

8.9 Differential Operators

Unit vectors along the coordinate axes: $\vec{i}$, $\vec{j}$, $\vec{k}$

Scalar functions (scalar fields): $f(x, y, z)$, $u(x_1, x_2, \dots, x_n)$

Gradient of a scalar field: $\text{grad } u$, ∇u

Directional derivative: $\frac{\partial f}{\partial l}$

Vector function (vector field): $\vec{F}(P, Q, R)$

Divergence of a vector field: $\text{div } \vec{F}$, $\nabla \cdot \vec{F}$

Curl of a vector field: $\text{curl } \vec{F}$, $\nabla \times \vec{F}$

Laplacian operator: ∇^2

856. Gradient of a Scalar Function

$$\text{grad } f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right),$$

$$\text{grad } u = \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right).$$

857. Directional Derivative

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta + \frac{\partial f}{\partial z} \cos \gamma,$$

where the direction is defined by the vector

$$\vec{l}(\cos \alpha, \cos \beta, \cos \gamma), \quad \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

858. Divergence of a Vector Field

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

859. Curl of a Vector Field

$$\begin{aligned}\operatorname{curl} \vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}\end{aligned}$$

860. Laplacian Operator

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

861. $\operatorname{div}(\operatorname{curl} \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) \equiv 0$

862. $\operatorname{curl}(\operatorname{grad} f) = \nabla \times (\nabla f) \equiv 0$

863. $\operatorname{div}(\operatorname{grad} f) = \nabla \cdot (\nabla f) = \nabla^2 f$

864. $\operatorname{curl}(\operatorname{curl} \vec{F}) = \operatorname{grad}(\operatorname{div} \vec{F}) - \nabla^2 \vec{F} = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$