CHAPTER 4

Determinants

Expansion of Determinants

- **Definition**: A determinant is a number (real or complex) that can be related to any square matrix A = [a_{ij}] of order n. It is denoted as det(A).

  Determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ can be given as:

  $$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

  Determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ by expanding along R_1 can be given as:

  $$\det(A) = |A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

  $$+ (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- **Minors:** The minor M_{ij} of a_{ij} in A is the determinant of the square sub matrix of order $(n - 1)$ obtained by deleting its i^{th} row and j^{th} column in which a_{ij} lies. It is denoted by M_{ij}.

  Minor of an element of a determinant of order n(for all $n \geq 2$) is a determinant of order $n - 1$.

- **Co-factors:** Co-factor of an element a_{ij} is defined by $A_{ij} = (-1)^{(i+j)} M_{ij}$, where M_{ij} is a minor of a_{ij}. Co-factor is denoted by A_{ij}.

Properties of Determinants

- **Properties of determinants**:
  - The value of the determinant remains unchanged if its rows and columns are interchanged.
  - If A is a square matrix, then $det(A) = det(A')$, where A'= transpose of A.
  - If any two columns (or rows) of a determinant are interchanged, then the sign of the determinant changes.
  - The determinant of the product of two square matrices of same order is equal to the product of their respective determinants, that is $|AB| = |A||B|$.
  - If any two rows or columns of a determinant are identical (i.e. all corresponding elements are same), then value of determinant is zero.
  - If each element of a row or a column of a determinant is multiplied by a constant k, then its value gets multiplied by k.
  - If some or all elements of a row or column of a determinant are expressed as sum of two or more terms, then the determinant can be expressed as sum of two or more determinants.

    For example:

    $$\begin{vmatrix} a_1 + x & a_2 + y & a_3 + z \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
  - If we multiply each element of a row or a column of a determinant, by a constant k, then the value of the determinant is also multiplied by k.
  - Multiplying a determinant by k means multiply elements of any one row or any one column by k.
  - Adding or subtracting each element of anycolumn or any row of a determinant withthe equimultiples of corresponding elements of any other row or column, does not changes the value of the determinant, i.e., the value of

determinant remain same if we apply the operation $R_i \to R_i \to kR_j$.

- **Area of triangle:** If a triangle is given with its vertices at points (x_1, y_1), (x_2, y_2) and (x_3, y_3), then its area can be calculated as:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

  - Area is a positive quantity, so we always take the absolute value of the determinant.
  - If the area of the triangle is already given, use both positive and negative values of the determinant for calculation.
  - Area of triangle formed by three collinear points is always zero.

Adjoint and Inverse of a Matrix

- **Adjoint of a square matrix:**
  The adjoint of a square matrix $A = [a_{ij}]_{n\times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n\times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by "$adj(A)$".

  Example: $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ then, $adj\,A = \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}$

  If A be any given square matrix of order n, then $A(adj\,A) = (adj\,A)A = AI$, where I is the identity matrix of order n.

- **Inverse of a Matrix**
  - **Singular matrix:** It is a matrix with zero determinant value. i.e. $|A| = 0$.
  - **Non-singular matrix:** It is a matrix with a non-zero determinant value. i.e. $|A| \neq 0$.
    If A and B are nonsingular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.
  - A square matrix A is invertible if and only if A is nonsingular matrix.
  - If A is a nonsingular matrix, then its **inverse** exists which is given by $A^{-1} = \frac{1}{|A|}adj(A)$
- **Consistent system:** A system of equations is said to be consistent if there exist one or more solution to the system of equation.
- **Inconsistent system:** If the solution to the system of equation does not exist, then it is termed as inconsistent system.
- For the square matrix A in the matrix equation $AX = B$
  - $|A| \neq 0$, there exists unique solution. The system of equation is consistent.
  - $|A| = 0$ and $(adj\ A)B \neq 0$ then there exists no solution. The system is inconsistent.
  - $|A| = 0$ and $(adj\ A)B = 0$, then system may or may not be consistent.

Exercise

1. The value of $\begin{vmatrix} \cos 15° & \sin 15° \\ \sin 15° & \cos 15° \end{vmatrix}$ is:

(*a*) 1 (*b*) $\frac{1}{2}$

(*c*) $\frac{\sqrt{3}}{2}$ (*d*) None

2. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, then what is the value of x?

(*a*) –2 (*b*) 2

(*c*) 4 (*d*) 3

3. The value of $\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$ is:

(*a*) –43 (*b*) 43

(*c*) 86 (*d*) 0

4. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, then the minor of the elements a_{23} is:

(*a*) 5 (*b*) 7

(*c*) 3 (*d*) 9

5. The value of the determinant $\begin{vmatrix} \cos 75° & -\sin 75° \\ \sin 30° & \cos 30° \end{vmatrix}$ is

(*a*) $-\frac{1}{\sqrt{2}}$ (*b*) $\frac{1}{\sqrt{2}}$

(*c*) $\sqrt{2}$ (*d*) None

6. If x ≠ 0 and $\begin{vmatrix} 1 & x & 2x \\ 1 & 3x & 5x \\ 1 & 3 & 4 \end{vmatrix} = 0$, then x = ?

(*a*) –1 (*b*) 1

(*c*) 2 (*d*) –2

7. The value of

$$\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}+\sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$$ is:

(*a*) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ (*b*) $\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$

(*c*) $\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}$ (*d*) None

8. If a, b, c be positive district real number, then the value of $\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}$ is

(*a*) Positive (*b*) Negative

(*c*) Perfect square (*d*) zero

9. The maximum value of $\begin{vmatrix}1 & 1 & 1\\ 1 & (1+\sin\theta) & 1\\ 1 & 1 & (1+\cos\theta)\end{vmatrix}$ is:

(*a*) 1 (*b*) $-\frac{1}{2}$

(*c*) $\frac{1}{2}$ (*d*) 0

10. The value of $\begin{vmatrix}a+ib & c+id\\ -c+id & a-ib\end{vmatrix}$ is:

(*a*) $(a^2-b^2+c^2-d^2)$ (*b*) $(a^2+b^2-c^2-d^2)$

(*c*) $(a^2+b^2+c^2+d^2)$ (*d*) None of these

11. Using properties of determinants, find the value of $\begin{vmatrix}-a^2 & ab & ac\\ ba & -b^2 & bc\\ ca & cb & -c^2\end{vmatrix}$ = ?

(*a*) $4a^2b^2c^2$ (*b*) $4a^2bc$

(*c*) $2a^2b^2c^2$ (*d*) None

12. If A is a 3 × 3 matrix and $|3A| = k|A|$, then the value of k is:

(*a*) 1 (*b*) 27

(*c*) 23 (*d*) None

13. If $\begin{vmatrix}(b+c) & (c+a) & (a+b)\\ (y+z) & (z+x) & (x+y)\\ (q+r) & (r+p) & (p+q)\end{vmatrix} = k\begin{vmatrix}a & b & c\\ x & y & z\\ p & q & r\end{vmatrix}$, then the value of k is:

(*a*) $\frac{1}{2}$ (*b*) 2

(*c*) 1 (*d*) –2

14. $\begin{vmatrix}0 & p-q & p-r\\ q-p & 0 & q-r\\ r-p & r-q & 0\end{vmatrix}$ = ?

(*a*) (p + q + r) (*b*) (p + q – r)

(*b*) 1 (*b*) 0

15. $\begin{vmatrix}(a-b-c) & 2b & 2c\\ 2a & (b-c-a) & 2c\\ 2a & 2b & (c-a-b)\end{vmatrix}$ = ?

(*a*) a + b + c (*b*) $(a+b+c)^2$

(*c*) $(a+b+c)^3$ (*d*) None

16. The value of k for which the equations 9x + 4y = 9 and 7x + ky = 5 have no solution, is:

(*a*) $\frac{9}{5}$ (*b*) $\frac{9}{7}$

(*c*) $\frac{28}{9}$ (*d*) $\frac{9}{28}$

17. If for any 2 × 2 square matrix A, A(adj A) = $\begin{bmatrix}8 & 0\\ 0 & 8\end{bmatrix}$, then write the value of |A|.

(*a*) 8 (*b*) 6

(*c*) 5 (*d*) 10

18. For what values of k, the system of linear equations x + y + z = 2, 2x + y – z = 3 and 3x + 2y + kz = 4 has a unique solution?

(*a*) R – {0} (*b*) R

(*c*) R – {0, 1} (*d*) None

19. The system of linear equations x + y + z = 0, 2x + y – z = 0 and 3x + 2y = 0 has:

(*a*) No solution

(*b*) A unique solution

(*c*) Infinitely many solutions

(*d*) None of these

20. If the system of equations x – ky – z = 0, kx – y – z = 0 and x + y – z = 0 has a non-zero solution, then the possible values of k are:

(*a*) –1, 2 (*b*) 1, 2

(*c*) 0, 1 (*d*) –1, 1

21. If

$$f(x) = \begin{vmatrix}1 & x & x+1\\ 2x & x(x-1) & (x+1)\\ 3x(x-1) & x(x-1)(x-2) & (x+1)(x-1)x\end{vmatrix},$$

then f(10) = ?

(*a*) 90 (*b*) 9000

(*c*) –100 (*d*) 0

22. If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, then $A^{-1} = ?$

(*a*) $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ (*b*) $\begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ (*d*) $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

23. For two determinants A and B of the same order $|AB| = ?$

(*a*) $|A| + |B|$ (*b*) $\frac{|A|}{|B|}$

(*c*) $|A|\,|B|$ (*d*) None of these

Answer Keys

1. (*c*) **2.** (*a*) **3.** (*a*) **4.** (*b*) **5.** (*b*) **6.** (*a*) **7.** (*a*) **8.** (*b*) **9.** (*c*) **10.** (*c*)
11. (*a*) **12.** (*b*) **13.** (*b*) **14.** (*d*) **15.** (*c*) **16.** (*c*) **17.** (*a*) **18.** (*a*) **19.** (*c*) **20.** (*d*)
21. (*d*) **22.** (*a*) **23.** (*c*)

Solutions

1. $\Delta = (\cos 15° \times \cos 15° - \sin 15° \times \sin 15°)$

$= (\cos^2 15° - \sin^2 15°)$

$= \cos 2 \times 15° = \cos 30° = \frac{\sqrt{3}}{2}$

2. Given, $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$

On expanding both determinants.

we get, $12\,x - (-14) = 8 \times 4 - 7 \times 6$

$\Rightarrow\ 12x + 14 = 32 - 42$

$\Rightarrow\ 12x = -10 - 14 = -24$

$\therefore\ x = \frac{-24}{12} = -2$

3. Given $\Delta = \begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$

By $C_1 \to C_1 - 3C_2$ and $C_3 \to C_3 - C_2$

We get, $\Delta = \begin{vmatrix} 67 - 3 \times 19 & 19 & (21 - 19) \\ 39 - 3 \times 13 & 13 & (14 - 13) \\ 81 - 3 \times 24 & 24 & (26 - 24) \end{vmatrix}$

$= \begin{vmatrix} 10 & 19 & 2 \\ 0 & 13 & 1 \\ 9 & 24 & 2 \end{vmatrix}$

$= 10\,(26 - 24) + 9(19 - 26)$

$= 10 \times 2 + 9 \times (-7)$

$= 20 - 63$

$= -43$

4. Minor of $a_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$

$= 10 - 3 = 7$

5. $\Delta = \cos 75° \cos 30° + \sin 75° \sin 30°$

$= \cos (75° - 30°)$

$= \cos 45° = \frac{1}{\sqrt{2}}$

6. Given, $\begin{vmatrix} 1 & x & 2x \\ 1 & 3x & 5x \\ 1 & 3 & 4 \end{vmatrix} = 0$

By $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

We get, $\begin{vmatrix} 1 & x & 2x \\ 0 & 2x & 3x \\ 0 & 3 - x & 4 - 2x \end{vmatrix} = 0$

$\Rightarrow\ 1[2x\,(4 - 2x) - 3x\,(3 - x)] = 0$

$\Rightarrow\ -4x^2 + 8x - 9x + 3x^2 = 0$

$\Rightarrow\ -x^2 - x = 0$

$\Rightarrow\ x\,(x + 1) = 0$

$x = 0$ or $1 + x = 0 \Rightarrow x = -1$

But $x \neq 0$, So, $x = -1$

7. $\cos\theta\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}+\sin\theta\begin{bmatrix}\sin\theta & -\cos\theta \\ \cos\theta & \sin\theta\end{bmatrix}$

$=\begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta\end{bmatrix}+$

$\begin{bmatrix}\sin^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}$

$=\begin{bmatrix}\cos^2\theta+\sin^2\theta & 0 \\ 0 & \sin^2\theta+\cos^2\theta\end{bmatrix}=\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$

8. Given $\Delta=\begin{vmatrix}a & b & c \\ b & c & a \\ c & a & b\end{vmatrix}$

By $R_1 \to R_1 + R_2 + R_3$

We get, $\Delta=\begin{vmatrix}a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b\end{vmatrix}$

$=(a+b+c)\begin{vmatrix}1 & 1 & 1 \\ b & c & a \\ c & a & b\end{vmatrix}$

[By $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$=(a+b+c)\begin{vmatrix}1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c\end{vmatrix}$

$= (a + b + c)\ [(c - b)(b - c) - (a - b)(a - c)]$

$= (a + b + c)(-a^2 - b^2 - c^2 + ab + bc + ca)$

$= -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{-1}{2}\ (a + b + c)\ [(a - b)^2 + (b - c)^2 + (c - a)^2]$

Which is negative.

9. Given, $\Delta=\begin{vmatrix}1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta\end{vmatrix}$

By ($R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$\Delta=\begin{vmatrix}0 & \sin\theta & 0 \\ 1 & 1+\sin\theta & 1 \\ 0 & -\sin\theta & \cos\theta\end{vmatrix}$

$= 1(\sin\theta\cos\theta)$

$=\frac{1}{2}\times 2\sin\theta\cos\theta$

$=\frac{1}{2}\sin 2\theta$

There, maximum value $=\frac{1}{2}\times 1=\frac{1}{2}$

10. Given, $\Delta=\begin{vmatrix}a+ib & c+id \\ -(c-id) & a-ib\end{vmatrix}$

$= (a + ib)\ (a - ib) + (c - id)\ (c + id)$

$= a^2 - (ib)^2 + c^2 - (id)^2$

$= a^2 + b^2 + c^2 + d^2$

11. $\Delta=\begin{vmatrix}-a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2\end{vmatrix}$

$=abc\begin{vmatrix}-a & b & c \\ a & -b & c \\ a & b & -c\end{vmatrix}$

$=a^2b^2c^2\begin{vmatrix}-1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1\end{vmatrix}$

By $c_1 \to c_1 + c_2$

$=a^2b^2c^2\begin{vmatrix}0 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & -1\end{vmatrix}$

$= a^2 b^2 c^2\ [2(1 + 1)] = 4a^2 b^2 c^2$

12. $\because$ A is a matrix of 3×3

$\therefore\ A=\begin{vmatrix}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix}$

$3A=\begin{vmatrix}3a_{11} & 3a_{12} & 3a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33}\end{vmatrix}$

$\therefore\ |3A| = 3 \times 3 \times 3\ |A| = 27|A|$

Which is given as $K|A| = 27|A|$

$\therefore\ K = 27$

13. Let $\begin{vmatrix}b+c & c+a & a+b \\ y+z & z+x & x+y \\ q+r & r+p & p+q\end{vmatrix}=\Delta$

By $c_1 \to c_1 + c_2 + c_3$

We get $\Delta=\begin{vmatrix}2(a+b+c) & c+a & a+b \\ 2(x+y+z) & z+x & x+y \\ 2(p+q+r) & r+p & p+q\end{vmatrix}$

$$= 2\begin{vmatrix} a+b+c & -b & -c \\ x+y+z & -y & -z \\ p+q+r & -q & -r \end{vmatrix}$$

$[c_2 \to c_2 - c_1 \text{ and } c_3 \to c_3 - c_1]$

$$= 2\begin{vmatrix} a & -b & -c \\ x & -y & -z \\ p & -q & -r \end{vmatrix}$$

$[c_1 \to c_1 + c_2 + c_3]$

$$= 2(-1)(-1)\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$$

Therefore, k = 2

14. Given $\Delta = \begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-p & 0 \end{vmatrix}$

By $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$

We get $\Delta = \begin{vmatrix} p-q & p-q & p-q \\ q-p & 0 & q-r \\ r-q & r-q & r-q \end{vmatrix}$

$$= (p-q)(r-q)\begin{vmatrix} 1 & 1 & 1 \\ q-p & 0 & q-r \\ 1 & 1 & 1 \end{vmatrix}$$

= 0 [$\because$ R_1 and R_3 are identical]

15. Given, $\Delta = \begin{vmatrix} (a-b-c) & 2b & 2c \\ 2a & (b-c-a) & 2c \\ 2a & 2b & (c-a-b) \end{vmatrix}$

By $c_1 \to c_1 + c_2 + c_3$

We get, $\Delta = \begin{vmatrix} a+b+c & 2b & 2c \\ a+b+c & b-c-a & 2c \\ a+b+c & 2b & c-a-b \end{vmatrix}$

$$= (a+b+c)\begin{vmatrix} 1 & 2b & 2c \\ 1 & b-c-a & 2c \\ 1 & 2b & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 2b & 2c \\ 0 & -(a+b+c) & 0 \\ 0 & 0 & -(a+b+c) \end{vmatrix}$$

$[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

$= (a + b + c)[(a + b + c)^2] = (a + b + c)^3$

16. Clearly, $\Delta_2 = \begin{vmatrix} 9 & 9 \\ 7 & 5 \end{vmatrix} = (45-63) = -18 \neq 0$

$\therefore$ Given system has no solution.

$\Rightarrow \begin{vmatrix} 9 & 4 \\ 7 & K \end{vmatrix} = 0 \Rightarrow 9K - 28 = 0$

$\Rightarrow$ 9 K = 28

$\therefore K = \dfrac{28}{9}$

17. We know that

A (adj A) = |A| I_n

$\Rightarrow |A|I_0 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$

$\Rightarrow |A|I_0 = 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ |A| = 8

18. Let x + y + z = 2

2x + y – z = 3

and 3x + 2y + kz = 4

$\because$ The system of linear equations has unique solution, then

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$$

$\Rightarrow$ 1(k + 2) – 1(2k + 3) + 1 (4 – 3) $\neq$ 0

$\Rightarrow$ k + 2 –2k – 3 + 1 $\neq$ 0

$\Rightarrow$ k $\neq$ 0

Hence, the value of k = R – {0}

19. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 0 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & -3 \\ 3 & -1 & -3 \end{vmatrix}$$

$[c_2 \to c_2 - c_1 \text{ and } c_3 \to c_3 - c_1] = 1(3 - 3) = 0$

Hence, non-zero solution of the system exist. Therefore, the given system has infinitely many solutions.

20. $\because$ Given system has non-zero solution.

$$\therefore \begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

By $c_1 \to c_1 + c_3$ and $c_3 \to c_3 + c_2$

$$\Rightarrow \begin{vmatrix} 0 & -k & -1-k \\ k-1 & -1 & -2 \\ 0 & 1 & 0 \end{vmatrix} = 0$$

$\Rightarrow$ $-(k - 1)(1 + k) = 0$

$\Rightarrow$ $(1 - k)(1 + k) = 0$

$\Rightarrow$ $1 - k^2 = 0$

$\Rightarrow$ $k^2 = 1$

$\therefore$ $k = \pm 1$

21. $$f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x+1 \\ 3x(x-1) & x(x-1)(x-2) & x(x+1)(x-1) \end{vmatrix}$$

Taking $x(x + 1)$ from R_3 and x from R_2.

$$= x^2(x-1)\begin{vmatrix} 1 & x & x+1 \\ 2 & x-1 & x+1 \\ 3 & x-2 & x+1 \end{vmatrix}$$

$$= x^2(x-1)(x+1)\begin{vmatrix} 1 & x & 1 \\ 2 & x-1 & 1 \\ 3 & x-2 & 1 \end{vmatrix}$$

$(R_3 \to R_2 - R_1$ and $R_3 \to R_3 - R_1)$

$$= x^2(x^2-1)\begin{vmatrix} 1 & x & 1 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{vmatrix}$$

$= x^2 (x^2 - 1) [1(-2 + 2)] = 0$ for all x

$\therefore$ $f(10) = 0$

22. Given $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$\therefore$ $|A| = 6 - 5 = 1$

adj $A = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{1}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

23. We know that

$|AB| = |A|\ |B|$