Chapter-6

Miscellaneous Exercise

Q1. Using differentials, find the approximate value of each of the following :-

(a) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ (b) $(33)^{-\frac{1}{5}}$.

Sol:- Let $y=(x)^{1/4}$, where $x=\frac{16}{81}$, $\Delta x=+\frac{1}{81}$.

$y+\Delta y=(x+\Delta x)^{1/4}=\left(17/81\right)^{1/4}$

$\therefore \Delta y=(x+\Delta x)^{1/4}-(x)^{\frac{1}{4}}$.

$\Rightarrow (x+\Delta x)^{1/4}=\Delta y+(x)^{\frac{1}{4}}$. —— (*)

Now $y=x^{1/4}$.

$\Rightarrow \frac{dy}{dx}=\frac{1}{4}(x)^{1/4-1}=\frac{1}{4}x^{-3/4}=\frac{1}{4x^{3/4}}$.

$\Rightarrow \frac{\Delta y}{\Delta x}=\frac{1}{4x^{3/4}}$.

$\Rightarrow \Delta y=\frac{1}{4(x)^{3/4}}\Delta x=\frac{1}{4\left(\frac{16}{81}\right)^{3/4}}\left(\frac{1}{81}\right)=\frac{1}{4\times\frac{8}{27}}\times\frac{1}{81}$

$\Delta y=\frac{27}{32}\times\frac{1}{81}=\frac{1}{96}$

put values in equation (*)

$(x+\Delta x)^{1/4}=\frac{1}{96}+\left(\frac{16}{81}\right)^{\frac{1}{4}}=\frac{1}{96}+\frac{2}{3}=\frac{1+64}{96}=\frac{65}{96}$

$\therefore \left(\frac{17}{81}\right)^{1/4}=\frac{65}{96}$ approximately.

(b) $(33)^{-1/5}$

Let $y=(x)^{-\frac{1}{5}}$ where $x=32$, $\Delta x=+1$

$y+\Delta y=(x+\Delta x)^{-1/5}=(33)^{-1/5}$

$\therefore \Delta y=(x+\Delta x)^{-\frac{1}{5}}-(x)^{-\frac{1}{5}}$

$\Rightarrow (x+\Delta x)^{-\frac{1}{5}}=\Delta y+(x)^{-\frac{1}{5}}$

$\Rightarrow (33)^{-\frac{1}{5}}=\Delta y+(x)^{-\frac{1}{5}}$ —— (*)

Now. $y = (x)^{-\frac{1}{5}}$

$$\frac{\Delta y}{\Delta x} = -\frac{1}{5}(x)^{-\frac{1}{5}-1} = -\frac{1}{5}(x)^{-\frac{6}{5}}$$

$$\Delta y = -\frac{1}{5}(x)^{-\frac{6}{5}}.\Delta x = -\frac{1}{5}(32)^{-\frac{6}{5}}.(1)$$

$$\Delta y = -\frac{1}{5}(2)^{-6} = -\frac{1}{5} \times \frac{1}{64} = -\frac{1}{320}$$

from equation — (*)

$$(33)^{-\frac{1}{5}} = (32)^{-\frac{1}{5}} + \left(-\frac{1}{320}\right) = \frac{1}{2} - \frac{1}{320} = \frac{160-1}{320} = \frac{159}{320}$$

$\therefore (33)^{-\frac{1}{5}} = \frac{159}{320}$ approximately.

Ques:- 2 Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$.

Sol:- $f(x) = \frac{\log x}{x}$.

$$f'(x) = \frac{d}{dx}\left(\frac{\log x}{x}\right) = \frac{x.\frac{1}{x} - (\log x).1}{x^2} = \frac{1-\log x}{x^2}$$

$$f''(x) = \frac{x^2\left(0-\frac{1}{x}\right) - (1-\log x)(2x)}{(x^2)^2} = \frac{-x - 2x(1-\log x)}{x^4} = \frac{-1-2(1-\log x)}{x^3}$$

$$f''(x) = \frac{-3+2\log x}{x^3}$$

Now. $f'(x) = 0 \Rightarrow \frac{1-\log x}{x^2} = 0 \Rightarrow 1 - \log x = 0$

$1 = \log x$

or $\log x = \log e \Rightarrow x = e$.

at $x = e$.

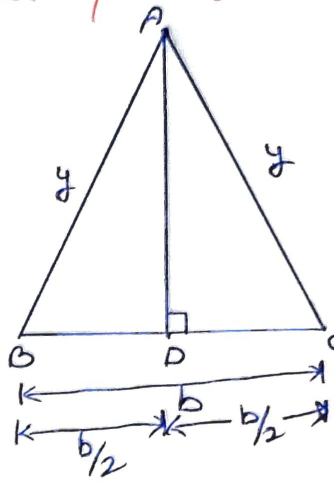
$$f''(e) = \frac{-3+2\log(e)}{(e)^3} = \frac{-3+2}{e^3} = \frac{-1}{e^3} < 0 \text{ (negative)}$$

Hence $f(x)$ has maximum at $x = e$.

Q.3. The two equal sides of an isosceles triangle with fixed base 'b' are decreasing at rate of 3 cm/sec. How fast is the area decreasing when the two equal sides are equal to base.

Sol:- Let base of triangle = 'b' cm.

equal sides of triangle = y cm.



Given $\frac{dy}{dt} = 3$ cm/sec. Find = $\frac{dA}{dt}$, when $y = b$

As Area of triangle $= \frac{1}{2} \times AD \times BC$

$$A = \frac{1}{2} \times \sqrt{y^2 - \frac{b^2}{4}} \times b = \frac{b}{4}\sqrt{4y^2 - b^2}$$

$$\frac{dA}{dt} = \frac{b}{4}\left[\frac{1}{2\sqrt{4y^2-b^2}} \times 8y\right] \times \frac{dy}{dt}$$

$$= \frac{b}{4}\left[\frac{4y}{\sqrt{4y^2-b^2}}\right](-3) = \frac{-3by}{\sqrt{4y^2-b^2}}$$

when $y = b$.

$$\frac{dA}{dt} = \frac{-3b^2}{\sqrt{4b^2-b^2}} = \frac{-3b^2}{\sqrt{3b^2}} = -\sqrt{3b^2} = -\sqrt{3}\,b.$$

∴ Area is decreasing at rate of $\sqrt{3}\,b$ cm²/sec.

Ques:-4. Find the equation of normal to curve $x^2 = 4y$ which passes through the point (1,2).

Sol:- Given equation of Curve is $x^2 = 4y$

Now. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{4}\right) = \frac{2x}{4} = x/2$.

slope of normal to curve $= -\frac{1}{x/2} = -\frac{2}{x}$. (say m)

at point (1,2) slope $= -\frac{2}{1} = -2$.

∴ Equation of normal with slope −2 and point (1,2)

$$(y - y_1) = m(x - x_1)$$

$$y - 2 = -2(x - 1)$$

$$y + 2x - 4 = 0.$$

Ques:-5 Show that the normal at any point θ to the curve $x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$ is at a constant distance from origin.

Sol:- $x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$

$$\frac{dx}{d\theta} = -a\sin\theta + a(\theta\cdot\cos\theta + \sin\theta) = -a\sin\theta + a\theta\cos\theta + a\sin\theta$$
$$= a\theta\cos\theta.$$

$$\frac{dy}{d\theta} = a\cos\theta - a(-\theta\sin\theta + \cos\theta) = a\cos\theta + a\theta\sin\theta - a\cos\theta = a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta.$$

$\therefore$ slope of normal $= -\frac{1}{\tan\theta} = -\frac{\cos\theta}{\sin\theta}$

Now. Equation of normal at (x, y). of slope $= -\frac{\cos\theta}{\sin\theta}$ is

$$y - a(\sin\theta - \theta\cos\theta) = -\frac{\cos\theta}{\sin\theta}(x - a(\cos\theta + \theta\sin\theta))$$

$$y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$$

$$x\cdot\cos\theta + y\sin\theta - a(\sin^2\theta + \cos^2\theta) = 0$$

$$x\cdot\cos\theta + y\sin\theta - a = 0$$

Now, Distance of origin from normal is

$$\frac{|0(\cos\theta) + 0(\sin\theta) - a|}{\sqrt{\cos^2\theta + \sin^2\theta}} = \frac{|-a|}{\sqrt{1}} = a, \text{ which is Constant}$$

Ques6: Find the intervals in which $f(x) = \frac{4\sin x - 2x - x\cos x}{2+\cos x}$ is (i) increasing (ii) decreasing.

Sol:- $f(x) = \frac{4\sin x - 2x - x\cos x}{2+\cos x} = \frac{4\sin x - x(2+\cos x)}{2+\cos x}$

$$\therefore f(x) = \frac{4\sin x}{2+\cos x} - \frac{x(2+\cos x)}{2+\cos x} = \frac{4\sin x}{2+\cos x} - x$$

differentiate w.r.t x

$$f'(x) = \frac{(2+\cos x)(4\cos x) - 4\sin x(-\sin x) - 1}{(2+\cos x)^2}$$

$$= \frac{8\cos x + 4\cos^2 x + 4\sin^2 x - (2+\cos x)^2}{(2+\cos x)^2}$$

$$f'(x) = \frac{8\cos x + 4(\cos^2 x + \sin^2 x) - 4 - \cos^2 x - 4\cos x}{(2+\cos x)^2}$$

$$= \frac{8\cos x + 4 - 4 - \cos^2 x - 4\cos x}{(2+\cos x)^2} = \frac{4\cos x - \cos^2 x}{(2+\cos x)^2}$$

$$\therefore f'(x) = \frac{\cos x(4-\cos x)}{(2+\cos x)^2} \quad \text{—— (1)}$$

We know $f(x)$ is increasing if $f'(x) \geq 0$.

As $-1 \leq \cos x \leq 1 \Rightarrow +1 \geq -\cos x \geq -1$

Adding 4 in all terms

$5 \geq 4 - \cos x \geq 3. \Rightarrow 4 - \cos x > 0$.

Also $(2+\cos x)^2 \geq 0$ (being square of real number).

$\therefore f(x)$ must increasing if $\cos x \geq 0$ (from eq. 1)

i.e. if x lies in Ist and IV quadrant.

$\therefore f(x)$ is increasing for $x \in [0, \pi/2]$ and $x \in [\frac{3\pi}{2}, 2\pi]$.

ii) $f(x)$ is decreasing if $f'(x) \leq 0$ i.e. $\cos x \leq 0$

$\Rightarrow x$ lies in II & III quadrant.

$\therefore f(x)$ is decreasing for $x \in [\pi/2, \frac{3\pi}{2}]$.

Ques:7 Find the intervals in which the function $f(x) = x^3 + \frac{1}{x^3}, x \neq 0$ is (i) increasing (ii) decreasing.

Sol:- $f(x) = x^3 + \frac{1}{x^3}$.

$f'(x) = 3x^2 + (-3/x^4) = 3x^2 - \frac{3}{x^4} = \frac{3x^6 - 3}{x^4} = 3\left[\frac{x^6 - 1}{x^4}\right]$

$f'(x) = \frac{3}{x^4}\left((x^2)^3 - 1^3\right) = \frac{3}{x^4}\left[(x^2-1)(x^4+x^2+1)\right] = \frac{3}{x^4}(1+x^2+x^4)(x-1)(x+1)$.

Now. $f(x)$ is increasing if $f'(x) \geq 0$

Here. $(1+x^2+x^4) > 0 \ \forall \ x \in R$.

Now $(x-1)(x+1) \geq 0$ if either $x \geq 1$ or $x \leq -1$.

$\therefore f(x)$ is increasing for $x \leq -1$ and for $x \geq 1$.

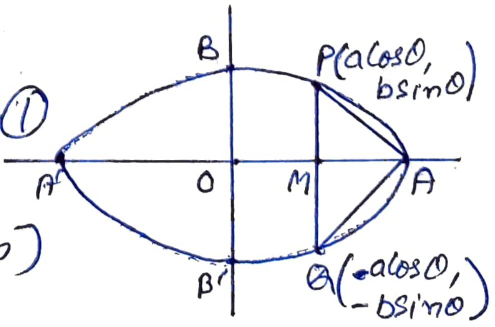
(ii) $f(x)$ is decreasing if $f'(x) \leq 0$.

Here. $(1+x^2+x^4) > 0 \ \forall \ x \in R$

Now $(x-1)(x+1) < 0$ if $-1 \leq x \leq 1$

$\therefore f(x)$ is decreasing for $-1 \leq x \leq 1$.

Ques:-8 Find the maximum area of an isosceles triangle incribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$. with its vertex at one end of the major axis.



Sol:- Equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. — ①

$OA = a$, $OB = b$, APQ is isosceles triangle with $PA = QA$. [∵ Ellipse is symm. about x and y axis]

from diagram $PM = QM$.

Area of isosceles triangle $APQ = \frac{1}{2} \times PQ \times AM. = \frac{1}{2} \times 2(PM) \times (OA - OM)$.

Compare equation ① with $\cos^2\theta + \sin^2\theta = 1$.

$\frac{x}{a} = \cos\theta \Rightarrow x = a\cos\theta$.

$y/b = \sin\theta \Rightarrow y = b\sin\theta$

Now. Area of Δ APQ $(\because \Delta) = PM(OA - OM)$

$$\Delta = b\sin\theta(a - a\cos\theta)$$

$$\Delta = ab(\sin\theta - \sin\theta\cos\theta) \quad \text{--- } (*)$$

$$\frac{d\Delta}{d\theta} = ab(\cos\theta - (\cos^2\theta - \sin^2\theta))$$

$$= ab(\cos\theta - \cos 2\theta).$$

$$\frac{d\Delta}{d\theta} = 0 \Rightarrow ab(\cos\theta - \cos 2\theta) = 0$$

$$\Rightarrow \cos\theta - \cos 2\theta = 0$$

$$\Rightarrow \cos\theta = \cos 2\theta$$

$$\cos\theta = \cos(360^\circ - 2\theta)$$

$$\Rightarrow \theta = 360^\circ - 2\theta$$

$$3\theta = 360^\circ \Rightarrow \theta = 120^\circ$$

$$\frac{d^2\Delta}{d\theta^2} = ab(-\sin\theta + 2\sin 2\theta)$$

$$= ab(-\sin 120 + 2\sin 240^\circ)$$

$$= ab\left(-\frac{\sqrt{3}}{2} + 2\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= ab\left(-\frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2}\right) = ab\left(\frac{-3\sqrt{3}}{2}\right) < 0.$$

$\therefore$ Δ is maximum. when $\theta = 120^\circ$

from equation --- $*$

$$\therefore \text{Area} = ab(\sin\theta - \sin\theta \cdot \cos\theta)$$

$$= ab(\sin 120^\circ - \sin 120^\circ \cos 120^\circ)$$

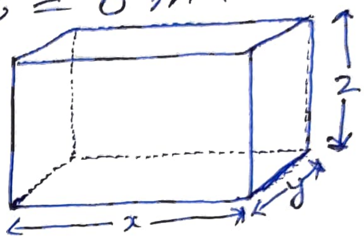
$$= ab\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\left(-\frac{1}{2}\right)\right)$$

$$= ab\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right) = ab\left(\frac{2\sqrt{3} + \sqrt{3}}{4}\right)$$

$$= \frac{3\sqrt{3}}{4}ab \quad \underline{\text{Ans.}}$$

Ques: 9. A tank with rectangular base and rectangular sides open at the top is to be constructed so that its depth is 2m and volume is $8m^3$. If building of tank costs Rs 70 per sq. meters for the base and Rs 45 per sq. meters for sides. What is the cost of least expensive test:

Sol:- Let x = length of tank, y = breadth of base of tank.
depth of tank = 2 m. Volume of tank = $8 m^3$.



$\therefore V = l \times b \times h = x \times y \times 2 = 2xy$

$8 = 2xy \Rightarrow xy = 4$. or $y = 4/x$. — (*)

Now cost of building tank means we need surface area of whole tank

$\therefore S$ = Surface area = 4 sides + base.

$S = (x\times2 + x\times2 + y\times2 + y\times2) + x.y = 4x + 4y + xy$

cost of building tank $C = 70(xy) + 45(4x+4y)$

$C = 70xy + 180x + 180y = 70\times4 + 180x + 180y$

$C = 280 + 180x + 180y = 280 + 180x + \frac{720}{x}$

$\therefore \frac{dc}{dx} = 0 + 180 - \frac{720}{x^2}$ & $\frac{d^2c}{dx^2} = 0 - 720x^{-2} = \frac{1440}{x^3}$ (+ive)

$\frac{dc}{dx} = 0 \Rightarrow 180 - \frac{720}{x^2} = 0$

$180 = \frac{720}{x^2}$

$\Rightarrow x^2 = \frac{720}{180}$ 4.

$x = +2, -2$ (rejected) →

$y = \frac{4}{x} = 2$

Hence minimum cost

$C = 280 + 180(2) + 180(2)$

$C = 280 + 360 + 360 = 1000$ Rs

Ques. 10 The sum of perimeter of a circle and square is k, where k is constant. Prove that sum of their areas is least when side of square is double the radius of circle.

Sol: Let radius of circle $= x$, side of square $= y$

Then $2\pi x + 4y = k$. (Perimeter)

$\Rightarrow y = \frac{k - 2\pi x}{4}$

Now Sum of areas of both $(S) = \pi x^2 + y^2$

$S = \pi x^2 + \left(\frac{k - 2\pi x}{4}\right)^2$

$\frac{ds}{dx} = \pi(2x) + 2\left(\frac{k - 2\pi x}{4}\right)\left(-\frac{2\pi}{4}\right) = 2\pi x - \frac{\pi}{4}(k - 2\pi x)$

$\frac{ds}{dx} = \frac{8\pi x - k\pi + 2\pi^2 x}{4}$ & $\frac{d^2s}{dx^2} = \frac{1}{4}\left[8\pi - 0 + 2\pi^2\right] = \frac{1}{4}(2\pi^2 + 8\pi)$ = +ive.

$\frac{ds}{dx} = 0$ gives.

$\frac{8\pi x - k\pi + 2\pi^2 x}{4} = 0$

$\Rightarrow 8\pi x + 2\pi^2 x - k\pi = 0$

$2\pi x(4 + \pi) = k\pi$

$x = \frac{k\pi}{2\pi(4+\pi)} = \frac{k}{2(4+\pi)}$

Now $y = \frac{1}{4}\left[\frac{k}{1} - 2\pi\left(\frac{k}{2(4+\pi)}\right)\right]$

$y = \frac{1}{4}\left[\frac{2k(4+\pi) - 2\pi k}{2(4+\pi)}\right]$

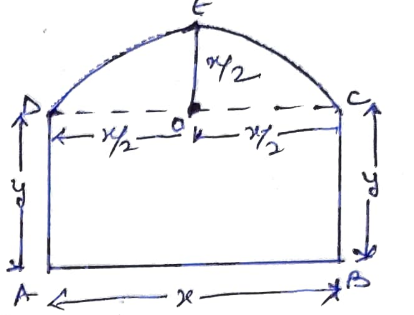
$= \frac{1}{8}\left[\frac{2k\pi + 8k - 2\pi k}{(4+\pi)}\right]$

$= \frac{k}{(4+\pi)} = 2\left[\frac{k}{2(4+\pi)}\right]$

or $\underline{y = 2x}$. $\Rightarrow$ side of square = double the radius of circle.

Ques:- 11 A window is in the form of rectangle surmounted by semicircular opening. The total perimeter of window is 10m. Find the dimensions of window so that maximum light is admit through the window.

Sol: Let x = length of window
y = height of window
from diagram radius of semicircle = $x/2$.



Perimeter. $P = y + x + y + \pi\left(\frac{x}{2}\right) = 10$

$P = 2y + x + \frac{\pi x}{2}$

$P = 2y + \frac{x(2+\pi)}{2} \Rightarrow y = \frac{1}{2}\left[P - \frac{x}{2}(2+\pi)\right] = \frac{1}{4}\left[2P - 2x - \pi x\right]$

Now Area of window = Area of rectangle + Area of semicircle

$A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

$= \frac{x}{4}\left[2P - 2x - \pi x\right] + \frac{\pi x^2}{8} = \frac{1}{4}\left[2Px - 2x^2 - \pi x^2\right] + \frac{\pi x^2}{8}$

$\frac{dA}{dx} = \frac{1}{4}\left[2P - 4x - 2\pi x\right] + \frac{\pi x}{4},$ $\quad \frac{d^2A}{dx^2} = \frac{1}{4}\left[0 - 4 - 2\pi\right) + \frac{\pi}{4}$

$= -1 - \frac{\pi}{2} + \frac{\pi}{4} = -1 - \frac{\pi}{4}$ (-ive)

$\frac{dA}{dx} = 0 \Rightarrow \frac{1}{4}\left[2P - 4x - 2\pi x\right] + \frac{\pi x}{4} = 0$

$\Rightarrow 2P - 4x - 2\pi x + \pi x = 0$

$\Rightarrow 2P - 4x - \pi x = 0$

$2P - x(4+\pi) = 0$

$\Rightarrow 2P = x(4+\pi)$

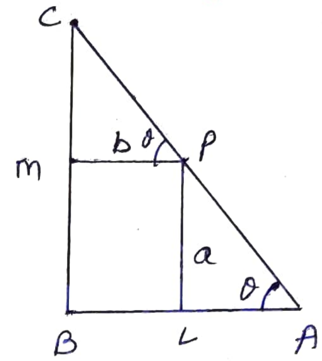
$\Rightarrow x = \frac{2P}{4+\pi} = \frac{2\times 10}{4+\pi} = \frac{20}{4+\pi}$

$y = \frac{1}{4}\left[2\times 10 - \frac{2\times 20}{4+\pi} - \frac{\pi \times 20}{4+\pi}\right]$

$= \frac{1}{4}\left[\frac{20(4+\pi) - 40 - 20\pi}{4+\pi}\right] = \frac{1}{4}\left[\frac{80 + 20\pi - 40 - 20\pi}{(4+\pi)}\right)$

$= \frac{10}{4+\pi}$ $\quad \Rightarrow \quad x = \frac{20}{4+\pi},\ y = \frac{10}{4+\pi}$ Ans

Ques. 12 A point on the hypotenuse of a triangle is at distance a and b from the sides of triangle. Show that minimum length of hypotenuse is $(a^{2/3}+b^{2/3})^{3/2}$.



Sol:- Let ABC be any triangle. Let P be a point on hypotenuse such that $PL = a$, $PM = b$ and $PM \perp BC$, $PL \perp AB$.

In ΔAPL $\frac{PL}{AP} = \sin\theta \Rightarrow AP = \frac{PL}{\sin\theta} = a.\operatorname{cosec}\theta$.

In ΔPMC, $\frac{PM}{PC} = \cos\theta \Rightarrow PC = \frac{PM}{\cos\theta} = b.\sec\theta$.

$AC = AP + PC = a.\operatorname{cosec}\theta + b\sec\theta$.

Let $h = AC = a\operatorname{cosec}\theta + b\sec\theta$. —— (*)

$\frac{dh}{d\theta} = -a\operatorname{cosec}\theta.\cot\theta + b\sec\theta.\tan\theta = -\frac{a\cos\theta}{\sin^2\theta} + \frac{b\sin\theta}{\cos^2\theta}$

$\frac{dh}{d\theta} = 0 \Rightarrow -\frac{a\cos\theta}{\sin^2\theta} + \frac{b\sin\theta}{\cos^2\theta} = 0$

$\Rightarrow \frac{b\sin\theta}{\cos^2\theta} = \frac{a\cos\theta}{\sin^2\theta}$

$\Rightarrow \frac{b}{a} = \frac{\cos^3\theta}{\sin^3\theta} = \cot^3\theta$

or $\frac{a}{b} = \tan^3\theta \Rightarrow \tan\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}$

$\frac{d^2h}{d\theta^2} = -a\left[\operatorname{cosec}\theta.(-\operatorname{cosec}^2\theta) + \cot\theta(-\operatorname{cosec}\theta.\cot\theta)\right] + b\left[\sec\theta(\sec^2\theta) + \tan\theta(\sec\theta.\tan\theta)\right]$

$= a\left[+\operatorname{cosec}^3\theta + \operatorname{cosec}\theta.\cot^2\theta\right) + b(\sec^3\theta + \sec\theta\tan^2\theta)]$

Now as $0 < \theta < \frac{\pi}{2} \Rightarrow$ All t-ratio's are positive, also, a and b are positive

$\therefore \frac{d^2h}{d\theta^2} > 0. \Rightarrow h$ is minimum.

Now. $\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{a}{b}\right)^{2/3} = \frac{b^{2/3} + a^{2/3}}{b^{2/3}}$

$\sec\theta = (b^{2/3} + a^{2/3})^{1/2} / (b)^{1/3}$ —— ①

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \left(\frac{b}{a}\right)^{2/3} = \frac{a^{2/3} + b^{2/3}}{a^{2/3}}$

$\Rightarrow \operatorname{cosec}\theta = \frac{(a^{2/3} + b^{2/3})^{1/2}}{(a)^{1/3}}$ —— ②

put values from eq. ① and ② in (*)

$$h = a\operatorname{cosec}\theta + b\sec\theta = \frac{a\left(a^{2/3}+b^{2/3}\right)^{1/2}}{a^{1/3}} + \frac{b\left(b^{2/3}+a^{2/3}\right)^{1/2}}{b^{1/3}}$$

$$= a^{2/3}\cdot\left(a^{2/3}+b^{2/3}\right)^{1/2} + b^{2/3}\left(b^{2/3}+a^{2/3}\right)^{\frac{1}{2}}$$

$$h. = \left(a^{2/3}+b^{2/3}\right)^{\frac{1}{2}}\left[a^{2/3}+b^{2/3}\right]^{1} = \left(a^{2/3}+b^{2/3}\right)^{3/2} \text{ Ans}$$

Ques:-13 find the points at which the function f given by $f(x) = (x-2)^4(x+1)^3$ has (i) local maxima (ii) local minima (iii) point of inflexion.

Sol:- $f(x) = (x-2)^4(x+1)^3$

$$f'(x) = (x-2)^4\, 3(x+1)^2 + (x+1)^3 . 4(x-2)^3$$

$$= (x-2)^3(x+1)^2\left[3(x-2)+4(x+1)\right] = (x-2)^3(x+1)^2(7x-2)$$

$$f'(x) = 0 \Rightarrow (x-2)^3(x+1)^2\left(3x-6+4x+4\right) = 0$$

or $(3x+4x-6+4) = 0$	$(x-2)^3 = 0$	$(x+1)^2 = 0$
$7x-2 = 0$	$x-2 = 0$	$x+1 = 0$
$7x = 2 \Rightarrow x = \frac{2}{7}$	$x = 2$	$x = -1$

Now as $x \to \frac{7}{2}^+$ $f'(x) = (-ive)(+ve)(+ive) = -ive$ i.e $f'(x) < 0$

& as $x \to \frac{7}{2}^-$ $f'(x) = (-ive)(+ve)(-ive) = +ive$ i.e $f'(x) > 0$

$\therefore f(x)$ has local maxima at $x = 7/2$.

ii) As $x \to 2^+$ $f'(x) = (+ive)(+ive)(+ive) = +ive$ i.e $f'(x) > 0$

& As $x \to 2^-$ $f'(x) = (-ive)(+ive)(+ive) = -ive$ i.e. $f'(x) < 0$

$\therefore f(x)$ has local minima at $x = 2$.

at $x = -1$:

as $x \to -1^+$ $f'(x) = (-ive)(+ive)(-ive) = +ive$ i.e. $f'(x) > 0$

$x \to -1^-$ $f'(x) = (-ive)(+ive)(-ive) = +ive$ i.e $f'(x) > 0$

as $f'(x)$ doesnot changes its sign $\therefore x = -1$ is the point of inflection.

Ques:- 14 Find the absolute maximum and minimum values of the function $f(x) = \cos^2 x + \sin x$; $x \in [0, \pi]$

Sol:- $f(x) = \cos^2 x + \sin x$

$f'(x) = 2\cos x \cdot (-\sin x) + \cos x$

$= -\sin 2x + \cos x \rightarrow$ or

$f'(x) = \cos x(1 - 2\sin x)$.

$f'(x) = 0 \Rightarrow \cos x(1 - 2\sin x) = 0$

$\Rightarrow \cos x = 0$ or $1 - 2\sin x = 0$

$x = \pi/2$ or $1 = 2\sin x$

$\sin x = \frac{1}{2}$.

$x = \pi/2$ or $x = \pi/6$

Now $f(\pi/6) = \cos^2 \pi/6 + \sin \pi/6 = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = 5/4$.

$f(\pi/2) = \cos^2(\pi/2) + \sin \pi/2 = 0 + 1 = 1$

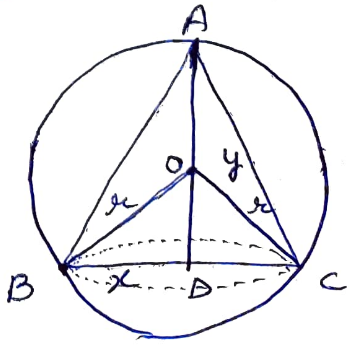
$f(0) = \cos^2(0) + \sin 0 = 1 + 0 = 1$

$f(\pi) = \cos^2(\pi) + \sin \pi = (-1)^2 + 0 = 1$.

$\therefore$ Absolute maximum is $= \frac{5}{4}$ at $x = \pi/6$

Absolute minimum is $= 1$ at $0, \pi/2, \pi$.

Ques:-15 Show that the altitude of right circular cone of maximum volume that can be incribed in a sphere of radius r is $\frac{4r}{3}$.



Sol:- Let x be the radius of base of cone and y be the height of the cone.

Volume of Cone $V = \frac{1}{3}\pi . x^2 y$ ——— (*)

$OD = AD - AO = y - r$,

$OB = x$, $AD = y$

Now In Δ OBD.

$BD^2 + OD^2 = OB^2$

$\Rightarrow x^2 + (y-r)^2 = r^2$

$\Rightarrow x^2 + y^2 + r^2 - 2yr = r^2$

$\Rightarrow x^2 = 2yr - y^2$

$\therefore V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi(2yr - y^2).y = \frac{1}{3}\pi(2y^2 r - y^3)$

$\frac{dV}{dy} = \frac{1}{3}\pi[4yr - 3y^2]$, $\frac{d^2V}{dy^2} = \frac{1}{3}\pi[4r - 6y]$

$\frac{dV}{dy} = 0 \Rightarrow \frac{1}{3}\pi[4yr - 3y^2] = 0$

$\Rightarrow y[4r - 3y] = 0$

$\Rightarrow y = 0$ or $y = \frac{4r}{3}$

but $y = 0$ is impossible as y is height of cone.

So take $y = \frac{4r}{3}$ and $\frac{d^2V}{dy^2} = \frac{1}{3}\pi[4r - 6 \times \frac{4r}{3}] = \frac{1}{3}\pi[-4r] < 0$

$\therefore$ volume of cone is maximum when height is $\frac{4r}{3}$.

Ques:- 16 Let f be a function defined on $[a,b]$ such that $f'(x) > 0 \ \forall \ x \in (a,b)$. Then prove that f is increasing on (a,b).

Sol:- Given $f'(x) > 0 \ \forall \ x \in (a,b)$.

Consider $x_1, x_2 \in (a,b)$ such that $x_1 < x_2$

By mean value theorem

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \quad \text{where } x_1 < c < x_2$$

$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1) \cdot f'(c)$

as $x_2 > x_1 \Rightarrow x_2 - x_1 > 0$

$\Rightarrow f'(c) > 0$ as $f'(x) > 0 \; \forall x \in (a,b)$

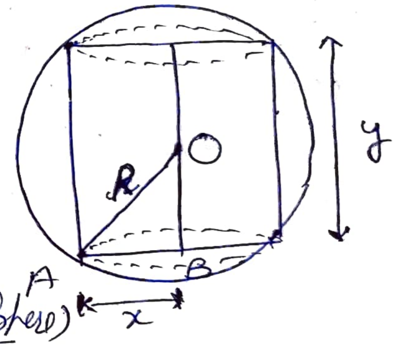
$\therefore f(x_2) - f(x_1) > 0$

$\Rightarrow f(x_2) > f(x_1)$.

i.e. for $x_1 < x_2$, $f(x_1) < f(x_2)$. $\forall \; x_1, x_2 \in (a,b)$.

which by def. show that f is increasing on (a,b).

Quest 17. Show that the height of cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find maximum volume.



Sol: Consider x be radius of cylinder and y be its height of cylinder in sphere of radius R & centre O.

$BO = y/2$ as O is (Centre of sphere)

In ΔOAB

$AB^2 + BO^2 = AO^2$

$\Rightarrow x^2 + (y/2)^2 = R^2$

$\Rightarrow x^2 = R^2 - \left(\frac{y}{2}\right)^2$

Volume of cylinder $V = \pi x^2 y = \pi\left(R^2 - \frac{y^2}{4}\right)y$

$V = \pi\left(R^2 y - y^3/4\right)$

$\frac{dV}{dy} = \pi\left(R^2 - \frac{3y^2}{4}\right)$ $\quad \frac{d^2V}{dy^2} = \pi\left(0 - \frac{6y}{4}\right) = -\frac{6\pi y}{4}$ (-ive).

Volume is maximum

$\frac{dV}{dy} = 0 \Rightarrow \pi\left(R^2 - \frac{3y^2}{4}\right) = 0$

$\Rightarrow R^2 - \frac{3y^2}{4} = 0$

$$R^2 = \frac{3y^2}{4}$$

or $y^2 = \frac{4R^2}{3}$.

$\Rightarrow \quad y = \sqrt{\frac{4R^2}{3}} = \frac{2R}{\sqrt{3}}$.

$\therefore$ height of cylinder $= \frac{2R}{\sqrt{3}}$.

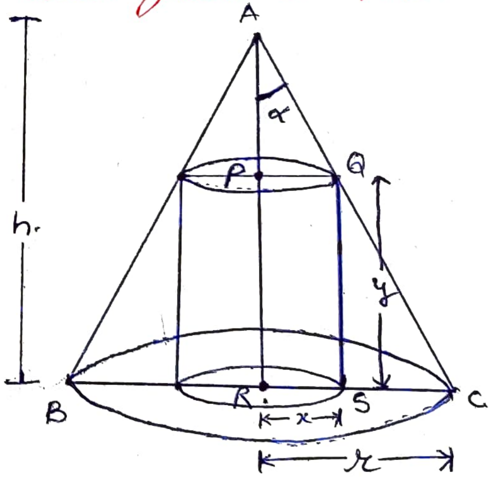
$\therefore$ Max. Volume $= \pi x^2 y = \pi x^2\left(\frac{2R}{\sqrt{3}}\right)$.

Now $x^2 = R^2 - \frac{y^2}{4} = R^2 - \left(\frac{4R^2}{3}\right)\cdot\frac{1}{4}$

$x^2 = R^2 - \frac{R^2}{3} = \frac{2R^2}{3}$.

Max. Volume $= \pi \cdot \frac{2R^2}{3} \cdot \frac{2R}{\sqrt{3}} = \frac{\pi 4R^3}{3\sqrt{3}}$ Ans.

Ques:-18 Show that height of cylinder of greatest volume inscribed in a right circular cone of height h, and semi-vertical angle α is one third that of cone. and greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2\alpha$.



Sol:- Let r be the radius of base of cone and height h.

Consider x be the radius of base of cylinder and y be the height of cylinder inscribed inside the cone.

from figure, $AP = AR - PR = h - y$.

$PQ = RS = x$.

In ΔAPQ & ΔARC [similar triangles].

$$\frac{PQ}{RC} = \frac{AP}{AR}$$

$$\Rightarrow \frac{x}{r} = \frac{h-y}{h}$$

$hx = hr - ry$ or. $ry = hr - hx$

$x = \frac{hr - ry}{h}$ $\quad y = \frac{h}{r}(r - x)$.

Ques:-19 A cylindrical tank of radius 10 m is filled with wheat at rate of 314 m^3/h. Then depth of wheat is increasing at rate of (A) 1 m/h. (B) 0.1 m/h. (C) 1.1 m/h (D) 0.5 m/h.

Sol: Let depth of cylinder = x, r = 10 m.

$V = \pi r^2 h = \pi (10)^2 h = 100\pi h.$

given $\frac{dV}{dt} = 314$, $\frac{dh}{dt} = ?$

$\frac{dV}{dt} = \pi \times 100 \times \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{314}{100 \times \pi} = \frac{314}{100} \times \frac{1}{3.14} = \frac{314 \times 100}{100 \times 314} = 1$ m/h.

$\therefore$ Ans is (A).

Ques: 20 The slope of tangent to curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at point (2, -1) is

(A) $\frac{22}{7}$ (B) $\frac{6}{7}$ (C) $\frac{7}{6}$ (D) $-\frac{6}{7}$

Sol: $\frac{dx}{dt} = 2t + 3$, $\frac{dy}{dt} = 4t - 2$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t-2}{2t+3}$.

Now at x = 2, $t^2 + 3t - 8 = 2$

$\Rightarrow t^2 + 3t - 10 = 0$

$t^2 + 5t - 2t - 10 = 0$

$(t+5)(t-2) = 0$

$t = -5, 2$

at y = -1, $2t^2 - 2t - 5 = -1$

$2t^2 - 2t - 4 = 0$

$t^2 - t - 2 = 0$

$(t-2)(t+1) = 0$

$t = 2, -1.$

$\therefore$ Common value of $t = 2$

$\therefore$ Slope of tangent to Curve $= \frac{4(2)-2}{2(2)+3} = \frac{8-2}{4+3} = \frac{6}{7}.$ (B) Correct.

Volume of cone $V = \frac{1}{3}\pi r^2 h$.

Volume of Cylinder $V' = \pi x^2 y = \pi x^2 \cdot \frac{h}{r}(r-x)$

$V' = \frac{\pi h}{r}(x^2 r - x^3)$

$\frac{dV'}{dx} = \frac{\pi h}{r}[2xr - 3x^2]$ & $\frac{d^2V'}{dx^2} = \frac{\pi h}{r}[2r - 6x]$

$\frac{dV'}{dx} = 0 \Rightarrow \frac{\pi h}{r}[2xr - 3x^2] = 0$

$2xr - 3x^2 = 0$

$\Rightarrow 2xr = 3x^2$

$\Rightarrow x = \frac{2r}{3}$

at $x = \frac{2r}{3}$, $\frac{d^2V'}{dx^2} = \frac{\pi h}{r}[2r - \frac{2r}{3} \times 6] = \frac{\pi h}{r}[2r - 4r] = -2\pi h$ (-ive).

$\therefore$ volume of cylinder is max.

Max. volume

$V' = \frac{\pi h}{r}\left[\left(\frac{2r}{3}\right)^2 \cdot r - \left(\frac{2r}{3}\right)^3\right]$

$= \frac{\pi h}{r}\left[\frac{4r^3}{9} - \frac{8}{27}r^3\right]$

$= \frac{\pi h r^3}{r}\left[\frac{4}{9} - \frac{8}{27}\right]$

$= \pi h r^2\left[\frac{12-8}{27}\right]$

$= \frac{4}{27}\pi h r^2$.

Now from diagram

~~PO~~. $\frac{RC}{AR} = \tan\alpha$

$\frac{r}{h} = \tan\alpha \Rightarrow r = h\tan\alpha$

$\therefore V' = \frac{4}{27}\pi h \cdot (h^2 \tan^2\alpha) = \frac{4}{27}\pi h^3 \tan^2\alpha$.

Question 21 The line $y = mx+1$ is tangent to curve $y^2 = 4x$ if value of m is (A) 1 (B) 2 (C) 3 (D) $\frac{1}{2}$

Sol:- Curve is $y^2 = 4x$.

$2y\frac{dy}{dx} = 4.$

$\Rightarrow \frac{dy}{dx} = \frac{2}{y}$

$\therefore$ slope $m = \frac{2}{y} \Rightarrow y = \frac{2}{m}.$

Also $y = mx+1$

$\frac{2}{m} = mx+1$

$\Rightarrow mx = \frac{2}{m} - 1 = \frac{2-m}{m}$

$x = \frac{1}{m^2}(2-m).$

put values of x & y

$\left(\frac{2}{m}\right)^2 = 4\left(\frac{2-m}{m^2}\right)$

$\frac{4}{m^2} = \frac{4}{m^2}(2-m)$

$1 = 2-m. \Rightarrow m = 2-1 = 1.$ Ans. (A) is correct.

Ques: 22 The normal at point (1, 1) on the curve $2y+x^2=3$ is

(A) $x+y=0$ (B) $x-y=0$ (C) $x+y+1=0$ (C) $x-y=1$

Sol: $2y+x^2=3 \Rightarrow y=\frac{1}{2}[3-x^2]$

$\frac{dy}{dx}=\frac{1}{2}[0-2x]=-x$

∴ Slope of tangent at (1,1) = −1

Slope of normal at (1,1) $= -\left(\frac{1}{-1}\right) = 1$

∴ equation of normal

$y-1=1(x-1) \Rightarrow y-1=x-1$

or $x-y=0$. (B) is correct

Ques! 23 The normal to the curve $x^2 = 4y$ passing $(1,2)$ is

(A) $x+y=3$ (B) $x-y=3$ (C) $x+y=1$ (d) $x-y=1$

Sol: $x^2 = 4y \Rightarrow y = x^2/4$ — (*)

$\frac{dy}{dx} = \frac{x}{2}$.

$\therefore$ slope of normal $= -\frac{1}{x/2} = -\frac{2}{x}$ — (1)

Now slope of normal at $(1,2)$ $= \frac{y-2}{x-1}$ — (2)

$\therefore$ from eq. (2) & (1)

$-\frac{2}{x} = \frac{y-2}{x-1}$

$-2x+2 = xy-2x$

$\Rightarrow y = \frac{2}{x}$

put in equation — (*)

$\frac{2}{x} = \frac{x^2}{4} \Rightarrow x^3 = 8$

$x = 2$

$y = \frac{2}{2} = 1.$

$\therefore$ equation of normal at $(2,1)$ & slope -1 is

$(y-1) = -1(x-2)$

$y-1 = -x+2$

$x+y = 3$ (A) is Correct

Ques: 24 The points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with axes are

(A) $\left(4, \pm\frac{8}{3}\right)$ (B) $\left(4, -\frac{8}{3}\right)$ (C) $\left(4, \pm\frac{3}{8}\right)$ (D) $\left(\pm4, \frac{3}{8}\right)$

Sol: Curve is $9y^2 = x^3$

$\Rightarrow 18y \cdot \frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} = \frac{x^2}{6y}$

$\therefore$ slope of normal $= -1/\frac{x^2}{6y} = \frac{-6y}{x^2}$.

Since normal to curve makes equal intercepts on axes

$\therefore$ slope of line with equal intercepts $= \pm 1$.

$\therefore \frac{-6y}{x^2} = \pm 1$

$\Rightarrow 6y = \pm x^2$

$6y = x^2$	
$y = \frac{x^2}{6}$	$y = -\frac{x^2}{6}$
$9\left(\frac{x^2}{6}\right)^2 = x^3$	$9\left(-\frac{x^2}{6}\right)^2 = x^3$
$\Rightarrow \frac{9x^4}{36} = x^3$	$\frac{9}{36}x^4 = x^3$
$x = \frac{36}{9} = 4.$	$\Rightarrow x = \frac{36}{9} = 4$
$y = \frac{16}{6} = \frac{8}{3}$	$y = -\frac{16}{6} = -\frac{8}{3}$

Hence points are $\left(4, \pm\frac{8}{3}\right)$. (A) is Correct.