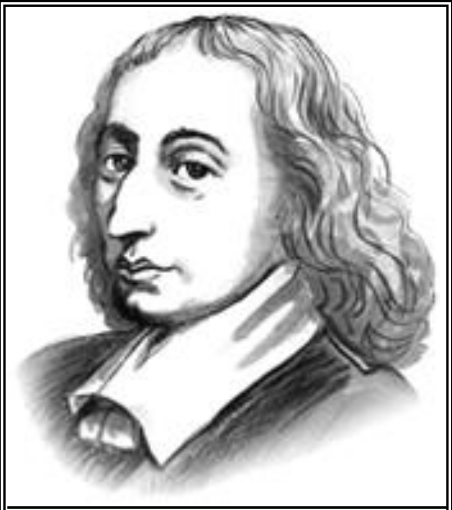
Chapter 6

Binomial Theorem and Mathematical Induction

Contents



Blaise Pascel

The ancient Indian mathematicians knew the coefficient in the expansion of $(x+y)^n, 0 \le n \le 7$. *The arrangement of these coefficient was in the form of a diagram called* Meru-Prastara, *provided by Pingla in his book* Chhanda-shastra (200 B.C.). *The term binomial coefficients was first introduced by the German mathematician.* Michael Stipel (1486-1567 A.D.)

The arithmetic triangle popularly known as pascal triangle was constructed by the French mathematician Blaise Pascal (1623-1662 A.D.) *He used the triangle to derive coefficients of a binomial expansion. It was printed in 1665 A.D. The present form of the binomial theorem for integral values of n appeared in Trate du triange arithmetic written by Pascal and published posthumously in 1665 A.D. The* ***generalization of the binomial theorem for negative integral and rational exponents is due to Sir Isaac Newton1*** **(642-1727 A.D)** ***in the same year 1665.***

6.1 Binomial Theorem

6.1.1 Binomial Expression

An algebraic expression consisting of two terms with $+ve$ or $-ve$ sign between them is called a binomial expression.

For example **:** $(a+b),(2x-3y),\left(\frac{p}{x^2}-\frac{q}{x^4}\right),\left(\frac{1}{x}+\frac{4}{y^3}\right)$ etc.

6.1.2 Binomial Theorem for Positive Integral Index

The rule by which any power of binomial can be expanded is called the binomial theorem.

If n is a positive integer and x, $y \in C$ then

$$(x+y)^n = {}^nC_0 x^{n-0} y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \ldots\ldots + {}^nC_r x^{n-r} y^r + \ldots\ldots + {}^nC_{n-1} x y^{n-1} + {}^nC_n x^0 y^n$$

i.e., $(x+y)^n = \sum_{r=0}^{n} {}^nC_r . x^{n-r} . y^r$ .....(i)

Here ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots {}^nC_n$ are called binomial coefficients and ${}^nC_r = \frac{n!}{r!(n-r)!}$ for $0 \le r \le n$.

Important Tips

- *The number of terms in the expansion of* $(x+y)^n$ *are (n + 1).*
- *The expansion contains decreasing power of x and increasing power of y. The sum of the powers of x and y in each term is equal to n.*
- *The binomial coefficients* ${}^nC_0, {}^nC_1, {}^nC_2 \ldots\ldots$ *equidistant from beginning and end are equal i.e.,* ${}^nC_r = {}^nC_{n-r}$.
- $(x+y)^n =$ *Sum of odd terms + sum of even terms.*

6.1.3 Some Important Expansions

(1) Replacing y by $-y$ in (i), we get,

$$(x-y)^n = {}^nC_0 x^{n-0} . y^0 - {}^nC_1 x^{n-1} . y^1 + {}^nC_2 x^{n-2} . y^2 \ldots + (-1)^r \, {}^nC_r x^{n-r} . y^r + \ldots + (-1)^n \, {}^nC_n x^0 . y^n$$

i.e., $(x-y)^n = \sum_{r=0}^{n} (-1)^r \, {}^nC_r x^{n-r} . y^r$ .....(ii)

The terms in the expansion of $(x-y)^n$ are alternatively positive and negative, the last term is positive or negative according as n is even or odd.

(2) Replacing x by 1 and y by x in equation (i) we get,

$$(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \ldots\ldots + {}^nC_r x^r + \ldots\ldots + {}^nC_n x^n \text{ i.e., } (1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$$

This is expansion of $(1+x)^n$ in ascending power of x.

(3) Replacing x by 1 and y by $-x$ in (i) we get, $(1-x)^n = {}^nC_0x^0 - {}^nC_1x^1 + {}^nC_2x^2 - ...... + (-1)^r\ {}^nC_rx^r + ... + (-1)^n\ {}^nC_nx^n$ *i.e.,* $(1-x)^n = \sum_{r=0}^{n}(-1)^r\ {}^nC_rx^r$

(4) $(x+y)^n + (x-y)^n = 2[{}^nC_0x^ny^0 + {}^nC_2x^{n-2}y^2 + {}^nC_4x^{n-4}y^4 + ...]$ and

$(x+y)^n - (x-y)^n = 2[{}^nC_1x^{n-1}y^1 + {}^nC_3x^{n-3}y^3 + {}^nC_5x^{n-5}y^5 + ...]$

(5) The coefficient of $(r+1)^{th}$ term in the expansion of $(1+x)^n$ is ${}^nC_r$.

(6) The coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$.

Note : ❑ If n is odd, then $(x+y)^n + (x-y)^n$ and $(x+y)^n - (x-y)^n$, both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$.

❑ If n is even, then $(x+y)^n + (x-y)^n$ has $\left(\frac{n}{2}+1\right)$ terms and $(x+y)^n - (x-y)^n$ has $\frac{n}{2}$ terms.

Example: 1 $x^5 + 10x^4a + 40x^3a^2 + 80x^2a^3 + 80xa^4 + 32a^5 =$

(a) $(x+a)^5$ (b) $(3x+a)^5$ (c) $(x+2a)^5$ (d) $(x+2a)^3$

Solution: (c) Conversely $(x+y)^n = {}^nC_0 + {}^nC_1x^{n-1}y^1 + {}^nC_2x^{n-2}y^2 + ... + {}^nC_nx^0y^n$

$$(x+2a)^5 = {}^5C_0x^5 + {}^5C_1x^4(2a)^1 + {}^5C_2x^3(2a)^2 + {}^5C_3x^2(2a)^3 + {}^5C_4x^1(2a)^4 + {}^5C_5x^0(2a)^5$$

$$= x^5 + 10x^4a + 40x^3a^2 + 80x^2a^3 + 80xa^4 + 32a^5.$$

Example: 2 The value of $(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$ will be

(a) – 198 (b) 198 (c) 98 (d) – 99

Solution: (b) We know that, $(x+y)^n + (x-y)^n = 2[x^n + {}^nC_2x^{n-2}y^2 + {}^nC_4x^{n-4}y^4 + ...]$

$(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 = 2[(\sqrt{2})^6 + {}^6C_2(\sqrt{2})^4(1)^2 + {}^6C_4(\sqrt{2})^2(1)^4 + {}^6C_6(\sqrt{2})^0(1)^6] = 2[8 + 15\times 4 + 30 + 1] = 198$

Example: 3 The larger of $99^{50} + 100^{50}$ and 101^{50} is **[IIT 1980]**

(a) $99^{50} + 100^{50}$ (b) Both are equal (c) 101^{50} (d) None of these

Solution: (c) We have, $101^{50} = (100+1)^{50} = 100^{50} + 50.100^{49} + \frac{50.49}{2.1}100^{48} +$ .....(i)

and $99^{50} = (100-1)^{50} = 100^{50} - 50.100^{49} + \frac{50.49}{2.1}100^{48} - ....$ ..... (ii)

Subtracting, $101^{50} - 99^{50} = 100^{50} + 2.\frac{50.49.48}{3.2.1}100^{47} + ... > 100^{50}$. Hence $101^{50} > 100^{50} + 99^{50}$.

Example: 4 Sum of odd terms is A and sum of even terms is B in the expansion of $(x+a)^n$, then

(a) $AB = \frac{1}{4}(x-a)^{2n} - (x+a)^{2n}$ (b) $2AB = (x+a)^{2n} - (x-a)^{2n}$

(c) $4AB = (x+a)^{2n} - (x-a)^{2n}$ (d) None of these

Solution: (c) $(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a^1 + {}^nC_2x^{n-2}a^2 + ... + {}^nC_nx^{n-n}.a^n = (x^n + {}^nC_2x^{n-2}a^2 + ..) + ({}^nC_1x^{n-1}a^1 + {}^nC_3x^{n-3}a^3 + ...) = A + B$ .....(i)

Similarly, $(x-a)^n = A - B$ .....(ii)

From (i) and (ii), we get $4AB = (x+a)^{2n} - (x-a)^{2n}$

Trick: Put $n=1$ in $(x+a)^n$. Then, $x+a = A+B$. Comparing both sides $A=x, B=a$.

Option (c) L.H.S. $4AB = 4xa$, R.H.S. $(x+a)^2 - (x-a)^2 = 4ax$. *i.e.,* L.H.S. = R.H.S

6.1.4 General Term

$$(x+y)^n = {}^nC_0x^ny^0 + {}^nC_1x^{n-1}y^1 + {}^nC_2x^{n-2}y^2 + ... + {}^nC_rx^{n-r}y^r + ... + {}^nC_nx^0y^n$$

The first term = ${}^nC_0x^ny^0$

The second term = ${}^nC_1x^{n-1}y^1$. The third term = ${}^nC_2x^{n-2}y^2$ and so on

The term ${}^nC_rx^{n-r}y^r$ is the $(r+1)^{th}$ term from beginning in the expansion of $(x+y)^n$.

Let T_{r+1} denote the $(r+1)^{\text{th}}$ term $\therefore$ $T_{r+1} = {}^nC_rx^{n-r}y^r$

This is called general term, because by giving different values to r, we can determine all terms of the expansion.

In the binomial expansion of $(x-y)^n, T_{r+1} = (-1)^r\ {}^nC_rx^{n-r}y^r$

In the binomial expansion of $(1+x)^n, T_{r+1} = {}^nC_rx^r$

In the binomial expansion of $(1-x)^n, T_{r+1} = (-1)^r\ {}^nC_rx^r$

Note : ❑ In the binomial expansion of $(x+y)^n$, the p^{th} term from the end is $(n-p+2)^{th}$ term from beginning.

Important Tips

☞ *In the expansion of* $(x+y)^n, n \in N$

$$\frac{T_{r+1}}{T_r} = \left(\frac{n-r+1}{r}\right)\frac{y}{x}$$

☞ *The coefficient of* x^{n-1} *in the expansion of* $(x-1)(x-2).......(x-n) = -\frac{n(n+1)}{2}$

☞ *The coefficient of* x^{n-1} *in the expansion of* $(x+1)(x+2).....(x+n) = \frac{n(n+1)}{2}$

Example: 5 If the 4^{th} term in the expansion of $(px+x^{-1})^m$ is 2.5 for all $x \in R$ then

(a) $p = 5/2, m = 3$ (b) $p = \frac{1}{2}, m = 6$ (c) $p = -\frac{1}{2}, m = 6$ (d) None of these

Solution: (b) We have $T_4 = \frac{5}{2} \Rightarrow T_{3+1} = \frac{5}{2} \Rightarrow {}^mC_3(px)^{m-3}\left(\frac{1}{x}\right)^3 = \frac{5}{2} \Rightarrow {}^mC_3\,p^{m-3}x^{m-6} = \frac{5}{2}$ ....(i)

Clearly, R.H.S. of the above equality is independent of x

$\therefore m-6=0$, $m=6$

Putting $m=6$ in (i) we get ${}^6C_3p^3 = \frac{5}{2} \Rightarrow p = \frac{1}{2}$. Hence $p = 1/2, m = 6$.

Example: 6 If the second, third and fourth term in the expansion of $(x+a)^n$ are 240, 720 and 1080 respectively, then the value of n is

[Kurukshetra CEE 1991; DCE 1995, 2001]

(a) 15 (b) 20 (c) 10 (d) 5

Solution: (d) It is given that $T_2 = 240, T_3 = 720, T_4 = 1080$

Now, $T_2 = 240 \Rightarrow T_2 = {}^nC_1x^{n-1}a^1 = 240$ .....(i) and $T_3 = 720 \Rightarrow T_3 = {}^nC_2x^{n-2}a^2 = 720$ .....(ii)

$T_4 = 1080 \Rightarrow T_4 = {}^nC_3x^{n-3}a^3 = 1080$ .....(iii)

To eliminate x, $\frac{T_2.T_4}{T_3^2} = \frac{240.1080}{720.720} = \frac{1}{2} \Rightarrow \frac{T_2}{T_3}.\frac{T_4}{T_3} = \frac{1}{2}$.

Now $\frac{T_{r+1}}{T_r} = \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$. Putting $r=3$ and 2 in above expression, we get $\frac{n-2}{3}.\frac{2}{n-1} = \frac{1}{2} \Rightarrow n = 5$

Example: 7 The 5^{th} term from the end in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^3}\right)^9$ is

(a) $63x^3$ (b) $-\frac{252}{x^3}$ (c) $\frac{672}{x^{18}}$ (d) None of these

Solution: (b) 5^{th} term from the end = $(9-5+2)^{th}$ term from the beginning in the expansion of $\left(\frac{x^3}{2}-\frac{2}{x^3}\right)^9 = T_6$

$\Rightarrow T_6 = T_{5+1} = {}^9C_5\left(\frac{x^3}{2}\right)^4\left(-\frac{2}{x^3}\right)^5 = -{}^9C_4.2.\frac{1}{x^3} = -\frac{252}{x^3}$.

Example: 8 If $\frac{T_2}{T_3}$ in the expansion of $(a+b)^n$ and $\frac{T_3}{T_4}$ in the expansion of $(a+b)^{n+3}$ are equal, then $n=$

[Rajasthan PET 1987, 96]

(a) 3 (b) 4 (c) 5 (d) 6

Solution: (c) $\because \frac{T_2}{T_3} = \frac{2}{n-2+1}.\frac{b}{a} = \frac{2}{n-1}\left(\frac{b}{a}\right)$ and $\frac{T_3}{T_4} = \frac{3}{n+3-3+1}.\left(\frac{b}{a}\right) = \frac{3}{n+1}\left(\frac{b}{a}\right)$

$\because \frac{T_2}{T_3} = \frac{T_3}{T_4}$ (given) ; $\therefore \frac{2}{n-1}\left(\frac{b}{a}\right) = \frac{3}{n+1}\left(\frac{b}{a}\right) \Rightarrow 2n+2 = 3n-3 \Rightarrow n=5$

6.1.5 Independent Term or Constant Term

Independent term or constant term of a binomial expansion is the term in which exponent of the variable is zero.

Condition : $(n-r)$ [Power of x] + r . [Power of y] = 0, in the expansion of $[x+y]^n$.

Example: 9 The term independent of x in the expansion of $\left(\sqrt{\frac{x}{3}}+\frac{3}{2x^2}\right)^{10}$ will be

[IIT 1965; BIT Ranchi 1993; Karnataka CET 2000; UPSEAT 2001]

(a) $\frac{3}{2}$ (b) $\frac{5}{4}$ (c) $\frac{5}{2}$ (d) None of these

Solution: (b) $(10-r)\left(\frac{1}{2}\right)+r(-2)=0 \Rightarrow r=2 \therefore T_3 = {}^{10}C_2\left(\frac{1}{3}\right)^{8/2}\left(\frac{3}{2}\right)^2 = \frac{5}{4}$

Example: 10 The term independent of x in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$ is **[EAMCET 1989]**

(a) $C_0^2+2C_1^2+......+(n+1)C_n^2$ (b) $(C_0+C_1+......+C_n)^2$ (c) $C_0^2+C_1^2+......+C_n^2$ (d)

Solution: (c) We know that, $(1+x)^n = {}^nC_0 + {}^nC_1x^1 + {}^nC_2x^2 + ... + {}^nC_nx^n$

$\left(1+\frac{1}{x}\right)^n = {}^nC_0 + {}^nC_1\frac{1}{x^1} + {}^nC_2\frac{1}{x^2} + ... + {}^nC_n\frac{1}{x^n}$

Obviously, the term independent of x will be ${}^nC_0.{}^nC_0 + {}^nC_1{}^nC_1 + ... + {}^nC_n.{}^nC_n = C_0^2 + C_1^2 + ... + C_n^2$

Trick : Put $n=1$ in the expansion of $(1+x)^1\left(1+\frac{1}{x}\right)^1 = 1+x+\frac{1}{x}+1 = 2+x+\frac{1}{x}$ .....(i)

We want coefficient of x^0. Comparing to equation (i). Then, we get 2 *i.e.,* independent of x.
Option (c) : $C_0^2+C_1^2+.....C_n^2$; Put $n=1$; Then ${}^1C_0^2+{}^1C_1^2 = 1+1 = 2$.

Example: 11 The coefficient of x^{-7} in the expansion of $\left(ax-\frac{1}{bx^2}\right)^{11}$ will be **[IIT 1967; Rajasthan PET 1996]**

(a) $\frac{462a^6}{b^5}$ (b) $\frac{462a^5}{b^6}$ (c) $\frac{-462a^5}{b^6}$ (d) $-\frac{462a^6}{b^5}$

Solution: (b) For coefficient of x^{-7}, $(11-r)(1)+(-2).r = -7 \Rightarrow 11-r-2r = -7 \Rightarrow r=6$; $T_7 = {}^{11}C_6(a)^5\left(-\frac{1}{b}\right)^6 = \frac{462a^5}{b^6}$

Example: 12 If the coefficients of second, third and fourth term in the expansion of $(1+x)^{2n}$ are in A.P., then $2n^2-9n+7$ is equal to

[AMU 2001]

(a) –1 (b) 0 (c) 1 (d) 3/2

Solution: (b) $T_2={}^{2n}C_1$, $T_3={}^{2n}C_2, T_4={}^{2n}C_3$ are in A.P. then, $2.{}^{2n}C_2={}^{2n}C_1+{}^{2n}C_3$

$$2.\frac{2n(.2n-1)}{2.1}=\frac{2n}{1}+\frac{2n(2n-1)(2n-2)}{3.2.1}$$

On solving, $2n^2-9n+7=0$

Example: 13 The coefficient of x^5 in the expansion of $(1+x^2)^5(1+x)^4$ is **[EAMCET 1996; UPSEAT 2001; Pb. CET 2002]**

(a) 30 (b) 60 (c) 40 (d) None of these

Solution: (b) We have $(1+x^2)^5(1+x)^4 = ({}^5C_0+{}^5C_1x^2+{}^5C_2x^4+.....)\ ({}^4C_0+{}^4C_1x^1+{}^4C_2x^2+.......)$

So coefficient of x^5 in $[(1+x^2)^5(1+x)^4] = {}^5C_2.{}^4C_1+{}^4C_3.{}^5C_1=60$

Example: 14 If A and B are the coefficient of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, then**[MP PET 1999**

(a) $A=B$ (b) $A=2B$ (c) $2A=B$ (d) None of these

Solution: (b) $A=$coefficient of x^n in $(1+x)^{2n} = {}^{2n}C_n=\frac{(2n)!}{n!n!}=\frac{2.(2n-1)!}{(n-1)!n!}$ .....(i)

$B=$ coefficient of x^n in $(1+x)^{2n-1}={}^{2n-1}C_n=\frac{(2n-1)!}{n!(n-1)!}$ .....(ii)

By (i) and (ii) we get, $A=2B$

Example: 15 The coefficient of x^n in the expansion of $(1+x)(1-x)^n$ is

(a) $(-1)^{n-1}n$ (b) $(-1)^n(1-n)$ (c) $(-1)^{n-1}(n-1)^2$ (d) $(n-1)$

Solution: (b) Coefficient of x^n in $(1+x)(1-x)^n$ = Coefficient of x^n in $(1-x)^n$ + coefficient of x^{n-1} in $(1-x)^n$

= Coefficient of x^n in $[{}^nC_n(-x)^n+x.{}^nC_{n-1}(-x)^{n-1}]$ = $(-1)^n\,{}^nC_n+(-1)^{n-1}.{}^nC_1$ = $(-1)^n+(-1)^n.(-n)=(-1)^n[1-n]$.

6.1.6 Number of Terms in the Expansion of $(a + b + c)^n$ and $(a + b + c + d)^n$

$(a+b+c)^n$ can be expanded as : $(a+b+c)^n=\{(a+b)+c\}^n$

$=(a+b)^n+{}^nC_1(a+b)^{n-1}(c)^1+{}^nC_2(a+b)^{n-2}(c)^2+.....+{}^nC_n\,c^n=(n+1)\text{term}+n\text{ term}+(n-1)\text{term}+...+1\text{term}$

$\therefore$ Total number of terms = $(n+1)+(n)+(n-1)+......+1=\frac{(n+1)(n+2)}{2}$.

Similarly, Number of terms in the expansion of $(a+b+c+d)^n=\frac{(n+1)(n+2)(n+3)}{6}$.

Example: 16 If the number of terms in the expansion of $(x-2y+3z)^n$ is 45, then $n=$

(a) 7 (b) 8 (c) 9 (d) None of these

Solution: (b) Given, total number of terms = $\frac{(n+1)(n+2)}{2}=45 \Rightarrow (n+1)(n+2)=90 \Rightarrow n=8$.

Example: 17 The number of terms in the expansion of $[(x+3y)^2(3x-y)^2]^3$ is **[Rajasthan PET 1986]**

(a) 14 (b) 28 (c) 32 (d) 56

Solution: (b) We have $[(x+3y)(3x-y)]^6$ = $[3x^2+8xy-3y^2]^6$; Number of terms = $\frac{(6+1)(6+2)}{2}=28$

6.1.7 Middle Term

The middle term depends upon the value of n.

(1) **When *n* is even,** then total number of terms in the expansion of $(x+y)^n$ is $n+1$ (odd). So there is only one middle term *i.e.,* $\left(\frac{n}{2}+1\right)^{\text{th}}$ term is the middle term. $T_{\left[\frac{n}{2}+1\right]} = {}^nC_{n/2}x^{n/2}y^{n/2}$

(2) **When *n* is odd**, then total number of terms in the expansion of $(x+y)^n$ is $n+1$ (even). So, there are two middle terms *i.e.,* $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$ are two middle terms. $T_{\left(\frac{n+1}{2}\right)} = {}^nC_{\frac{n-1}{2}}x^{\frac{n+1}{2}}y^{\frac{n-1}{2}}$ and $T_{\left(\frac{n+3}{2}\right)} = {}^nC_{\frac{n+1}{2}}x^{\frac{n-1}{2}}y^{\frac{n+1}{2}}$

Note : ❑ When there are two middle terms in the expansion then their binomial coefficients are equal.

❑ Binomial coefficient of middle term is the greatest binomial coefficient.

Example: 18 The middle term in the expansion of $\left(x+\frac{1}{x}\right)^{10}$ is **[BIT Ranchi 1991; Rajasthan PET 2002; Pb. CET 1991]**

(a) ${}^{10}C_4\frac{1}{x}$ (b) ${}^{10}C_5$ (c) ${}^{10}C_5x$ (d) ${}^{10}C_7x^4$

Solution: (b) $\because$ *n* is even so middle term $T_{\left(\frac{10}{2}+1\right)} = T_6 \Rightarrow T_6 = T_{5+1} = {}^{10}C_5\,x^5.\frac{1}{x^5} = {}^{10}C_5$

Example: 19 The middle term in the expansion of $(1+x)^{2n}$ is **[Pb. CET 1998]**

(a) $\frac{1.3.5......(2n-1)}{n!}x^{2n+1}$ (b) $\frac{2.4.6......2n}{n!}x^{2n+1}$ (c) $\frac{1.3.5......(2n-1)}{n!}x^n$ (d) $\frac{1.3.5......(2n-1)}{n!}x^n.2^n$

Solution: (d) Since 2*n* is even, so middle term = $T_{\frac{2n}{2}+1} = T_{n+1} \Rightarrow T_{n+1} = {}^{2n}C_nx^n = \frac{(2n)!}{n!.n!}x^n = \frac{1.3.5........(2n-1)}{n!}.2^nx^n$.

6.1.8 To Determine a Particular Term in the Expansion

In the expansion of $\left(x^\alpha \pm \frac{1}{x^\beta}\right)^n$, if x^m occurs in T_{r+1}, then *r* is given by $n\alpha - r(\alpha+\beta) = m \Rightarrow r = \frac{n\alpha - m}{\alpha+\beta}$

Thus in above expansion if constant term which is independent of *x*, occurs in T_{r+1} then *r* is determined by

$$n\alpha - r(\alpha+\beta) = 0 \Rightarrow r = \frac{n\alpha}{\alpha+\beta}$$

Example: 20 The term independent of *x* in the expansion of $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$ is

(a) 7/12 (b) 7/18 (c) – 7/12 (d) – 7/16

Solution: (b) *n* =9, $\alpha = 2$, $\beta = 1$. Then $r = \frac{9(2)}{1+2} = 6$. Hence, $T_7 = {}^9C_6\left(\frac{3}{2}\right)^3\left(-\frac{1}{3}\right)^6 = \frac{9\times 8\times 7}{3\times 2\times 1}.\frac{1}{2^3.3^3} = \frac{7}{18}$.

Example: 21 If the coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ is equal to the coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$ then *ab* =

[MP PET 1999; AMU 2001; Pb. CET 2002]

(a) 1 (b) 1/2 (c) 2 (d) 3

Solution: (a) For coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$; n = 11, $\alpha=2,\beta=1$, $m=7$

$r=\frac{11.2-7}{2+1}=\frac{15}{3}=5$

Coefficient of x^7 in $T_6={}^{11}C_5\,a^6.\frac{1}{b^5}$ ......(i)

and for coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$; $n=11,\alpha=1,\beta=2$, $m=-7$; $r=\frac{11.1+7}{3}=6$

Coefficient of x^{-7} in $T_7={}^{11}C_6.a^5.\frac{1}{b^6}$ .....(ii)

From equation (i) and (ii), we get $ab=1$

6.1.9 Greatest Term and Greatest Coefficient

(1) **Greatest term :** If T_r and T_{r+1} be the r^{th} and $(r+1)^{th}$ terms in the expansion of $(1+x)^n$, then

$$\frac{T_{r+1}}{T_r}=\frac{{}^nC_rx^r}{{}^nC_{r-1}x^{r-1}}=\frac{n-r+1}{r}x$$

Let numerically, T_{r+1} be the greatest term in the above expansion. Then $T_{r+1}\geq T_r$ or $\frac{T_{r+1}}{T_r}\geq 1$

$\therefore \quad \frac{n-r+1}{r}|x|\geq 1$ or $r\leq\frac{(n+1)}{(1+|x|)}|x|$ ......(i)

Now substituting values of n and x in (i), we get $r\leq m+f$ or $r\leq m$

where m is a positive integer and f is a fraction such that $0<f<1$.

When n is even T_{m+1} is the greatest term, when n is odd T_m and T_{m+1} are the greatest terms and both are equal.

Short cut method : To find the greatest term (numerically) in the expansion of $(1+x)^n$.

(i) Calculate m = $\left|\frac{x(n+1)}{x+1}\right|$

(ii) If m is integer, then T_m and T_{m+1} are equal and both are greatest term.

(iii) If m is not integer, there $T_{[m]+1}$ is the greatest term, where [.] denotes the greatest integral part.

(2) **Greatest coefficient**

(i) If n is even, then greatest coefficient is ${}^nC_{n/2}$

(ii) If n is odd, then greatest coefficient are ${}^nC_{\frac{n+1}{2}}$ and ${}^nC_{\frac{n+3}{2}}$

Important Tips

☞ *For finding the greatest term in the expansion of* $(x+y)^n$. *we rewrite the expansion in this form* $(x+y)^n=x^n\left[1+\frac{y}{x}\right]^n$.

Greatest term in (x + y)^n = x^n. Greatest term in $\left(1+\frac{y}{x}\right)^n$

Example: 22 The largest term in the expansion of $(3+2x)^{50}$, where $x=\frac{1}{5}$ is

(a) 5^{th} (b) 8^{th} (c) 7^{th} (d) 6^{th}

Solution: (c,d) $(3+2x)^{50}=3^{50}\left[1+\frac{2x}{3}\right]^{50}$, Now greatest term in $\left(1+\frac{2x}{3}\right)^{50}$

$$r=\left|\frac{x(n+1)}{1+x}\right|=\left|\frac{\frac{2x}{3}(50+1)}{\frac{2x}{3}+1}\right|=\frac{\frac{2.\frac{1}{5}}{3}(51)}{\frac{2}{15}+1}=6 \text{ (an integer)}$$

$\therefore$ T_r and $T_{[r]+1}=T_6$ and $T_{[6]+1}=T_6$ and T_7 are numerically greatest terms

Example: 23 The greatest coefficient in the expansion of $(1+x)^{2n+2}$ is

(a) $\frac{(2n)!}{n!^2}$ (b) $\frac{(2n+2)!}{[(n+1)!]^2}$ (c) $\frac{(2n+2)!}{n!(n+1)!}$ (d) $\frac{(2n)!}{n!.(n+1)!}$

Solution: (b) $\because$ n is even so greatest coefficient in $(1+x)^{2n+2}$ is = ${}^{2n+2}C_{n+1}=\frac{(2n+2)!}{[(n+1)!]^2}$

Example: 24 The interval in which x must lie so that the greatest term in the expansion of $(1+x)^{2n}$ has the greatest coefficient is

(a) $\left(\frac{n-1}{n},\frac{n}{n-1}\right)$ (b) $\left(\frac{n}{n+1},\frac{n+1}{n}\right)$ (c) $\left(\frac{n}{n+2},\frac{n+2}{n}\right)$ (d) None of these

Solution: (b) Here the greatest coefficient is ${}^{2n}C_n$

$\therefore$ ${}^{2n}C_nx^n>{}^{2n}C_{n+1}x^{n-1}\Rightarrow x>\frac{n}{n+1}$ and ${}^{2n}C_nx^n>{}^{2n}C_{n-1}x^{n+1}\Rightarrow x<\frac{n+1}{n}$. Hence the result is (b)

6.1.10 Properties of Binomial Coefficients

In the binomial expansion of $(1+x)^n$, $(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+.....+{}^nC_rx^r+....+{}^nC_nx^n$.

where ${}^nC_0,{}^nC_1,{}^nC_2,......,{}^nC_n$ are the coefficients of various powers of x and called binomial coefficients, and they are written as $C_0,C_1,C_2,.....C_n$.

Hence, $(1+x)^n=C_0+C_1x+C_2x^2+.....+C_rx^r+.....+C_nx^n$ .....(i)

(1) The sum of binomial coefficients in the expansion of $(1+x)^n$ is 2^n.

Putting $x=1$ in (i), we get $2^n=C_0+C_1+C_2+.....+C_n$ .....(ii)

(2) Sum of binomial coefficients with alternate signs : Putting $x=-1$ in (i)

We get, $0=C_0-C_1+C_2-C_3+......$ .....(iii)

(3) Sum of the coefficients of the odd terms in the expansion of $(1+x)^n$ is equal to sum of the coefficients of even terms and each is equal to 2^{n-1}.

From (iii), we have $C_0+C_2+C_4+.....=C_1+C_3+C_5+........$ ......(iv)

i.e., sum of coefficients of even and odd terms are equal.

From (ii) and (iv), $C_0+C_2+C_4+.....=C_1+C_3+C_5+.....=2^{n-1}$ ......(v)

(4) ${}^nC_r=\frac{n}{r}\,{}^{n-1}C_{r-1}=\frac{n}{r}.\frac{n-1}{r-1}\,{}^{n-2}C_{r-2}$ and so on.

(5) Sum of product of coefficients : Replacing x by $\frac{1}{x}$ in (i) we get $\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ... \frac{C_n}{x^n} + ...$ (vi)

Multiplying (i) by (vi), we get $\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1x + C_2x^2 + ...)\left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ...\right)$

Now comparing coefficient of x^r on both sides. We get, ${}^{2n}C_{n+r} = C_0C_r + C_1C_{r+1} + ...C_{n-r}.C_n$ .....(vii)

(6) Sum of squares of coefficients : Putting $r=0$ in (vii), we get ${}^{2n}C_n = C_0^2 + C_1^2 + ...C_n^2$

(7) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Example: 25 The value of $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + ...$ is equal to **[Karnataka CET 2000]**

(a) $\frac{2^n - 1}{n+1}$ (b) $n.2^n$ (c) $\frac{2^n}{n}$ (d) $\frac{2^n + 1}{n+1}$

Solution: (a) We have $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + ...$ = $\frac{n}{2.1} + \frac{n(n-1)(n-2)}{4.3.2.1} + \frac{n(n-1)(n-2)(n-3)(n-4)}{6.5.4.3.2.1} + ...$

= $\frac{1}{n+1}\left[\frac{(n+1)n}{2!} + \frac{(n+1)(n)(n-1)(n-2)}{4!} + ...\right]$ = $\frac{1}{n+1}[2^{(n+1)-1} - 1] = \frac{2^n - 1}{n+1}$

Trick: For $n=1$, = $\frac{C_1}{2} = \frac{{}^1C_1}{2} = \frac{1}{2}$

Which is given by option (a) $\frac{2^n - 1}{n+1} = \frac{2^1 - 1}{1+1} = \frac{1}{2}$.

Example: 26 The value of $C_0 + 3C_1 + 5C_2 + ... + (2n+1)C_n$ is equal to

(a) 2^n (b) $2^n + n.2^{n-1}$ (c) $2^n(n+1)$ (d) None of these

Solution: (c) We have $C_0 + 3C_1 + 5C_2 + ... + (2n+1)C_n = \sum_{r=0}^{n}(2r+1)C_r = \sum_{r=0}^{n}(2r+1)\,{}^nC_r$ = $\sum_{r=0}^{n} 2r\ {}^nC_r + \sum_{r=0}^{n} {}^nC_r$

= $2.\sum_{r=1}^{n} r.\frac{n}{r}.{}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r$ = $2n\sum_{r=1}^{n} {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r = 2n[(1+1)^{n-1}] + [1+1]^n$ = $2n.2^{n-1} + 2^n = 2^n.[n+1]$.

Trick: Put $n=1$ in given expansion ${}^1C_0 + 3.{}^1C_1 = 1+3 = 4$.

Which is given by option (c) $2^n.(n+1) = 2^1(1+1) = 4$.

Example: 27 If $S_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$ and $t_n = \sum_{r=0}^{n} \frac{r}{{}^nC_r}$. Then $\frac{t_n}{S_n}$ is equal to

(a) $\frac{2n-1}{2}$ (b) $\frac{1}{2}n - 1$ (c) $n-1$ (d) $\frac{n}{2}$

Solution: (d) Take $n = 2m$, then, $S_n = \frac{1}{{}^{2m}C_0} + \frac{1}{{}^{2m}C_1} + ... + \frac{1}{{}^{2m}C_{2m}}$ $= 2\left[\frac{1}{{}^{2m}C_0} + \frac{1}{{}^{2m}C_1} + ... + \frac{1}{{}^{2m}C_{m-1}}\right] + \frac{1}{{}^{2m}C_m}$

$t_n = \sum_{r=0}^{n} \frac{r}{{}^nC_r} = \sum_{r=0}^{2m} \frac{r}{{}^{2m}C_r} = \frac{1}{{}^{2m}C_1} + \frac{2}{{}^{2m}C_2} + ... + \frac{2m}{{}^{2m}C_{2m}}$

$$t_n = \left(\frac{1}{^{2m}C_1} + \frac{2m-1}{^{2m}C_{2m-1}}\right) + \left(\frac{2}{^{2m}C_2} + \frac{2m-2}{^{2m}C_{2m-2}}\right) + ... + \left(\frac{m-1}{^{2m}C_{m-1}} + \frac{m+1}{^{2m}C_{m+1}}\right) + \frac{m}{^{2m}C_m} + \frac{2m}{^{2m}C_{2m}}$$

$$= 2m\left[\frac{1}{^{2m}C_1} + \frac{1}{^{2m}C_2} + ...\right] + \frac{m}{^{2m}C_m} + 2m = 2m\left[\frac{1}{^{2m}C_0} + \frac{1}{^{2m}C_1} + ... + \frac{1}{^{2m}C_{m-1}}\right] + \frac{m}{^{2m}C_m} = m\left[S_n - \frac{1}{^{2m}C_m}\right] + \frac{m}{^{2m}C_m} = mSn$$

$$t_n = mS_n \Rightarrow \frac{t_n}{S_n} = m = \frac{n}{2}$$

Example: 28 If $(1-x+x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$. Then $a_0 + a_2 + a_4 + ... + a_{2n} =$

[MNR 1992; DCE 1996; AMU 1998; Rajasthan PET 1999; Karnataka CET 1999; UPSEAT 1999]

(a) $\frac{3^n+1}{2}$ (b) $\frac{3^n-1}{2}$ (c) $\frac{1-3^n}{2}$ (d) $3^n + \frac{1}{2}$

Solution: (a) $(1-x+x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$

Putting $x=1$, we get $(1-1+1)^n = a_0 + a_1 + a_2 + ... + a_{2n}$; $1 = a_0 + a_1 + a_2 + ... + a_{2n}$ .....(i)

Again putting $x=-1$, we get $3^n = a_0 - a_1 + a_2 - ...... + a_{2n}$ ......(ii)

Adding (i) and (ii), we get, $3^n + 1 = 2[a_0 + a_2 + a_4 + ... + a_{2n}]$

$$\frac{3^n+1}{2} = a_0 + a_2 + a_4 + ... + a_{2n}$$

Example: 29 If $(1+x)^n = \sum_{r=0}^{n} C_r x^r$, then $\left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right)......\left(1+\frac{C_n}{C_{n-1}}\right) =$

(a) $\frac{n^{n-1}}{(n-1)!}$ (b) $\frac{(n+1)^{n-1}}{(n-1)!}$ (c) $\frac{(n+1)^n}{n!}$ (d) $\frac{(n+1)^{n+1}}{n!}$

Solution: (c) We have $\left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right)........\left(1+\frac{C_n}{C_{n-1}}\right) = \left(1+\frac{n}{1}\right)\left(1+\frac{n(n-1)/2!}{n}\right)......\left(1+\frac{1}{n}\right)$

$$= \left(\frac{1+n}{1}\right)\left(\frac{1+n}{2}\right)\left(\frac{1+n}{3}\right).......\left(\frac{1+n}{n}\right) = \frac{(n+1)^n}{n!}$$

Trick : Put $n = 1,2,3......., S_1 = 1 + \frac{^1C_1}{^1C_0} = 2, S_2 = \left(1+\frac{^2C_1}{^2C_1}\right)\left(1+\frac{^2C_2}{^2C_1}\right) = \frac{9}{2}$

Which is given by option (c) $n=1$, $\frac{(1+1)^1}{1!} = 2$; For $n=2$, $\frac{(2+1)^2}{2!} = \frac{9}{2}$

Example: 30 In the expansion of $(1+x)^5$, the sum of the coefficient of the terms is **[Rajasthan PET 1992, 97; Kurukshetra CEE**

(a) 80 (b) 16 (c) 32 (d) 64

Solution: (c) Putting $x=1$ in $(1+x)^5$, the required sum of coefficient = $(1+1)^5 = 2^5 = 32$

Example: 31 If the sum of coefficient in the expansion of $(\alpha^2x^2 - 2\alpha x + 1)^{51}$ vanishes, then the value of α is **[IIT 1991; Pb. CET 1988]**

(a) 2 (b) –1 (c) 1 (d) – 2

Solution: (c) The sum of coefficient of polynomial $(\alpha^2x^2 - 2\alpha x + 1)^{51}$ is obtained by putting $x=1$ in $(\alpha^2x^2 - 2\alpha x + 1)^{51}$.

Therefore by hypothesis $(\alpha^2 - 2\alpha + 1)^{51} = 0 \Rightarrow \alpha = 1$

Example: 32 If C_r stands for ^nC_r, the sum of given series $\frac{2(n/2)!(n/2)!}{n!}[C_0^2 - 2C_1^2 + 3C_2^2 - ...... + (-1)^n(n+1)C_n^2]$ where n is an even positive integer, is

(a) 0 (b) $(-1)^{n/2}(n+1)$ (c) $(-1)^n(n+2)$ (d) $(-1)^{n/2}(n+2)$

Solution: (d) We have $C_0^2 - 2C_1^2 + 3C_2^2 - .... + (-1)^n(n+1)C_n^2 = [C_0^2 - C_1^2 + C_2^2 - .... + (-1)^n C_n^2] - [C_1^2 - 2C_2^2 + 3C_3^2 ...... + (-1)^n n.C_n^2]$

$$= (-1)^{n/2}.^nC_{n/2} - (-1)^{n/2-1}.\frac{1}{2}n.^nC_{n/2} = (-1)^{n/2}\left[1+\frac{n}{2}\right]^nC_{n/2}$$

Therefore the value of given expression $= \dfrac{2.\frac{n}{2}!\frac{n}{2}!}{n!}\left[(-1)^{n/2}.\left(1+\frac{n}{2}\right)\dfrac{n!}{\frac{n}{2}!\frac{n}{2}!}\right] = (-1)^{n/2}(n+2)$

Example: 33 If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then the value of $C_0 + 2C_1 + 3C_2 + ... + (n+1)C_n$ will be

[MP PET 1996; Rajasthan PET 1997; DCE 1995; IIT 1971; AMU 1995; EAMCET 2001]

(a) $(n+2)2^{n-1}$ (b) $(n+1)2^n$ (c) $(n+1)2^{n-1}$ (d) $(n+2)2^n$

Solution: (a) **Trick:** Put $n=1$ the expansion is equivalent to ${}^1C_0 + 2.{}^1C_1 = 1+2 = 3$.

Which is given by option (a) = $(n+2)2^{n-1} = (1+2)2^0 = 3$

(1) **Use of Differentiation :** This method applied only when the numericals occur as the product of binomial coefficients.

Solution process : (i) If last term of the series leaving the plus or minus sign be *m*, then divide *m* by *n* if *q* be the quotient and *r* be the remainder. *i.e.,* $m = nq + r$

Then replace *x* by x^q in the given series and multiplying both sides of expansion by x^r.

(ii) After process (i), differentiate both sides, w.r.t. *x* and put $x=1$ or -1 or *i* or – *i* etc. according to given series.

(iii) If product of two numericals (or square of numericals) or three numericals (or cube of numerical) then differentiate twice or thrice.

Example: 34 $C_1 + 2C_2 + 3C_3 + ...{}^nC_n =$ **[Rajasthan PET 1995; MP PET 2002]**

(a) 2^n (b) $n.2^n$ (c) $n.2^{n-1}$ (d) $n.2^{n+1}$

Solution: (c) We know that, $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ ......(i)

Differentiating both sides w.r.t. *x*, we get $n(1+x)^{n-1} = 0 + C_1 + 2.C_2x + 3C_3x^2 + ... + nC_nx^{n-1}$

Putting $x=1$, we get, $n.2^{n-1} = C_1 + 2C_2 + 3C_3 + ... + nC_n$.

Example: 35 If *n* is an integer greater than 1, then $a - {}^nC_1(a-1) + {}^nC_2(a-2) - .... + (-1)^n(a-n) =$

(a) a (b) 0 (c) a^2 (d) 2^n

Solution: (b) We have $a[C_0 - C_1 + C_2......] + [C_1 - 2C_2 + 3C_3.....] = a[C_0 - C_1 + C_2......] - [-C_1 + 2C_2 - 3C_3 + ...]$

We know that $(1-x)^n = C_0 - C_1x + C_2x^2...... + (-1)^nC_nx^n$; Put $x=1$, $0 = C_0 - C_1 + C_2 - ....$

Then differentiating both sides w.r.t. to x, we get $n(1-x)^{n-1} = 0 - C_1 + 2C_2x - 3C_3x^2 + ...$

Put $x=1$, $0 = -C_1 + 2C_2 - 3C_3 + ... = a[0] - [0] = 0$.

(2) **Use of Integration :** This method is applied only when the numericals occur as the denominator of the binomial coefficients.

Solution process : If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then we integrate both sides between the suitable limits which gives the required series.

(i) If the sum contains $C_0, C_1, C_2, ... C_n$ with all positive signs, then integrate between limit 0 to 1.

(ii) If the sum contains alternate signs (*i.e.* +, –) then integrate between limit – 1 to 0.

(iii) If the sum contains odd coefficients *i.e.,* (C_0, C_2, C_4.....) then integrate between –1 to 1.

(iv) If the sum contains even coefficients (*i.e.,* $C_1, C_3, C_5.....$) then subtracting (ii) from (i) and then dividing by 2.

(v) If in denominator of binomial coefficients is product of two numericals then integrate two times, first taking limit between 0 to *x* and second time take suitable limits.

Example: 36 $\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}+.....+\frac{C_n}{n+1}=$ **[Rajasthan PET 1996]**

(a) $\frac{2^n}{n+1}$ (b) $\frac{2^n-1}{n+1}$ (c) $\frac{2^{n+1}-1}{n+1}$ (d) None of these

Solution: (c) Consider the expansion $(1+x)^n=C_0+C_1x+C_2x^2+....+C_nx^n$ .....(i)

Integrating both sides of (i) within limits 0 to 1 we get,

$$\int_0^1(1+x)^n dx=\int_0^1 C_0+\int_0^1 C_1x+\int_0^1 C_2x^2+.......+\int_0^1 C_nx^n dx$$

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1=C_0[x]_0^1+C_1\left[\frac{x^2}{2}\right]_0^1+........+C_n\left[\frac{x^{n+1}}{n+1}\right]_0^1$$

$$\frac{2^{n+1}}{n+1}-\frac{1}{n+1}=C_0[1]+C_1\frac{1}{2}+C_2\frac{1}{3}+.......+C_n.\frac{1}{n+1};\quad C_0+\frac{C_1}{2}+\frac{C_2}{3}+....+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}.$$

Example: 37 $2C_0+\frac{2^2}{2}C_1+\frac{2^3}{3}C_2+......+\frac{2^{11}}{11}C_{10}=$ **[MP PET 1999; EAMCET 1992]**

(a) $\frac{3^{11}-1}{11}$ (b) $\frac{2^{11}-1}{11}$ (c) $\frac{11^3-1}{11}$ (d) $\frac{11^2-1}{11}$

Solution: (a) It is clear that it is a expansion of $(1+x)^{10}=C_0+C_1x+C_2x^2+.....+C_{10}x^{10}$

Integrating w.r.t. x both sides between the limit 0 to 2.

$$\left[\frac{(1+x)^{11}}{11}\right]_0^2=C_0[x]_0^2+C_1\left[\frac{x^2}{2}\right]_0^2+C_2\left[\frac{x^3}{3}\right]_0^2+......+C_{10}\left[\frac{x^{11}}{11}\right]_0^2\Rightarrow\frac{3^{11}-1}{11}=2C_0+\frac{2^2}{2}.C_1+\frac{2^3}{2}.C_2+......+\frac{2^{11}}{11}C_{10}.$$

Example: 38 The sum to $(n+1)$ terms of the following series $\frac{C_0}{2}-\frac{C_1}{3}+\frac{C_2}{4}-\frac{C_3}{5}+.....$ is

(a) $\frac{1}{n+1}$ (b) $\frac{1}{n+2}$ (c) $\frac{1}{n(n+1)}$ (d) None of these

Solution: (d) $(1-x)^n=C_0-C_1x+C_2x^2-C_3x^3+.......$

$$\Rightarrow x(1-x)^n=C_0x-C_1x^2+C_2x^3-C_3x^4+.....\Rightarrow\int_0^1 x(1-x)^n dx=C_0\left[\frac{x^2}{2}\right]_0^1-C_1\left[\frac{x^3}{3}\right]_0^1+C_2\left[\frac{x^4}{4}\right]_0^1-.......\text{ (i)}$$

The integral on L.H.S. of (i) = $\int_1^0(1-t)t^n(-dt)$ by putting $1-x=t$, $\Rightarrow\int_0^1(t^n-t^{n+1})dt=\frac{1}{n+1}-\frac{1}{n+2}$

Whereas the integral on the R.H.S. of (i)

$$=C_0\left[\frac{1}{2}\right]-C_1\left[\frac{1}{3}\right]+\frac{C_2}{4}-.......=\frac{C_0}{2}-\frac{C_1}{3}+\frac{C_2}{4}-.......\text{ to }(n+1)\text{ terms }=\frac{1}{n+1}-\frac{1}{n+2}=\frac{1}{(n+1)(n+2)}$$

Trick : Put $n=1$ in given series = $\frac{{}^1C_0}{2}-\frac{{}^1C_1}{3}=\frac{1}{6}$. Which is given by option (d).

6.1.11 An Important Theorem

If $(\sqrt{A}+B)^n=I+f$ where I and n are positive integers, n being odd and $0\le f<1$ then $(I+f).f=K^n$ where $A-B^2=K>0$ and $\sqrt{A}-B<1$.

Note: ❑ If n is even integer then $(\sqrt{A}+B)^n+(\sqrt{A}-B)^n=I+f+f'$

Hence L.H.S. and I are integers.

$\therefore\ f+f'$ is also integer; $\Rightarrow f+f'=1$; $\therefore\ f'=(1-f)$

Hence $(I+f)(1-f)=(I+f)f' = (\sqrt{A}+B)^n(\sqrt{A}-B)^n = (A-B^2)^n = K^n$.

Example: 39 Let $R=(5\sqrt{5}+11)^{2n+1}$ and $f=R-[R]$ where [.] denotes the greatest integer function. The value of R.f is **[IIT 19**

(a) 4^{2n+1} (b) 4^{2n} (c) 4^{2n-1} (d) 4^{-2n}

Solution: (a) Since $f=R-[R]$, $R=f+[R]$

$[5\sqrt{5}+11]^{2n+1}=f+[R]$, where $[R]$ is integer

Now let $f'=[5\sqrt{5}-11]^{2n+1}, 0<f'<1$

$f+[R]-f' = [5\sqrt{5}+11]^{2n+1}-[5\sqrt{5}-11]^{2n+1} = 2\left[{}^{2n+1}C_1(5\sqrt{5})^{2n}(11)^1+{}^{2n+1}C_3(5\sqrt{5})^{2n-2}(11)^3+.......\right]$

$= 2.(\text{Integer}) = 2K\ (K\in N)$ = Even integer

Hence $f-f'$ = even integer – $[R]$, but $-1<f-f'<1$. Therefore, $f-f'=0 \quad \therefore f=f'$

Hence R.f = $R.f'=(5\sqrt{5}+11)^{2n+1}(5\sqrt{5}-11)^{2n+1}=4^{2n+1}$.

6.1.12 Multinomial Theorem (For positive integral index)

If n is positive integer and $a_1,a_2,a_3,....a_n\in C$ then $(a_1+a_2+a_3+...+a_m)^n=\sum\frac{n!}{n_1!n_2!n_3!...n_m!}a_1^{n_1}a_2^{n_2}...a_m^{n_m}$

Where $n_1,n_2,n_3,.....n_m$ are all non-negative integers subject to the condition, $n_1+n_2+n_3+.....n_m=n$.

(1) The coefficient of $a_1^{n_1}.a_2^{n_2}.....a_m^{n_m}$ in the expansion of $(a_1+a_2+a_3+....a_m)^n$ is $\frac{n!}{n_1!n_2!n_3!....n_m!}$

(2) The greatest coefficient in the expansion of $(a_1+a_2+a_3+....a_m)^n$ is $\frac{n!}{(q!)^{m-r}[(q+1)!]^r}$

Where q is the quotient and r is the remainder when n is divided by m.

(3) If n is $+ve$ integer and $a_1,a_2,.....a_m\in C$, $a_1^{n_1}.a_2^{n_2}.........a_m^{n_m}$ then coefficient of x^r in the expansion of $(a_1+a_2x+.....a_mx^{m-1})^n$ is $\sum\frac{n!}{n_1!n_2!n_3!.....n_m!}$

Where $n_1,n_2.....n_m$ are all non-negative integers subject to the condition: $n_1+n_2+.....n_m=n$ and $n_2+2n_3+3n_4+....+(m-1)n_m=r$.

(4) The number of distinct or dissimilar terms in the multinomial expansion $(a_1+a_2+a_3+....a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

Example: 40 The coefficient of x^5 in the expansion of $(x^2-x-2)^5$ is

(a) – 83 (b) – 82 (c) – 81 (d) 0

Solution: (c) Coefficient of x^5 in the expansion of $(x^2-x-2)^5$ is $\sum\frac{5!}{n_1!.n_2!n_3!}(1)^{n_1}(-1)^{n_2}(-2)^{n_3}$.

where $n_1+n_2+n_3=5$ and $n_2+2n_3=5$. The possible value of n_1,n_2 and n_3 are shown in margin

n_1	n_2	n_3
1	3	1
2	1	2
0	5	0

$\therefore$ The coefficient of x^5 = $\frac{5!}{1!3!1!}(1)^1(-1)^3(-2)^1+\frac{5!}{2!1!2!}(1)^2(-1)^1(-2)^2+\frac{5!}{0!5!0!}(1)^0(-1)^5(-2)^0 = 40-120-1=-81$

Example: 41 Find the coefficient of $a^3b^4c^5$ in the expansion of $(bc+ca+ab)^6$

(a) 0 (b) 60 (c) – 60 (d) None of these

Solution: (b) In this case, $a^3b^4c^5=(ab)^x(bc)^y(ca)^z=a^{x+z}.b^{x+y}.c^{y+z}$

$z+x=3$, $x+y=4, y+z=5$; $2(x+y+z)=12$; $x+y+z=6$. Then $x=1, y=3, z=2$

Therefore the coefficient of $a^3b^4c^5$ in the expansion of $(bc+ca+ab)^6$ = $\dfrac{6!}{1!3!2!}=60$.

6.1.13 Binomial Theorem for any Index

Statement : $(1+x)^n=1+nx+\dfrac{n(n-1)x^2}{2!}+\dfrac{n(n-1)(n-2)}{3!}x^3+....+\dfrac{n(n-1)......(n-r+1)}{r!}x^r+...\text{terms up to }\infty$

When n is a negative integer or a fraction, where $-1<x<1$, otherwise expansion will not be possible.

If $x<1$, the terms of the above expansion go on decreasing and if x be very small a stage may be reached when we may neglect the terms containing higher power of x in the expansion, then $(1+x)^n=1+nx$.

Important Tips

- *Expansion is valid only when* $-1<x<1$.
- ${}^nC_r$ *can not be used because it is defined only for natural number, so* ${}^nC_r$ *will be written as* $\dfrac{(n)(n-1)......(n-r+1)}{r!}$
- *The number of terms in the series is infinite.*
- *If first term is not 1, then make first term unity in the following way,* $(x+y)^n=x^n\left[1+\dfrac{y}{x}\right]^n$, *if* $\left|\dfrac{y}{x}\right|<1$.

General term : $T_{r+1}=\dfrac{n(n-1)(n-2)......(n-r+1)}{r!}x^r$

Some important expansions:

(i) $(1+x)^n=1+nx+\dfrac{n(n-1)}{2!}x^2+.......+\dfrac{n(n-1)(n-2)......(n-r+1)}{r!}x^r+......$

(ii) $(1-x)^n=1-nx+\dfrac{n(n-1)}{2!}x^2-.......+\dfrac{n(n-1)(n-2).....(n-r+1)}{r!}(-x)^r+.......$

(iii) $(1-x)^{-n}=1+nx+\dfrac{n(n+1)}{2!}x^2+\dfrac{n(n+1)(n+2)}{3!}x^3+.....+\dfrac{n(n+1)......(n+r-1)}{r!}x^r+.....$

(iv) $(1+x)^{-n}=1-nx+\dfrac{n(n+1)}{2!}x^2-\dfrac{n(n+1)(n+2)}{3!}x^3+.....+\dfrac{n(n+1)......(n+r-1)}{r!}(-x)^r+......$

(a) **Replace *n* by 1 in (iii) :** $(1-x)^{-1}=1+x+x^2+.....+x^r+......\infty$, General term, $T_{r+1}=x^r$

(b) **Replace *n* by 1 in (iv) :** $(1+x)^{-1}=1-x+x^2-x^3+.....+(-x)^r+......\infty$, General term, $T_{r+1}=(-x)^r$.

(c) **Replace *n* by 2 in (iii) :** $(1-x)^{-2}=1+2x+3x^2+.....+(r+1)x^r+.....\infty$, General term, $T_{r+1}=(r+1)x^r$.

(d) **Replace *n* by 2 in (iv) :** $(1+x)^{-2}=1-2x+3x^2-4x^3+......+(r+1)(-x)^r+.....\infty$

General term, $T_{r+1}=(r+1)(-x)^r$.

(e) **Replace *n* by 3 in (iii) :** $(1-x)^{-3}=1+3x+6x^2+10x^3+.....+\dfrac{(r+1)(r+2)}{2!}x^r+..........\infty$

General term, $T_{r+1}=(r+1)(r+2)/2!.x^r$

(f) **Replace *n* by 3 in (iv) :** $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + ... + \frac{(r+1)(r+2)}{2!}(-x)^r + ...\infty$

General term, $T_{r+1} = \frac{(r+1)(r+2)}{2!}(-x)^r$

Example: 42 To expand $(1+2x)^{-1/2}$ as an infinite series, the range of *x* should be **[AMU 2002]**

(a) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (b) $\left(-\frac{1}{2}, \frac{1}{2}\right)$ (c) $[-2, 2]$ (d) (– 2, 2)

Solution: (b) $(1+2x)^{-1/2}$ can be expanded if $|2x| < 1$ *i.e.,* if $|x| < \frac{1}{2}$ *i.e.,* if $-\frac{1}{2} < x < \frac{1}{2}$ *i.e.,* if $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

Example: 43 If the value of *x* is so small that x^2 and higher power can be neglected, then $\frac{\sqrt{1+x} + \sqrt[3]{(1-x)^2}}{1 + x + \sqrt{1+x}}$ is equal to

[Roorkee 1962]

(a) $1 + \frac{5}{6}x$ (b) $1 - \frac{5}{6}x$ (c) $1 + \frac{2}{3}x$ (d) $1 - \frac{2}{3}x$

Solution: (b) Given expression can be written as $= \frac{(1+x)^{1/2} + (1-x)^{2/3}}{1 + x + (1+x)^{1/2}} =$

$$\frac{\left(1 + \frac{1}{2}x + \left(-\frac{1}{8}\right)x^2 + ...\right) + \left(1 - \frac{2}{3}x - \frac{1}{9}x^2 - ....\right)}{1 + x + \left[1 + \frac{1}{2}x - \frac{1}{8}x^2 + ...\right]}$$

$$= \frac{1 - \frac{1}{12}x - \frac{1}{144}x^2 + ...}{1 + \frac{3}{4}x - \frac{1}{16}x^2 + ...} = 1 - \frac{5}{6}x + ... = 1 - \frac{5}{6}x$$, when $x^2, x^3....$ are neglected.

Example: 44 If $(1+ax)^n = 1 + 8x + 24x^2 + ...$ then the value of *a* and *n* is

(a) 2, 4 (b) 2, 3 (c) 3, 6 (d) 1, 2

Solution: (a) We know that $(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + ...$

$$(1+ax)^n = 1 + \frac{n(ax)}{1!} + \frac{n(n-1)(ax)^2}{2!} + ... \Rightarrow 1 + 8x + 24x^2 + ... = 1 + \frac{n(ax)}{1!} + \frac{n(n-1)(ax)^2}{2!} + ...$$

Comparing coefficients of both sides we get, $na = 8$, and $\frac{n(n-1)a^2}{2!} = 24$ on solving, $a = 2$, b = 4.

Example: 45 Coefficient of x^r in the expansion of $(1-2x)^{-1/2}$ **[Kurukshetra CEE 2001]**

(a) $\frac{(2r)!}{(r!)^2}$ (b) $\frac{(2r)!}{2^r.(r!)^2}$ (c) $\frac{(2r)!}{(r!)^2.2^{2r}}$ (d) $\frac{(2r)!}{2^r.(r+1)!(r-1)!}$

Solution: (b) Coefficient of

$$x^r = \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)....\left(-\frac{1}{2}-r+1\right)}{r!}(-2)^r = \frac{1.3.5...(2r-1).(-1)^r.(-1)^r.2^r}{2^r\, r!} = \frac{1.3.5...(2r-1)}{r!} = \frac{(2r)!}{r!r!2^r}$$

Example: 46 The coefficient of x^{25} in $(1 + x + x^2 + x^3 + x^4)^{-1}$ is

(a) 25 (b) – 25 (c) 1 (d) – 1

Solution: (c) Coefficient of x^{25} in $(1 + x + x^2 + x^3 + x^4)^{-1}$

= Coefficient of x^{25} in $\left[\frac{1(1-x^5)}{1-x}\right]^{-1}$ = Coefficient of x^{25} in $(1-x^5)^{-1}.(1-x)$

= Coefficient of x^{25} in $[(1-x^5)^{-1} - x(1-x^5)^{-1}] = [1+(x^5)^1+(x^5)^2+......] - x[1+(x^5)^1+(x^5)^2]+......]$

= Coefficient of x^{25} in $[1+x^5+x^{10}+x^{15}+.....]$ – Coefficient of x^{24} in $[1+x^5+x^{10}+x^{15}+.....]$ = $1-0=1$.

Example: 47 $1-\frac{1}{8}+\frac{1}{8}.\frac{3}{16}-......... =$ **[EAMCET 1990]**

(a) $\frac{2}{5}$ (b) $\frac{\sqrt{2}}{5}$ (c) $\frac{2}{\sqrt{5}}$ (d) None of these

Solution: (c) We know that $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+......\infty$

Here $nx=-\frac{1}{8}, \frac{n(n-1)}{2}x^2 = \frac{3}{8.16} \Rightarrow x=\frac{1}{4}, n=-\frac{1}{2} \Rightarrow 1-\frac{1}{8}+\frac{1}{8}.\frac{3}{16}-...... = \left(1+\frac{1}{4}\right)^{-1/2} = \frac{2}{\sqrt{5}}$.

Example: 48 If x is so small that its two and higher power can be neglected and $(1-2x)^{-1/2}(1-4x)^{-5/2} = 1+kx$ then k =

[Rajasthan PET 1993]

(a) 1 (b) – 2 (c) 10 (d) 11

Solution: (d) $(1-2x)^{-1/2}(1-4x)^{-5/2} = 1+kx$

$$\left[1+\frac{(-1/2)(-2x)}{1!}+\frac{(-1/2)(-3/2)(-2x)^2}{2!}+......\right]\left[1+\frac{(-5/2)(-4x)}{1!}+\frac{(-5/2)(-7/2)(-4x)^2}{2!}+......\right] = 1+kx$$

Higher power can be neglected. Then $\left[1+\frac{x}{1!}\right]\left[1+\frac{10x}{1!}\right] = 1+kx$; $1+10x+x=1+kx$; $k=11$

Example: 49 The cube root of $1+3x+6x^2+10x^3+.....$ is

(a) $1-x+x^2-x^3+.....\infty$ (b) $1+x^3+x^6+x^9+......$ (c) $1+x+x^2+x^3+....$ (d) None of these

Solution: (c) We have $(1+3x+6x^2+10x^3+.....)^{1/3} = [(1-x)^{-3}]^{1/3}$; $[\because (1-x)^{-3} = 1+3x+6x^2+...\infty] \Rightarrow$ $(1-x)^{-1} = 1+x+x^2+...\infty$

Example: 50 The coefficient of x^n in the expansion of $\left(\frac{1}{1-x}\right)\left(\frac{1}{3-x}\right)$ is

(a) $\frac{3^{n+1}-1}{2.3^{n+1}}$ (b) $\frac{3^{n+1}-1}{3^{n+1}}$ (c) $2\left(\frac{3^{n+1}-1}{3^{n+1}}\right)$ (d) None of these

Solution: (a) $\frac{1}{(1-x)(3-x)} = (1-x)^{-1}(3-x)^{-1} = 3^{-1}(1-x)^{-1}\left(1-\frac{x}{3}\right)^{-1} = \frac{1}{3}[1+x+x^2+.....x^n]\left[1+\frac{x}{3}+\frac{x^2}{3^2}+.....+\frac{x^{n-1}}{3^{n-1}}+\frac{x^n}{3^n}\right]$

Coefficient of $x^n = \frac{1}{3^{n+1}}+\frac{1}{3^n}+\frac{1}{3^{n-1}}+.....(n+1)\text{ terms} = \frac{1}{3^{n+1}}\frac{[3^{n+1}-1]}{3-1} = \frac{3^{n+1}-1}{2.3^{n+1}}$.

Trick: Put $n=1,2,3......$ and find the coefficients of $x, x^2, x^3......$ and comparing with the given option as :

Coefficient of x^2 is $= \frac{1}{3^3}+\frac{1}{3^2}+\frac{1}{3^1} = \frac{1}{3^3}\frac{[3^3-1]}{3-1} = \frac{13}{27}$; Which is given by option (a) $\frac{3^{n+1}-1}{2.(3^{n+1})} = \frac{3^3-1}{2.3^3} = \frac{13}{27}$.

6.1.14 Three / Four Consecutive terms or Coefficients

(1) **If consecutive coefficients are given:** In this case divide consecutive coefficients pair wise. We get equations and then solve them.

(2) **If consecutive terms are given :** In this case divide consecutive terms pair wise *i.e.* if four consecutive terms be $T_r, T_{r+1}, T_{r+2}, T_{r+3}$ then find $\frac{T_r}{T_{r+1}}, \frac{T_{r+1}}{T_{r+2}}, \frac{T_{r+2}}{T_{r+3}} \Rightarrow \lambda_1, \lambda_2, \lambda_3$ (say) then divide λ_1 by λ_2 and λ_2 by λ_3 and solve.

Example: 51 If a_1, a_2, a_3, a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, then $\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} =$

[IIT 1975]

(a) $\frac{a_2}{a_2+a_3}$ (b) $\frac{1}{2}\frac{a_2}{a_2+a_3}$ (c) $\frac{2a_2}{a_2+a_3}$ (d) $\frac{2a_3}{a_2+a_3}$

Solution: (c) Let a_1, a_2, a_3, a_4 be respectively the coefficients of $(r+1)^{th}, (r+2)^{th}, (r+3)^{th}, (r+4)^{th}$ terms in the expansion of $(1+x)^n$. Then $a_1 = {}^nC_r, a_2 = {}^nC_{r+1}, a_3 = {}^nC_{r+2}, a_4 = {}^nC_{r+3}$.

Now, $\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}} + \frac{{}^nC_{r+2}}{{}^nC_{r+2} + {}^nC_{r+3}} = \frac{{}^nC_r}{{}^{n+1}C_{r+1}} + \frac{{}^nC_{r+2}}{{}^{n+1}C_{r+3}} = \frac{{}^nC_r}{\frac{n+1}{r+1}\,{}^nC_r} + \frac{{}^nC_{r+2}}{\frac{n+1}{r+3}\,{}^nC_{r+2}}$

$= \frac{r+1}{n+1} + \frac{r+3}{n+1} = \frac{2(r+2)}{n+1} = 2.\frac{{}^nC_{r+1}}{{}^{n+1}C_{r+2}} = 2.\frac{{}^nC_{r+1}}{{}^nC_{r+1} + {}^nC_{r+2}} = \frac{2a_2}{a_2+a_3}$

6.1.15 Some Important Points

(1) **Pascal's Triangle :**

1	$(x+y)^0$
1 1	$(x+y)^1$
1 2 1	$(x+y)^2$
1 3 3 1	$(x+y)^3$
1 4 6 4 1	$(x+y)^4$
1 5 10 10 5 1	$(x+y)^5$

Pascal's triangle gives the direct binomial coefficients.

Example : $(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

(2) **Method for finding terms free from radical or rational terms in the expansion of $(a^{1/p} + b^{1/q})^N \;\forall\, a, b \in$ prime numbers :** Find the general term $T_{r+1} = {}^NC_r (a^{1/p})^{N-r}(b^{1/q})^r = {}^NC_r\, a^{\frac{N-r}{p}}.b^{\frac{r}{q}}$

Putting the values of $0 \le r \le N$, when indices of *a* and *b* are integers.

Note : ❑ Number of irrational terms = Total terms – Number of rational terms.

Example: 52 The number of integral terms in the expansion of $(\sqrt{3} + \sqrt[8]{5})^{256}$ is **[AIEEE 2003]**

(a) 32 (b) 33 (c) 34 (d) 35

Solution: (b) $T_{r+1} = {}^{256}C_r . 3^{\frac{256-r}{2}} . 5^{\frac{r}{8}}$

First term = ${}^{256}C_0\, 3^{128} 5^0 =$ integer and after eight terms, *i.e.*, 9th term = ${}^{256}C_8\, 3^{124}.5^1 =$ integer

Continuing like this, we get an A.P., $1^{st}, 9^{th}$....... 257^{th} ; $T_n = a + (n-1)d \Rightarrow 257 = 1 + (n-1)8 \Rightarrow n = 33$

Example: 53 The number of irrational terms in the expansion of $\left(\sqrt[8]{5}+\sqrt[6]{2}\right)^{100}$ is

(a) 97 (b) 98 (c) 96 (d) 99

Solution: (a) $T_{r+1}={}^{100}C_r\,5^{\frac{100-r}{8}}.2^{\frac{r}{6}}$

As 2 and 5 are co-prime. T_{r+1} will be rational if $100-r$ is multiple of 8 and r is multiple of 6 also $0\le r\le 100$

$\therefore r=0,6,12.......96$; $\therefore 100-r=4,10,16.....100$ ......(i)

But $100-r$ is to be multiple of 8.

So, $100-r$ = 0, 8, 16, 24,......96 .....(ii)

Common terms in (i) and (ii) are 16, 40, 64, 88.

$\therefore$ r = 84, 60, 36, 12 give rational terms $\therefore$ The number of irrational terms = 101 – 4 = 97.

***

6.2 Mathematical Induction

6.2.1 First Principle of Mathematical Induction

The proof of proposition by mathematical induction consists of the following three steps :

Step I : (Verification step) **:** Actual verification of the proposition for the starting value "i"

Step II : (Induction step) **:** Assuming the proposition to be true for "k", $k \geq i$ and proving that it is true for the value $(k + 1)$ which is next higher integer.

Step III : (Generalization step) **:** To combine the above two steps

Let $p(n)$ be a statement involving the natural number n such that

(i) $p(1)$ is true *i.e.* $p(n)$ is true for $n = 1$.

(ii) $p(m + 1)$ is true, whenever $p(m)$ is true *i.e.* $p(m)$ is true $\Rightarrow p(m + 1)$ is true.

Then $p(n)$ is true for all natural numbers n.

6.2.2 Second Principle of Mathematical Induction

The proof of proposition by mathematical induction consists of following steps :

Step I : (Verification step) : Actual verification of the proposition for the starting value i and $(i + 1)$.

Step II : (Induction step) : Assuming the proposition to be true for $k - 1$ and k and then proving that it is true for the value $k + 1$; $k \geq i + 1$.

Step III : (Generalization step) : Combining the above two steps.

Let $p(n)$ be a statement involving the natural number n such that

(i) $p(1)$ is true *i.e.* $p(n)$ is true for $n = 1$ and

(ii) $p(m + 1)$ is true, whenever $p(n)$ is true for all n, where $i \leq n \leq m$

Then $p(n)$ is true for all natural numbers.

For $a \neq b$, The expression $a^n - b^n$ is divisible by

(a) $a + b$ if n is even. (b) $a - b$ is n if odd or even.

6.2.3 Some Formulae based on Principle of Induction

For any natural number n

(i) $\sum n = 1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$

(ii) $\sum n^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(iii) $\sum n^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \dfrac{n^2(n+1)^2}{4} = \left(\sum n\right)^2$

Example: 1 The smallest positive integer n, for which $n!<\left(\frac{n+1}{2}\right)^n$ hold is

(a) 1 (b) 2 (c) 3 (d) 4

Solution: (b) Let $P(n)$: $n!<\left(\frac{n+1}{2}\right)^n$

Step I : For n = 2 $\Rightarrow$ $2!<\left(\frac{2+1}{2}\right)^2$ $\Rightarrow$ $2<\frac{9}{4}$ $\Rightarrow$ $2<2.25$ which is true. Therefore, $P(2)$ is true.

Step II : Assume that $P(k)$ is true, then $p(k)$: $k!<\left(\frac{k+1}{2}\right)^k$

Step III : For n = k + 1,

$$P(k+1):(k+1)!<\left(\frac{k+2}{2}\right)^{k+1} \Rightarrow k!<\left(\frac{k+1}{2}\right)^k \Rightarrow (k+1)k!<\frac{(k+1)^{k+1}}{2^k}$$

$$\Rightarrow (k+1)!<\frac{(k+1)^{k+1}}{2^k} \quad .....(i) \quad \text{and} \quad \frac{(k+1)^{k+1}}{2^k}<\left(\frac{k+2}{2}\right)^{k+1} \quad .....(ii)$$

$$\Rightarrow \left(\frac{k+2}{k+1}\right)^{k+1}>2 \Rightarrow \left[1+\frac{1}{k+1}\right]^{k+1}>2 \Rightarrow 1+(k+1)\frac{1}{k+1}+{}^{k+1}C_2\left(\frac{1}{k+1}\right)^2+........>2$$

$$\Rightarrow 1+1+{}^{k+1}C_2\left(\frac{1}{k+1}\right)^2+......>2$$

Which is true, hence (ii) is true.

From (i) and (ii), $(k+1)!<\frac{(k+1)^{k+1}}{2^k}<\left(\frac{k+2}{2}\right)^{k+1}$ $\Rightarrow$ $(k+1)!<\left(\frac{k+2}{2}\right)^{k+1}$

Hence $P(k+1)$ is true. Hence by the principle of mathematical induction $P(n)$ is true for all $n \in N$

Trick : By check option

(a) For n = 1, $1!<\left(\frac{1+1}{2}\right)^1$ $\Rightarrow 1<1$ which is wrong (b) For n = 2, $2!<\left(\frac{3}{2}\right)^2$ $\Rightarrow 2<\frac{9}{4}$ which is correct

(c) For n = 3, $3!<\left(\frac{3+1}{2}\right)^3$ $\Rightarrow$ 6 < 8 which is correct

(d) For n = 4, $4!<\left(\frac{4+1}{2}\right)^4$ $\Rightarrow$ $24<\left(\frac{5}{2}\right)^4$ $\Rightarrow$ 24 < 39.0625 which is correct.

But smallest positive integer n is 2.

Example: 2 Let $S(k)=1+3+5+.......+(2k-1)=3+k^2$. Then which of the following is true. **[AIEEE 2004]**

(a) Principle of mathematical induction can be used to prove the formula

(b) $S(k)\Rightarrow S(k+1)$

(c) $S(k)\nRightarrow S(k+1)$

(d) $S(1)$ is correct

Solution: (c) We have $S(k)=1+3+5+......+(2k-1)=3+k^2$, $S(1)\Rightarrow 1=4$, Which is not true and $S(2)\Rightarrow 3=7$, Which is not true.

Hence induction cannot be applied and $S(k)\nRightarrow S(k+1)$

Example: 3 When P is a natural number, then $P^{n+1}+(P+1)^{2n-1}$ is divisible by **[IIT 1994]**

(a) P (b) P^2+P (c) P^2+P+1 (d) P^2-1

Solution: (c) For n =1, we get, $P^{n+1}+(P+1)^{2n-1}=P^2+(P+1)^1=P^2+P+1$,

Which is divisible by P^2+P+1, so result is true for n =1

Let us assume that the given result is true for $n=m \in N$

i.e. $P^{m+1}+(P+1)^{2m-1}$ is divisible by P^2+P+1 *i.e.* $P^{m+1}+(P+1)^{2m-1}=k(P^2+P+1) \quad \forall \quad k \in N$ .....(i)

Now, $P^{(m+1)+1}+(P+1)^{2(m+1)-1} = P^{m+2}+(P+1)^{2m+1}=P^{m+2}+(P+1)^2(P+1)^{2m-1}$

$= P^{m+2} + (P+1)^2[k(P^2+P+1) - P^{m+1}]$ by using (i)

$= P^{m+2} + (P+1)^2 \cdot k(P^2+P+1) - (P+1)^2(P)^{m+1} = P^{m+1}[P-(P+1)^2] + (P+1)^2 \cdot k(P^2+P+1)$

$= P^{m+1}[P - P^2 - 2P - 1] + (P+1)^2 \cdot k(P^2+P+1) = -P^{m+1}[P^2+P+1] + (P+1)^2 \cdot k(P^2+P+1)$

$= (P^2+P+1)[k \cdot (P+1)^2 - P^{m+1}]$

Which is divisible by p^2+p+1, so the result is true for $n = m+1$. Therefore, the given result is true for all $n \in N$ by induction.

Trick : For n = 2, we get, $P^{n+1} + (P+1)^{2n-1} = P^3 + (P+1)^3 = P^3 + P^3 + 1 + 3P^2 + 3P = 2P^3 + 3P^2 + 3P + 1$

Which is divisible by P^2+P+1. Given result is true for all $n \in N$

Example: 4 Given $U_{n+1} = 3U_n - 2U_{n-1}$ and $U_0 = 2$, $U_1 = 3$, the value of U_n for all $n \in N$ is

(a) $2^n - 1$ (b) $2^n + 1$ (c) 0 (d) None of these

Solution: (b) $\because U_{n+1} = 3U_n - 2U_{n-1}$ .....(i)

Step I : Given $U_1 = 3$

For n =1, $U_{1+1} = 3U_1 - 2U_0$, $U_2 = 3.3 - 2.2 = 5$

Option (b) $U_n = 2^n + 1$

For n = 1, $U_1 = 2^1 + 1 = 3$ which is true. For n = 2, $U_2 = 2^2 + 1 = 5$ which is true

Therefore, the result is true for n = 1 and n = 2

Step II : Assume it is true for n = k then it is also true for n = k – 1

Then $U_k = 2^k + 1$ .....(ii) and $U_{k-1} = 2^{k-1} + 1$ .....(iii)

Step III : Putting n = k in (i), we get

$U_{k+1} = 3U_k - 2U_{k-1} = 3[2^k+1] - 2[2^{k-1}+1] = 3.2^k + 3 - 2.2^{k-1} - 2 = 3.2^k + 1 - 2.2^{k-1}$

$\Rightarrow 3.2^k - 2^k + 1 = 2.2^k + 1 = 2^{k+1} + 1 \Rightarrow U_{k+1} = 2^{k+1} + 1$

This shows that the result is true for $n = k+1$, by the principle of mathematical induction the result is true for all $n \in N$.

6.2.4 Divisibility Problems

To show that an expression is divisible by an integer

(i) If a, p, n, r are positive integers, then first of all we write $a^{pn+r} = a^{pn}.a^r = (a^p)^n.a^r$.

(ii) If we have to show that the given expression is divisible by c.S

Then express, $a^p = [1 + (a^p - 1]$, if some power of $(a^p - 1)$ has c as a factor.

$a^p = [2 + (a^p - 2)]$, if some power of $(a^p - 2)$ has c as a factor.

$a^p = [K + (a^p - K)]$, if some power of $(a^p - K)$ has c as a factor.

Example: 5 $(1+x)^n - nx - 1$ is divisible by (where $n \in N$)

(a) $2x$ (b) x^2 (c) $2x^3$ (d) All of these

Solution: (b) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... \Rightarrow (1+x)^n - nx - 1 = x^2\left[\frac{n(n-1)}{2!} + \frac{n(n-1)(n-3)}{3!}x + ...\right]$

From above it is clear that $(1+x)^n - nx - 1$ is divisible by x^2.

Trick : $(1+x)^n - nx - 1$. Put $n = 2$ and $x = 3$; Then $4^2 - 2.3 - 1 = 9$

Is not divisible by 6, 54 but divisible by 9. Which is given by option (c) = $x^2 = 9$.

Example: 6 The greatest integer which divides the number $101^{100} - 1$ is

(a) 100 (b) 1000 (c) 10000 (d) 100000

Solution: (c) $(1+100)^{100} = 1 + 100.100 + \frac{100.99}{1.2}(100)^2 + ... \Rightarrow 101^{100} - 1 = 100.100\left[1 + \frac{100.99}{1.2} + \frac{100.99.98}{3.2.1}100 + ...\right]$

From above it is clear that, $101^{100} - 1$ is divisible by $(100)^2 = 10000$

***