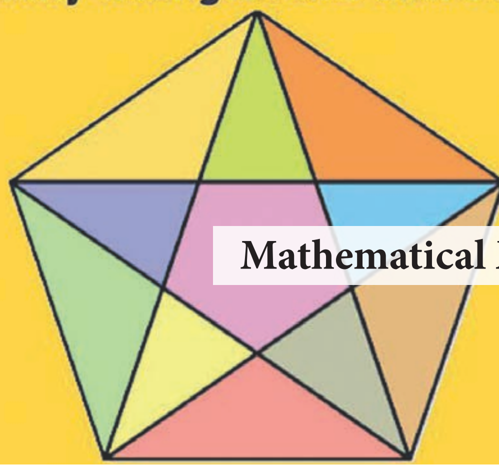
Chapter **7**

Mathematical Methods

How many triangles are there below?



Srinivasa Ramanujan

(22 December, 1887 - 26 April, 1920)

Srinivasa Ramanujan, was an Indian Mathematician born on December 22, 1887, at Erode inTamilnadu, whose contributions to the **theory of numbers** include pioneering discoveries of the properties of the partition function. In 1911 Ramanujan published the first of his papers in the ***Journal of the Indian Mathematical Society.*** His genius slowly gained recognition, and in 1913 he began a correspondence with the British mathematician **Godfrey H. Hardy** that led to a special scholarship from the University of Madras and a grant from Trinity College, Cambridge.

Overcoming his religious objections, Ramanujan traveled to England in 1914, where Hardy tutored him and collaborated with him in some research.In England Ramanujan made further advances, especially in the **partition of numbers.** His papers were published in English and European journals, and in 1918 he was elected to **the Royal Society of London**.

In 1917 Ramanujan had contracted tuberculosis, but his condition improved sufficiently for him to return to India in 1919. He died the following year at Kumbakonam in Tamilnadu on April 26,1920, generally unknown to the world at large but recognized by mathematicians as a phenomenal genius. Ramanujan left behind three notebooks and a sheaf of pages (also called the **"lost notebook"**) containing many unpublished results that mathematicians continued to verify long after his death..

"Numbers are my friends".

- Srinivasa Ramanujan

Learning Objectives

- Understands permutations and combinations
- Knows about the Binomial, Exponential and Logarithmic expansions
- Understands the concept of differentiation and applies in solving problems
- Understands the concept of integration and applies in solving problems

Introduction

Before we proceed to further statistical concepts and calculations, we need to have a very good knowledge on some more new mathematical concepts, rules and formulae to understand in a better way the theory and problems in statistics. Hence we introduce some algebraic methods and elementary calculus in this chapter.

7.1 Fundamental Principles of counting

We use two fundamental principles of counting in solving the problems. They are addition rule on counting and multiplication rule on counting.

Fundamental Principle of addition on counting:

If an operation can be performed in m ways and if another operation can be performed in n ways and only one operation can be done at a time, then either of the two operations can be done at a time can be performed in $m + n$ ways.

Example 7.1

In a box there are 5 red balls and 6 green balls. A person wants to select either a red ball or a green ball. In how many ways can the person make this selection?.

Solution:

Selection of a red ball from 5 balls in 5 ways.

Selection of a green ball from 6 balls in 6 ways.

By the fundamental Principle of addition, selection of a red ball or a green ball in (5+6) =11 ways.

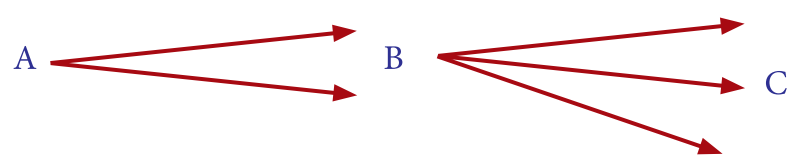
Fundamental Principle of multiplication on counting:

If an operation can be performed in m ways and if another operation in n ways independent of the first, then the number of ways of performing both the operations simultaneously in m × n ways.

Example 7.2

A person has to travel from a place A to C through B. From A to B there are two routes and from B to C three routes. In how many ways can he travel from A to C?.

Solution:



The person can travel from A to B in 2 ways and the person can travel from B to C in 3 ways.

By the Fundamental Principle of multiplication, the person can travel from A to C simultaneously in $2 \times 3 = 6$ ways.

[Note: Observe the answer part and the diagram carefully.]

Example 7.3

A company allots a code on each different product they sell. The code is made up of one English letter and two digit numbers. How many different codes are possible?

Solution:

There are 26 English Letters (A to Z) and other two digit numbers (0 to 9) are given.

Letter	Number	Number
26 ways	10 ways	10 ways

The letter place can be filled in 26 ways with the 26 alphabets *A* to Z.

The ten's place can be filled in 10 ways with the digits 0 to 9.

The unit's place also can be filled in 10 ways with the digits 0 to 9.

So the number of product codes can be formed in $26 \times 10 \times 10$ ways $= 2600$ ways.

Example 7.4

How many four digit numbers can be formed by using the digits 2, 5, 7, 8, 9, if the repetition of the digits is not allowed?.

Solution:

Thousand	Hundred	Ten	One
5 ways	4 ways	3 ways	2 ways

The thousand's place can be filled with the 5 digits in 5 ways.

Since the repetition is not allowed, the hundred's place can be filled with the remaining 4 ways.

Similarly, for the ten's place can be filled with the remaining 3 digits in 3 ways and the unit's place can be filled with the remaining 2 digits in 2 ways.

Therefore the number of numbers formed in $5 \times 4 \times 3 \times 2 = 120$ ways.

$\therefore$ 120 four digit numbers can be formed.

Factorial:

The consecutive product of first n natural numbers is known as factorial n and is denoted as n ! or $\underline{|n}$

That is $n! = n \times (n-1) \times \ldots \times 3 \times 2 \times 1$

$3! = 3 \times 2 \times 1$

$4! = 4 \times 3 \times 2 \times 1$

$5! = 5 \times 4 \times 3 \times 2 \times 1$

$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

Also $6! = 6 \times (5 \times 4 \times 3 \times 2 \times 1) = 6 \times (5!)$

This can be algebraically expressed as $n! = n\,(n-1)!$

Note that $1! = 1$ and $0! = 1$.

7.2 Permutations

Permutation means arrangement of things in different ways. Let us take 3 things A, B, C for an arrangement. Out of these three things two at a time, we can arrange them in the following manner.

AB	*AC*	*BC*
BA	*CA*	*CB*

Here we find 6 arrangements. In these arrangements, order of arrangement is considered. Note that the arrangement AB and the arrangement BA are different.

The number of arrangements of the above is given as the number of permutation of 3 things taken 2 at a time which gives the value 6.

This is written symbolically $3P_2 = 6$

Thus the number of arrangements that can be formed out of n things taken r at a time is known as the number of permutation of n things taken r at a time and is denoted as nP_r or $P(n, r)$

We write nP_r as $nP_r = n\,(n-1)\,(n-2) \ldots [n-(r-1)]$

The same nP_r can be written in factorial notation as follows:

$$nP_r = \frac{n!}{(n-r)!}$$

For example to find $10\,P_3$, we write in

factorial notation as $\quad 10\,P_3 = \frac{10!}{(10-3)!}$

$$10\,P_3 = \frac{10!}{7!} = \frac{10 \times 9 \times 8 \times (7!)}{(7!)}$$

$$= 10 \times 9 \times 8$$

$$= 720$$

Also we get the value for $10P_3$ as follows:

$$10\,P_3 = 10 \times (10\text{-}1) \times (10\text{-}2)$$

$$= 10 \times 9 \times 8$$

$$= 720$$

[To find $10\,P_3$, Start with 10, write the product of 3 consecutive natural numbers in the descending order]

Example 7.5

In how many ways can five students stand for a photograph in a row?

Solution:

The number of ways in which 5 students can stand in a row is same as the number of arrangements of 5 different things taken all at a time.

NOTE

(1) Number of all permutations of n things taken all at a time is n!

(2) $n\,P_0 = 1$, $\quad n\,P_1 = n$ and $\quad n\,P_n = n!$

This can be done in $5P_5$ ways and $\quad 5P_5 = 5! = 120$ ways.

Permutation of objects not all distinct:

The number of permutations of n objects, where p_1 objects are of one kind, p_2 are of second kind … p_k are of k^{th} kind and the rest, if any, are of different kind is given by

$$\frac{n!}{p_1!\,p_2!\ldots.p_k!}$$

Example 7.6

Find the number of permutations of the letters in the word 'STATISTICS'

Solution:

Here there are 10 objects (letters) of which there are 3S, 3 T, 2 I and 1A and 1C

Therefore the required number of arrangements is

$$= \frac{10!}{3!2!2!1!1!}$$

$$= 50400$$

Example 7.7

If $10\,P_r = 720$ find the value of r.

$$10\,P_r = 720$$

$$10\,P_r = 10 \times 9 \times 8$$

$$10\,P_r = 10\,P_3$$

$$r = 3$$

7.3 Combinations

Combination is a selection of objects without considering the order of arrangements. For example out of three things A, B, C we have to select two things at a time. This can be selected in three different ways as follows.

AB AC BC

Here the selection of object AB and BA are one and the same. The order of arrangement is not considered in combination. Hence the number of combinations from 3 different things taken 2 at a time is 3.

This is written symbolically $3C_2 = 3$.

Now we use the formula to find combination.

The number of combination of n different things, taken r at a time is given by

$$nC_r = \frac{np_r}{r!} \quad \text{or} \quad nC_r = \frac{n!}{(n-r)!r!}$$

NOTE

(i) nC_r is also denoted by $C(n, r)$ or $\binom{n}{r}$

(ii) $nC_0 = 1$, $nC_1 = n$, $nC_n = 1$

Example 7.8

Find $10C_3$ and $8C_4$

Solution:

$$10C_3 = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$$

$$8C_4 = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 70$$

[To find 10 C_3 : In the numerator, first write the product of 3 natural numbers starting from 10 in the descending order and in the denominator write the factorial 3 and then simplify].

Compare $10C_8$ and $10C_2$

$$10\,C_8 = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3}{4 \times 3 \times 2 \times 1} = \frac{10 \times 9}{2 \times 1} = 45$$

$$10\,C_2 = \frac{10 \times 9}{2 \times 1} = 45$$

From the above we find $10\,C_8 = 10\,C_2$

Therefore this is also written as $10\,C_8 = 10\,C_{(10\text{-}8)} = 10\,C_2$

This is very useful, when the difference between n and r is very high in $n\,C_r$.

This property of combination is given as a result, $n\,C_r = n\,C_{n-r}$

To find 200 C_{198}, we can use the above formula as follows:

$$\begin{aligned} 200\,C_{198} &= 200\,C_{200-198} \\ &= 200\,C_2 \\ &= \frac{200 \times 199}{2 \times 1} \\ &= 19900. \end{aligned}$$

Example 7.9

Out of 13 players, 11 Players are to be selected for a cricket team. In how many ways can this be done?

Solution:

Out of 13 Players, 11 Players are selected in $13C_{11}$ ways

i.e., $13C_{11} = 13C_2 = \dfrac{13 \times 12}{2 \times 1} = 78$

Example 7.10

In how many ways a committee of 5 members can be selected from 6 men and 5 women, consisting of 2 men and 3 women?

Solution:

For a committee, 2 men and 3 women members are to be selected. From 6 men, 2 men are selected in $6C_2$ ways. From 5 women, 3 women are selected in $5C_3$ ways.

Hence a committee of 5 members (2 men and 3 women) is selected in

$6C_2 \times 5C_3$ ways

i.e., $\dfrac{6 \times 5}{2 \times 1} \times \dfrac{5 \times 4 \times 3}{3 \times 2 \times 1} = 150$ ways.

Example 7.11

How many triangles can be formed by joining the vertices of a pentagon of five sides.

Solution:

There are 5 vertices in a pentagon. One triangle is formed by selecting a group of 3 vertices from given 5 vertices. This can be done in $5C_3$ ways.

i.e., Number of triangles $= 5C_3 = \dfrac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$

Example 7.12

A question paper contains section A with 5 questions and section B with 7 questions. A student is required to attempt 8 questions in all, selecting at least 3 from each section. In how many ways can a student select the questions?

Solution:

Selection of 8 questions from 12 questions and at least 3 from each section is given below

Section A 5Questions	Section B 7Questions	Combinations	NO. of ways
3	5	$5C_3 \times 7C_5$	210
4	4	$5C_4 \times 7C_4$	175
5	3	$5C_5 \times 7C_3$	35
		Total	420

Therefore total number of selection is 420

Example 7.13

If $6P_r = 360$ and $6C_r = 15$ find r.

Solution:

From the formula,

$$nC_r = \frac{nP_r}{r!}$$

$$6C_r = \frac{6p_r}{r!}$$

Here, $15 = \frac{360}{r!}$

i.e., $r! = \frac{360}{15} = 24$

$r! = 4 \times 3 \times 2 \times 1$

$r! = 4!$

$\therefore$ $r = 4$

Example 7.14

If $nC_8 = n\,C_{7,}$ find $n\,C_{15}$

Solution:

$$nC_8 = n\,C_7$$
$$n\,C_{n-8} = nC_7$$
$$n-8 = 7$$
$$\text{n} = 15$$

Now, $nC_{15} = 15C_{15} = 1.$

7.4 Introduction to Binomial, Exponential and Logarithmic series

7.4.1 Binomial series

A binomial is an algebraic expression of two terms. Now let us see the following binomial expansion and the number pattern we get adjacent to it.

Binomials	Expansions	Number pattern
$(x+a)^0$	1	1
$(x+a)^1$	$x+a$	1 1
$(x+a)^2$	$x^2+2xa+a^2$	1 2 1
$(x+a)^3$	$x^3+3x^2a+3xa^2+a^3$	1 3 3 1
$(x+a)^4$	$x^4+4x^3a+6x^2a^2+4xa^3+a^4$	1 4 6 4 1
$(x+a)^5$	$x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5$	1 5 10 10 5 1

From the above, we observe that the binomial coefficients form a number pattern which is in a triangular form. This pattern is known as Pascal's triangle. [In pascal's triangle, the binomial coefficients appear as each entry is the sum of the two above it.]

Binomial theorem for a positive integral index:

For any natural number n

$$(x+a)^n = nc_0 x^n a^0 + nc_1 x^{n-1} a^1 + nc_2 x^{n-2} a^2 + \dots + nc_r x^{n-r} a^r + \dots + nc_n a^n$$

In the above binomial expansion, we observe,

(i) The $(r+1)^{th}$ term is denoted by $T_{r+1} = nc_r x^{n-r} a^r$

(ii) The degree of 'x' in each term decreases while that of 'a' increases such that the sum of the power in each term equal to n

(iii) $nc_0, nc_1, nc_2, \dots nc_r, \dots nc_n$ are binomial coefficients they are also written as $c_0, c_1, c_2 \dots , c_n$

(iv) From the relation $nc_r = nc_{n-r}$ we see that the coefficients of term equidistant from the beginning and the end are equal.

Example 7.15

Expand $(2x+y)^5$ using binomial theorem.

Solution:

$$(x+a)^n = nc_0 x^n a^0 + nc_1 x^n-1_a + nc_2 x^{n-2} a^2 + \dots + nc_r x^{n-r} a^r + \dots + nc_n a^n$$

Here, $n = 5$, $X = 2x$, $a = y$

$$\begin{aligned}(2x+y)^5 &= 5C_0\ (2x)^5 (y)^0 + 5C_1\ (2x)^4 (y)^1 + 5C_2 (2x)^3 (y)^2 + \\ &\quad 5C_3\ (2x)^2 (y)^3 + 5C_4 (2x)^1 (y)^4 + 5C_5\ (2x)^0 (y)^5 \\ &= (1)2^5x^5 + (5)\ 2^4x^4 y + (10)\ 2^3x^3 y^2 + (10)\ 2^2x^2 y^3 + (5)\ 2^1x^1 y^4 + (1)\ y^5 \\ &= 32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5 \\ (2x+y)^5 &= 32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5\end{aligned}$$

Example 7.16

Find the middle terms of expansion $(3x+y)^5$

Solution:

In the expansion of $(3x+y)^5$ we have totally 6 terms. From this the middle terms are

T_3 and T_4

To find T_3 put r = 2 in T_{r+1}

Here $n = 5$, $x = 3x$, $a = y$

$$T_{r+1} = nC_r\, x^{n-r}\, a^r$$

$$T_{2+1} = 5C_2\, (3x)^{5-2}\, (y)^2$$

$$= 5C_2\, 3^3\, x^3\, y^2$$

$$= 270x^3y^2$$

Similarly we can find by putting $r = 3$ in T_{r+1} to get T_4, then $T_4 = 90x^2y^3$

Binomial theorem for a rational index:

For any rational number other than positive integer

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots, \quad \text{provided } |x| < 1.$$

NOTE

(i) If $n \in N$, $(1+x)^n$ is desired for all value of x and if n is a rational number other than the natural number then $(1+x)^n$ is desired only when $|x| < 1$.

(ii) If $n \in$ N, then the expansion of $(1+x)^n$ contains only $(n+1)$ terms. If n is a rational number, then the expansion of $(1+x)^n$ contain infinitely many terms.

Some important expansions:

$$(1-x)^n = 1 - \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$(1+x)^{-n} = 1 - \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots$$

$$(1-x)^{-n} = 1 + \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots$$

Special cases of infinite series:

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

Example 7.17

Find the approximate value of $\sqrt[3]{1002}$ (correct to 3 decimal places) using Binomial series.

Solution:

$$\sqrt[3]{1002} = \sqrt[3]{1000+2}$$

$$= \sqrt[3]{1000\left(1+\frac{2}{1000}\right)}$$

$$= [(10^3)^{\frac{1}{3}}(1+0.002)^{\frac{1}{3}}]$$

$$= 10[1+0.002]^{\frac{1}{3}}$$

$$= 10[1+\frac{1}{3}[0.002]+\ldots]$$

$$= 10[1+0.00066+\ldots]$$

$$= 10[1+0.00066]$$

$$= [10+0.0066]$$

$$= 10.007$$

7.4.2 Exponential series

For all real values of x, $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$ is known as exponential series, where e is an irrational number and the value of e to six decimal places is

$e = 2.718282 \ldots$ and

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\infty\,, \quad e^{-1}=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots$$

The other exponential expansions are

$$e^{-x}= 1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots$$

$$\frac{e^x+e^{-x}}{2} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots \quad \text{and} \quad \frac{e^x-e^{-x}}{2} \; x+\frac{x_3}{3!}+\frac{x^5}{5!}+\ldots$$

$$\frac{e+e^{-1}}{2} = 1+\frac{1!}{2!}+\frac{1}{4!}+\ldots \quad \text{and} \quad \frac{e-e^{-1}}{2} \; 1+\frac{1}{3!}+\frac{1}{5!}+\ldots$$

Example 7.18

Show that $\dfrac{\frac{1}{1!}+\frac{1}{3}+\frac{1}{5!}+\ldots}{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots} = \dfrac{e+1}{e-1}$

Solution:

$$\frac{\frac{1}{1!}+\frac{1}{3}+\frac{1}{5!}+\ldots}{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots} = \frac{\frac{e-e^{-1}}{2}}{\left[\frac{e+e^{-1}}{2}-1\right]}$$

$$= \left[\frac{e-e^{-1}}{2}\right] \div \left[\frac{e+e^{-1}-2}{2}\right]$$

$$= \frac{e-e^{-1}}{e+e^{-1}-2}$$

$$= \left[e-\frac{1}{e}\right] \div \left[e+\frac{1}{e}-2\right]$$

$$= \left[\frac{e^2-1}{e}\right] \div \left[\frac{e^2-2e+1}{e}\right]$$

$$= \frac{(e^2-1)}{(e^2-2e+1)}$$

$$= \frac{(e^2-1)}{(e-1)^2}$$

$$= \frac{(e-1)(e+1)}{(e-1)^2}$$

$$= \frac{e+1}{e-1}$$

7.4.3 Logarithmic series:

If $|x| < 1$, the series, $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ converges to $\log(1+x)$

Some important deductions from the above series are

(i) $\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$

(ii) $\log(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\ldots$

(iii) $\log(1+x)-\log(1-x) = 2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right)$

or $\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots$

7.5 Introduction to Elementary calculus

Before going to understand the problems on continuous random variables, we need to know some fundamental knowledge about differentiation and integration, which are part of calculus in higher mathematics.

Hence, we introduce some simple concepts, techniques and formulae to calculate problems in statistics, which involve calculus.

7.5.1 Differentiation

We Studied about functions and functional values in earlier classes. Functional value is an exact value. For some function $f(x)$, when $x = a$, we obtain the *functional value* as $f(a) = k$.

Another type of approximation gives the very nearest value to the functional value is known as *limiting value*. So the limiting value is an approximate value. This limiting value approaches the nearest to the exact value k.

Suppose the exact value is 4, the limiting value may be 4.00000001 or 3.999999994. Here we observe that the functional value and the limiting value are more or less the same and there is no significant difference between them.

Hence in many occasions we use the limiting values for some critical problems.

The limiting value of $f(x)$ when x approaches a number 2 is denoted by $\lim_{x\to 2} f(x) = f(2) = l$ (some existing value)

The special type of any existing limit, $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ is called the *derivative* of the function f with respect to x and is denoted by $f'(x)$. If y is a function of x, and has a derivative, then the differential coefficient of y with respect to x is denoted by $\frac{dy}{dx}$ .This process of finding the limiting value is known as *differentiation*.

Some rules on differentiation:

(i) Derivative of a constant function is zero.
i.e., $f'(c) = 0$, where c is some constant.

(ii) If u is a function of x and k is some constant and dash denotes the differentiation, $[k\,u]' = k[u]'$

(iii) $(u \pm v)' = u' \pm v'$

(iv) $(u\,v)' = u'v + u\,v'$ (product rule)

(v) $\left[\frac{u}{v}\right]' = \frac{u'v - uv'}{v^2}$ (quotient rule)

Important formulae:

(i) $(x^n)' = n\,x^{n-1}$

(ii) $(e^x)' = e^x$

(iii) $(\log x)' = \frac{1}{x}$

Example 7.19

Evaluate the following limits:

(i) $\lim\limits_{x\to 5} \dfrac{x^3-25}{x-3}$ (ii) $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$

Solution:

(i) $\lim\limits_{x\to 5} \dfrac{x^3-25}{x-3} = \dfrac{5^3-25}{5-3} = \dfrac{100}{2} = 50$

(ii) $\lim_{x\to 2} \dfrac{x^2-4}{x-2} = \dfrac{0}{0}$, an indeterminate form. Therefore first factorise and simplify and then apply the same limit to get the limiting value.

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2}$$

$$= \lim_{x\to 2} (x+2)$$

$$= 2+2 = 4$$

$$\therefore \quad \lim_{x\to 2} \frac{x^2-4}{x-2} = 4$$

Example 7.20

Find the derivative of the following with respect to x:

(i) x^{15}-10 (ii) x^3+3x^2-6 (iii) x^4e^x (iv) $\dfrac{x^2-1}{x+3}$

Solution:

(i) Let $y = x^{15}-10$

$$\frac{dy}{dx} = 15x^{15-1} - 0$$

$$= 15x^{14}$$

(ii) Let $y = x^3+3x^2-6$

$$\frac{dy}{dx} = 3x^2+3(2x)-0$$

$$= 3x^2+6x$$

(iii) Let $y = x^4e^x$

This is of the type

$$[uv]' = u'v + uv'$$

$$= [x^4]'\ (e^x) + (x^4)\ [e^x]'$$

$$= 4x^3e^x + x^4e^x$$

$$= (4x^3 + x^4)e^x$$

(iv) Let $\quad y = \dfrac{x^2-1}{x+3}$

This is of the type

$$\left[\frac{u}{v}\right]' = \frac{u'v - uv'}{v^2}$$

$$\frac{dy}{dx} = \left[\frac{x^2-1}{x+3}\right]'$$

$$= \frac{(x^2-1)'(x+3) - (x^2-1)(x+3)'}{(x+3)^2}$$

$$= \frac{(2x)(x+3) - (x^2-1)\times 1}{(x+3)^2}$$

$$= \frac{2x^2+6x-x^2+1}{(x+3)^2}$$

$$= \frac{x^2+6x+1}{(x+3)^2}$$

Repeated differentiation:

If the derivative of a function is again differentiated with respect to the same variable, we say that the differentiation is the second order differentiation and is denoted as $\dfrac{d}{dx}(\dfrac{dy}{dx}) = \dfrac{d^2y}{dx^2}$ *or* D^2y *or* Y_2

For example if $y = (x^3+4x^2+7)$, then $\dfrac{dy}{dx} = (3x^2+8x)$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(3x^2+8x)$$

$$= 6x+8$$

NOTE

Repeated or successive differentiation may be extended to any higher order derivation.

7.5.2 Integration

Integration is the reverse process of differentiation. It is also called anti-derivative.

Suppose the derivative of x^5 is $5x^4$. Then the integration of $5x^4$ with respect to x is x^5. we use this in symbol as follows:

$$\frac{d}{dx}(x^5) = 5x^4 \Rightarrow \int 5x^4\,dx = x^5$$

Similarly,

$$\frac{d}{dx}(x^7) = 7x^6 \Rightarrow \int 7x^6\,dx = x^7$$

$$\frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x\,dx = e^x$$

$$\frac{d}{dx}(\log x) = \frac{1}{x} \Rightarrow \int \frac{1}{x}\,dx = \log x$$

and so on.

NOTE

While differentiating the constant term we get zero. But in the reverse process, that is on integration, unless you know the value of the constant we cannot include. That is why we include an arbitrary constant c to each integral value.

Therefore for the above examples, we usually write

$$\int 7x^6\,dx = x^7 + c$$

$$\int e^x\,dx = e^x + c$$

These integrals are also called improper integrals or indefinite integrals

Rules and some formulae on integration:

(i) $\int k\,dx = kx$

(ii) $\int x^n\,dx = \frac{x^{n+1}}{n+1}$

(iii) $\int e^x\,dx = e^x$

(iv) $\int \frac{1}{x}\,dx = \log x$

(v) $\int (u \pm v)\,dx = \int u dx \pm \int v dx$

Example 7.21

Integrate the following with respect to x.

(i) x^7 (ii) $\frac{1}{x^6}$ (iii) $\sqrt{x}$ (iv) $x^5 - 4x^2 + 3x + 2$

Solutions:

(i) $$\int x^7\,dx = \frac{x^{7+1}}{7+1} = \frac{x^8}{8} + c$$

(ii) $$\int \frac{1}{x^6}\,dx$$

$$= \int x^{-6}\,dx$$

$$= \frac{x^{-6+1}}{-6+1} = \frac{x^{-5}}{-5}$$

$$= -\frac{1}{5x^5} + c$$

(iii) $$\int \sqrt{x}\,dx$$

$$= \int x^{1/2}\,dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= \frac{2}{3}x^{\frac{3}{2}} + c$$

(iv) $$\int (x^5 - 4x^2 + 3x + 2)\,dx$$

$$= \left(\frac{x^6}{6}\right) - 4\left(\frac{x^3}{3}\right) + 3\left(\frac{x^2}{2}\right) + 2x + c$$

The above discussed integrals are known as improper integrals or indefinite integrals . For the proper or definite integrals we have the limiting points at both sides. These are called the lower limit and the upper limit of the integral.

This integral $\int f(x)\,dx$ is an indefinite integral. Integrating the same function with in the given limits a and b is known as the definite integral. We write this in symbol as

$$\int_a^b f(x)\,dx = \text{k(a constant value)}$$

In a definite integral where a is known as the lower limit and b is known as the upper limit of the definite integral. To find the value of definite integral, we do as follows:

Suppose $\int f(x)\,dx = F(x)$

Then $\int_a^b f(x)\,dx = F(b) - F(a)$

Example 7.22

Evaluate the following definite integrals:

(i) $\int_1^3 x^3\,dx$ (ii) $\int_{-1}^{+1} 5x^4\,dx$ (iii) $\int_1^2 \frac{5}{x}\,dx$

Solutions:

(i)
$$\int_1^3 x^3\,dx = \left[\frac{x^4}{4}\right]_1^3$$
$$= \frac{1}{4}[3^4 - 1^4]$$
$$= \frac{1}{4}[81 - 1]$$
$$= \frac{1}{4}[80]$$
$$= 20$$

(ii)
$$\int_{-1}^{+1} 5x^4\,dx = 5\int_{-1}^{1} x^4\,dx$$
$$= 5\left[\frac{x^5}{5}\right]_{-1}^{1}$$
$$= [x^5]_{-1}^{1}$$
$$= [1^5 - (-1)^5]$$
$$= 1-(-1)$$
$$= 1+1$$
$$= 2$$

(iii)
$$\int_1^2 \frac{5}{x}\,dx$$
$$= 5\int_1^2 \frac{1}{x}\,dx$$
$$= 5\,[\log x]_1^2$$
$$= 5[log2 - log1]$$
$$= 5log2$$

7.5.3 Double integrals

A double integral is an integral of two variable function $f(x,y)$ over a region R If $R=[a, b] \times [c, d]$ then the double integral can be done by iterated Integration(integrate first with respect to y and then with respect to x)

The notation used for double integral is $\int_a^b \int_c^d f(x,y)\,dy\,dx$

Here the function $f(x,y)$ is integrated with respect to y first and treat $f(x)$ constant and then integrate with respect to x and apply limits of x and simplify

Example 7.23

Evaluate: $\int_0^1 \int_1^2 x^2 y dx dy$, for $0 \le x \le 1,\ 1 \le y \le 2$

Solutions:

Let us first integrate with respect to y and then with respect to x. Hence the double integral is written as

$$\int_0^1 x^2 \left[\int_1^2 y dy\right] dx$$

Now integrate the inner integral only and simplify.

$$= \int_0^1 x^2 \left[\frac{y^2}{2}\right]_1^2 dx$$

$$= \int_0^1 \frac{x^2}{2}[4-1]\,dx$$

Again integrate with respect to x

$$= \frac{3}{2}\int_0^1 x^2\,dx$$

$$= \frac{3}{2}\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{2}[1^3-0]$$

$$= \frac{1}{2}$$

NOTE

We can also change the order of integration

Example 7.24

Evaluate: $\int_0^2 \int_0^1 [2y^2x^2+3]\,dydx$

Solution:

$$= \int_0^2 \int_0^1 [2y^2x^2+3]\,dydx = \int_0^2 \left(\int_0^1 \left(2y^2x^2+3\right)dy\right)dx$$

$$= \int_0^2 \left[\frac{2x^2}{3}y^3+3y\right]_0^1 dx = \int_0^2 \left[2x^2\left(\frac{1}{3}\right)+3(1)-0\right]dx$$

$$= \left[\frac{2}{3}\left(\frac{x^3}{3}\right)+3x\right]_0^2 = \left[\frac{16}{9}+6\right]$$

$$= \frac{70}{9}$$

Example 7.25

Evaluate $\int_0^1 \int_0^1 16abxydx\, dy$

Solution:

$$\int_0^1 \int_0^1 16abxydx\, dy$$

$$= 16ab \int_0^1 \left[\int_0^1 xydx \right] dy$$

$$= 16ab \int_0^1 \left[\frac{x^2}{2} y \right]_0^1 dy$$

$$= 16ab \int_0^1 \left[\frac{y}{2} \right] dy$$

$$= 8ab \left[\frac{y^2}{2} \right]_0^1 \quad = 4ab[1-0]$$

$$= 4ab$$

Points to Remember

- Fundamental Principle of multiplication on counting: If an operation can be performed in m ways and if another operation in n ways independent of the first, then the number of ways of performing both the operations simultaneously in $m \times n$ ways.
- Factorial: The consecutive product of first n natural numbers is known as factorial n and is denoted as $n\,!$ or $\underline{|n}$
- Permutations : Thus the number of arrangements that can be formed out of n things taken r at a time is known as the number of permutation of n things taken r at a time and is denoted as nP_r; $nP_r = n(n-1)(n-2) \text{ - - - - } [n-(r-1)] \quad = \frac{n!}{(n-r)!}$
- Combinations : The number of combination of n different things, taken r at a time is given by $nC_r = \frac{np_r}{r!} \quad = \frac{n!}{(n-r)!r!}$, $nC_r = nC_{n-r}$
- Binomial series : For any natural number n
  $(x+a)^n = nc_0 x^n a^0 + nc_1 x^{n-1} a + nc_2 x^{n-2} a^2 + \ldots + nc_r x^{n-r} a^r + \ldots + nc_n a^n$
  For any rational number other than positive integer
  $(1+x)^n = 1 + \frac{n}{1!} x + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots$, provided $|x| < 1$.
- Exponential series : For all real values of x, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ is known as exponential series, Here, e= 2.718282

- Differentiation : The special type of any existing limit, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ is called the derivative of the function f with respect to x and is denoted by $f'(x)$. If y is a function of x, and has a derivative, then the differential coefficient of y with respect to x is denoted by $\frac{dy}{dx}$.

- Some rules on differentiation:

(i) Derivative of a constant function is zero. $f'(c) = 0$, where c is some constant.

(ii) If u is a function of x and k is some constant and *dash* denotes the differentiation, $[k\, u]' = k\,[\, u\,]'$

(iii) $(u \pm v)' = u' \pm v'$

(iv) $(u\, v)' = u'v + u\,v'$

(v) $\left[\frac{u}{v}\right]' = \frac{u'v - uv'}{v^2}$

- Important formulae:

(i) $(x^n)' = n\, x^{n-1}$

(ii) $(e^x)' = e^x$

(iii) $(\log x)' = \frac{1}{x}$

- Integration

Integration is the reverse process of differentiation. It is also called anti-derivative.

Rules and some formulae on integration:

(i) $\int k\, dx = kx$

(ii) $\int x^n\, dx = \frac{x^{n+1}}{n+1}$

(iii) $\int e^x\, dx = e^x$

(iv) $\int \frac{1}{x} dx = \log x$

(v) $\int (u \pm v)\, dx = \int u dx \pm \int v dx$

- Definite Integral : $\int_a^b f(x)\, dx = F(b) - F(a)$

- Double integrals : The double integral defined on the function $f(x,y)$ is denoted as $\int_a^b \int_c^d f(x,y)\, dy\, dx$. The function $f(x,y)$ is integrated with respect to y first by treating $f(x)$ as constant and then integrate with respect to x and apply limits of x and simplify we get the value of integral.

EXERCISE 7

I. Choose the best answer:

1. The factorial n is also written as

(a) $n(n\text{-}1)!$ (b) $n(n+1)!$ (c) $(n\text{-}1)!$ (d) $(n+1)!$

2. The value of $10P_2$ is

(a) 10 (b) 45 (c) 90 (d) 20

3. The number of different four letter words can be formed with the words 'DATE' is

(a) 4 (b) 8 (c) 24 (d)48

4. The value of $50C_{50}$ is equal to

(a) 50 (b) 25 (c)1 (d)0

5. The value of $20C_{18}$ is

(a) 190 (b) 180 (c) 360 (d) 95

6. $\lim_{x \to 0} \dfrac{x^2 - 1}{x - 1}$ is equal to

(a) -1 (b) 0 (c) 1 (d) 2

7. Derivative of $\log x$ is equal to

(a) 1 (b) $\dfrac{1}{x}$ (c) e^x (d) $\log x$

8. Derivative of x^9 is

(a) x^8 (b) $9x^8$ (c) $8x^9$ (d) $8x^8$

9. The integral value of x^{11} is

(a) $\dfrac{1}{12}x^{12}$ (b) x^{12} (c) $11x^{10}$ (d) $10x^{11}$

10. $\int e^x\, dx$ is equal to

(a) $e\,x^2$ (b)e^x (c) e^{-x} (d) xe^x

11. $\int_0^1 x^3\, dx$ is

(a) $\dfrac{1}{4}$ (b) $\dfrac{1}{2}$ (c) 1 (d) 3

II. Fill in the blanks:

12. The number of all permutations of n distinct things taken all at a time is ________

13. Number of ways of 4 students can stand in a queue is ________

14. From a group of 10 students 2 students are selected in ________ ways

15. The value of nC_n is equal to ________

16. The value of $21C_3$ is equal to ________

17. $\lim_{x \to 5} \frac{2x+3}{x+5}$ is equal to ________

18. The derivative of e^x is equal to ________

19. The integral of $\frac{1}{x}$ is ________

20. The value of $\int 6x^5 \, dx$ is ________

21. The value of $\int_0^1 x^{10} \, dx$ is ________

III . Very Short Answer Questions :

22. A boy has 6 pants and 10 shirts. In how many ways can he wear them?

23. Evaluate (i) $4P_4$ (ii) $10P_4$ (iii) $100P_2$

24. How many different four letter code words can be formed with the letter 'SWIPE' no letter is not repeated.

25. In a competition, in how many ways can first and second place be awarded to 10 people?

26. Evaluate $10C_4$, $22C_3$, $100C_{98}$

27. Evaluate (i) $\lim_{x \to 2} \frac{x^2-4x+7}{x+8}$ (ii) $\lim_{x \to 5} \frac{x^2-25}{x-5}$

28. Find the derivative of x^2e^x

IV. Short Answer Questions :

29. How many different words can be formed with letters of the word 'PROBABILITY'?

30. A bag contains 7 blue balls and 5 red balls. Determine the number of ways in which 3 blue and 2 red balls can be selected.

31. How many triangles can be formed by joining the vertices of a hexagon?

32. In how many ways can 3 vowels and 2 consonants be chosen from {E,Q,U,A,T,I,O,N}?

33. Different the following with respect to x:

(i) $e^x(x^2-5)$ (ii) $\frac{x^2-1}{x+7}$

34. Find $\int (3x^3-2x^2+6x-7)\,dx$

35. Find $\int (\frac{2}{x}+\frac{2}{x^2}-e^x+3)\,dx$

36. Find $\int_0^2 \frac{3}{4}(2-x)\,\mathrm{d}x$

37. Find $\int_{-1}^{1} 2x\,dx$

38. Evaluate $\int_0^2 \int_0^1 [2x+3y]\,dydx$

V. Calculate the following :

39. A class contains 12 boys and 10 girls. From the class, 10 students are to be chosen for a competition under the condition that at least 4 boys and at least 4 girls must be represented. In how many ways can the selection are made

40. What is the number of ways of choosing 4cards from a pack of 52 playing cards? In how many of these (i) 3 cards are of the same suit (ii) 4 cards are belong to different suits (iii) two are red cards and two are black cards

41. There are 3 questions in the first section, 3 questions in the second section and 2 questions in the third section in a question paper of an exam. The student has to answer any 5 questions, choosing atleast one from each section. In how many ways can the student answer the exam?

42. Differentiate the following with respect to x:

(i) $(x^3-4x^2+2x)(e^x+5)$ (ii) $\frac{x^2-7x}{x^2+8}$

43. Integrate the following with respect to x: (i) $\frac{x^2-5x+6}{x}$ (ii) $2\sqrt{x}+\frac{3}{x^4}$

44. Evaluate: (i) $\int_0^2 \frac{3}{4}x(2-x)\,dx$ (ii) $\int_1^2 \frac{12}{x}\,dx$

45. Evaluate $\int_0^1 \int_0^1 x^2\, y dx dy$

46. Evaluate $\int_1^2 \int_2^4 6x\, y^2\, dx dy$

47. Evaluate $\int_0^2 \int_0^2 4mnxy dx dy$

48. $\int_1^2 \int_1^3 (4x-2y)\, dx dy$

ANSWERS:

I. 1. (a) , 2. (c) , 3. (c) , 4. (c) , 5. (a) , 6. (c) , 7. (b) , 8. (b) , 9. (a) ,

10. (b) , 11. (a)

II. 12. n! 13. 24 , 14. 45, 15.1 , 16. 1330 , 17. $\frac{13}{10}$, 18. e^x ,

19. $\log x$, 20. x^6+c, 21. $\frac{1}{11}$

III. 22. 60, 23. (i) 24 , (ii) 5040 , (iii) 9900 , 24.120 , 25. 90,

26. (i) 210 , (ii) 1540 , (iii) 4950 , 27. (i) $\frac{3}{10}$, (ii) 10 , 28. $e^x\ (x^2+2x)$

IV. 29. $\frac{11}{2!2!}$, 30. 350 , 31. 20 , 32. 30 , 33. (i) $e^x\ (x^2+2x-5)$,

(ii) $\frac{x^2+14x+1}{(x+7)^2}$, 34. $\frac{3}{4}x^4 - \frac{2}{3}x^3 + 3x^2 - 7x + c$ 35. $2\log x - \frac{2}{x} - e^x + 3x + c$

36. $\frac{3}{2}$, 37. 0, 38. 7

V. 39. 497574 , 40. (i) $4(13C_3.39C_1)$, (ii) $(13C_1)^4$, (iii) $26C_2.26C_2$, 41. 48,

42. (i) $e^x\ (x^3 - x^2 - 6x + 2) + 5(3x^2 - 8x + 2)$, (ii) $\frac{7x^2-16x-56}{(x^2+8)^2}$

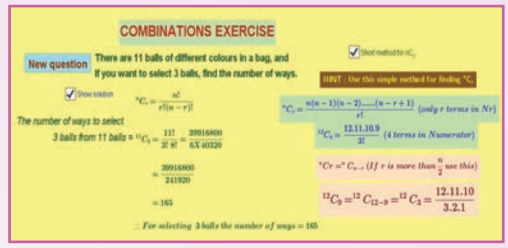
43. (i) $\frac{x^2}{2} - 5x + 6\log x + c$, (ii) $\frac{4}{3}x^{\frac{3}{2}} - \frac{3}{x} + c$, 44. (i) 1, (ii) 12 log2

45. $\frac{1}{6}$, 46. 84 , 47. $16\, mn$, 48. 24

ICT CORNER

SOME MATHEMATICAL METHODS-COMBINATION

This activity helps to practice combinations. Which is an important fundamental for probability

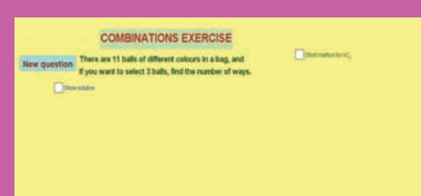


Steps:

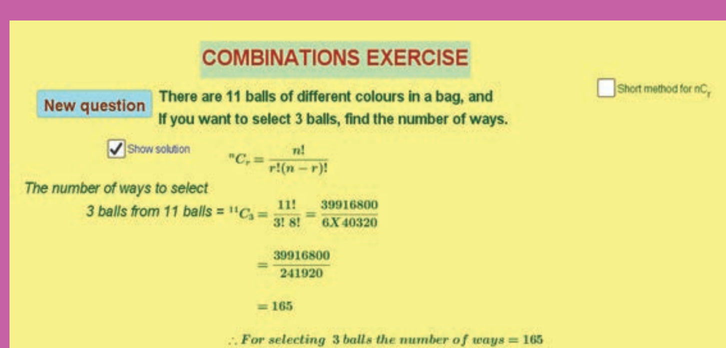
- Open the browser and type the URL given (or) scan the QR code.
- GeoGebra work book called "**11th Standard Statistics**" will appear. In this several work sheets for statistics are given, open the worksheet named "**Problems in Combination**"
- A problem is given in the work sheet. Solve it. To check your answer, click on the box "**Show Solution**". You can change the question by pressing on "**NEW QUESTION**".
- Before going to the new question click on "**Short method for ^nC_r**" and follow the steps given carefully and practice it. Now go to new question and practice till you are thorough as it is important for doing probability.

Step-1

Step-2



Step-3



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