

INDEFINITE INTEGRATION [JEE ADVANCED PREVIOUS YEAR SOLVED PAPER]

JEE ADVANCED

Single Correct Answer Type

1. The value of the integral $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx$ is

- a. $\sin x - 6 \tan^{-1}(\sin x) + C$
- b. $\sin x - 2(\sin x)^{-1} + C$
- c. $\sin x - 2(\sin x)^{-1} - 6 \tan^{-1}(\sin x) + C$
- d. $\sin x - 2(\sin x)^{-1} + 5 \tan^{-1}(\sin x) + C$

(IIT-JEE 1995)

2. $\int \frac{x^2 - 1}{x^3 \sqrt{2x^4 - 2x^2 + 1}} dx$ is equal to

- a. $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^3} + C$
- b. $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x} + C$
- c. $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^2} + C$
- d. $\frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C$

(IIT-JEE 2006)

3. Let $f(x) = \frac{x}{(1+x^n)^{1/n}}$ for $n \geq 2$ and $g(x) = \underbrace{(f \circ f \circ \dots \circ f)}_{f \text{ occurs } n \text{ times}}(x)$.

Then $\int x^{n-2} g(x) dx$ equals

- a. $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}} + K$
- b. $\frac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}} + K$
- c. $\frac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}} + K$
- d. $\frac{1}{n+1}(1+nx^n)^{1+\frac{1}{n}} + K$

(IIT-JEE 2007)

4. Let $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1} dx$, $J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} dx$. Then

for an arbitrary C , the value of $J - I$ equals

- a. $\frac{1}{2} \log \left(\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1} \right) + C$
- b. $\frac{1}{2} \log \left(\frac{e^{2x} + e^x + 1}{e^{2x} - e^x + 1} \right) + C$

c. $\frac{1}{2} \log \left(\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1} \right) + C$

d. $\frac{1}{2} \log \left(\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1} \right) + C$

(IIT-JEE 2008)

5. The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$ equals (for some arbitrary constant K)

- a. $-\frac{1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} - \frac{1}{7} (\sec x + \tan x)^2 \right\} + K$
- b. $\frac{1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} - \frac{1}{7} (\sec x + \tan x)^2 \right\} + K$
- c. $-\frac{1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + K$
- d. $\frac{1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + K$

(IIT-JEE 2012)

Assertion-Reasoning Type

1. Let $F(x)$ be an indefinite integral of $\sin^2 x$.

Statement 1: The function $F(x)$ satisfies $F(x + \pi) = F(x)$ for all real x .

Statement 2: $\sin^2(x + \pi) = \sin^2 x$ for all real x .

- a. Statement 1 is true, statement 2 is true; statement 2 is the correct explanation for statement 1.
- b. Statement 1 is true, statement 2 is true; statement 2 is NOT the correct explanation for statement 1.
- c. Statement 1 is true, statement 2 is false.
- d. Statement 1 is false, statement 2 is true.

(IIT-JEE 2007)

Fill in the Blanks Type

1. $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B \log(9e^{2x} - 4) + C$, then

$A = \underline{\hspace{1cm}}$, $B = \underline{\hspace{1cm}}$, $C = \underline{\hspace{1cm}}$. (IIT-JEE 1990)

Subjective Type

1. Evaluate $\int \frac{\sin x}{\sin x - \cos x} dx$.

(IIT-JEE 1978)

2. Evaluate $\int \frac{x^2}{(a+bx)^2} dx$.

(IIT-JEE 1979)

3. Evaluate the following integrals:

a. $\int \sqrt{1 + \sin\left(\frac{x}{2}\right)} dx$ b. $\int \frac{x^2}{\sqrt{1-x}} dx.$

(IIT-JEE 1980)

4. Evaluate $\int (e^{\log x} + \sin x) \cos x dx.$ (IIT-JEE 1981)

5. Evaluate $\int \frac{(x-1)e^x}{(x+1)^3} dx.$ (IIT-JEE 1983)

6. Evaluate $\int \frac{dx}{x^2(x^4+1)^{3/4}}.$ (IIT-JEE 1984)

7. Evaluate $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx.$ (IIT-JEE 1985)

8. Evaluate $\int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx.$ (IIT-JEE 1986)

9. Evaluate $\int \frac{\sqrt{\cos 2x}}{\sin x} dx.$ (IIT-JEE 1987)

10. Evaluate $\int (\sqrt{\tan x} + \sqrt{\cot x}) dx.$ (IIT-JEE 1989)

11. Evaluate $\int \left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} + \frac{\ln(1 + \sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}} \right) dx.$ (IIT-JEE 1992)

12. Evaluate $\int \cos 2\theta \ln \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) d\theta.$ (IIT-JEE 1994)

13. Evaluate $\int \frac{x+1}{x(1+xe^x)^2} dx.$ (IIT-JEE 1996)

14. Evaluate $\int \frac{1}{x} \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx.$ (IIT-JEE 1997)

15. Evaluate $\int \frac{x^3 + 3x + 2}{(x^2 + 1)^2(x+1)} dx.$ (IIT-JEE 1999)

16. Evaluate $\int \sin^{-1} \left(\frac{2x+2}{\sqrt{4x^2+8x+13}} \right) dx.$ (IIT-JEE 2001)

17. Evaluate for, $m \in N$,
 $\int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m} dx, x > 0.$ (IIT-JEE 2002)

Answer Key

JEE Advanced

Single Correct Answer Type

1. c. 2. d. 3. a. 4. c. 5. c.

Assertion-Reasoning Type

1. d

Fill in the Blanks Type

1. $A = -3/2, B = \frac{35}{36}$; C can have any real value.

Subjective Type

1. $\frac{1}{2} \log |\sin x - \cos x| + \frac{x}{2} + C$

2. $\frac{1}{b^3} \left[a + bx - 2a \log |a + bx| - \frac{a^2}{a + bx} \right] + C$

3. a. $\pm 4 \left[\sin \frac{x}{4} - \cos \frac{x}{4} \right] + C$

b. $-2 \left[\frac{(1-x)^{5/2}}{5} - \frac{2(1-x)^{3/2}}{3} + \sqrt{1-x} \right] + C$

4. $x \sin x + \cos x - \frac{1}{4} \cos 2x + C$

5. $\frac{e^x}{(x+1)^2} + C$

6. $-\left(1 + \frac{1}{x^4}\right)^{1/4} + C$

7. $-2 \sqrt{1-x} + \cos^{-1} \sqrt{x} + \sqrt{x} \sqrt{1-x} + C$

8. $\frac{2}{\pi} [\sqrt{x-x^2} - (1-2x) \sin^{-1} \sqrt{x}] - x + C$

9. $-\log |y + \sqrt{y^2+1}| - \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2y^2+2}-y}{\sqrt{2y^2+2}+y} \right| + C$

where $\cot^2 x = 1 + y^2$

10. $\sqrt{2} \sin^{-1}(\sin x - \cos x) + C$

12. $\sin 2\theta \ln \sqrt{\frac{\sin \theta + \cos \theta}{\cos \theta - \sin \theta}} - \frac{1}{2} \ln |\sec 2\theta| + C$

13. $-\log \left(\frac{1 + xe^x}{xe^x} \right) - \frac{1}{1 + xe^x} + C$

14. $-4 \left[\log \left| \frac{1 + \sqrt{1-x}}{\sqrt{x}} \right| - \cos^{-1} \sqrt{x} \right] + C$

15. $\frac{1}{4} \log \left| \frac{x^2 + 1}{(x+1)^2} \right| + \frac{3}{2} \tan^{-1} x + \frac{x}{1+x^2} + C$

16. $\frac{3}{2} \left\{ \frac{2}{3} (x+1) \tan^{-1} \left(\frac{2}{3} (x+1) \right) - \log \sqrt{4x^2 + 8x + 13} \right\} + C$

17. $\frac{1}{6(m+1)} (2x^{3m} + 3x^{2m} + 6x^m)^{(m+1)/m} + C$

Hints and Solutions

JEE Advanced

Single Correct Answer Type

$$\begin{aligned}
 1. \text{ c. Let } I &= \int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx \\
 &= \int \frac{(\cos^2 x + \cos^4 x) \cos x}{\sin^2 x (1 + \sin^2 x)} dx \\
 &= \int \frac{[1 - \sin^2 x + (1 - \sin^2 x)^2] \cos x}{\sin^2 x (1 + \sin^2 x)} dx \\
 &= \int \frac{(2 - 3 \sin^2 x + \sin^4 x) \cos x}{\sin^2 x (1 + \sin^2 x)} dx
 \end{aligned}$$

Put $\sin x = t$ or $\cos x dx = dt$

$$\begin{aligned}
 \therefore I &= \int \frac{2 - 3t^2 + t^4}{t^4 + t^2} dt \\
 &= \int \left(1 + \frac{2}{t^2} - \frac{6}{t^2 + 1} \right) dt \\
 &= t - \frac{2}{t} - 6 \tan^{-1}(t) + C \\
 &= \sin x - 2(\sin x)^{-1} - 6 \tan^{-1}(\sin x) + C
 \end{aligned}$$

$$\begin{aligned}
 2. \text{ d. } I &= \int \frac{x^2 - 1}{x^3 \sqrt{2x^4 - 2x^2 + 1}} dx \\
 &= \frac{1}{4} \int \frac{\frac{4}{x^3} - \frac{4}{x^5}}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}} dx \\
 \text{Put } 2 - \frac{2}{x^2} + \frac{1}{x^4} &= t \quad \therefore \left(\frac{4}{x^3} - \frac{4}{x^5} \right) dx = dt \\
 \therefore I &= \frac{1}{4} \int \frac{dt}{\sqrt{t}} = \frac{2\sqrt{t}}{4} + C \\
 &= \frac{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}{2} + C \\
 &= \frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 3. \text{ a. Here } f(f(x)) &= \frac{f(x)}{[1 + f(x)^n]^{1/n}} \\
 &= \frac{x}{(1 + 2x^n)^{1/n}} \\
 f(f(f(x))) &= \frac{x}{(1 + 3x^n)^{1/n}} \\
 \Rightarrow g(x) &= (\text{fofo} \dots \text{of})(x) = \frac{x}{(1 + nx^n)^{1/n}}
 \end{aligned}$$

$$\begin{aligned}
 \text{Hence, } I &= \int x^{n-2} g(x) dx \\
 &= \int \frac{x^{n-1} dx}{(1+nx^n)^{1/n}} \\
 &= \frac{1}{n^2} \int \frac{n^2 x^{n-1} dx}{(1+nx^n)^{1/n}} \\
 &= \frac{1}{n^2} \int \frac{d(1+nx^n)}{(1+nx^n)^{1/n}} dx \\
 &= \frac{1}{n(n-1)} (1+nx^n)^{1-\frac{1}{n}} + K
 \end{aligned}$$

$$\begin{aligned}
 4. \text{ c. } J - I &= \int \frac{e^x(e^{2x}-1)}{e^{4x}+e^{2x}+1} dx \\
 &= \int \frac{(z^2-1)}{z^4+z^2+1} dz \text{ (where } z=e^x) \\
 &= \int \frac{\left(1-\frac{1}{z^2}\right) dz}{\left(z+\frac{1}{z}\right)^2-1} \\
 &= \int \frac{dt}{t^2-1} \text{ (where } z+\frac{1}{z}=t) \\
 &= \frac{1}{2} \log_e \left(\frac{t-1}{t+1} \right) + C \\
 &= \frac{1}{2} \log_e \left(\frac{e^x+e^{-x}-1}{e^x+e^{-x}+1} \right) + C \\
 &= \frac{1}{2} \log_e \left(\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1} \right) + C
 \end{aligned}$$

$$\begin{aligned}
 5. \text{ c. } I &= \int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx \\
 \text{Let } \sec x + \tan x &= t \\
 \therefore \sec x - \tan x &= 1/t \\
 \text{Now, } (\sec x \tan x + \sec^2 x) dx &= dt \\
 \text{or } \sec x (\sec x + \tan x) dx &= dt \\
 \text{or } \sec x dx &= \frac{dt}{t}, \frac{1}{2} \left(t + \frac{1}{t} \right) = \sec x \\
 \therefore I &= \frac{1}{2} \int \frac{\left(t + \frac{1}{t} \right) dt}{t^{9/2}} \\
 &= \frac{1}{2} \int (t^{-9/2} + t^{-13/2}) dt \\
 &= \frac{1}{2} \left[\frac{t^{-9/2+1}}{-\frac{9}{2}+1} + \frac{t^{-13/2+1}}{-\frac{13}{2}+1} \right] + K \\
 &= \frac{1}{2} \left[\frac{t^{-7/2}}{-\frac{7}{2}} + \frac{t^{-11/2}}{-\frac{11}{2}} \right] + K
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{7} t^{-7/2} - \frac{1}{11} t^{-11/2} + K \\
 &= -\frac{1}{7} \frac{1}{t^{7/2}} - \frac{1}{11} \frac{1}{t^{11/2}} + K \\
 &= -\frac{1}{t^{11/2}} \left(\frac{1}{11} + \frac{t^2}{7} \right) = -\frac{1}{(\sec x + \tan x)^{11/2}} \\
 &\quad \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + K
 \end{aligned}$$

Assertion-Reasoning Type

$$1. \text{ d. } F(x) = \int \sin^2 x dx = \int \frac{1 - \cos 2x}{2} dx$$

$$\Rightarrow F(x) = \frac{1}{4} (2x - \sin 2x) + c$$

Since $F(x + \pi) \neq F(x)$

Hence, statement 1 is false.

But statement 2 is true as $\sin^2 x$ is periodic with period π .

Fill in the Blanks Type

$$1. \text{ We have } \int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B \ln(9e^{2x} - 4) + C$$

Differentiating both sides w.r.t. x , we get

$$\frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} = A + \frac{18B e^{2x}}{9e^{2x} - 4} = A + \frac{18B e^x}{9e^x - 4e^{-x}}$$

$$\text{or } \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} = \frac{(9A + 18B)e^x - 4Ae^{-x}}{9e^x - 4e^{-x}}$$

$$\text{or } 9A + 18B = 4, -4A = 6$$

$$\text{or } A = -3/2, B = \left(4 + \frac{27}{2}\right) \frac{1}{18} = \frac{35}{36}$$

C can have any real value.

Subjective Type

$$\begin{aligned}
 1. \quad I &= \int \frac{\sin x}{\sin x - \cos x} dx \\
 &= \frac{1}{2} \int \frac{2 \sin x}{\sin x - \cos x} dx \\
 &= \frac{1}{2} \int \frac{\sin x + \cos x + \sin x - \cos x}{\sin x - \cos x} dx \\
 &= \int \frac{\cos x + \sin x}{\sin x - \cos x} dx + \frac{1}{2} \int dx \\
 &= \frac{1}{2} \log |\sin x - \cos x| + \frac{x}{2} + C
 \end{aligned}$$

$$\begin{aligned}
 2. \quad I &= \int \frac{x^2 dx}{(a + bx)^2} \\
 \text{Let } a + bx &= t \text{ or } x = \left(\frac{t - a}{b} \right)
 \end{aligned}$$

$$\text{or } dx = \frac{dt}{b}$$

$$\begin{aligned}\therefore I &= \frac{1}{b^3} \int \frac{t^2 - 2at + a^2}{t^2} dt \\ &= \frac{1}{b^3} \int \left(1 - \frac{2a}{t} + \frac{a^2}{t^2} \right) dt \\ &= \frac{1}{b^3} \left[t - 2a \log |t| - \frac{a^2}{t} \right] + C \\ &= \frac{1}{b^3} \left[a + bx - 2a \log |a + bx| - \frac{a^2}{a + bx} \right] + C\end{aligned}$$

$$\begin{aligned}3. \text{ a. } \int \sqrt{1 + \sin\left(\frac{x}{2}\right)} dx &= \int \sqrt{\sin^2 \frac{x}{4} + \cos^2 \frac{x}{4} + 2 \sin \frac{x}{4} \cos \frac{x}{4}} dx \\ &= \pm \int \left(\sin \frac{x}{4} + \cos \frac{x}{4} \right) dx \\ &= \pm \left[\frac{-\cos x/4}{1/4} + \frac{\sin x/4}{1/4} \right] + C \\ &= \pm 4 \left[\sin \frac{x}{4} - \cos \frac{x}{4} \right] + C\end{aligned}$$

$$\text{b. } I = \int \frac{x^2}{\sqrt{1-x}} dx$$

$$\text{Let } 1-x = t^2 \text{ or } dx = -2t dt$$

$$\begin{aligned}\therefore I &= \int \frac{(1-t^2)^2}{t} (-2t) dt \\ &= -2 \int (t^4 - 2t^2 + 1) dt \\ &= -2 \left[\frac{t^5}{5} - \frac{2t^3}{3} + t \right] + C \\ &= -2 \left[\frac{(1-x)^{5/2}}{5} - \frac{2(1-x)^{3/2}}{3} + \sqrt{1-x} \right] + C\end{aligned}$$

$$\begin{aligned}4. \int (e^{\log x} + \sin x) \cos x dx &= \int (x + \sin x) \cos x dx \quad [\text{Using } e^{\log x} = x] \\ &= \int x \cos x + \frac{1}{2} \int \sin 2x dx \\ &= x \sin x - \int \sin x dx + \frac{1}{2} \left(\frac{-\cos 2x}{2} \right) + C \\ &= x \sin x + \cos x - \frac{1}{4} \cos 2x + C\end{aligned}$$

Alternative Method:

$$\begin{aligned}\int (e^{\log x} + \sin x) \cos x dx &= \int (x + \sin x) \cos x dx \\ &= \int (x + \sin x)(1 + \cos x - 1) dx \\ &= \int (x + \sin x)(1 + \cos x) dx - \int (x + \sin x) dx\end{aligned}$$

$$\begin{aligned}&= \frac{(x + \sin x)^2}{2} - \frac{x^2}{2} + \cos x + C \\ &= \frac{x^2 + 2x \sin x + \sin^2 x}{2} - \frac{x^2}{2} + \cos x + C \\ &= \frac{2x \sin x + \sin^2 x}{2} + \cos x + C \\ &= x \sin x + \frac{1 - \cos 2x}{4} + \cos x + C \\ &= x \sin x - \frac{\cos 2x}{4} + \cos x + C\end{aligned}$$

$$\begin{aligned}5. I &= \int \frac{(x-1)e^x}{(x+1)^3} dx \\ &= \int \frac{(x+1-2)e^x}{(x+1)^3} dx \\ &= \int \left[\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3} \right] e^x dx \\ &= \int \left[\frac{1}{(x+1)^2} + \left(\frac{1}{(x+1)^2} \right)' \right] e^x dx \\ &= \frac{e^x}{(x+1)^2} + C\end{aligned}$$

$$6. \text{ Let } I = \frac{dx}{x^2(x^4+1)^{3/4}} = \int \frac{dx}{x^3 \cdot x^2 \left(1 + \frac{1}{x^4} \right)^{3/4}}$$

$$\text{Put } 1 + \frac{1}{x^4} = t \text{ or } \frac{-4}{x^5} dx = dt \text{ or } \frac{dx}{x^5} = -\frac{dt}{4}$$

$$\begin{aligned}\therefore I &= \int \frac{-dt}{4 t^{3/4}} = \frac{-1}{4} \frac{t^{-3/4+1}}{-3/4+1} + C \\ &= -t^{1/4} + C = -\left(1 + \frac{1}{x^4} \right)^{1/4} + C\end{aligned}$$

$$7. I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx$$

$$\text{Put } x = \cos^2 \theta \text{ or } dx = -2 \cos \theta \sin \theta d\theta$$

$$\begin{aligned}\therefore I &= -\int \sqrt{\frac{1-\cos \theta}{1+\cos \theta}} 2 \sin \theta \cos \theta d\theta \\ &= -\int \frac{\sin \theta/2}{\cos \theta/2} 2 \cdot 2 \sin \theta/2 \cos \theta/2 \cos \theta d\theta \\ &= -2 \int (1 - \cos \theta) \cos \theta d\theta \\ &= -2 \int (\cos \theta - \cos^2 \theta) d\theta \\ &= -2 \int \left(\cos \theta - \frac{1 + \cos 2\theta}{2} \right) d\theta\end{aligned}$$

$$\begin{aligned}
&= -2 \left[\sin \theta - \frac{1}{2} \left(\theta + \frac{\sin 2\theta}{2} \right) \right] + C \\
&= -2\sqrt{1-x} + \frac{2}{2} \left[\cos^{-1} \sqrt{x} + \sqrt{x} \sqrt{1-x} \right] + C \\
&\quad \text{[Using } \sin \theta = \sqrt{1-x} \text{]} \\
&= -2\sqrt{1-x} + \cos^{-1} \sqrt{x} + \sqrt{x} \sqrt{1-x} + C
\end{aligned}$$

8. Let $I = \int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx$

We know that

$$\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x} = \pi/2 \quad (1)$$

$$\text{Also, } \cos^{-1} \sqrt{x} = \pi/2 - \sin^{-1} \sqrt{x} \quad (2)$$

Using equations (1) and (2), we get

$$\begin{aligned}
I &= \int \frac{\sin^{-1} \sqrt{x} - (\pi/2 - \sin^{-1} \sqrt{x})}{\pi/2} dx \\
&= \frac{2}{\pi} \int (2 \sin^{-1} \sqrt{x} - \pi/2) dx \\
&= \frac{4}{\pi} \int \sin^{-1} \sqrt{x} dx - \int 1 dx
\end{aligned}$$

Let $x = \sin^2 \theta$ or $dx = 2 \sin \theta \cos \theta d\theta$

$$\begin{aligned}
\therefore I &= \frac{4}{\pi} \int \sin^{-1} (\sin \theta) 2 \sin \theta \cos \theta d\theta - x + C \\
&= \frac{4}{\pi} \int \theta \sin 2\theta d\theta - x + C \\
&= \frac{4}{\pi} \left[\frac{-\theta \cos 2\theta}{2} + \int 1 \times \frac{\cos 2\theta}{2} d\theta \right] - x + C \\
&\quad \text{[Integrating by parts]} \\
&= \frac{4}{\pi} \left[\frac{-\theta \cos 2\theta}{2} + \frac{\sin 2\theta}{4} \right] - x + C \\
&= \frac{4}{4 \times \pi} [-2 \sin^{-1} \sqrt{x} (1-2x) + 2 \sqrt{x} \sqrt{1-x}] - x + C \\
&= \frac{2}{\pi} [\sqrt{x-x^2} - (1-2x) \sin^{-1} \sqrt{x}] - x + C
\end{aligned}$$

9. $I = \int \frac{\sqrt{\cos 2x}}{\sin x} dx$

$$= \int \frac{\sqrt{\cos^2 x - \sin^2 x}}{\sin x} dx$$

$$= \int \sqrt{\cot^2 x - 1} dx$$

$$\text{Put } \cot^2 x - 1 = y^2$$

$$\text{or } \cot^2 x = 1 + y^2$$

$$\text{or } -2 \cot x \operatorname{cosec}^2 x dx = 2y dy$$

$$\text{or } dx = \frac{-y dy}{\sqrt{1+y^2} (2+y^2)}$$

$$\therefore I = - \int \frac{y \times y dy}{\sqrt{1+y^2} (2+y^2)}$$

$$= - \int \frac{1}{\sqrt{y^2+1}} dy + 2 \int \frac{dy}{(y^2+2)\sqrt{y^2+1}}$$

$$= -\log |y + \sqrt{y^2+1}| + 2I_1$$

(1)

$$\text{where } I_1 = \int \frac{dy}{(y^2+2)\sqrt{y^2+1}}$$

$$\text{Put } y = \frac{1}{t} \text{ or } dy = -\frac{dt}{t^2}$$

$$\begin{aligned}
\therefore I_1 &= \int \frac{-\frac{dt}{t^2}}{\left(\frac{1}{t^2}+2\right)\sqrt{\frac{1}{t^2}+1}} \\
&= - \int \frac{t dt}{(1+2t^2)\sqrt{t^2+1}}
\end{aligned}$$

$$\text{Now let } t^2 + 1 = z^2$$

$$\text{or } t dt = z dz$$

$$\therefore I_1 = - \int \frac{z dz}{(1+2(z^2-1))z}$$

$$= - \int \frac{dz}{2z^2-1}$$

$$= -\frac{1}{2} \int \frac{dz}{z^2 - \frac{1}{2}}$$

$$= -\frac{1}{2\sqrt{2}} \log \left| \frac{z - \frac{1}{\sqrt{2}}}{z + \frac{1}{\sqrt{2}}} \right|$$

$$= -\frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{\frac{1}{y^2+1}} - \frac{1}{\sqrt{2}}}{\sqrt{\frac{1}{y^2+1}} + \frac{1}{\sqrt{2}}} \right| + C$$

$$= -\frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2y^2+2}-y}{\sqrt{2y^2+2}+y} \right| + C$$

$$\therefore I = -\log |y + \sqrt{y^2+1}| - \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2y^2+2}-y}{\sqrt{2y^2+2}+y} \right| + C,$$

$$\text{where } \cot^2 x = 1 + y^2$$

10. $\int (\sqrt{\tan x} + \sqrt{\cot x}) dx$

$$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx$$

$$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

$$\begin{aligned}
&= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}} dx \\
&= \sqrt{2} \int \frac{dt}{\sqrt{1-t^2}}, \text{ where } t = \sin x - \cos x \\
&= \sqrt{2} \sin^{-1} t + C \\
&= \sqrt{2} \sin^{-1} (\sin x - \cos x) + C
\end{aligned}$$

$$11. \int \left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} + \frac{\ln(1 + \sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}} \right) dx$$

Let $I = \underbrace{\int \frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} dx}_{I_1} + \underbrace{\int \frac{\ln(1 + \sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}} dx}_{I_2}$ (1)

$$I_1 = \int \frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} dx$$

$$\text{Let } x = y^{12} \text{ so that } dx = 12 y^{11} dy$$

$$\begin{aligned}
\therefore I_1 &= \int \frac{12 y^{11}}{y^4 + y^3} dy = 12 \int \frac{y^8}{1+y} dy \\
&= 12 \int \left(y^7 - y^6 + y^5 - y^4 + y^3 - y^2 + y - 1 + \frac{1}{y+1} \right) dy \\
&= 12 \left[\frac{y^8}{8} - \frac{y^7}{7} + \frac{y^6}{6} - \frac{y^5}{5} + \frac{y^4}{4} - \frac{y^3}{3} + \frac{y^2}{2} - y + \log|y+1| \right] + C \\
&= \frac{2}{3} x^{2/3} - \frac{12}{7} x^{7/12} + 2x^{1/2} - \frac{12}{5} x^{5/12} + 3x^{1/3} \\
&\quad - 4x^{1/4} + 6x^{1/6} - 12x^{1/12} + 12 \log|x^{1/12} + 1| + C_1 \quad (2)
\end{aligned}$$

$$I_2 = \int \frac{\ln(1 + x^{1/6})}{x^{1/3} + x^{1/2}} dx$$

$$\text{Let } x = z^6 \text{ so that } dx = 6z^5 dz$$

$$\begin{aligned}
\therefore I_2 &= \int \frac{\ln(1+z)}{z^2 + z^3} 6z^5 dz \\
&= \int \frac{6z^3 \ln(z+1)}{z+1} dz
\end{aligned}$$

$$\text{Put } z+1 = t \text{ or } dz = dt$$

$$\begin{aligned}
\therefore I_2 &= \int \frac{6(t-1)^3 \ln t}{t} dt \\
&= 6 \int \left(t^2 - 3t + 3 - \frac{1}{t} \right) \ln t dt \\
&= 6 \left[\int (t^2 - 3t + 3) \ln t dt - \int \frac{1}{t} \ln t dt \right] \\
&= 6 \left[\left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t \right) \ln t - \int \left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t \right) \frac{1}{t} dt - \frac{(\ln t)^2}{2} \right] + C \\
&= 6 \left[\left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t \right) \ln t - \left(\frac{t^3}{9} - \frac{3t^2}{4} + 3t \right) - \frac{(\ln t)^2}{2} \right] + C \quad (3)
\end{aligned}$$

Thus, we get the value of I on substituting the values of I_1 and I_2 from equations (2) and (3) in equation (1).

$$\begin{aligned}
12. \text{ Let } \int \cos 2\theta \ln \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) d\theta \\
&= \ln \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) \int \cos 2\theta d\theta \\
&\quad - \int \frac{(\sin 2\theta)(\cos \theta - \sin \theta)}{2(\sin \theta + \cos \theta)} \frac{2}{(\cos \theta - \sin \theta)^2} d\theta \\
&= \frac{\sin 2\theta}{2} \ln \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) - I_1 \\
I_1 &= \int \frac{(\sin 2\theta)}{(\sin \theta + \cos \theta)(-\sin \theta + \cos \theta)} d\theta \\
&= \int \frac{\sin 2\theta}{\cos 2\theta} d\theta = \frac{1}{2} \ln |\sec 2\theta| \\
\therefore I &= \sin 2\theta \ln \sqrt{\frac{\sin \theta + \cos \theta}{\cos \theta - \sin \theta}} - \frac{1}{2} \ln |\sec 2\theta| + C
\end{aligned}$$

$$\begin{aligned}
13. I &= \int \frac{(x+1)}{x(1+xe^x)^2} dx \\
&= \int \frac{e^x(x+1)}{x e^x (1+xe^x)^2} dx
\end{aligned}$$

$$\text{Put } 1 + xe^x = t \text{ or } (xe^x + e^x) dx = dt$$

$$\begin{aligned}
&= \int \frac{dt}{(t-1)t^2} \\
&= \int \left(\frac{1}{1-t} + \frac{1}{t} + \frac{1}{t^2} \right) dt \quad (\text{using partial fractions}) \\
&= -\log|1-t| + \log|t| - \frac{1}{t} + C \\
&= -\log \left| \frac{t}{1-t} \right| - \frac{1}{t} + C \\
&= -\log \left| \frac{1+xe^x}{-xe^x} \right| - \frac{1}{1+xe^x} + C \\
&= -\log \left(\frac{1+xe^x}{xe^x} \right) - \frac{1}{1+xe^x} + C
\end{aligned}$$

$$14. \text{ Put } x = \cos^2 \theta \text{ or } dx = -2 \cos \theta \sin \theta d\theta$$

$$\begin{aligned}
\therefore \int \frac{1}{x} \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx &= \int \left(\frac{1-\cos \theta}{1+\cos \theta} \right)^{1/2} \left(\frac{-2 \cos \theta \sin \theta d\theta}{\cos^2 \theta} \right) dx \\
&= -\int \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \cdot \frac{4 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos \theta} d\theta \\
&= -\int \frac{4 \sin^2 \frac{\theta}{2}}{\cos \theta} d\theta
\end{aligned}$$

$$\begin{aligned}
&= -4 \int \frac{1 - \cos \theta}{\cos \theta} d\theta \\
&= -4 \int (\sec \theta - 1) d\theta \\
&= -4 [\log |\sec \theta + \tan \theta| - \theta] + C \\
&= -4 \left[\log \left| \frac{1}{\sqrt{x}} + \frac{\sqrt{1-x}}{\sqrt{x}} \right| - \cos^{-1} \sqrt{x} \right] + C \\
&= -4 \left[\log \left| \frac{1 + \sqrt{1-x}}{\sqrt{x}} \right| - \cos^{-1} \sqrt{x} \right] + C
\end{aligned}$$

$$\begin{aligned}
15. \int \frac{x^3 + 3x + 2}{(x^2 + 1)^2 (x + 1)} dx \\
&= \int \frac{x(x^2 + 1) + 2(x + 1)}{(x^2 + 1)^2 (x + 1)} dx \\
&= \int \frac{x}{(x^2 + 1)(x + 1)} dx + 2 \int \frac{dx}{(x^2 + 1)^2} \\
&= \int \left(\frac{x+1}{2(x^2 + 1)} - \frac{1}{2(x + 1)} \right) dx + \int \frac{2}{(x^2 + 1)^2} dx \\
&= \frac{1}{4} \int \frac{2x}{1 + x^2} dx + \frac{1}{2} \int \frac{dx}{x^2 + 1} - \int \frac{dx}{2(x + 1)} + \int \frac{2}{(x^2 + 1)^2} dx \\
&= \frac{1}{4} \log |x^2 + 1| + \frac{1}{2} \tan^{-1} x - \frac{1}{2} \log |x + 1| + 2I
\end{aligned}$$

where $I = \int \frac{dx}{(x^2 + 1)^2}$. Put $x = \tan \theta$ or $dx = \sec^2 \theta d\theta$. Then

$$\begin{aligned}
I &= \int \frac{\sec^2 \theta d\theta}{(1 + \tan^2 \theta)^2} \\
&= \int \cos^2 \theta d\theta \\
&= \int \frac{1 + \cos 2\theta}{2} d\theta \\
&= \frac{1}{2} \left(\theta + \frac{\sin 2\theta}{2} \right) + C \\
&= \frac{1}{2} \left(\theta + \frac{\tan \theta}{1 + \tan^2 \theta} \right) + C \\
&= \frac{1}{2} \tan^{-1} x + \frac{1}{2} \frac{x}{1 + x^2} + C
\end{aligned}$$

$\therefore$ Given integral

$$\begin{aligned}
&= \frac{1}{4} \log |x^2 + 1| + \frac{1}{2} \tan^{-1} x - \frac{1}{2} \log |x + 1| \\
&\quad + \tan^{-1} x + \frac{x}{1 + x^2} + C \\
&= \frac{1}{4} \log \left| \frac{x^2 + 1}{(x + 1)^2} \right| + \frac{3}{2} \tan^{-1} x + \frac{x}{1 + x^2} + C
\end{aligned}$$

$$\begin{aligned}
16. I &= \int \sin^{-1} \left(\frac{2x + 2}{\sqrt{4x^2 + 8x + 13}} \right) dx \\
&= \int \sin^{-1} \left(\frac{2x + 2}{\sqrt{(2x + 2)^2 + 3^2}} \right) dx \\
&\quad \text{[Put } 2x + 2 = 3 \tan \theta \text{ or } 2 dx = 3 \sec^2 \theta d\theta]
\end{aligned}$$

$$\begin{aligned}
&= \int \sin^{-1} \left(\frac{3 \tan \theta}{3 \sec \theta} \right) \frac{3}{2} \sec^2 \theta d\theta \\
&= \frac{3}{2} \int \theta \sec^2 \theta d\theta \\
&= \frac{3}{2} \{ \theta \tan \theta - \int \tan \theta d\theta \} \\
&= \frac{3}{2} \{ \theta \tan \theta - \log |\sec \theta| \} + C \\
&= \frac{3}{2} \left\{ \frac{2x + 2}{3} \tan^{-1} \left(\frac{2x + 2}{3} \right) - \log \left(\sqrt{1 + \left(\frac{2x + 2}{3} \right)^2} \right) \right\} + C \\
&= \frac{3}{2} \left\{ \frac{2}{3} (x + 1) \tan^{-1} \left(\frac{2}{3} (x + 1) \right) - \log \sqrt{4x^2 + 8x + 13} \right\} + C
\end{aligned}$$

$$\begin{aligned}
17. I &= \int (x^{3m} + x^{2m} + x^m) (2x^{2m} + 3x^m + 6)^{1/m} dx \\
&= \int \left(\frac{x^{3m} + x^{2m} + x^m}{x} \right) (x) (2x^{2m} + 3x^m + 6)^{1/m} dx \\
&= \int (x^{3m-1} + x^{2m-1} + x^{m-1}) (2x^{2m} + 3x^m + 6)^{1/m} dx \\
&\quad \text{Let } 2x^{3m} + 3x^{2m} + 6x^m = t \text{ or } dt = 6m(x^{3m-1} + x^{2m-1} + x^{m-1}) dx \\
&\therefore I = \int t^{1/m} \frac{dt}{6m} = \frac{1}{6m} \frac{t^{\frac{1}{m} + 1}}{\frac{1}{m} + 1} + C \\
&= \frac{1}{6(m+1)} (2x^{3m} + 3x^{2m} + 6x^m)^{(m+1)/m} + C
\end{aligned}$$