5.5 DISPERSION AND ABSORPTION OF LIGHT

5.200 In a travelling plane electromagnetic wave the intensity is simply the time averaged magnitude of the Poynting vector :-

$$I = \langle |\vec{E} \times \vec{H}| \rangle = \left\langle \sqrt{\frac{\varepsilon_0}{\mu_0}} E^2 \right\rangle = \langle c\, \varepsilon_0 E^2 \rangle$$

on using $$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}, \quad E\sqrt{\varepsilon_0} = H\sqrt{\mu_0}.$$

(see chapter 4.4 of the book).

Now time averaged value of E^2 is $E_0^2/2$ so

$$I = \frac{1}{2} c\, \varepsilon_0 E_0^2 \quad \text{or} \quad E_0 = \sqrt{\frac{2I}{c\, \varepsilon_0}},$$

(a) Represent the electric field at any point by $E = E_0 \sin \omega t$. Then for the electron we have the equation.

$$m\ddot{x} = e E_0 \sin \omega t$$

so $$x = -\frac{e E_0}{m \omega^2} \sin \omega t$$

The ampitude of the forced oscillation is

$$\frac{e E_0}{m \omega^2} = \frac{e}{m \omega^2} \sqrt{\frac{2I}{c\, \varepsilon_0}} = 5{\cdot}1 \times 10^{-16} \text{ cm}$$

The velocity amplitude is clearly

$$\frac{e E_0}{m \omega} = 5{\cdot}1 \times 10^{-16} \times 3{\cdot}4 \times 10^{15} = 1{\cdot}73 \text{ cm/sec}$$

(b) For the electric force

$F_e =$ amplitude of the electric force

$= e E_0$

For the magnetic force (which we have neglected above), it is

$$(e \text{v} B) = (e \text{v} \mu_0 H)$$

$$= e \text{v} E \sqrt{\varepsilon_0 \mu_0} = e \text{v} \frac{E}{c}$$

writing $\text{v} = -\text{v}_0 \cos \omega t$

where $$\text{v}_0 = \frac{e E_0}{m \omega}$$

we see that the magnetic force is apart from a sign

$$\frac{e \text{v}_0 E_0}{2c} \sin 2\omega t$$

Hence $\dfrac{F_m}{F_e}$ = Ratio of amplitudes of the two forces

$$= \frac{v_0}{2c} = 2{\cdot}9 \times 10^{-11}$$

This is negligible and justifies the neglect of magnetic field of the electromagnetic wave in calculating v_0.

5.201 (a) It turns out that one can neglect the spatial dependence of the electric field as well as the magnetic field. Thus for a typical electron

$$m\ddot{\vec{r}} = e\vec{E}_0 \sin \omega t$$

so $\vec{r} = -\dfrac{e\vec{E}_0}{m\omega^2} \sin \omega t$ (neglecting any nonsinusoidal part).

The ions will be practically unaffected. Then

$$\vec{P} = n_o e\vec{r} = -\frac{n_0 e^2}{m\omega^2}\vec{E}$$

and
$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \left(1 - \frac{n_0 e^2}{\varepsilon_0 m \omega^2}\right)\vec{E}$$

Hence the permittivity
$$\varepsilon = 1 - \frac{n_0 e^2}{\varepsilon_0 m \omega^2}.$$

(b) The phase velocity is given by

$$v = \omega/K = \frac{c}{\sqrt{\varepsilon}}$$

So
$$ck = \omega\sqrt{1 - \frac{\omega_P^2}{\omega^2}}, \quad \omega_P^2 = \frac{n_0 e^2}{\varepsilon_0 m}$$

$$\omega^2 = c^2 k^2 + \omega_P^2$$

Thus
$$v = c\sqrt{1 + \frac{\omega_P^2}{c^2 k^2}} = c\sqrt{1 + \left(\frac{n_0 e^2}{4\pi^2 m c^2 \varepsilon_0}\right)\lambda^2}$$

5.202 From the previous problem

$$n^2 = 1 - \frac{n_0 e^2}{\varepsilon_0 m \omega^2}$$

$$= 1 - \frac{n_0 e^2}{4\pi^2 \varepsilon_0 m v^2}$$

Thus $n_0 = (4\pi^2 v^2 m \varepsilon_0 / e^2)(1 - n^2) = 2{\cdot}36 \times 10^7\ \text{cm}^{-3}$

5.203 For hard x– rays, the electrons in graphite will behave as if nearly free and the formula of previous problem can be applied. Thus

$$n^2 = 1 - \frac{n_0 e^2}{\varepsilon_0 m \omega^2}$$

and
$$n = 1 - \frac{n_0 e^2}{2 \varepsilon_0 m \omega^2}$$

on taking square root and neglecting higher order terms.

So
$$n - 1 = -\frac{n_0 e^2}{2 \varepsilon_0 m \omega^2} = -\frac{n_0 e^2 \lambda^2}{8 \pi^2 \varepsilon_0 m e^2}$$

We calculate n_0 as follows : There are $6 \times 6{\cdot}023 \times 10^{23}$ electrons in 12 gms of graphite of density $1{\cdot}6$ gm/c.c. Thus

$$n_0 = \frac{6 \times 6{\cdot}023 \times 10^{23}}{(12/1{\cdot}6)} \text{ per c.c}$$

Using the values of other constants and $\lambda = 50 \times 10^{-12}$ metre we get

$$n - 1 = -5{\cdot}4 \times 10^{-7}$$

5.204 (a) The equation of the electron can (under the stated conditions) be written as

$$m \ddot{x} + \gamma \dot{x} + k x = e E_0 \cos \omega t$$

To solve this equation we shall find it convenient to use complex displacements. Consider the equation

$$m \ddot{z} + \gamma \dot{z} + k z = e E_0 e^{-i \omega t}$$

Its solution is

$$z = \frac{e E_0 e^{-i \omega t}}{-m \omega^2 - i \gamma \omega + k}$$

(we ignore transients.)

Writing
$$\beta = \frac{\gamma}{2m}, \quad \omega_0^2 = \frac{k}{m}$$

we find
$$z = \frac{e E_0}{m} e^{-i \omega t} \Big/ (\omega_o^2 - \omega^2 - 2 i \beta \omega)$$

Now x = Real part of z

$$= \frac{e E_0}{m} \cdot \frac{\cos(\omega t + \varphi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4 \beta^2 \omega^2}} = a \cos(\omega t + \varphi)$$

where
$$\tan \varphi = \frac{2 \beta \omega}{\omega^2 - \omega_0^2}$$

$$\left(\sin \varphi = -\frac{2 \beta \omega}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4 \beta^2 \omega^2}} \right).$$

(b) We calculate the power absorbed as

$$P = \langle F\dot{x} \rangle = \langle eE_0 \cos\omega t\,(-\omega a \sin(\omega t + \varphi)) \rangle$$

$$= eE_0 \cdot \frac{eE_0}{m}\frac{1}{2} \cdot \frac{2\beta\omega}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \cdot \omega = \left(\frac{eE_0}{m}\right)^2 \frac{\beta m \omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

This is clearly maximum when $\omega_0 = \omega$ because P can be written as

$$P = \left(\frac{eE_0}{m}\right)^2 \frac{\beta m}{\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + 4\beta^2}$$

and $$P_{max} = \frac{m}{4\beta}\left(\frac{eE_0}{m}\right)^2 \text{ for } \omega = \omega_0 .$$

P can also be calculated from $P = \langle \gamma \dot{x} \cdot \dot{x} \rangle$

$$= (\gamma \omega^2 a^2/2) = \frac{\beta m \omega^2 (eE_0/m)^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}.$$

5.205 Let us write the solutions of the wave equation in the form

$$A = A_0 e^{i(\omega t - kx)}$$

where $k = \frac{2\pi}{\lambda}$ and λ is the wavelength in the medium. If $n' = n + i\chi$, then

$$k = \frac{2\pi}{\lambda_0} n'$$

(λ_0 is the wavelength in vacuum) and the equation becomes

$$A = A_0 e^{\chi' x} \exp(i(\omega t_1 - k' x))$$

where $\chi' = \frac{2\pi}{\lambda_0}\chi$ and $k' = \frac{2\pi}{\lambda_0} n$. In real form,

$$A = A_0 e^{\chi' x} \cos(\omega t - k' x)$$

This represents a plane wave whose amplitude diminishes as it propagates to the right (provided $\chi' < 0$).

when $n' = i\chi$, then similarly

$$A = A_0 e^{\chi' x} \cos\omega t$$

(on putting $n = 0$ in the above equation).

This represents a standing wave whose amplitude diminishes as one goes to the right (if $\chi' < 0$). The wavelength of the wave is infinite ($k' = 0$).

Waves of the former type are realized inside metals as well as inside dielectrics when there is total reflection. (penetration of wave).

5.206 In the plasma radio waves with wavelengths exceeding λ_0 are not propagated. We interpret this to mean that the permittivity becomes negative for such waves. Thus

$$0 = 1 - \frac{n_0 e^2}{\varepsilon_0 m \omega^2} \quad \text{if} \quad \omega = \frac{2\pi c}{\lambda_0}$$

Hence
$$\frac{n_0 e^2 \lambda_0^2}{4\pi^2 \varepsilon_0 m c^2} = 1$$

or
$$n_0 = \frac{4\pi^2 \varepsilon_0 m c^2}{e^2 \lambda_0^2} = 1{\cdot}984 \times 10^9 \text{ per c.c}$$

5.207 By definition

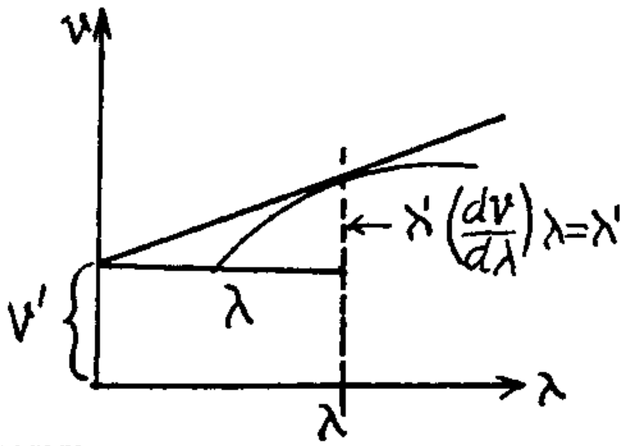
$$u = \frac{d\omega}{dk} = \frac{d}{dk}(\text{v}\,k) \quad \text{as } \omega = \text{v}\,k \;= \text{v} + k\frac{d\text{v}}{dk}$$

Now $\quad k = \dfrac{2\pi}{\lambda} \quad$ so $\; dk = -\dfrac{2\pi}{\lambda^2}d\lambda$

Thus $\quad u = \text{v} - \lambda \dfrac{d\text{v}}{d\lambda}.$

Its interpretation is the following :

$\left(\dfrac{d\text{v}}{d\lambda}\right)_{\lambda = \lambda'}$ is the slope of the v – λ curve at $\lambda = \lambda'$.



Thus as is obvious from the diagram

$\text{v}' = \text{v}(\lambda') - \lambda'\left(\dfrac{d\text{v}}{d\lambda}\right)_{\lambda = \lambda'}$ is the group velocity for $\lambda = \lambda'$.

5.208 (a) $\text{v} = a/\sqrt{\lambda}$, a = constant

Then
$$u = \text{v} - \lambda\frac{d\text{v}}{d\lambda}$$
$$= \frac{a}{\sqrt{\lambda}} - \lambda\left(-\frac{1}{2}a\lambda^{-3/2}\right) = \frac{3}{2}\cdot\frac{a}{\sqrt{\lambda}} = \frac{3}{2}\text{v}.$$

(b) $\text{v} = b\,k = \omega\,k$, b = constant

so $\quad \omega = b\,k^2 \quad$ and $\quad u = \dfrac{d\omega}{dk} = 2\,b\,k = 2\,\text{v}.$

(c) $\text{v} = \dfrac{c}{\omega^2}$, c = constant $= \dfrac{\omega}{k}$.

so $\quad \omega^3 = c\,k \quad$ or $\quad \omega = c^{1/3}k^{1/3}$

Thus
$$u = \frac{d\omega}{dk} = c^{1/3}\frac{1}{3}k^{-2/3} = \frac{1}{3}\frac{\omega}{k} = \frac{1}{3}\text{v}$$

5.209 We have

$$u\,\mathrm{v} = \frac{\omega}{k}\frac{d\omega}{dk} = c^2$$

Integrating we find

$$\omega^2 = A + c^2 k^2, \quad A \text{ is a constant.}$$

so
$$k = \frac{\sqrt{\omega^2 - A}}{c}$$

and
$$\mathrm{v} = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \dfrac{A}{\omega^2}}}$$

writing this as $c/\sqrt{\varepsilon(\omega)}$ we get $\varepsilon(\omega) = 1 - \dfrac{A}{\omega^2}$

(A can be +ve or negative)

5.210 The phase velocity of light in the vicinity of $\lambda = 534\,nm = \lambda_0$ is obtained as

$$\mathrm{v}(\lambda_0) = \frac{c}{n(\lambda_0)} = \frac{3 \times 10^8}{1{\cdot}640} = 1{\cdot}829 \times 10^8 \text{ m/s}$$

To get the group velocity we need to calculate

$\left(\dfrac{dn}{d\lambda}\right)_{\lambda = \lambda_0}$. We shall use linear

interpolation in the two intervals. Thus

$$\left(\frac{dn}{d\lambda}\right)_{\lambda = 521{\cdot}5} = -\frac{{\cdot}007}{25} = -28 \times 10^{-5} \text{ per nm}$$

$$\left(\frac{dn}{d\lambda}\right)_{\lambda = 561{\cdot}5} = -\frac{{\cdot}01}{55} = -18{\cdot}2 \times 10^{-5} \text{ per nm}$$

There $(dn/d\lambda)$ values have been assigned to the mid-points of the two intervals. Interpolating again we get

$$\left(\frac{dn}{d\lambda}\right)_{\lambda = 534} = \left[-28 + \frac{9{\cdot}8}{40} \times 12{\cdot}5\right] \times 10^{-5} \text{ per mm} = -24{\cdot}9 \times 10^{-5} \text{ per n m}.$$

Finally

$$u = \frac{c}{n} - \lambda \frac{d}{d\lambda}\left(\frac{c}{n}\right) = \frac{c}{n}\left[1 + \frac{\lambda}{n}\left(\frac{dn}{d\lambda}\right)\right]$$

At $\lambda = 534$

$$u = \frac{3 \times 10^8}{1{\cdot}640}\left[1 - \frac{534}{1{\cdot}640} \times 24{\cdot}9 \times 10^{-5}\right] \text{m/s} = 1{\cdot}59 \times 10^8 \text{ m/s}$$

5.211 We write

$$v = \frac{\omega}{k} = a + b\lambda$$

so $$\omega = k(a + b\lambda) = 2\pi b + ak.$$

$\left(\text{since } k = \frac{2\pi}{\lambda}\right)$. Suppose a wavetrain at time $t = 0$ has the form

$$F(x,0) = \int f(k) e^{ikx} dk$$

Then at time t it will have the form

$$F(x,t) = \int f(k) e^{ikx - i\omega t} dk$$

$$= \int f(k) e^{ikx - i(2\pi b + ak)t} = \int f(k)\, e^{ik(x-at)}\, e^{-i2\pi bt} dk$$

At $t = \frac{1}{b} = \tau$

$$F(x,\tau) = F(x - a\tau, 0)$$

so at time $t = \tau$ the wave train has regained its shape though it has advanced by $a\tau$.

5.212 On passing through the first (polarizer) Nicol the intensity of light becomes $\frac{1}{2}I_0$ because one of the components has been cut off. On passing through the solution the plane of polarization of the light beam will rotate by

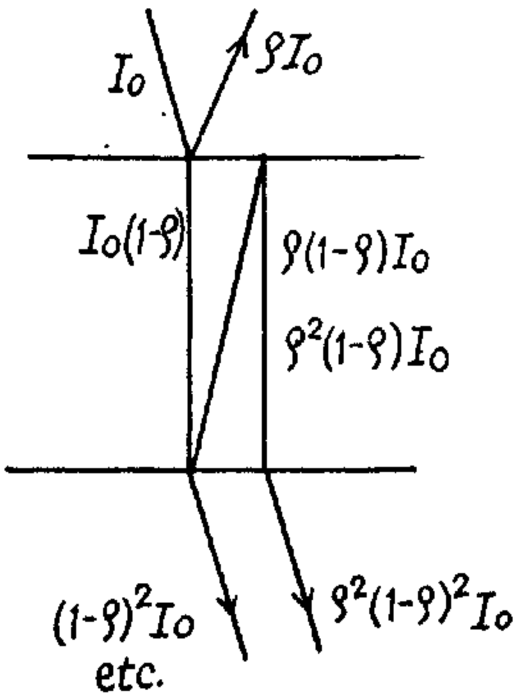
$\varphi = VlH$

and its intensity will also decrease by a factor $e^{-\chi l}$. The plane of vibraton of the light wave wll then make an angle $90° - \varphi$ with the principal direction of the analyzer Nicol. Thus by Malus' law the intensity of light coming out of the second Nicol will be

$$\frac{1}{2}I_0 \cdot e^{-\chi l} \cdot \cos^2(90° - \varphi)$$

$$= \frac{1}{2}I_0 e^{-\chi l} \sin^2\varphi.$$

5.213 (a) The multiple reflections are shown below. Transmission gives a factor $(1 - \rho)$ while reflections give factors of ρ. Thus the transmitted intensity assuming incoheren light is



$$(1-\rho)^2 I_0 + (1-\rho)^2 \rho^2 I_0 + (1-\rho)^2 \rho^4 I_0 + \ldots$$

$$= (1-\rho)^2 I_0 (1 + \rho^2 + \rho^4 + \rho^6 + \ldots)$$

$$= (1-\rho)^2 I_0 \times \frac{1}{1-\rho^2} = I_0 \frac{1-\rho}{1+\rho}.$$

(b) When there is absorption, we pick up a factor $\sigma = e^{-\chi d}$ in each traversal of the plate. Thus we get

$$(1-\rho)^2 \sigma I_0 + (1-\rho)^2 \sigma^3 \rho^2 I_0 + (1-\rho)^2 \sigma^5 \rho^4 I_0 + \ldots$$
$$= (1-\rho)^2 \sigma I_0 (1 + \sigma^2 \rho^2 + \sigma^4 \rho^4 + \ldots)$$
$$= I_0 \frac{\sigma(1-\rho)^2}{1-\sigma^2 \rho^2}$$

5.214 We have

$$\tau_1 = e^{-\chi d_1} (1-\rho)^2$$
$$\tau_2 = e^{-\chi d_2} (1-\rho)^2$$

where ρ is the reflectivity; see previous problem, multiple reflection have been ignored.

Thus
$$\frac{\tau_1}{\tau_2} = e^{\chi(d_2-d_1)}$$

or
$$\chi = \frac{\ln\left(\frac{\tau_1}{\tau_2}\right)}{d_2-d_1} = 0.35 \text{ cm}^{-1}.$$

5.215 On each surface we pick up a factor $(1-\rho)$ from reflection and a factor $e^{-\chi l}$ due to absorption in each plate.

Thus
$$\tau = (1-\rho)^{2N} e^{-\chi N l}$$

Thus
$$\chi = \frac{1}{N l} \ln \frac{(1-\rho)^{2N}}{\tau} = 0.034 \text{ cm}^{-1}.$$

5.216 Apart from the factor $(1-\rho)$ on each end face of the plate, we shall get a factor due to absorptions. This factor can be calculated by assuming the plate to consist of a large number of very thin slab within each of which the absorption coefficient can be assumed to be constant. Thus we shall get a product like

$$\ldots\ldots e^{-\chi(x)dx}\, e^{-\chi(x+dx)dx}\, e^{-\chi(x+2dx)dx} \ldots\ldots$$

This product is nothing but

$$e^{-\int_0^l \chi(x)\,dx}$$

Now $\chi(0) = \chi_1$, $\chi(l) = \chi_2$ and variation

with x is linear so $\chi(x) = \chi_1 + \frac{x}{l}(\chi_2 - \chi_1)$

Thus the factor becomes

$$e^{-\int_0^l \left[\chi_1 + \frac{x}{l}(\chi_2-\chi_1)\right]dx} = e^{-\frac{1}{2}(\chi_1+\chi_2)l}$$

5.217 The spectral density of the incident beam (i.e. intensity of the components whose wave length lies in the interval λ & $\lambda + d\lambda$) is

$$\frac{I_0}{\lambda_2 - \lambda_1} d\lambda, \; \lambda_1 \le \lambda \le \lambda_2$$

The absorption factor for this component is

$$e^{-\left[\chi_1 + \frac{\lambda + \lambda_1}{\lambda_2 - \lambda_1}(\chi_2 - \chi_1)\right] l}$$

and the transmission factor due to reflection at the surface is $(1 - \rho)^2$. Thus the intensity of the transmitted beam is

$$(1-\rho)^2 \frac{I_0}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} d\lambda \; e^{-l\left[\chi_1 + \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1}(\chi_2 - \chi_1)\right]}$$

$$= (1-\rho)^2 \frac{I_0}{\lambda_2 - \lambda_1} e^{-\chi_1 l} \left(\frac{1 - e^{-(\chi_2 - \chi_1) l}}{(\chi_2 - \chi_1)' me} \right) \chi(\lambda_2 - \lambda_1) = (1-\rho)^2 I_0 \frac{e^{-\chi_1 l} - e^{-\chi_2 l}}{(\chi_2 - \chi_1) l}$$

5.218 At the wavelength λ_0, the absorption coefficient vanishes and loss in transmission is entirely due to reflection. This factor is the same at all wavelengths and therefore cancels out in calculating the pass band and we need not worry about it. Now

$$T_0 = (\text{transmissivity at } \lambda = \lambda_0) = (1 - \rho)^2$$

$$T = \text{transmissivity at } \lambda = (1 - \rho)^2 e^{-\chi(\lambda) d}$$

The edges of the passband are $\lambda_0 \pm \dfrac{\Delta\lambda}{2}$ and at the edge

$$\frac{T}{T_0} = e^{-\alpha d \left(\frac{\Delta\lambda}{2\lambda_0}\right)^2} = \eta$$

Thus
$$\frac{\Delta\lambda}{2\lambda_0} = \sqrt{\left(\ln\frac{1}{\eta}\right) / \alpha d}$$

or
$$\Delta\lambda = 2\lambda_0 \sqrt{\frac{1}{\alpha d}\left(\ln\frac{1}{\eta}\right)}$$

5.219 We have to derive the law of decrease of intensity in an absorbing medium taking in to account the natural geometrical fall-off (inverse sequare law) as well as absorption.
Consider a thin spherical shell of thickness dx and internal radius x. Let $I(x)$ and $I(x + dx)$ be the intersities at the inner and outer surfaces of this shell.

Then
$$4\pi x^2 I(x) \; e^{-\chi dx} = 4\pi (x + dx)^2 I\,(x + dx)$$

Except for the factor $e^{-\chi dx}$ this is the usual equation. We rewrite this as

$$x^2 I(x) = I(x + dx)(x + dx)^2 (1 + \chi\, dx)$$

$$= \left(I + \frac{dI}{dx} dx \right) (x^2 + 2x\,dx) (1 + \chi\,dx)$$

or $$x^2 \frac{dI}{dx} + \chi x^2 I + 2xI = 0$$

Hence $$\frac{d}{dx} (x^2 I) + \chi (x^2 I) = 0$$

so $$x^2 I = C e^{-\chi x}$$

where C is a constant of integration.

In our case we apply this equation for $a \le x \le b$

For $x \le a$ the usual inverse square law gives

$$I(a) = \frac{\Phi}{4 \pi a^2}$$

Hence $$C = \frac{\Phi}{4\pi} e^{\chi a}$$

and $$I(b) = \frac{\Phi}{4 \pi b^2} e^{-\chi (b-a)}$$

This does not take into account reflections. When we do that we get

$$I(b) = \frac{\Phi}{4 \pi b^2} (1 - \rho)^2 e^{-\chi (b-a)}$$

5.220 The transmission factor is $e^{-\mu d}$ and so the intensity will decrease

$$e^{\mu d}$$

$$= e^{3\cdot6 \times 11\cdot3 \times 0\cdot1} = 58\cdot4 \text{ timestimes}$$

(we have used $\mu = (\mu/\rho) \times \rho$ and used the known value of density of lead).

5.221 We require $\mu_{Pb}\, d_{Pb} = \mu_{Al}\, d_{Al}$

or $$\left(\frac{\mu_{Pb}}{\rho_{Pb}} \right) \rho_{Pb}\, d_{Pb} = \left(\frac{\mu_{Al}}{\rho_{Al}} \right) \rho_{Al}\, d_{Al}$$

$$72\cdot0 \times 11\cdot3 \times d_{Pb} = 3\cdot48 \times 2\cdot7 \times 2\cdot6$$

$$d_{Pb} = 0\cdot3 \text{ mm}$$

5.222 $\frac{1}{2} = e^{-\mu d}$

or $$d = \frac{\ln 2}{\mu} = \frac{\ln 2}{\left(\frac{\mu}{\rho} \right) \rho} = 0\cdot80 \text{ cm}$$

5.223 We require N plates where

$$\left(\frac{1}{2} \right)^N = \frac{1}{50} \quad \text{So } N = \frac{\ln 50}{\ln 2} = 5\cdot6$$