CHAPTER XXIV.

RECURRING SERIES.

320. A series $u_0 + u_1 + u_2 + u_3 + \ldots\ldots,$
in which from and after a certain term each term is equal to the sum of a fixed number of the preceding terms multiplied respectively by certain constants is called a **recurring series.**

321. In the series

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots\ldots,$$

each term after the second is equal to the sum of the two preceding terms multiplied respectively by the *constants* $2x$, and $-x^2$; these quantities being called constants because they are the same for all values of n. Thus

$$5x^4 = 2x \,.\, 4x^3 + (-x^2) \,.\, 3x^2;$$

that is,

$$u_4 = 2xu_3 - x^2u_2;$$

and generally when n is greater than 1, each term is connected with the two that immediately precede it by the equation

$$u_n = 2xu_{n-1} - x^2u_{n-2},$$

or

$$u_n - 2xu_{n-1} + x^2u_{n-2} = 0.$$

In this equation the coefficients of u_n, u_{n-1}, and u_{n-2}, taken with their proper signs, form what is called the **scale of relation.**

Thus the series

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots\ldots$$

is a recurring series in which the scale of relation is

$$1 - 2x + x^2.$$

322. If the scale of relation of a recurring series is given, any term can be found when a sufficient number of the preceding

terms are known. As the method of procedure is the same however many terms the scale of relation may consist of, the following illustration will be sufficient.

If $$1 - px - qx^2 - rx^3$$

is the scale of relation of the series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$$

we have

$$a_nx^n = px \,.\, a_{n-1}x^{n-1} + qx^2 \,.\, a_{n-2}x^{n-2} + rx^3 \,.\, a_{n-3}x^{n-3},$$

or $$a_n = pa_{n-1} + qa_{n-2} + ra_{n-3};$$

thus any coefficient can be found when the coefficients of the three preceding terms are known.

323. Conversely, if a sufficient number of the terms of a series be given, the scale of relation may be found.

Example. Find the scale of relation of the recurring series

$$2 + 5x + 13x^2 + 35x^3 + \ldots\ldots$$

Let the scale of relation be $1 - px - qx^2$; then to obtain p and q we have the equations $13 - 5p - 2q = 0$, and $35 - 13p - 5q = 0$; whence $p = 5$, and $q = -6$, thus the scale of relation is

$$1 - 5x + 6x^2.$$

324. If the scale of relation consists of 3 terms it involves 2 constants, p and q; and we must have 2 equations to determine p and q. To obtain the first of these we must know at least 3 terms of the series, and to obtain the second we must have one more term given. Thus to obtain a scale of relation involving two constants we must have at least 4 terms given.

If the scale of relation be $1 - px - qx^2 - rx^3$, to find the 3 constants we must have 3 equations. To obtain the first of these we must know at least 4 terms of the series, and to obtain the other two we must have two more terms given; hence to find a scale of relation involving 3 constants, at least 6 terms of the series must be given.

Generally, to find a scale of relation involving m constants, we must know at least $2m$ consecutive terms.

Conversely, if $2m$ consecutive terms are given, we may assume for the scale of relation

$$1 - p_1x - p_2x^2 - p_3x^3 - \ldots\ldots - p_mx^m.$$

325. *To find the sum of* n *terms of a recurring series.*

The method of finding the sum is the same whatever be the scale of relation; for simplicity we shall suppose it to contain only two constants.

Let the series be

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

and let the sum be S; let the scale of relation be $1 - px - qx^2$; so that for every value of n greater than 1, we have

$$a_n - pa_{n-1} - qa_{n-2} = 0.$$

$$\text{Now } \quad S = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1},$$

$$-px\,S = -pa_0 x - pa_1 x^2 - \dots - pa_{n-2} x^{n-1} - pa_{n-1} x^n,$$

$$-qx^2\,S = -qa_0 x^2 - \dots - qa_{n-3} x^{n-1} - qa_{n-2} x^n - qa_{n-1} x^{n+1}.$$

$$\therefore\ (1 - px - qx^2)\,S = a_0 + (a_1 - pa_0)\,x - (pa_{n-1} + qa_{n-2})\,x^n - qa_{n-1} x^{n+1},$$

for the coefficient of every other power of x is zero in consequence of the relation

$$a_n - pa_{n-1} - qa_{n-2} = 0.$$

$$\therefore\ S = \frac{a_0 + (a_1 - pa_0)\,x}{1 - px - qx^2} - \frac{(pa_{n-1} + qa_{n-2})\,x^n + qa_{n-1} x^{n+1}}{1 - px - qx^2}.$$

Thus the sum of a recurring series is a fraction whose denominator is the scale of relation.

326. If the second fraction in the result of the last article decreases indefinitely as n increases indefinitely, the sum of an infinite number of terms reduces to $\dfrac{a_0 + (a_1 - pa_0)\,x}{1 - px - qx^2}$.

If we develop this fraction in ascending powers of x as explained in Art. 314, we shall obtain as many terms of the original series as we please; for this reason the expression

$$\frac{a_0 + (a_1 - pa_0)\,x}{1 - px - qx^2}$$

is called the *generating function* of the series.

327. From the result of Art. 325, we obtain

$$\frac{a_0 + (a_1 - pa_0)\,x}{1 - px - qx^2} = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n+1}$$
$$+ \frac{(pa_{n-1} + qa_{n-2})\,x^n + qa_{n-1} x^{n+1}}{1 - px - qx^2};$$

from which we see that although the generating function

$$\frac{a_0 + (a_1 - pa_0)\, x}{1 - px - qx^2}$$

may be used to obtain as many terms of the series as we please, it can be regarded as the true equivalent of the infinite series

$$a_0 + a_1 x + a_2 x^2 + \dots\dots,$$

only if the remainder

$$\frac{(pa_{n-1} + qa_{n-2})\, x^n + qa_{n-1} x^{n+1}}{1 - px - qx^2}$$

vanishes when n is indefinitely increased; in other words only when the series is convergent.

328. When the *generating function* can be expressed as a group of partial fractions the general term of a recurring series may be easily found. Thus, suppose the generating function can be decomposed into the partial fractions

$$\frac{A}{1 - ax} + \frac{B}{1 + bx} + \frac{C}{(1 - cx)^2}.$$

Then the general term is

$$\{Aa^r + (-1)^r Bb^r + (r+1)\, Cc^r\}\, x^r.$$

In this case the sum of n terms may be found without using the method of Art. 325.

Example. Find the generating function, the general term, and the sum to n terms of the recurring series

$$1 - 7x - x^2 - 43x^3 - \dots\dots$$

Let the scale of relation be $1 - px - qx^2$; then

$$-1 + 7p - q = 0, \quad -43 + p + 7q = 0;$$

whence $p = 1$, $q = 6$; and the scale of relation is

$$1 - x - 6x^2.$$

Let S denote the sum of the series; then

$$\begin{aligned} S &= 1 - 7x - x^2 - 43x^3 - \dots\dots \\ -xS &= \quad\ - x + 7x^2 + x^3 + \dots\dots \\ -6x^2S &= \qquad\quad - 6x^2 + 42x^3 + \dots\dots \end{aligned}$$

$$\therefore \quad (1 - x - 6x^2)\, S = 1 - 8x,$$

$$S = \frac{1 - 8x}{1 - x - 6x^2};$$

which is the generating function.

If we separate $\dfrac{1-8x}{1-x-6x^2}$ into partial fractions, we obtain $\dfrac{2}{1+2x}-\dfrac{1}{1-3x}$; whence the $(r+1)^{\text{th}}$ or general term is

$$\{(-1)^r\,2^{r+1}-3^r\}\,x^r.$$

Putting $$r=0,\ 1,\ 2,\ldots n-1,$$

the sum to n terms

$$=\{2-2^2x+2^3x^2-\ldots+(-1)^{n-1}\,2^n x^{n-1}\}-(1+3x+3^2x^2+\ldots+3^{n-1}\,x^{n-1})$$

$$=\frac{2+(-1)^{n-1}\,2^{n+1}\,x^n}{1+2x}-\frac{1-3^n\,x^n}{1-3x}.$$

329. To find the general term and sum of n terms of the recurring series $a_0+a_1+a_2+\ldots\ldots$, we have only to find the general term and sum of the series $a_0+a_1x+a_2x^2+\ldots\ldots$, and put $x=1$ in the results.

Example. Find the general term and sum of n terms of the series

$$1+6+24+84+\ldots\ldots$$

The scale of relation of the series $1+6x+24x^2+84x^3+\ldots$ is $1-5x+6x^2$, and the generating function is $\dfrac{1+x}{1-5x+6x^2}$.

This expression is equivalent to the partial fractions

$$\frac{4}{1-3x}-\frac{3}{1-2x}.$$

If these expressions be expanded in ascending powers of x the general term is $$(4\,.\,3^r-3\,.\,2^r)\,x^r.$$

Hence the general term of the given series is $4\,.\,3^r-3\,.\,2^r$; and the sum of n terms is $$2\,(3^n-1)-3\,(2^n-1).$$

330. We may remind the student that in the preceding article the generating function cannot be taken as the sum of the series

$$1+6x+24x^2+84x^3+\ldots\ldots$$

except when x has such a value as to make the series convergent. Hence when $x=1$ (in which case the series is obviously divergent) the generating function is not a true equivalent of the series. But the general term of

$$1+6+24+84+\ldots\ldots$$

is independent of x, *and whatever value* x *may have* it will always be the coefficient of x^n in

$$1+6x+24x^2+84x^3+\ldots\ldots.$$

We therefore treat this as a convergent series and find its general term in the usual way, and then put $x=1$.

EXAMPLES. XXIV.

Find the generating function and the general term of the following series:

1. $1+5x+9x^2+13x^3+\ldots\ldots$
2. $2-x+5x^2-7x^3+\ldots\ldots$
3. $2+3x+5x^2+9x^3+\ldots\ldots$
4. $7-6x+9x^2+27x^4+\ldots\ldots$
5. $3+6x+14x^2+36x^3+98x^4+276x^5+\ldots\ldots$

Find the n^{th} term and the sum to n terms of the following series:

6. $2+5+13+35+\ldots\ldots$
7. $-1+6x^2+30x^3+\ldots\ldots$
8. $2+7x+25x^2+91x^3+\ldots\ldots$
9. $1+2x+6x^2+20x^3+66x^4+212x^5+\ldots\ldots$
10. $-\frac{3}{2}+2+0+8+\ldots\ldots$

11. Shew that the series

$$1^2+2^2+3^2+4^2+\ldots\ldots+n^2,$$
$$1^3+2^3+3^3+4^3+\ldots\ldots+n^3,$$

are recurring series, and find their scales of relation.

12. Shew how to deduce the sum of the first n terms of the recurring series

$$a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots$$

from the sum to infinity.

13. Find the sum of $2n+1$ terms of the series

$$3-1+13-9+41-53+\ldots\ldots$$

14. The scales of the recurring series

$$a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots,$$
$$b_0+b_1x+b_2x^2+b_3x^3+\ldots\ldots,$$

are $1+px+qx^2$, $1+rx+sx^2$, respectively; shew that the series whose general term is $(a_n+b_n)x^n$ is a recurring series whose scale is

$$1+(p+r)x+(q+s+pr)x^2+(qr+ps)x^3+qsx^4.$$

15. If a series be formed having for its n^{th} term the sum of n terms of a given recurring series, shew that it will also form a recurring series whose scale of relation will consist of one more term than that of the given series.