

2. Matrices

- The various elementary operations or transformations on a matrix are as follows:
 - $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
 - $R_i \leftrightarrow kR_i$ or $C_i \leftrightarrow kC_i$, where k is a non-zero constant
 - $R_i \leftrightarrow R_i + kR_j$ or $C_i \leftrightarrow C_i + kC_j$, where k is a constant.

For example, by applying $R_1 \rightarrow R_1 - 7R_3$ to the matrix $\begin{bmatrix} -9 & 5 & 8 \\ 5 & 6 & 11 \\ 2 & -1 & 0 \end{bmatrix}$, we obtain $\begin{bmatrix} -23 & 12 & 8 \\ 5 & 6 & 11 \\ 2 & -1 & 0 \end{bmatrix}$.

- If A and B are the square matrices of same order such that $AB = BA = I$, then B is called the inverse of A and A is called the inverse of B . i.e., $A^{-1} = B$ and $B^{-1} = A$
- If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1} A^{-1}$
- If the inverse of a square matrix exists, then it is unique.
- If the inverse of a matrix exists, then it can be calculated either by using elementary row operations or by using elementary column operations.

Example: Find the inverse of the matrix: $A = \begin{bmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{bmatrix}$

Solution:

We know that $A = IA$. Therefore, we have

$$\begin{bmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_1 \rightarrow \sin \theta R_1$ and $R_2 \rightarrow \cos \theta R_2$, we have

$$\begin{bmatrix} \sin^2 \theta & \sin \theta \cos \theta \\ -\cos^2 \theta & \sin \theta \cos \theta \end{bmatrix} = \begin{bmatrix} \sin \theta & 0 \\ 0 & \cos \theta \end{bmatrix} A$$

Applying $R_1 \rightarrow R_1 - R_2$, we have

$$\begin{bmatrix} \sin^2 \theta + \cos^2 \theta & 0 \\ -\cos^2 \theta & \sin \theta \cos \theta \end{bmatrix} = \begin{bmatrix} \sin \theta & -\cos \theta \\ 0 & \cos \theta \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ -\cos^2 \theta & \sin \theta \cos \theta \end{bmatrix} = \begin{bmatrix} \sin \theta & -\cos \theta \\ 0 & \cos \theta \end{bmatrix} A$$

Applying $R_2 \rightarrow R_2 + \cos^2 \theta R_1$, we have

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & \sin \theta \cos \theta \end{bmatrix} = \begin{bmatrix} \sin \theta & -\cos \theta \\ \sin \theta \cos^2 \theta & \cos \theta (1 - \cos^2 \theta) \end{bmatrix} A$$

- If A is a square matrix, then $A (adj A) = (adj A) A = |A| I$
- A square matrix A is said to be singular, if $|A| = 0$
- A square matrix A is said to be non-singular, if $|A| \neq 0$
- If A and B are square matrices of same order, then $|AB| = |A||B|$

Therefore, if A and B are non-singular matrices of same order, then AB and BA are also non-singular matrices of same order.

- If A is a non-singular matrix of order n , then $(adj A)(adj A) = |A|^{n-1} I$
- A square matrix A is invertible, if and only if A is non-singular and inverse of A is given by the formula:

$$A^{-1} = \frac{1}{|A|} (adj A)$$

- The system of following linear equations $a_1x + b_1y + c_1z = d_1$
 $a_2x + b_2y + c_2z = d_2$
 $a_3x + b_3y + c_3z = d_3$ can be written as $AX = B$, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- A system of linear equations is said to be consistent, if its solution (one or more) exists.
- A system of linear equations is said to be inconsistent, if its solution does not exist.
- Unique solution of the equation $AX = B$ is given by $X = A^{-1} B$, where $|A| \neq 0$
- For a square matrix A in equation $AX = B$, if
 - $|A| \neq 0$, then there exists a unique solution
 - $|A| = 0$ and $(adj A) B \neq O$, then no solution exists
 - $|A| = 0$ and $(adj A) B = O$, then the system may or may not be consistent

Example 2:

Solve the following system of linear equations:

$$x - 3y + 4z = 12$$

$$2x + 2y - 3z = -7$$

$$6x - y + 2z = 13$$

Solution:

The given system of equations can be written in the form $AX = B$, where

$$A = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 2 & -3 \\ 6 & -1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 12 \\ -7 \\ 13 \end{bmatrix}$$

$$\text{Now, } |A| = 1[2 \times 2 - (-1)(-3)] + 3[2 \times 2 - 6(-3)] + 4[2 \times (-1) - 6 \times 2] = 11 \neq 0$$

Therefore, A is a non-singular matrix and hence, the given system of linear equations has only one solution.

Now,

$$\begin{aligned}
A_{11} &= [2 \times 2 - (-1)(-3)] = 1 \\
A_{12} &= -[2 \times 2 - 6(-3)] = -22 \\
A_{13} &= [2(-1) - 6 \times 2] = -14 \\
A_{21} &= -[(-3) \times 2 - (-1) \times 4] = 2 \\
A_{22} &= [1 \times 2 - 6 \times 4] = -22 \\
A_{23} &= -[1(-1) - 6(-3)] = -17 \\
A_{31} &= [(-3)(-3) - 4 \times 2] = 1 \\
A_{32} &= -[1(-3) - 2 \times 4] = 11 \\
A_{33} &= [1 \times 2 - 2(-3)] = 8 \\
\therefore A^{-1} &= \frac{1}{|A|}(\text{adj } A) = \frac{1}{11} \begin{bmatrix} 1 & 2 & 1 \\ -22 & -22 & 11 \\ -14 & -17 & 8 \end{bmatrix}
\end{aligned}$$

$$\text{Now, } X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 1 & 2 & 1 \\ -22 & -22 & 11 \\ -14 & -17 & 8 \end{bmatrix} \begin{bmatrix} 12 \\ -7 \\ 13 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 11 \\ 33 \\ 55 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$x = 1, y = 3, \text{ and } z = 5$$

Solution of System of Linear Equations (Method of Reduction)

The method of reduction can be used to solve a system of linear equations.

Steps involved:

- Write the given system of linear equations in the matrix equation form $AX = B$.
- Perform a suitable row transformation on matrix A to reduce it to an upper triangular matrix or a lower triangular matrix. The same row operations need to be simultaneously performed on matrix B .
- Rewrite the equations in the form of system of linear equations that can be solved by the elimination method.

Note: Matrix B is a column matrix, so elementary column transformations cannot be used to reduce matrix A to an upper triangular matrix or a lower triangular matrix.