# 4. Pair of Straight Lines

## Pair of straight Lines

## Equation of a pair of straight lines

The general form of equation of second degree is:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

# Quadratic form of equation of a pair of straight lines

The equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  can also be written as:

$$by^2 + 2(hx + f)y + ax^2 + 2gx + c = 0$$

By quadratic formula, we get:

$$y = \frac{-2(hx+f) \pm 2\sqrt{(hx+f)^2 - b(ax^2 + 2gx + c)}}{2b}$$

$$\Rightarrow y = \frac{-(hx+f) \pm \sqrt{(h^2 - ab)x^2 + 2(hf - bg)x + f^2 - bc}}{b}$$

The nature of lines can be determined w.r.t. the value of  $h^2 - ab$  as follows:

Case	Nature of lines
$h^2 - ab > 0$	Pair of non-parallel lines
$h^2 - ab = 0$	
$(i) f^2 - bc > 0$	Parallel non-coincident lines
$(ii) f^2 - bc = 0$	Coincident lines
$(iii) f^2 - bc < 0$	Imaginary lines
$h^2 - ab < 0$	Pair of imaginary lines

### General form of a pair of straight lines passing through the origin

The homogeneous equation of a pair of straight lines passing through the origin is

$$ax^2 + 2hxy + by^2 = 0.$$

# Angle between the lines

Let  $y - m_1 x = 0$  and  $y - m_2 x = 0$  be the lines represented by the general equation  $ax^2 + 2hxy + by^2 = 0$ .

Then the angle  $\theta$  between these lines is given as follows:

$$\tan\theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2} = \pm \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2} = \pm \frac{2\sqrt{h^2 - ab}}{a + b}$$

#### Points to be remembered

Equation of the pair of straight lines passing through the origin and the perpendicular to pair of lines  $ax^2 + 2hxy + by^2 = 0$  is  $bx^2 - 2hxy + ay^2 = 0$ .

Area of the triangle formed by the pair of lines  $ax^2 + 2hxy + by^2 = 0$  and lx + my + n = 0 is  $\left| \frac{n^2 \sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2} \right|$ .

Point of intersection of the pair of lines  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  is  $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$ 

Distance between  $\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$  and the origin is  $\sqrt{\frac{g^2+f^2-c(a+b)}{h^2-ab}}$ .

Equation of the pair of lines passing through  $(x_1, y_1)$  and parallel to the lines represented by  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  is  $a(x - x_1)^2 + 2h(x - x_1)(y - y_1) + b(y - y_1)^2 = 0$ .

Equation of the pair of lines passing through  $(x_1, y_1)$  and perpendicular to the lines represented by  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  is  $b(x - x_1)^2 - 2h(x - x_1)(y - y_1) + a(y - y_1)^2 = 0$ .

Equation of the pair of lines passing through the origin and at a distance of d units from  $(x_1, y_1)$  is  $d^2(x^2 + y^2) = (y_1x - x_1y)^2$ .

If  $d_1$  and  $d_2$  are the perpendicular distances from  $(x_1, y_1)$  to the pair of lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
, then  $d_1d_2 = \frac{\left|ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c\right|}{\sqrt{(a-b)^2 + 4h^2}}$ 

If  $d_1$  and  $d_2$  are the perpendicular distances from  $(x_1, y_1)$  to the pair of lines represented by  $ax^2 + 2hxy + by^2 = 0$ , then  $d_1d_2 = \frac{\left|ax_1^2 + 2hx_1y_1 + by_1^2\right|}{\sqrt{(a-b)^2 + 4h^2}}$ .

If the lines represented by  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  are equidistant from the origin, then  $f^4 - g^4 = c$   $(bf^2 - ag^2)$ .

If the lines represented by  $ax^2 + 2hxy + by^2 = 0$  bisect the angle between the co-ordinate axes, then  $(a + b)^2 = 4h^2$ .