3}{4}\int \cos x\,dx$$

$$= \frac{\sin 3x}{12} + \frac{3\sin x}{4} + c$$

Exercise 2.4

Integrate the following with respect to x.

1. $2\cos x - 3\sin x + 4\sec^2 x - 5\operatorname{cosec}^2 x$
2. $\sin^3 x$
3. $\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$
4. $\frac{1}{\sin^2 x\cos^2 x}$ $\left[\text{Hint}: \sin^2 x + \cos^2 x = 1\right]$
5. $\sqrt{1-\sin 2x}$

2.1.4 Integration by parts

Type: V

(1) We know that, if u and v are two differentiable functions of x, then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrate on both sides with respect to x.

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$\Rightarrow \int d(uv) = \int u\,dv + \int v\,du$$

$$uv = \int udv + \int vdu$$

$$\therefore \int udv = uv - \int vdu$$

This method is very useful when the integrand is a product of two different types of functions or a function which is not directly integrable. The success of this method depends on the proper choice of u. So we can choose the function u by using the following guidelines.

(i) If the integrand contains only a function which is directly not integrable, then take this as u.

(ii) If the integrand contains both directly integrable and non integrable functions, then take non integrable function as u.

(iii) If the integrand contains both the functions are integrable and one of them is of the form x^n, n is a positive integer, then take this x^n as u.

(iv) for all other cases, the choice of u is ours (Or) we can also choose u as the function which comes first in the word **"I L A T E"** Where,

I stands for the inverse trigonometric function

L stands for the logarithmic function

A stands for the algebraic function

T stands for the trigonometric function

E stands for the exponential function and take the remaining part of the function and dx as dv.

(2) If u and v are functions of x, then $\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$

where $u', u'', u''', \ldots$ are the successive derivatives of u and $v_1, v_2, v_3, \ldots$ are the repeated integrals of v.

Note

- The above mentioned formula is well known as Bernoulli's formula.
- Bernoulli's formula is applied when $u = x^n$ where n is a positive integer.

Example 2.25

Evaluate $\int xe^x \, dx$

Solution:

Take $u = x$ Differentiate $du = dx$	and $dv = e^x dx$ Integrate $v = e^x$

$$\begin{aligned} \int xe^x \, dx &= \int udv \\ &= uv - \int vdu \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + c \\ &= e^x (x-1) + c \end{aligned}$$

Example 2.26

Evaluate $\int x^3 e^x dx$

Solution:

Successive derivatives	Repeated integrals
Take $u = x^3$ $u' = 3x^2$ $u'' = 6x$ $u''' = 6$	and $dv = e^x dx$ $v = e^x$ $v_1 = e^x$ $v_2 = e^x$ $v_3 = e^x$

$$\int x^3 e^x dx = \int udv$$

$$= uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$$

$$= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + c$$

$$= e^x \left(x^3 - 3x^2 + 6x - 6\right) + c$$

Example 2.27

Evaluate $\int x^3 \log x \, dx$

Solution:

Take $u = \log x$ Differentiate $du = \frac{1}{x} dx$	and $dv = x^3 dx$ Integrate $v = \frac{x^4}{4}$

$$\int x^3 \log x \; dx = \int u dv$$
$$= uv - \int v du$$
$$= \frac{x^4}{4} \log x - \frac{1}{4} \int x^3 \, dx$$
$$= \frac{x^4}{4} \log x - \frac{1}{4}\left(\frac{x^4}{4}\right) + c$$
$$= \frac{x^4}{4}\left[\log x - \frac{1}{4}\right] + c$$

Example 2.28

Evaluate $\int (\log x)^2 \; dx$

Solution:

Take $u = (\log x)^2$ Differentiate $du = (2\log x)\left(\frac{1}{x} dx\right)$	and $dv = dx$ Integrate $v = x$

$$\int (\log x)^2 \; dx = \int u dv$$
$$= uv - \int v du$$
$$= x(\log x)^2 - 2\int \log x \, dx \; \dots (*)$$

For $\int \log x \, dx$ in (*)	
Take $u = (\log x)$ Differentiate $du = \frac{1}{x} dx$	and $dv = dx$ Integrate $v = x$

$$= x(\log x)^2 - 2\int u dv$$
$$= x(\log x)^2 - 2\left[uv - \int v du\right]$$
$$= x(\log x)^2 - 2\left[x \log x - \int dx\right]$$
$$= x(\log x)^2 - 2x \log x + 2x + c$$
$$= x\left[(\log x)^2 - \log x^2 + 2\right] + c$$

Example 2.29

Evaluate $\int (x^2 - 2x + 5) e^{-x} dx$

Solution:

Successive derivatives	Repeated integrals
Take $u = x^2 - 2x + 5$ $u' = 2x - 2$ $u'' = 2$	and $dv = e^{-x} dx$ $v = -e^{-x}$ $v_1 = e^{-x}$ $v_2 = -e^{-x}$

$$\int (x^2 - 2x + 5) e^{-x} dx$$
$$= \int u dv$$
$$= uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$
$$= (x^2 - 2x + 5)(-e^{-x}) - (2x - 2)e^{-x} + 2(-e^{-x}) + c$$
$$= e^{-x}(-x^2 - 5) + c$$

Exercise 2.5

Integrate the following with respect to x.

1. xe^{-x} 2. $x^3 e^{3x}$ 3. $\log x$

4. $x \log x$ 5. $x^n \log x$ 6. $x^5 e^{x^2}$

2.1.5 Integration by substitution (or) change of variable method

Integrals of certain functions cannot be obtained directly, because they are not in any one of the standard forms as discussed above, but may be reduced to a standard form by suitable substitution. The method of evaluating an integral by reducing it to a standard form by suitable substitution is called integration by substitution.

Type: VI

1. $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c, n \neq -1$
2. $\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c$
3. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$
4. $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$
5. $\int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$

Example 2.30

Evaluate $\int \frac{x}{x^2+1} dx$

Solution:

Take $f(x) = x^2+1$

$\therefore \; f'(x) = 2x$

$$\int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx$$

$$= \frac{1}{2} \int \frac{f'(x)}{f(x)} dx$$

$$= \frac{1}{2} \log\left[f(x)\right] + c$$

$$= \frac{1}{2} \log\left|x^2+1\right| + c$$

DO YOU KNOW?

$$\int \frac{nax^{n-1}}{ax^n+b} dx = \int \frac{d\left(ax^n+b\right)}{ax^n+b}$$

$$= \log\left|ax^n+b\right| + c$$

Example 2.31

Evaluate $\int \frac{x}{\sqrt{x^2+1}} dx$

Solution:

Take $f(x) = x^2+1$

$\therefore \; f'(x) = 2x$

$$\int \frac{x}{\sqrt{x^2+1}} dx = \frac{1}{2} \int \frac{2x}{\sqrt{x^2+1}} dx$$

$$= \frac{1}{2} \int \frac{f'(x)}{\sqrt{f(x)}} dx$$

$$= \frac{1}{2}\left[2\sqrt{f(x)}\right] + c$$

$$= \sqrt{x^2+1} + c$$

DO YOU KNOW?

$$\int \frac{nax^{n-1}}{\sqrt{ax^n+b}} dx = \int \frac{d\left(ax^n+b\right)}{\sqrt{ax^n+b}}$$

$$= 2\sqrt{ax^n+b} + c$$

Example 2.32

Evaluate $\int x\sqrt{x^2+1} \; dx$

Solution:

Take $f(x) = x^2+1$

$\therefore \; f'(x) = 2x$

$$\int x\sqrt{x^2+1} \; dx = \frac{1}{2} \int \left(x^2+1\right)^{\frac{1}{2}} (2x) dx$$

$$= \frac{1}{2} \int \left[f(x)\right]^{\frac{1}{2}} f'(x) dx$$

$$= \frac{1}{2} \frac{\left[f(x)\right]^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}} + c$$

Example 2.33

Evaluate $\int \frac{x^3}{\left(x^2+1\right)^3} dx$

Solution:

Take $z = x^2+1$

$\therefore x^2 = z-1$ *and*

$dz = 2x\,dx$

$\Rightarrow \frac{dz}{2} = x\,dx$

$$\int \frac{x^3}{\left(x^2+1\right)^3} dx = \int \frac{x^2}{\left(x^2+1\right)^3} x\,dx$$

$$= \frac{1}{2} \int \frac{z-1}{z^3} dz$$

$$= \frac{1}{2}\int\left[\frac{1}{z^2} - \frac{1}{z^3}\right]dz$$

$$= \frac{1}{2}\left[-\frac{1}{z} + \frac{1}{2z^2}\right] + c$$

$$= \frac{1}{4\left(x^2+1\right)^2} - \frac{1}{2\left(x^2+1\right)} + c$$

Example 2.34

Evaluate $\int \frac{dx}{x\left(x^3+1\right)}$

Solution:

Take $z = x^3$ $\therefore\ dz = 3x^2\,dx$ $\Rightarrow \frac{dz}{3} = x^2\,dx$	By partial fractions, $\frac{1}{z(z+1)} = \frac{A}{z} + \frac{B}{z+1}$ $\Rightarrow \frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1}$

$$\int \frac{dx}{x\left(x^3+1\right)} = \int \frac{x^2}{x^3\left(x^3+1\right)}dx$$

$$= \frac{1}{3}\int \frac{1}{z(z+1)}dz$$

$$= \frac{1}{3}\int\left[\frac{1}{z} - \frac{1}{z+1}\right]dz$$

$$= \frac{1}{3}\left[\log|z| - \log|z+1|\right] + c$$

$$= \frac{1}{3}\log\left|\frac{z}{z+1}\right| + c$$

$$= \frac{1}{3}\log\left|\frac{x^3}{x^3+1}\right| + c$$

Example 2.35

Evaluate $\int x^3 e^{x^2}\,dx$

Solution:

Take $z = x^2$ $\therefore\ dz = 2x\,dx$ $\Rightarrow \frac{dz}{2} = x\,dx$

$$\int x^3 e^{x^2}\,dx = \int x^2 e^{x^2}\,(x\,dx)$$

$$= \frac{1}{2}\int z\,e^z dz$$

$$= \frac{1}{2}\left[e^z(z-1)\right] + c \quad \text{[By applying Integration by parts]}$$

$$= \frac{1}{2}\left[e^{x^2}(x^2-1)\right] + c$$

Example 2.36

Evaluate $\int e^x\left(x^2+2x\right)dx$

Solution:

Take $f(x) = x^2$ $\therefore\ f'(x) = 2x$

$$\int e^x\left(x^2+2x\right)dx = \int e^x\left[f(x) + f'(x)\right]dx$$

$$= e^x f(x) + c$$

$$= e^x x^2 + c$$

Example 2.37

Evaluate $\int \frac{xe^x}{(1+x)^2}dx$

Solution:

By partial fractions, $\frac{x}{(1+x)^2} = \frac{A}{1+x} + \frac{B}{(1+x)^2}$ $\Rightarrow \frac{x}{(1+x)^2} = \frac{1}{1+x} + \frac{-1}{(1+x)^2}$	Take $f(x) = \frac{1}{1+x}$ $f'(x) = \frac{-1}{(1+x)^2}$

$$\int \frac{xe^x}{(1+x)^2}dx$$

$$= \int e^x\left[\frac{1}{1+x} + \frac{-1}{(1+x)^2}\right]dx$$

$$= \int e^x\left[f(x) + f'(x)\right]dx$$

$$= e^x f(x) + c$$

$$= \frac{e^x}{1+x} + c$$

Example 2.38

Evaluate $\int e^{2x}\left[\frac{2x-1}{4x^2}\right]dx$

Solution:

$\frac{2x-1}{4x^2} = \frac{1}{2x}+\frac{-1}{4x^2}$ $=\frac{1}{4}\left[2\left(\frac{1}{x}\right)+\frac{-1}{x^2}\right]$	Take $a = 2$, $f(x)=\frac{1}{x}$ $\therefore f'(x) = -\frac{1}{x^2}$

$$\int e^{2x}\left[\frac{2x-1}{4x^2}\right]dx$$

$$=\frac{1}{4}\int e^{2x}\left[2\left(\frac{1}{x}\right)+\frac{-1}{x^2}\right]dx$$

$$=\frac{1}{4}\int e^{ax}\left[a\,f(x)+f'(x)\right]dx$$

$$=\frac{1}{4}\left[e^{ax}\ f(x)\right]+c$$

$$=\frac{1}{4x}\ e^{2x}\ +c$$

Example 2.39

Evaluate $\int\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right]dx$

Solution:

Take $z=\log x$

$\therefore dz=\frac{1}{x}dx$

$\Rightarrow dx=e^z dz\ \left[\because x=e^z\right]$

and $f(z)=\frac{1}{z}$

$\therefore f'(z)=-\frac{1}{z^2}$

$$\int\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right]dx$$

$$=\int\left[\frac{1}{z}-\frac{1}{z^2}\right]e^z dz$$

$$=\int e^z\left[f(z)+f'(z)\right]dz$$

$$=e^z f(z)+c$$

$$=e^z\left[\frac{1}{z}\right]+c$$

$$=\frac{x}{\log x}\ +c$$

Exercise 2.6

Integrate the following with respect to *x*.

1. $\frac{2x+5}{x^2+5x-7}$
2. $\frac{e^{3\log x}}{x^4+1}$
3. $\frac{e^{2x}}{e^{2x}-2}$
4. $\frac{(\log x)^3}{x}$
5. $\frac{6x+7}{\sqrt{3x^2+7x-1}}$
6. $(4x+2)\sqrt{x^2+x+1}$
7. $x^8\left(1+x^9\right)^5$
8. $\frac{x^{e-1}+e^{x-1}}{x^e+e^x}$
9. $\frac{1}{x\log x}$
10. $\frac{x}{2x^4-3x^2-2}$
11. $e^x(1+x)\log\left(xe^x\right)$
12. $\frac{1}{x\left(x^2+1\right)}$
13. $e^x\left[\frac{1}{x^2}-\frac{2}{x^3}\right]$
14. $e^x\left[\frac{x-1}{(x+1)^3}\right]$
15. $e^{3x}\left[\frac{3x-1}{9x^2}\right]$

2.1.6 Some special types of Integrals

Type: VII

To evaluate the integrals of the form $\int\frac{dx}{ax^2+bx+c}$, $\int\frac{dx}{\sqrt{ax^2+bx+c}}$ and $\int\sqrt{ax^2+bx+c}\,dx$, first we have to express ax^2+bx+c as the sum or difference of two square terms [completing the squares], that is $(x+\alpha)^2+\beta^2$ (or) $(x+\alpha)^2-\beta^2$ (or) $\beta^2-(x+\alpha)^2$ and apply the suitable formula from the formulae given below.

1. $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$

2. $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$

3. $\int\frac{dx}{\sqrt{x^2-a^2}}=\log\left|x+\sqrt{x^2-a^2}\right|+c$

4. $\int\frac{dx}{\sqrt{x^2+a^2}}=\log\left|x+\sqrt{x^2+a^2}\right|+c$

5. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$

6. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$

Example 2.40

Evaluate $\int \frac{dx}{16-x^2}$

Solution:

$$\int \frac{dx}{16-x^2} = \int \frac{dx}{4^2-x^2}$$
$$= \frac{1}{2(4)}\log\left|\frac{4+x}{4-x}\right| + c$$
$$= \frac{1}{8}\log\left|\frac{4+x}{4-x}\right| + c$$

Example 2.41

Evaluate $\int \frac{dx}{1-25x^2}$

Solution:

$$\int \frac{dx}{1-25x^2} = \frac{1}{25}\int \frac{dx}{\left(\frac{1}{5}\right)^2 - x^2}$$
$$= \frac{1}{25}\left[\frac{1}{2\left(\frac{1}{5}\right)}\log\left|\frac{\frac{1}{5}+x}{\frac{1}{5}-x}\right|\right] + c$$
$$= \frac{1}{10}\log\left|\frac{1+5x}{1-5x}\right| + c$$

DO YOU KNOW?

$$\int \frac{m}{a^2-(mx)^2}\,dx = \int \frac{d(mx)}{a^2-(mx)^2}$$
$$= \frac{1}{2a}\log\left|\frac{a+mx}{a-mx}\right| + c$$

Example 2.42

Evaluate $\int \frac{dx}{2+x-x^2}$

Solution:

By completing the squares
$2+x-x^2 = 2-\left[x^2-x\right]$
$= 2-\left[\left(x-\frac{1}{2}\right)^2-\frac{1}{4}\right]$
$= \left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2$

$$\int \frac{dx}{2+x-x^2} = \int \frac{dx}{\left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$
$$= \frac{1}{2\left(\frac{3}{2}\right)}\log\left|\frac{\frac{3}{2}+\left(x-\frac{1}{2}\right)}{\frac{3}{2}-\left(x-\frac{1}{2}\right)}\right| + c$$
$$= \frac{1}{3}\log\left|\frac{2+2x}{4-2x}\right| + c$$
$$= \frac{1}{3}\log\left|\frac{1+x}{2-x}\right| + c$$

Example 2.43

Evaluate $\int \frac{dx}{4x^2-1}$

Solution:

$$\int \frac{dx}{4x^2-1} = \int \frac{dx}{4\left(x^2-\frac{1}{4}\right)}$$
$$= \frac{1}{4}\int \frac{dx}{x^2-\left(\frac{1}{2}\right)^2}$$
$$= \frac{1}{4}\left[\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{x-\frac{1}{2}}{x+\frac{1}{2}}\right|\right] + c$$
$$= \frac{1}{4}\log\left|\frac{2x-1}{2x+1}\right| + c$$

DO YOU KNOW?

$$\int \frac{m}{(mx)^2-a^2}\,dx = \int \frac{d(mx)}{(mx)^2-a^2}$$
$$= \frac{1}{2a}\log\left|\frac{mx-a}{mx+a}\right| + c$$

Example 2.44

Evaluate $\int \frac{x^2}{x^2-25}dx$

Solution:

$$\int \frac{x^2}{x^2-25}dx = \int \frac{\left(x^2-25\right)+25}{x^2-25}dx$$

$$= \int \left\{1+\frac{25}{x^2-25}\right\}dx$$

$$= \int dx + 25\int \frac{dx}{x^2-25}$$

$$= x + 25\left[\frac{1}{2(5)}\log\left|\frac{x-5}{x+5}\right|\right]+c$$

$$= x + \frac{5}{2}\log\left|\frac{x-5}{x+5}\right|+c$$

Example 2.45

Evaluate $\int \frac{dx}{x^2-3x+2}$

Solution:

By completing the squares
$x^2-3x+2 = \left(x-\frac{3}{2}\right)^2-\frac{9}{4}+2$ $= \left(x-\frac{3}{2}\right)^2-\frac{1}{4}$ $= \left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$

$$\int \frac{dx}{x^2-3x+2} = \int \frac{dx}{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{\left(x-\frac{3}{2}\right)-\frac{1}{2}}{\left(x-\frac{3}{2}\right)+\frac{1}{2}}\right|+c$$

$$= \log\left|\frac{2x-4}{2x-2}\right|+c$$

$$= \log\left|\frac{x-2}{x-1}\right|+c$$

Example 2.46

Evaluate $\int \frac{dx}{\sqrt{4x^2-9}}$

Solution:

$$\int \frac{dx}{\sqrt{4x^2-9}} = \int \frac{dx}{\sqrt{4\left[x^2-\frac{9}{4}\right]}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{x^2-\left(\frac{3}{2}\right)^2}}$$

$$= \frac{1}{2}\log\left|x+\sqrt{x^2-\left(\frac{3}{2}\right)^2}\right|+c$$

$$= \frac{1}{2}\log\left|x+\sqrt{x^2-\frac{9}{4}}\right|+c$$

$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2-9}\right|+c$$

DO YOU KNOW?

$$\int \frac{m}{\sqrt{(mx)^2-a^2}}dx = \int \frac{d(mx)}{\sqrt{(mx)^2-a^2}}$$

$$= \log\left|mx+\sqrt{(mx)^2-a^2}\right|+c$$

Note

$m\log n \pm k = c$

Example 2.47

Evaluate $\int \frac{dx}{\sqrt{x^2-3x+2}}$

Solution:

By completing the squares
$x^2-3x+2 = \left(x-\frac{3}{2}\right)^2-\frac{9}{4}+2$ $= \left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$

$$\int \frac{dx}{\sqrt{x^2-3x+2}} = \int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+c$$

$$=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+c$$

Example 2.48

Evaluate $\int \frac{dx}{\sqrt{x^2+25}}$

Solution:

$$\int \frac{dx}{\sqrt{x^2+25}} = \int \frac{dx}{\sqrt{x^2+5^2}}$$

$$= \log\left|x+\sqrt{x^2+5^2}\right|+c \quad = \log\left|x+\sqrt{x^2+25}\right|+c$$

Example 2.49

Evaluate $\int \frac{dx}{\sqrt{x^2+4x+8}}$

Solution:

By completing the squares
$x^2+4x+8=(x+2)^2-4+8$ $=(x+2)^2+2^2$

$$\int \frac{dx}{\sqrt{x^2+4x+8}} = \int \frac{dx}{\sqrt{(x+2)^2+2^2}}$$

$$=\log\left|(x+2)+\sqrt{(x+2)^2+2^2}\right|+c$$

$$=\log\left|(x+2)+\sqrt{x^2+4x+8}\right|+c$$

DO YOU KNOW?

$$\int \frac{m}{\sqrt{(mx)^2+a^2}}dx = \int \frac{d(mx)}{\sqrt{(mx)^2+a^2}}$$

$$=\log\left|mx+\sqrt{(mx)^2+a^2}\right|+c$$

Example 2.50

Evaluate $\int \frac{x^3dx}{\sqrt{x^8+1}}$

Solution:

$$\int \frac{x^3dx}{\sqrt{x^8+1}} = \frac{1}{4}\int \frac{4x^3dx}{\sqrt{\left(x^4\right)^2+1^2}}$$

$$=\frac{1}{4}\log\left|x^4+\sqrt{\left(x^4\right)^2+1^2}\right|+c$$

$$=\frac{1}{4}\log\left|x^4+\sqrt{x^8+1}\right|+c$$

Example 2.51

Evaluate $\int \sqrt{x^2-16}\ dx$

Solution:

$$\int \sqrt{x^2-16}\ dx = \int \sqrt{x^2-4^2}\ dx$$

$$=\frac{x}{2}\sqrt{x^2-4^2}-\frac{4^2}{2}\log\left|x+\sqrt{x^2-4^2}\right|+c$$

$$=\frac{x}{2}\sqrt{x^2-16}-8\log\left|x+\sqrt{x^2-16}\right|+c$$

Example 2.52

Evaluate $\int \sqrt{x^2+5}\ dx$

Solution:

$$\int \sqrt{x^2+5}\ dx = \int \sqrt{x^2+\left(\sqrt{5}\right)^2}\ dx$$

$$=\frac{x}{2}\sqrt{x^2+\left(\sqrt{5}\right)^2}+\frac{\left(\sqrt{5}\right)^2}{2}\log\left|x+\sqrt{x^2+\left(\sqrt{5}\right)^2}\right|+c$$

$$=\frac{x}{2}\sqrt{x^2+5}+\frac{5}{2}\log\left|x+\sqrt{x^2+5}\right|+c$$

DO YOU KNOW?

$$\int m\sqrt{(mx)^2+a^2}\ dx = \int \sqrt{(mx)^2+a^2}\ d(mx)$$

$$=\frac{mx}{2}\sqrt{(mx)^2+a^2}+\frac{a^2}{2}\log\left|mx+\sqrt{(mx)^2+a^2}\right|+c$$

Example 2.53

Evaluate $\int \sqrt{4x^2+9}\ dx$

Solution:

$$\int \sqrt{4x^2+9}\ dx$$

$$= \frac{1}{2}\int \sqrt{(2x)^2 + 3^2}\, d(2x)$$

$$= \frac{1}{2}\left[\frac{2x}{2}\sqrt{(2x)^2+3^2} + \frac{3^2}{2}\log\left|2x+\sqrt{(2x)^2+3^2}\right|\right] + c$$

$$= \frac{x}{2}\sqrt{4x^2+9} + \frac{9}{4}\log\left|2x+\sqrt{4x^2+9}\right| + c$$

Example 2.54

Evaluate $\int \sqrt{x^2-4x+3}\, dx$

Solution:

By completing the squares
$x^2-4x+3 = (x-2)^2 - 4 + 3$ $= (x-2)^2 - 1^2$ $\therefore \sqrt{x^2-4x+3} = \sqrt{(x-2)^2-1^2}$

$$\int \sqrt{x^2-4x+3}\, dx$$

$$= \int \sqrt{(x-2)^2 - 1^2}\, dx$$

$$= \frac{(x-2)}{2}\sqrt{(x-2)^2-1^2} - \frac{1}{2}\log\left|(x-2)+\sqrt{(x-2)^2-1^2}\right| + c$$

$$= \frac{(x-2)}{2}\sqrt{x^2-4x+3} - \frac{1}{2}\log\left|(x-2)+\sqrt{x^2-4x+3}\right| + c$$

Example 2.55

Evaluate $\int \frac{1}{x-\sqrt{x^2-1}}\, dx$

Solution:

By rationalisation,
$\frac{1}{x-\sqrt{x^2-1}} = \frac{1}{x-\sqrt{x^2-1}} \times \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$ $= x+\sqrt{x^2-1}$

$$\int \frac{1}{x-\sqrt{x^2-1}}\, dx$$

$$= \int \left[x+\sqrt{x^2-1}\right] dx$$

$$= \int x\, dx + \int \sqrt{x^2-1}\, dx$$

$$= \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\log\left|x+\sqrt{x^2-1}\right| + c$$

Exercise 2.7

Integrate the following with respect to *x*

1. $\frac{1}{9-16x^2}$
2. $\frac{1}{9-8x-x^2}$
3. $\frac{1}{2x^2-9}$
4. $\frac{1}{x^2-x-2}$
5. $\frac{1}{x^2+3x+2}$
6. $\frac{1}{2x^2+6x-8}$
7. $\frac{e^x}{e^{2x}-9}$
8. $\frac{1}{\sqrt{9x^2-7}}$
9. $\frac{1}{\sqrt{x^2+6x+13}}$
10. $\frac{1}{\sqrt{x^2-3x+2}}$
11. $\frac{x^3}{\sqrt{x^8-1}}$
12. $\sqrt{1+x+x^2}$
13. $\sqrt{x^2-2}$
14. $\sqrt{4x^2-5}$
15. $\sqrt{2x^2+4x+1}$
16. $\frac{1}{x+\sqrt{x^2-1}}$

2.2 Definite integrals

So for we learnt about indefinite integrals on elementary algebraic, exponential, trigonometric and logarithmic functions. Now we are going to study about the definite integrals.

Geometrically, definite integral $\int_a^b f(x)\,dx$ represents the limit of a sum. It is also represented as the area bounded the curve $y = f(x)$, the axis of *x*, and the ordinates $x = a$ and $x = b$.

2.2.1 The fundamental theorems of Integral Calculus

Theorem 2.1 First fundamental theorem of Integral Calculus:

If $f(x)$ is a continuous function and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$.

Theorem 2.2 Second fundamental theorem of Integral Calculus:

Let $f(x)$ be a continuous function on [a,b], if $F(x)$ is anti derivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Here a and b are known as the lower limit and upper limit of the definite integral.

Note

$\int_a^b f(x)\,dx$ is a definite constant, whereas $\int_a^x f(t)\,dt$ which is a function of the variable x.

DO YOU KNOW?

$$\int f(x)\,dx = F(x) + c$$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$$

Example 2.56

Evaluate $\int_0^1 \left(x^3 + 7x^2 - 5x\right) dx$

Solution:

We have already learnt about the evaluation of the integral $\int \left(x^3 + 7x^2 - 5x\right) dx$ in the previous section.

$$\therefore \int \left(x^3 + 7x^2 - 5x\right) dx = \frac{x^4}{4} + \frac{7x^3}{3} - \frac{5x^2}{2} + c$$

Now, $\int_0^1 \left(x^3 + 7x^2 - 5x\right) dx$

$$= \left[\frac{x^4}{4} + \frac{7x^3}{3} - \frac{5x^2}{2}\right]_0^1$$

$$= \left[\frac{1}{4} + \frac{7}{3} - \frac{5}{2}\right] - \left[\frac{0}{4} + \frac{0}{3} - \frac{0}{2}\right]$$

$$= \left[\frac{1}{4} + \frac{7}{3} - \frac{5}{2}\right] = \frac{1}{12}$$

Example 2.57

Find the integration for $\frac{dy}{dx} = \frac{2x}{5x^2 + 1}$ with limiting values as 0 and 1

Solution:

Here, $\frac{dy}{dx} = \frac{2x}{5x^2+1}$

$$\therefore\ y = \int_0^1 \frac{2x}{5x^2+1}\,dx$$

$$= \frac{1}{5}\int_0^1 \frac{10x}{5x^2+1}\,dx$$

$$= \frac{1}{5}\left[\log\left(5x^2+1\right)\right]_0^1$$

$$= \frac{1}{5}\left[\log 6 - \log 1\right]$$

$$= \frac{1}{5}\log 6$$

Example 2.58

Evaluate $\int_0^1 \left(e^x - 4a^x + 2 + \sqrt[3]{x}\right) dx$

Solution:

$$\int_0^1 \left(e^x - 4a^x + 2 + \sqrt[3]{x}\right) dx = \left[e^x - 4\frac{a^x}{\log a} + 2x + 3\frac{x^{\frac{4}{3}}}{4}\right]_0^1$$

$$= e - \frac{4a}{\log a} + 2 + \frac{3}{4} - 1 + \frac{4}{\log a}$$

$$= e + \frac{4(1-a)}{\log a} + \frac{7}{4}$$

Example 2.59

Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\,dx$

Solution:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\,dx = \left[-\cos x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= -\left(\cos\frac{\pi}{3} - \cos\frac{\pi}{6}\right)$$

$$= \frac{1}{2}\left(\sqrt{3} - 1\right)$$

Example 2.60

Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$

Solution:

> Change into simple integrands
>
> $$\cos 2x = 2\cos^2 x - 1$$
>
> $$\therefore \cos^2 x = \frac{1}{2}[1+\cos 2x]$$

$$\int_0^{\frac{\pi}{2}} \cos^2 x\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}[1+\cos 2x]\, dx$$

$$= \frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+0\right] = \frac{\pi}{4}$$

Note

$$e^{a\log x} = x^a$$

Example 2.61

Evaluate $\int_0^1 \left[e^{a\log x} + e^{x\log a}\right]dx$

Solution:

$$\int_0^1 \left[e^{a\log x} + e^{x\log a}\right]dx = \int_0^1 \left(x^a + a^x\right)dx$$

$$= \left[\frac{x^{a+1}}{a+1} + \frac{a^x}{\log a}\right]_0^1$$

$$= \left(\frac{1}{a+1} + \frac{a}{\log a}\right) - \left(0 + \frac{1}{\log a}\right)$$

$$= \frac{1}{a+1} + \frac{a}{\log a} - \frac{1}{\log a}$$

$$= \frac{1}{a+1} + \frac{(a-1)}{\log a}$$

Example 2.62

Evaluate $\int_2^3 \frac{x^4+1}{x^2}dx$

Solution:

$$\int_2^3 \frac{x^4+1}{x^2}dx = \int_2^3 \left(x^2 + x^{-2}\right)dx$$

$$= \left[\frac{x^3}{3} - \frac{1}{x}\right]_2^3$$

$$= \left(9-\frac{1}{3}\right) - \left(\frac{8}{3}-\frac{1}{2}\right) = \frac{13}{2}$$

Example 2.63

Evaluate $\int_{-1}^1 \left(x^3+3x^2\right)^3\left(x^2+2x\right)dx$

Solution:

$$\int_{-1}^1 \left(x^3+3x^2\right)^3\left(x^2+2x\right)dx = \left[\frac{1}{3}\frac{\left(x^3+3x^2\right)^4}{4}\right]_{-1}^1$$

$$\left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$$

$$= \frac{1}{3}(64-4)$$

$$= 20$$

Example 2.64

Evaluate $\int_a^b \frac{\sqrt{\log x}}{x}dx \quad a,b>0$

Solution:

$$\int_a^b \frac{\sqrt{\log x}}{x}dx = \int_a^b (\log x)^{\frac{1}{2}}\frac{dx}{x}$$

$$= \left[2\frac{(\log x)^{\frac{3}{2}}}{3}\right]_a^b$$

$$\left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$$

$$= \frac{2}{3}\left[(\log b)^{\frac{3}{2}} - (\log a)^{\frac{3}{2}}\right]$$

DO YOU KNOW?

$$\log b^{\frac{3}{2}} = \frac{3}{2}\log b \text{ but,}$$

$$(\log b)^{\frac{3}{2}} \neq \frac{3}{2}\log b \text{ Further,}$$

$$(\log b)^{\frac{3}{2}} - (\log a)^{\frac{3}{2}} \neq \left(\log\frac{b}{a}\right)^{\frac{3}{2}}$$

Example 2.65

Evaluate $\int_{-1}^{1} x\sqrt{x+1}\, dx$

Solution:

Take $t = x+1$, $dt = dx$ and		
x	-1	1
t	0	2

$$\int_{-1}^{1} x\sqrt{x+1}\, dx = \int_{0}^{2} (t-1)\sqrt{t}\, dt$$
$$= \int_{0}^{2} \left(t^{\frac{3}{2}} - t^{\frac{1}{2}} \right) dt$$
$$= \left[\frac{2t^{\frac{5}{2}}}{5} - \frac{2t^{\frac{3}{2}}}{3} \right]_0^2$$
$$= \frac{8\sqrt{2}}{5} - \frac{4\sqrt{2}}{3}$$
$$= \frac{4\sqrt{2}}{15}$$

Example 2.66

Evaluate $\int_{0}^{\infty} e^{-\frac{x}{2}}\, dx$

Solution:

$$\int_{0}^{\infty} e^{-\frac{x}{2}}\, dx = -2\left[e^{-\frac{x}{2}} \right]_0^{\infty}$$
$$= -2[0-1] = 2$$

Example 2.67

Evaluate $\int_{0}^{\infty} x^2 e^{-x^3}\, dx$

Solution:

Take $x^3 = t$, $3x^2\, dx = dt$, $\Rightarrow x^2\, dx = \frac{dt}{3}$ and		
x	0	∞
t	0	∞

$$\int_{0}^{\infty} x^2 e^{-x^3}\, dx = \int_{0}^{\infty} e^{-t}\frac{dt}{3}$$
$$= \frac{1}{3}\left[-e^{-t} \right]_0^{\infty}$$
$$= \frac{-1}{3}[0-1]$$
$$= \frac{1}{3}$$

Example 2.68

Evaluate $\int_{1}^{2} \frac{1}{(x+1)(x+2)}\, dx$

Solution:

By partial fractions,

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$
$$\Rightarrow \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$\int_{1}^{2} \frac{1}{(x+1)(x+2)}\, dx = \int_{1}^{2} \left[\frac{1}{x+1} - \frac{1}{x+2} \right] dx$$
$$= \left[\log|x+1| - \log|x+2| \right]_1^2$$
$$= \log\frac{3}{4} - \log\frac{2}{3}$$
$$= \log\frac{9}{8}$$

Example 2.69

Evaluate $\int_{1}^{e} \log x\, dx$

Solution:

Take $u = \log x$	$dv = dx$
Differentiate	Integrate
$du = \frac{1}{x}dx$	$v = x$

$$\int_{1}^{e} \log x\, dx = \int_{1}^{e} u\, dv$$
$$= [uv]_1^e - \int_{1}^{e} v\, du$$
$$= [x\log x]_1^e - \int_{1}^{e} x\frac{1}{x}\, dx$$

$$= \left(e\log e - 1\log 1\right) - [x]_1^e$$
$$= \left(e-0\right) - \left(e-1\right)$$
$$= 1$$

Example 2.70

Evaluate $\int_0^{\frac{\pi}{2}} x \sin x \, dx$

Solution:

Take $u = x$	and $dv = \sin x \, dx$
Differentiate	Integrate
$du = dx$	$v = -\cos x$

$$\int_0^{\frac{\pi}{2}} x \sin x \, dx = \int_0^{\frac{\pi}{2}} u dv$$
$$= [uv]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} v du$$
$$= [-x\cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, dx$$
$$= 0 + [\sin x]_0^{\frac{\pi}{2}} = 1$$

Example 2.71

If $\int_1^a 3x^2 \, dx = -1$, then find the value of a, $a \in R$.

Solution:

Given that
$$\int_1^a 3x^2 \, dx = -1$$
$$\left[x^3\right]_1^a = -1$$
$$a^3 - 1 = -1$$
$$a^3 = 0 \Rightarrow a = 0$$

Example 2.72

If $\int_a^b dx = 1$ and $\int_a^b x\,dx = 1$, then find a and b

Solution:

Given that
$$\int_a^b dx = 1$$
$$[x]_a^b = 1$$
$$b - a = 1 \quad \ldots (1)$$

Now,
$$\int_a^b x\,dx = 1$$
$$\left[\frac{x^2}{2}\right]_a^b = 1$$
$$b^2 - a^2 = 2$$
$$(b+a)(b-a) = 2$$
$$b + a = 2 \ldots (2) \quad [\because b - a = 1]$$
$$(1) + (2) \Rightarrow \quad 2b = 3$$
$$\therefore b = \frac{3}{2}$$

Now,
$$\frac{3}{2} - a = 1 \quad [\because \text{from } (1)]$$
$$\therefore a = \frac{1}{2}$$

Example 2.73

Evaluate $\int_1^4 f(x)\,dx$, where

$$f(x) = \begin{cases} 7x+3, & \text{if } 1 \le x \le 3 \\ 8x, & \text{if } 3 \le x \le 4 \end{cases}$$

Solution:

$$\int_1^4 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^4 f(x)\,dx$$
$$= \int_1^3 (7x+3)\,dx + \int_3^4 8x\,dx$$
$$= \left[\frac{7x^2}{2} + 3x\right]_1^3 + \left[\frac{8x^2}{2}\right]_3^4$$
$$= \frac{63}{2} + 9 - \frac{13}{2} + 64 - 36$$
$$= 62$$

Example 2.74

If $f(x) = \begin{cases} x^2, & -2 \le x < 1 \\ x, & 1 \le x < 2 \\ x-4, & 2 \le x \le 4 \end{cases}$,

then find the following

(i) $\int_{-2}^{1} f(x)dx$ (ii) $\int_{1}^{2} f(x)dx$

(iii) $\int_{2}^{3} f(x)dx$ (iv) $\int_{-2}^{1.5} f(x)dx$

(v) $\int_{1}^{3} f(x)dx$

Solution:

(i) $\int_{-2}^{1} f(x)dx = \int_{-2}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_{-2}^{1} = \frac{1}{3} - \left(\frac{-8}{3}\right) = 3$

(ii) $\int_{1}^{2} f(x)dx = \int_{1}^{2} x dx = \left[\frac{x^2}{2}\right]_{1}^{2} = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$

(iii) $\int_{2}^{3} f(x)dx$ $= \int_{2}^{3} (x-4)dx = \left[\frac{x^2}{2} - 4x\right]_{2}^{3}$

$= \left(\frac{9}{2} - 12\right) - \left(\frac{4}{2} - 8\right)$

$= -\frac{15}{2} + 6 = \frac{-3}{2}$

(iv) $\int_{-2}^{1.5} f(x)dx$ $= \int_{-2}^{1} f(x)dx + \int_{1}^{1.5} f(x)dx$

$= 3 + \int_{1}^{1.5} x dx$ using (i)

$= 3 + \left[\frac{x^2}{2}\right]_{1}^{1.5}$

$= 3 + \frac{2.25}{2} - \frac{1}{2} = 3 + \frac{1.25}{2} = 3.625$

(v) $\int_{1}^{3} f(x)dx$ $= \int_{1}^{2} f(x)dx + \int_{2}^{3} f(x)dx$

$= \frac{3}{2} + \left(\frac{-3}{2}\right) = 0$ using (ii) and (iii)

Exercise 2.8

I. Using second fundamental theorem, evaluate the following:

1. $\int_{0}^{1} e^{2x} dx$ 2. $\int_{0}^{\frac{1}{4}} \sqrt{1-4x}\, dx$ 3. $\int_{1}^{2} \frac{x dx}{x^2+1}$

4. $\int_{0}^{3} \frac{e^x dx}{1+e^x}$ 5. $\int_{0}^{1} x e^{x^2} dx$ 6. $\int_{1}^{e} \frac{dx}{x(1+\log x)^3}$

7. $\int_{-1}^{1} \frac{2x+3}{x^2+3x+7} dx$ 8. $\int_{0}^{\frac{\pi}{2}} \sqrt{1+\cos x}\, dx$

9. $\int_{1}^{2} \frac{x-1}{x^2} dx$

II. Evaluate the following:

1. $\int_{1}^{4} f(x)\, dx$ where $f(x) = \begin{cases} 4x+3, & 1 \le x \le 2 \\ 3x+5, & 2 < x \le 4 \end{cases}$

2. $\int_{0}^{2} f(x)\, dx$ where $f(x) = \begin{cases} 3-2x-x^2, & x \le 1 \\ x^2+2x-3, & 1 < x \le 2 \end{cases}$

3. $\int_{-1}^{1} f(x)\, dx$ where $f(x) = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

4. $f(x) = \begin{cases} cx, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

Find 'c' if $\int_{0}^{1} f(x)dx = 2$

DO YOU KNOW?

$\int_{a}^{b} dx = b - a$

2.2.2 Properties of definite integrals

(i) $\int_{a}^{b} f(x)dx = \int_{a}^{b} f(t)dt$

(ii) $\int_{a}^{b} f(x)dx = -\int_{b}^{a} f(x)dx$

(iii) If $f(x)$ and $g(x)$ are continuous functions in $[a, b]$, then

$\int_{a}^{b} [f(x) \pm g(x)]dx = \int_{a}^{b} f(x)dx \pm \int_{a}^{b} g(x)dx$

(iv) If $f(x)$ is a continuous function in $[a, b]$ and a < c < b, then

$\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx$

(v) $\int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx$

Proof:

Take $a - x = t \Rightarrow dx = -dt$		
x	0	a
t	a	0

R.H.S. $= \int_0^a f(a-x)dx$

$= \int_a^0 f(t)(-dt)$

$= \int_0^a f(t)dt$

$= \int_0^a f(x)dx$ [using the Property (i)]

= L.H.S.

(vi) (a) If $f(x)$ is an even function, then

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

(b) If $f(x)$ is an odd function, then

$$\int_{-a}^{a} f(x)dx = 0$$

Proof :

(a) If $f(x)$ is an even function, then $f(x) = f(-x)$ …(1)

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$$

$$= \int_{-a}^{0} f(-x)dx + \int_0^a f(x)dx$$

using (1)

Take $-x = t \Rightarrow dx = -dt$		
x	$-a$	0
t	a	0

Apply the above mentioned substitution only in first part of integral of the R.H.S.

$$\therefore \int_{-a}^{a} f(x)dx = \int_a^0 f(t)(-dt) + \int_0^a f(x)dx$$

$$= \int_0^a f(t)\,dt + \int_0^a f(x)dx$$

$$= \int_0^a f(x)dx + \int_0^a f(x)dx$$

[using the Property (i)]

$$= 2\int_0^a f(x)dx$$

b) If $f(x)$ is an odd function, then $f(-x) = -f(x)$ …(2)

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$$

$$= \int_{-a}^{0} -f(-x)dx + \int_0^a f(x)dx \text{ using (2)}$$

Take $-x = t \Rightarrow dx = -dt$		
x	$-a$	0
t	a	0

Apply the above mentioned substitution only in first part of integral of the R.H.S.

$$\therefore \int_{-a}^{a} f(x)dx = -\int_a^0 f(t)(-dt) + \int_0^a f(x)dx$$

$$= -\int_0^a f(t)dt + \int_0^a f(x)dx$$

$$= -\int_0^a f(x)dx + \int_0^a f(x)dx$$ [using the Property (i)]

$= 0$

(vii) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Proof:

Let $a+b-x = t$ and

$-dx = dt$

$dx = -dt$

$t = a+b-x$		
x	a	b
t	b	a

$$\therefore \int_a^b f(a+b-x)dx = -\int_b^a f(t)dt = \int_a^b f(t)dt$$

$$= \int_a^b f(x)dx$$ [using the Property (i)]

$$\therefore \int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$$

Evaluate the following using properties of definite integrals:

Example 2.75

Evaluate $\int_{-1}^{1} \frac{x^5\,dx}{a^2-x^2}$

Solution:

Let $f(x) = \frac{x^5}{a^2-x^2}$

$$f(-x) = \frac{(-x)^5}{a^2-(-x)^2} = \frac{-x^5}{a^2-x^2} = -f(x)$$

Here $f(-x) = -f(x)$

$\therefore f(x)$ is an odd function

$$\Rightarrow \int_{-1}^{1} \frac{x^5\,dx}{a^2-x^2} = 0$$

Example 2.76

Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$

Solution:

Let $f(x) = \cos x$

$f(-x) = \cos(-x) = \cos x$

$\Rightarrow f(-x) = f(x)$

$\therefore f(x)$ is an even function

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = 2\int_0^{\frac{\pi}{2}} \cos x\,dx$$

$$= 2\left[\sin x\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= 2$$

Example 2.77

Evaluate $\int_{-1}^{1} (x^2+x)\,dx$

Solution:

$$\int_{-1}^{1} (x^2+x)\,dx = \int_{-1}^{1} x^2\,dx + \int_{-1}^{1} x\,dx$$

$= 2\int_0^1 x^2\,dx + 0$ [$\because x^2$ is an even function and x is an odd function]

$$= 2\left[\frac{x^3}{3}\right]_0^1 = 2\left[\frac{1}{3}-0\right]$$

$$= \frac{2}{3}$$

Example 2.78

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}dx$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}dx$ ... (1)

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right) + \cos\left(\frac{\pi}{2}-x\right)}dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}dx \quad \ldots(2)$$

$(1)+(2) \Rightarrow$

$$2I = \int_0^{\frac{\pi}{2}} \left[\frac{\sin x}{\sin x + \cos x} + \frac{\cos x}{\cos x + \sin x}\right]dx$$

$$= \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{4}$$

Example 2.79

Evaluate $\int_{2}^{5} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}\,dx$

Solution:

Let $I = \int_{2}^{5} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}\,dx \quad \ldots(1)$

$I = \int_{2}^{5} \frac{\sqrt{2+5-x}}{\sqrt{2+5-x}+\sqrt{7-(2+5-x)}}\,dx$

$\left[\because \int_{a}^{b} f(x)dx = \int_{a}^{b} f(a+b-x)dx\right]$

$I = \int_{2}^{5} \frac{\sqrt{7-x}}{\sqrt{7-x}+\sqrt{x}}\,dx \ldots(2)$

$(1)+(2) \Rightarrow$

$$2I = \int_{2}^{5}\left[\frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}} + \frac{\sqrt{7-x}}{\sqrt{7-x}+\sqrt{x}}\right]dx$$

$$= \int_{2}^{5}\left[\frac{\sqrt{x}+\sqrt{7-x}}{\sqrt{x}+\sqrt{7-x}}\right]dx$$

$$= \int_{2}^{5} dx = [x]_{2}^{5} = 3$$

$\therefore \quad I = \frac{3}{2}$

Exercise 2.9

Evaluate the following using properties of definite integrals:

1. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^3 \cos^3 x\,dx$

2. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\,d\theta$

3. $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right)dx$

4. $\int_{0}^{\frac{\pi}{2}} \frac{\sin^7 x}{\sin^7 x+\cos^7 x}\,dx$

5. $\int_{0}^{1} \log\left(\frac{1}{x}-1\right)dx$

6. $\int_{0}^{1} \frac{x}{(1-x)^{\frac{3}{4}}}\,dx$

2.2.3 Gamma Integral

Gamma integral is an important result which is very useful in the evaluation of a particular type of an improper definite integrals.

First, let us know about the concepts of indefinite integrals, proper definite integrals and improper definite integrals.

Indefinite integral:

An integral function which is expressed without limits, and so containing an arbitrary constant is called an indefinite integral

Example: $\int e^{-t}dt$

Proper definite integral:

Proper definite integral is an integral function, which has both the limits a and b are finite and the integrand $f(x)$ is continuous in $[a, b]$.

Example: $\int_{0}^{1} e^{-t}dt$

Improper definite integral:

An improper definite integral is an integral function, in which the limits either a or b or both are infinite, or the integrand $f(x)$ becomes infinite at some points of the interval $[a, b]$.

Example: $\int_{0}^{\infty} e^{-t}dt$

Definition 2.3

For $n > 0$, $\int_{0}^{\infty} x^{n-1}e^{-x}dx$ is known known as Gamma function and is denoted by $\Gamma(n)$ [read as Gamma of n].

Note

If n is a positive integer, then $\int_{0}^{\infty} x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$ is the particular case of Gamma Integral.

Properties:

1. $\Gamma(n) = (n-1)\Gamma(n-1),\ n>1$
2. $\Gamma(n+1) = n\Gamma(n),\ n>0$

3. $\Gamma(n+1) = n!$, n is a positive integer.

4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

DO YOU KNOW?

$$\Gamma(1) = 1$$

Example 2.80

Evaluate (i) $\Gamma(6)$ (ii) $\Gamma\left(\frac{7}{2}\right)$

(iii) $\int_0^\infty e^{-2x} x^5 dx$ (iv) $\int_0^\infty e^{-x^2} dx$

Solution:

(i) $\Gamma(6) = 5!$

$= 120$

(ii) $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right)$

$= \frac{5}{2}\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$

$= \frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi} = \frac{15}{8}\sqrt{\pi}$

(iii) we know that

$$\int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}$$

$$\therefore \int_0^\infty e^{-2x} x^5\, dx = \frac{5!}{2^{5+1}} = \frac{5!}{2^6}$$

(iv) $\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt$

Put $t = x^2 \Rightarrow dt = 2x dx$

$$\therefore \quad \Gamma(n) = \int_0^\infty e^{-x^2} \left(x^2\right)^{n-1} 2x\, dx$$

$$= \int_0^\infty e^{-x^2} x^{2n-2} 2x\, dx$$

$$= 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$$

Put $n = \frac{1}{2}$, we have

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx$$

$$\Rightarrow \sqrt{\pi} = 2\int_0^\infty e^{-x^2} dx$$

$$\therefore \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Exercise 2.10

1. Evaluate the following:

(i) $\Gamma(4)$ (ii) $\Gamma\left(\frac{9}{2}\right)$ (iii) $\int_0^\infty e^{-mx} x^6 dx$

(iv) $\int_0^\infty e^{-4x} x^4 dx$ (v) $\int_0^\infty e^{-\frac{x}{2}} x^5 dx$

2. If $f(x) = \begin{cases} x^2 e^{-2x}, x \geq 0 \\ 0, \text{ otherwise} \end{cases}$, then evaluate $\int_0^\infty f(x) dx$

2.2.4 Definite integral as the limit of a sum

Let $f(x)$ be a continuous real valued function in $[a, b]$, which is divided into n equal parts of width h, then

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ h\to 0}} h\left[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+(n-1)h)\right] \text{ (or)}$$

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{r=1}^{n} h\, f(a+rh),$$

where $h = \frac{b-a}{n}$

The following results are very useful in evaluating definite integral as the limit of a sum

(i) $1+2+3+\ldots\ldots n = \frac{n(n+1)}{2} = \sum_{r=1}^{n} r$

(ii) $1^2+2^2+3^2+\ldots\ldots n^2 = \frac{n(n+1)(2n+1)}{6} = \sum_{r=1}^{r=n} r^2$

(iii) $1^3+2^3+3^3+\ldots\ldots n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \sum_{r=1}^{r=n} r^3$

XII Std - Business Maths & Stat EM Chapter 2.indd 51 08-12-2021 11:43:11

Example 2.81

Evaluate the integral as the limit of a sum: $\int_0^1 x\,dx$

Solution:

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{r=1}^{n} h\, f(a+rh)$$

Here $a=0,\ b=1,\ h=\frac{b-a}{n}=\frac{1-0}{n}=\frac{1}{n}$

and $f(x)=x$

Now $f(a+rh)=f\left(0+\frac{r}{n}\right)=f\left(\frac{r}{n}\right)=\frac{r}{n}$

On substituting in (1) we have

$$\int_0^1 x\,dx = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}.\frac{r}{n}$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\sum_{r=1}^{n} r$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\cdot\frac{n^2\left(1+\frac{1}{n}\right)}{2}$$

$$= \frac{1+0}{2}=\frac{1}{2}$$

$$\therefore\ \int_0^1 x\,dx = \frac{1}{2}$$

Example 2.82

Evaluate the integral as the limit of a sum: $\int_1^2 (2x+1)\,dx$

Solution:

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{r=1}^{n} h\, f(a+rh)$$

Here $a=1,\ b=2,\ h=\frac{b-a}{n}=\frac{2-1}{n}=\frac{1}{n}$

and $f(x)=2x+1$, $f(a+rh) = f\left(1+\frac{r}{n}\right)$

$$= 2\left(1+\frac{r}{n}\right)+1$$

$$= 2+\frac{2r}{n}+1$$

$$f(a+rh) = 3+\frac{2r}{n}$$

$$\int_1^2 f(x)\,dx = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\left(3+\frac{2r}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{3}{n}+\frac{2r}{n^2}\right)$$

$$= \lim_{n\to\infty}\left[\frac{3}{n}\sum_{r=1}^{n}1 + \frac{2}{n^2}\sum_{r=1}^{n} r\right]$$

$$= \lim_{n\to\infty}\left[\frac{3}{n}(n)+\frac{2}{n^2}\frac{n(n+1)}{2}\right]$$

$$= 3+\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$$\int_1^2 f(x)\,dx = 3+1=4$$

Example 2.83

Evaluate the integral as the limit of a sum: $\int_1^2 x^2\,dx$

Solution:

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{r=1}^{n} h\, f(a+rh)$$

Here $a=1,\ b=2,\ h=\frac{b-a}{n}=\frac{2-1}{n}=\frac{1}{n}$ and

$f(x)=x^2$

Now $f(a+rh)=f\left(1+\frac{r}{n}\right) = \left(1+\frac{r}{n}\right)^2$

$$= 1+\frac{2r}{n}+\frac{r^2}{n^2}$$

$$\therefore\ \int_1^2 x^2\,dx = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\left(1+\frac{2r}{n}+\frac{r^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{1}{n}+\frac{2r}{n^2}+\frac{r^2}{n^3}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{r=1}^{n}1+\frac{2}{n^2}\sum_{r=1}^{n}r+\frac{1}{n^3}\sum_{r=1}^{n}r^2\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}(n)+\frac{2}{n^2}\frac{n(n+1)}{2}+\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty}\left[1+\left(1+\frac{1}{n}\right)+\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}\right]$$

$$= \left[1+1+\frac{(1)(2)}{6}\right]$$

$$\therefore \int_1^2 x^2\, dx = \frac{7}{3}$$

Exercise 2.11

Evaluate the following integrals as the limit of the sum:

1. $\int_0^1 (x+4)\, dx$
2. $\int_1^3 x\, dx$
3. $\int_1^3 (2x+3)\, dx$
4. $\int_0^1 x^2\, dx$

Exercise 2.12

Choose the correct answer:

1. $\int \frac{1}{x^3} dx$ is

(a) $\frac{-3}{x^2}+c$ (b) $\frac{-1}{2x^2}+c$

(c) $\frac{-1}{3x^2}+c$ (d) $\frac{-2}{x^2}+c$

2. $\int 2^x dx$ is

(a) $2^x \log 2 + c$ (b) $2^x + c$

(c) $\frac{2^x}{\log 2}+c$ (d) $\frac{\log 2}{2^x}+c$

3. $\int \frac{\sin 2x}{2\sin x} dx$ is

(a) $\sin x + c$ (b) $\frac{1}{2}\sin x + c$

(c) $\cos x + c$ (d) $\frac{1}{2}\cos x + c$

4. $\int \frac{\sin 5x - \sin x}{\cos 3x} dx$ is

(a) $-\cos 2x + c$ (b) $-\cos 2x + c$

(c) $-\frac{1}{4}\cos 2x + c$ (d) $-4\cos 2x + c$

5. $\int \frac{\log x}{x} dx,\ x>0$ is

(a) $\frac{1}{2}(\log x)^2 + c$ (b) $-\frac{1}{2}(\log x)^2$

(c) $\frac{2}{x^2}+c$ (d) $\frac{2}{x^2}+c$

6. $\int \frac{e^x}{\sqrt{1+e^x}} dx$ is

(a) $\frac{e^x}{\sqrt{1+e^x}}+c$ (b) $2\sqrt{1+e^x}+c$

(c) $\sqrt{1+e^x}+c$ (d) $e^x\sqrt{1+e^x}+c$

7. $\int \sqrt{e^x}\, dx$ is

(a) $\sqrt{e^x}+c$ (b) $2\sqrt{e^x}+c$

(c) $\frac{1}{2}\sqrt{e^x}+c$ (d) $\frac{1}{2\sqrt{e^x}}+c$

8. $\int e^{2x}[2x^2+2x]dx$

(a) $e^{2x}x^2+c$ (b) $xe^{2x}+c$

(c) $2x^2e^2+c$ (d) $\frac{x^2e^x}{2}+c$

9. $\int \frac{e^x}{e^x+1} dx$ is

(a) $\log\left|\frac{e^x}{e^x+1}\right|+c$ (b) $\log\left|\frac{e^x+1}{e^x}\right|+c$

(c) $\log\left|e^x\right|+c$ (d) $\log\left|e^x+1\right|+c$

10. $\int \left[\frac{9}{x-3}-\frac{1}{x+1}\right] dx$ is

(a) $\log|x-3| - \log|x+1| + c$

(b) $\log|x-3| + \log|x+1| + c$

(c) $9\log|x-3| - \log|x+1| + c$

(d) $9\log|x-3| + \log|x+1| + c$

11. $\int \frac{2x^3}{4+x^4} dx$ is

(a) $\log|4+x^4|+c$ (b) $\frac{1}{2}\log|4+x^4|+c$

(c) $\frac{1}{4}\log|4+x^4|+c$ (d) $\log\left|\frac{2x^3}{4+x^4}\right|+c$

12. $\int \frac{dx}{\sqrt{x^2-36}}$ is

(a) $\sqrt{x^2-36+c}$

(b) $\log\left|x+\sqrt{x^2-36}\right|+c$

(c) $\log\left|x-\sqrt{x^2-36}\right|+c$

(d) $\log\left|x^2+\sqrt{x^2-36}\right|+c$

13. $\int \frac{2x+3}{\sqrt{x^2+3x+2}}\,dx$ is

(a) $\sqrt{x^2+3x+2}\ +c$

(b) $2\sqrt{x^2+3x+2}+c$

(c) $\log\left(x^2+3x+2\right)+c$

(d) $\frac{2}{3}\left(x^2+3x+2\right)^{\frac{3}{2}}+c$

14. $\int_0^1 (2x+1)\,dx$ is

(a) 1 (b) 2 (c) 3 (d) 4

15. $\int_2^4 \frac{dx}{x}$ is

(a) log 4 (b) 0 (c) log 2 (d) log 8

16. $\int_0^\infty e^{-2x}\,dx$ is

(a) 0 (b) 1 (c) 2 (d) $\frac{1}{2}$

17. $\int_{-1}^1 x^3\,e^{x^4}\,dx$ is

(a) 1 (b) $2\int_0^1 x^3\,e^{x^4}\,dx$ (c) 0 (d) e^{x^4}

18. If $f(x)$ is a continuous function and $a<c<b$, then $\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$ is

(a) $\int_a^b f(x)\,dx-\int_a^c f(x)\,dx$

(b) $\int_a^c f(x)\,dx-\int_a^b f(x)\,dx$

(c) $\int_a^b f(x)\,dx$ (d) 0

19. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$ is

(a) 0 (b) 2 (c) 1 (d) 4

20. $\int_0^1 \sqrt{x^4(1-x)^2}\,dx$ is

(a) $\frac{1}{12}$ (b) $\frac{-7}{12}$ (c) $\frac{7}{12}$ (d) $\frac{-1}{12}$

21. If $\int_0^1 f(x)\,dx=1$, $\int_0^1 x\,f(x)\,dx=a$ and $\int_0^1 x^2\,f(x)\,dx=a^2$, then $\int_0^1 (a-x)^2\,f(x)\,dx$ is

(a) $4a^2$ (b) 0 (c) $2a^2$ (d) 1

22. The value of $\int_2^3 f(5-x)\,dx\ -\ \int_2^3 f(x)\,dx$ is

(a) 1 (b) 0 (c) −1 (d) 5

23. $\int_0^4\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$ is

(a) $\frac{20}{3}$ (b) $\frac{21}{3}$ (c) $\frac{28}{3}$ (d) $\frac{1}{3}$

24. $\int_0^{\frac{\pi}{3}} \tan x\,dx$ is

(a) log 2 (b) 0 (c) $\log\sqrt{2}$ (d) 2 log 2

25. Using the factorial representation of the gamma function, which of the following is the solution for the gamma function $\Gamma(n)$ when $n=8$

(a) 5040 (b) 5400 (c) 4500 (d) 5540

26. $\Gamma(n)$ is

(a) $(n-1)!$ (b) $n!$

(c) $n\Gamma(n)$ (d) $(n-1)\Gamma(n)$

27. $\Gamma(1)$ is

(a) 0 (b) 1 (c) n (d) $n!$

28. If $n>0$, then $\Gamma(n)$ is

(a) $\int_0^1 e^{-x}x^{n-1}dx$ (b) $\int_0^1 e^{-x}x^{n}dx$

(c) $\int_0^\infty e^{x}x^{-n}dx$ (d) $\int_0^\infty e^{-x}x^{n-1}dx$

29. $\Gamma\left(\frac{3}{2}\right)$

(a) $\sqrt{\pi}$ (b) $\frac{\sqrt{\pi}}{2}$ (c) $2\sqrt{\pi}$ (d) $\frac{3}{2}$

30. $\int_0^\infty x^4 e^{-x} dx$ is

(a) 12 (b) 4 (c) 4! (d) 64

Miscellaneous problems

Evaluate the following integrals:

1. $\int \frac{1}{\sqrt{x+2}-\sqrt{x+3}} dx$
2. $\int \frac{dx}{2-3x-2x^2}$
3. $\int \frac{dx}{e^x+6+5e^{-x}}$
4. $\int \sqrt{2x^2-3}\, dx$
5. $\int \sqrt{9x^2+12x+3}\, dx$
6. $\int (x+1)^2 \log x\, dx$
7. $\int \log\left(x-\sqrt{x^2-1}\right) dx$
8. $\int_0^1 \sqrt{x(x-1)}\, dx$
9. $\int_{-1}^1 x^2 e^{-2x} dx$
10. $\int_0^3 \frac{x\, dx}{\sqrt{x+1}+\sqrt{5x+1}}$

Summary

In this chapter, we have acquired the knowledge of

- **The relation between the Primitive function and the derived function:**

  A function $F(x)$ is said to be a primitive function of the derived function $f(x)$, if $\frac{d}{dx}[F(x)] = f(x)$

- **Integration of a function:**

  The process of determining an integral of a given function is defined as integration of a function

- **Properties of indefinite integrals:**

  $\int a f(x)\, dx = a\int f(x)\, dx$

  $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

- **Standard results of indefinite integrals:**

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
2. $\int \frac{1}{x} dx = \log|x| + c$
3. $\int e^x dx = e^x + c$
4. $\int a^x dx = \frac{1}{\log a} a^x + c, a > 0$ and $a \neq 1$
5. $\int \sin x\, dx = -\cos x + c$
6. $\int \cos x\, dx = \sin x + c$
7. $\int \sec^2 x\, dx = \tan x + c$
8. $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$
9. $\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c, n \neq -1$
10. $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$
11. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

12. $\int u\,dv = uv - \int v\,du$

13. $\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$

14. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

15. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$

16. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + c$

17. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + c$

18. $\int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + c$

19. $\int e^{ax}\left[a\,f(x) + f'(x)\right]dx = e^{ax} f(x) + c$

20. $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$

21. $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + c$

- **Definite integral:**

  Let $f(x)$ be a continuous function on [a,b] and if $F(x)$ is anti derivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

- **Properties of definite integrals:**

  (i) $\int_a^b f(x)dx = \int_a^b f(t)dt$

  (ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

  (iii) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

  (iv) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

  (v) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

  (vi) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

  (vii) a) If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

  b) If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

- **Particular case of Gamma Integral:**

  If n is a positive integer, then $\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$

- **Properties of gamma function:**

  (i) $\Gamma(n) = (n-1)\Gamma(n-1),\ n > 1$

  (ii) $\Gamma(n+1) = n\Gamma(n),\ n > 0$

  (iii) $\Gamma(n+1) = n!$, n is a positive integer

  (iv) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

- **Definite integral as the limit of a sum:**

  Let $f(x)$ be a continuous real valued function in $[a, b]$, which is divided into n equal parts each of width h, then

  $$\int_a^b f(x)dx = \lim_{\substack{h\to 0\\ n\to\infty}} \sum_{r=1}^{n} h\, f(a+rh) \quad \text{where } h = \frac{b-a}{n}$$

- **Results:**

(i) $1+2+3+......n = \dfrac{n(n+1)}{2} = \sum_{r=1}^{n} r$

(ii) $1^2+2^2+3^2+......n^2 = \dfrac{n(n+1)(2n+1)}{6} = \sum_{r=1}^{r=n} r^2$

(iii) $1^3+2^3+3^3+......n^3 = \left[\dfrac{n(n+1)}{2}\right]^2 = \sum_{r=1}^{r=n} r^3$

GLOSSARY (கலைச்சொற்கள்)

Abscissa	மட்டாயம் / கிடை தூரம்
Abstract value	நுண்ம மதிப்பு / புலனாகாத மதிப்பு
Algebraic function	அறமச் சார்பு / இயற் கணிதச் சார்பு
Anti derivative	எதிர்மறை வகையீடு
Approaches	அணுகுதல் / நெருங்குதல்
Approximate	தோராயம்
Arbitrary	ஏதேனுமொரு / தன்னிச்சையான
Between the specified limits	குறிப்பிடப்பட்ட எல்லைகளுக்குள்
Bounded region	வரம்புள்ள பகுதி
Certain	நிச்சயமான / உறுதியான
Change of variable	மாறியின் மாற்றம்
Closed interval	மூடிய இடைவெளி
Completing the square	வர்க்க நிறைவாக்கல்
Concept	கருத்துரு
Constant of integration	தொகையிடல் மாறிலி
Constant term	மாறா உறுப்பு
Continuous function	தொடர்ச்சியான சார்பு
Decomposition	பிரித்தல் / கூறாக்கல்
Definite integral	வரையறுத்த தொகை / வரையறுத்த தொகையீடு
Derived function	வருவித்தச் சார்பு
Differentiable function	வகையிடத்தக்கச் சார்பு
Differential coefficient	வகைக்கெழு / வகையீட்டு கெழு
Differentiation	வகையிடல்
Directly integrate	நேரிடையாக தொகையிட
Exponential function	அடுக்கைச் சார்பு / அடுக்குகுறிச் சார்பு
Family of curves	வளைவரைகளின் தொகுதி
Indefinite integral	அறுதியிடப்படாத தொகையீடு / வரையறாத் தொகையீடு
Infinity	முடிவிலி / கந்தழி / எண்ணிலி
Inscribe	உள்வரை
Instantaneous	உடனடியாக

Integrable function	தொகையிடத்தக்கச் சார்பு
Integral	தொகை / தொகையீடு
Integrand	தொகைக்காண் சார்பு / தொகைக் காண்பான்
Integrate	தொகையிடு / தொகையிட
Integration	தொகையிடல்
Integration by parts	பகுதிப் படுத்தித் தொகையிடல் / பகுதித் தொகையிடல்
Integration by substitution	பிரதியிட்டுத் தொகையிடல்
Integrator	தொகைப்பான்
Inverse trigonometric function	நேர் மாற்று கோணவியல் சார்பு / நேர்மாறு திரிகோணமிதி சார்பு
Logarithmic function	மடக்கை சார்பு
Lower limit	கீழ் எல்லை
Marginal function	இறுதிநிலைச் சார்பு
Natural logarithm	இயல் மடக்கை
Open interval	திறந்த இடைவெளி
Ordinate	குத்தாயம் / நிலைத் தூரம்
Parallel tangents	இணைத் தொடுகோடுகள்
Partial fraction	பகுதிப் பின்னம்
Positive integer	மிகை முழு எண்
Power rule	அடுக்கு விதி
Primitive function	மூலமுதலான சார்பு / தொடக்க நிலைச் சார்பு
Quantity	கணியம் / அளவு
Rationalisation method	காரணக்காரீய முறை
Reduction	குறைப்பு / சுருக்கல்
Repeated integral	தொடர் முறைத் தொகையீடு / மீண்டும் மீண்டும் தொகையிடப்பட்ட
Reverse process	எதிர் மறை செயல்
Standard form	திட்ட வடிவம்
Substitution	பிரதியிடல் / ஈடு செய்தல்
Successive	அடுத்தடுத்த / தொடர்ச்சியான
Successive derivatives	தொடர் வகையிடல்கள் / அடுத்தடுத்த வகையிடல்கள்
Suitable	ஏற்புடைய / பொருத்தமான
Summation	கூடுதல் / கூட்டல் தொகை
Technique	உத்தி / நுட்பம்
Trigonometric function	முக்கோண கணிப்பு சார்பு / திரிகோணமிதிச்சார்பு
Unique function	ஒருமைத் தன்மை கொண்ட சார்பு
Upper limit	மேல் எல்லை
Variable of integration	தொகையிடல் மாறி