Exercise 15.2

Chapter 15 Multiple Integrals 15.2 1E

Given that $f(x, y) = 12x^2 y^3$

Now $\int_0^5 f(x, y)\,dx = \int_0^5 12x^2 y^3\,dx$

$$= 12\int_0^5 x^2 y^3\,dx$$

$$= 12y^3 \int_0^5 x^2\,dx$$

$$= 12y^3\left[\frac{x^3}{3}\right]_0^5$$

$$= 4y^3\left[5^3 - 0^3\right]$$

$$= 4y^3.125$$

$$= 500y^3$$

and $\int_0^1 f(x, y)\,dy = \int_0^1 12x^2 y^3\,dy$

$$= 12x^2 \int_0^1 y^3\,dy$$

$$= 12x^2\left[\frac{y^4}{4}\right]_0^1$$

$$= 3x^2\left[1^4 - 0^4\right]$$

$$= 3x^2$$

Chapter 15 Multiple Integrals 15.2 2E

Given that $f(x, y) = y + xe^y$

Now $\int_0^5 f(x, y)\, dx = \int_0^5 (y + xe^y)\, dx$

$$= \int_0^5 y\, dx + \int_0^5 xe^y\, dx$$

$$= y[x]_0^5 + \left[\frac{x^2}{2} e^y\right]_0^5$$

$$= y[5-0] + \left[\frac{25}{2} e^y - 0\right]$$

$$= 5y + \frac{25}{2} e^y$$

and $\int_0^1 f(x, y)\, dy = \int_0^1 (y + xe^y)\, dy$

$$= \left[\frac{y^2}{2}\right]_0^1 + x\left[e^y\right]_0^1$$

$$= \frac{1}{2} + x(e^1 - e^0)$$

$$= \frac{1}{2} + x(e - 1)$$

Chapter 15 Multiple Integrals 15.2 3E

We have the double integral $\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx$.

Let us start by removing the innermost integral.

$$\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx = \int_1^4 \left[3x^2y^2 - 2xy\right]_0^2 dx$$

$$= \int_1^4 \left[3x^2(2)^2 - 2x(2)\right] - \left[3x^2(0)^2 - 2(0)y\right] dx$$

$$= \int_1^4 12x^2 - 4x\, dx$$

The integral is simplified to $\int_1^4 12x^2 - 4x\, dx$.

Now, let us evaluate the outer integral and apply the limits.

$$\int_1^4 12x^2 - 4x\, dx = \left(4x^3 - 2x^2\right)_1^4$$

$$= \left[4(4)^3 - 2(4)^2\right] - \left[4(1)^3 - 2(1)^2\right]$$

$$= 222$$

Thus, the iterated integral evaluates to $\boxed{222}$.

Chapter 15 Multiple Integrals 15.2 4E

$$\int_0^1 \int_1^2 \left(4x^3 - 9x^2y^2\right) dy\, dx = \int_0^1 \left[\frac{4x^3y}{1} - 9x^2\frac{y^3}{3}\right]_{y=1}^{y=2} dx$$

$$= \int_0^1 4x^3[2-1] - 3x^2[8-1]\, dx$$

$$= \int_0^1 \left(4x^3 - 21x^2\right) dx$$

$$= \left[\frac{4x^4}{4} - \frac{21x^3}{3}\right]_0^1$$

$$= [1 - 7] = -6$$

Chapter 15 Multiple Integrals 15.2 5E

Consider the double integral $\int_0^2\int_0^4 y^3 e^{2x}\,dy\,dx$.

Let us start by removing the innermost integral.

$$\begin{aligned}\int_0^2\int_0^4 y^3 e^{2x}\,dy\,dx &= \int_0^2\left[\frac{y^4}{4}e^{2x}\right]_0^4 dx \\ &= \frac{1}{4}\int_0^2\left[(4)^4 e^{2x}-(0)^4 e^{2x}\right]dx \\ &= \frac{(4)^4}{4}\int_0^2 e^{2x}\,dx \\ &= 64\int_0^2 e^{2x}\,dx\end{aligned}$$

Now, let us evaluate the outer integral and apply the limits.

$$\begin{aligned}\int_0^2\int_0^4 y^3 e^{2x}\,dy\,dx &= 64\int_0^2 e^{2x}\,dx \\ &= 64\left(\frac{e^{2x}}{2}\right)_0^2 \\ &= 32\left[e^{2(2)}-e^{2(0)}\right] \\ &= \boxed{32\left(e^4-1\right)}\end{aligned}$$

Chapter 15 Multiple Integrals 15.2 6E

$$\int_{\pi/6}^{\pi/2}\int_{-1}^{5}\cos y\,dx\,dy = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos y[x]_{-1}^{5}\,dy$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos y[5-(-1)]\,dy$$

$$= 6\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos y\,dy$$

$$= 6[\sin y]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 6\left[\sin\frac{\pi}{2} - \sin\frac{\pi}{6}\right]$$

$$= 6\left[1 - \frac{1}{2}\right]$$

$$= 3$$

Chapter 15 Multiple Integrals 15.2 7E

We have the double integral $\int_{-3}^{3}\int_{0}^{\frac{\pi}{2}}\left(y + y^2\cos x\right)dx\,dy$.

Let us start by removing the innermost integral.

$$\int_{-3}^{3}\int_{0}^{\frac{\pi}{2}}\left(y + y^2\cos x\right)dx\,dy = \int_{-3}^{3} xy + y^2\sin x\Big]_0^{\frac{\pi}{2}}dy$$

$$= \int_{-3}^{3}\left[\left(\frac{\pi}{2}\right)y + y^2\sin\left(\frac{\pi}{2}\right)\right] - \left[(0)y + y^2\sin(0)\right]dy$$

$$= \int_{-3}^{3}\frac{\pi y}{2} + y^2\,dy$$

The integral is simplified to $\int_{-3}^{3}\frac{\pi y}{2} + y^2\,dy$.

Now, let us evaluate the outer integral and apply the limits.

$$\int_{-3}^{3} \frac{\pi y}{2} + y^2 \, dy = \left(\frac{\pi y^2}{4} + \frac{y^3}{3}\right)_{-3}^{3}$$

$$= \left[\frac{\pi(-3)^2}{4} + \frac{(-3)^3}{3}\right] - \left[\frac{\pi(-3)^2}{4} + \frac{(-3)^3}{3}\right]$$

$$= 18$$

Thus, the iterated integral evaluates to $\boxed{18}$.

Chapter 15 Multiple Integrals 15.2 8E

Consider the double integral $\int_1^3 \int_1^5 \frac{\ln y}{xy} \, dy \, dx$.

Calculate the iterated integral.

Put $\ln y = t$ then $\frac{1}{y} dy = dt$

Let us start by removing the innermost integral.

$$\int_1^3 \int_1^5 \frac{\ln y}{xy} \, dy \, dx = \int_1^3 \int_1^5 \frac{1}{x}\left(\ln y \cdot \frac{1}{y} dy\right) dx$$

$$= \int_1^3 \int_1^5 \frac{1}{x}(t \cdot dt) \, dx$$

$$= \int_1^3 \frac{1}{x}\left[\frac{t^2}{2}\right]_1^5 dx$$

$$= \int_1^3 \frac{1}{x}\left[\frac{(\ln y)^2}{2}\right]_1^5 dx \qquad \text{Since } t = \ln y$$

$$= \int_1^3 \frac{1}{x}\left[\frac{(\ln 5)^2}{2} - \frac{(\ln 1)^2}{2}\right] dx$$

$$= \int_1^3 \frac{1}{x}\left[\frac{(\ln 5)^2}{2} - 0\right] dx \qquad \text{Since } \ln 1 = 0$$

$$= \frac{(\ln 5)^2}{2} \int_1^3 \left(\frac{1}{x}\right) dx$$

Now, let us evaluate the outer integral and apply the limits.

$$\int_1^3\int_1^5 \frac{\ln y}{xy}\,dy\,dx = \frac{(\ln 5)^2}{2}\int_1^3\left(\frac{1}{x}\right)dx$$

$$= \frac{(\ln 5)^2}{2}\left[\ln|x|\right]_1^3$$

$$= \frac{(\ln 5)^2}{2}\left[\ln|3| - \ln|1|\right]$$

$$= \frac{(\ln 5)^2 \ln 3}{2} \qquad (\text{Since } \ln 1 = 0)$$

Hence, $\boxed{\int_1^3\int_1^5 \frac{\ln y}{xy}\,dy\,dx = \frac{(\ln 5)^2 \ln 3}{2}}$

Chapter 15 Multiple Integrals 15.2 9E

We have to evaluate the iterated integral $\int_1^4\int_1^2\left(\frac{x}{y}+\frac{y}{x}\right)dy\,dx$.

$$\int_1^4\int_1^2\left(\frac{x}{y}+\frac{y}{x}\right)dy\,dx$$

$$= \int_1^4\int_1^2\left(\frac{x}{y}\right)dy\,dx + \int_1^4\int_1^2\left(\frac{y}{x}\right)dy\,dx$$

$$= \int_1^4 x\left(\int_1^2 \frac{1}{y}.dy\right)dx + \int_1^4 \frac{1}{x}\left(\int_1^2 y\,dy\right)dx$$

$$= \int_1^4 x\left[\ln y\right]_{y=1}^{y=2} dx + \int_1^4 \frac{1}{x}\left(\frac{y^2}{2}\right)_{y=1}^{y=2} dx$$

$$= \int_1^4 \left[x(\ln 2 - \ln 1)\right]dx + \frac{1}{2}\int_1^4 \frac{1}{x}(4-1)dx$$

$$= \ln 2\int_1^4 x\,dx + \frac{3}{2}\int_1^4 \frac{1}{x}dx$$

i.e. $\int_1^4\int_1^2\left(\frac{x}{y}+\frac{y}{x}\right)dy\,dx = \ln 2\left(\frac{x^2}{2}\right)_1^4+\frac{3}{2}(\ln x)_1^4$

$$= \frac{15}{2}\ln 2+\frac{3}{2}\ln 4$$

(since $\ln 1 = 0$)

$$= \frac{15}{2}\ln 2+3\ln 2\left(\because \frac{1}{2}\ln(4)=\ln(4)^{\frac{1}{2}}=\ln(2)\right)$$

$$= \boxed{\frac{21}{2}\ln 2}$$

Chapter 15 Multiple Integrals 15.2 10E

Given Integral is $\int_0^1\int_0^3 e^{x+3y}\,dx\,dy = \int_0^3 e^x\,dx.\int_0^1 e^{3y}\,dy$

$$= \left[e^x\right]_0^3.\left[\frac{e^{3y}}{3}\right]_0^1$$

$$= \left[e^3-e^0\right].\frac{1}{3}\left[e^{3.1}-e^{3.0}\right]$$

$$= \frac{1}{3}\left(e^3-1\right)^2$$

Chapter 15 Multiple Integrals 15.2 11E

Consider the integral:

$$\int_0^1\int_0^1 v(u+v^2)^4\,du\,dv$$

Now, the integral can be evaluated as follows:

$$\int_0^1\int_0^1 v\left(u+v^2\right)^4 du\,dv = \int_0^1\left(\int_0^1 v\left(u+v^2\right)^4 du\right)dv$$

$$= \int_0^1 v\left(\int_0^1\left(u+v^2\right)^4 du\right)dv$$

Start by evaluating the innermost integral.

$$\int_0^1 \left(u + v^2\right)^4 du$$

In this integral the variable v is constant.

Take a substitution

$$u + v^2 = t$$
$$du = dt$$
$$u = 0 \Rightarrow t = 0 + v^2 = v^2$$
$$u = 1 \Rightarrow t = 1 + v^2$$

$$\begin{aligned}\int_0^1 \left(u + v^2\right)^4 du &= \int_{v^2}^{1+v^2} (t)^4 dt \\ &= \left[\frac{t^{4+1}}{4+1}\right]_{v^2}^{1+v^2} \\ &= \frac{1}{5}\left[t^5\right]_{v^2}^{1+v^2} \\ &= \frac{1}{5}\left[\left(1+v^2\right)^5 - \left(v^2\right)^5\right] \\ &= \frac{1}{5}\left[\left(1+v^2\right)^5 - v^{10}\right]\end{aligned}$$

Now evaluate the outer integral and apply the limits.

$$\begin{aligned}\int_0^1\int_0^1 v\left(u + v^2\right)^4 du\,dv &= \int_0^1 \left(v\int_0^1 \left(u + v^2\right)^4 du\right)dv \\ &= \frac{1}{5}\int_0^1 \left[v\left(\left(1+v^2\right)^5 - v^{10}\right)\right]dv \\ &= \int_0^1 \left(v^9 + 2v^7 + 2v^5 + v^3 + \frac{1}{5}v\right) dv\end{aligned}$$

$$= \left(\frac{1}{10}v^{10} + \frac{1}{4}v^8 + \frac{1}{3}v^6 + \frac{1}{4}v^4 + \frac{1}{10}v^2\right)_0^1$$

$$= \left[\frac{1^{10}}{10} + \frac{1^8}{4} + \frac{1^6}{3} + \frac{1^4}{4} + \frac{1^2}{10}\right] - \left[\frac{0^{10}}{10} + \frac{0^8}{4} + \frac{0^6}{3} + \frac{0^4}{4} + \frac{0^2}{10}\right]$$

$$= \frac{31}{30} - 0$$

$$= \frac{31}{30}$$

Hence, $\boxed{\int_0^1\int_0^1 v\left(u + v^2\right)^4 du\,dv = \frac{31}{30}}$.

Chapter 15 Multiple Integrals 15.2 12E

Consider the following iterate integral:

$$\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dydx$$

$$\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dydx = \int_0^1 x\left(\int_0^1 y\sqrt{x^2+y^2}\,dy\right)dx$$

In the integral $\int_0^1 y\sqrt{x^2+y^2}\,dy$, the varibale x is constant.

Let,

$$\sqrt{x^2+y^2} = u$$

$$\frac{2ydy}{2\sqrt{x^2+y^2}} = du$$

$$\frac{ydy}{\sqrt{x^2+y^2}} = du$$

$$ydy = u\,du$$

Find the limits of integration:

$$y=0 \Rightarrow u=\sqrt{x^2+0^2} \Rightarrow u=\sqrt{x^2} \Rightarrow u=x$$

$$y=1 \Rightarrow u=\sqrt{x^2+y^2} \Rightarrow u=\sqrt{x^2+1}$$

Substitute values of u in the integral to obtain that,

$$\int_0^1 y\sqrt{x^2+y^2}\,dy = \int_0^1 \sqrt{x^2+y^2}\,(y\,dy)$$

$$= \int_x^{\sqrt{x^2+1}} u\,(u\,du)$$

$$= \int_x^{\sqrt{x^2+1}} u^2 du$$

$$= \left[\frac{u^3}{3}\right]_x^{\sqrt{x^2+1}}$$

$$= \frac{1}{3}\left[\left(\sqrt{x^2+1}\right)^3 - x^3\right]$$

$$\int_0^1 y\sqrt{x^2+y^2}\,dy = \frac{1}{3}\left[\left(x^2+1\right)^{3/2} - x^3\right]$$

The iterated integral is,

$$\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dydx = \int_0^1 x\left(\frac{1}{3}\left[\left(x^2+1\right)^{3/2} - x^3\right]\right)dx$$

$$= \frac{1}{3}\int_0^1 x\left(x^2+1\right)^{3/2} dx - \frac{1}{3}\int_0^1 x^4 dx$$

$$= \frac{1}{3}\int_0^1 x\left(x^2+1\right)^{3/2} dx - \frac{1}{3}\left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{3}\int_0^1 x\left(x^2+1\right)^{3/2} dx - \frac{1}{3}\cdot\frac{1}{5}[1-0]$$

$$= \frac{1}{3}\int_0^1 x\left(x^2+1\right)^{3/2} dx - \frac{1}{15} \qquad \cdots\cdots(1)$$

Use the following subsitution to find the integral $\frac{1}{3}\int_0^1 x\left(x^2+1\right)^{3/2}dx$.

$$u = x^2+1$$
$$du = 2x\,dx$$
$$\frac{du}{2} = x\,dx$$

Find the limits of integration:

If $x=0$ then $u=1$.

If $x=1$ then $u=2$.

Substitute these values in the above integral to get,

$$\begin{aligned}\frac{1}{3}\int_0^1 x\left(x^2+1\right)^{3/2}dx &= \frac{1}{3}\int_0^1 \left(x^2+1\right)^{3/2}\left(x\,dx\right)\\ &= \frac{1}{3}\int_1^2 u^{3/2}\left(\frac{du}{2}\right)\\ &= \frac{1}{6}\int_1^2 u^{3/2}du\end{aligned}$$

$$\begin{aligned}&= \frac{1}{6}\left(\frac{2}{5}u^{5/2}\right)_1^2\\ &= \frac{1}{6}\cdot\frac{2}{5}\left(2^{5/2}-1\right)\\ &= \frac{1}{15}\left(4\sqrt{2}-1\right) \qquad \cdots\cdots(2)\end{aligned}$$

Use equation (2) in (1) to obtain that,

$$\begin{aligned}\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dydx &= \frac{1}{3}\int_0^1 x\left(x^2+1\right)^{3/2}dx-\frac{1}{15}\\ &= \frac{1}{15}\left[4\sqrt{2}-1\right]-\frac{1}{15}\\ &= \frac{1}{15}\left[4\sqrt{2}-2\right]\\ &= \frac{2}{15}\left[2\sqrt{2}-1\right]\end{aligned}$$

Thus, the value of the iterated integral is,

$$\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dydx = \boxed{\frac{2}{15}\left[2\sqrt{2}-1\right]}.$$

Chapter 15 Multiple Integrals 15.2 13E

Consider the following integral:

$$\int_0^2 \int_0^\pi r\sin^2\theta \, d\theta \, dr$$

Integrate with respect to θ.

$$\begin{aligned}\int_0^2 \int_0^\pi r\sin^2\theta \, d\theta \, dr &= \int_0^2 \int_0^\pi r\frac{1}{2}(1-\cos 2\theta)\, d\theta \, dr \\ &= \frac{1}{2}\int_0^2 \int_0^\pi r \, d\theta \, dr - \frac{1}{2}\int_0^2 \int_0^\pi r\cos 2\theta d\theta \, dr \\ &= \frac{1}{2}\int_0^2 r[\theta]_0^\pi \, dr - \frac{1}{2}\int_0^2 r\left[\frac{\sin 2\theta}{2}\right]_0^\pi dr \\ &= \frac{1}{2}\int_0^2 r[\pi-0]\, dr - \frac{1}{2}\int_0^2 r\left[\frac{\sin 2\pi}{2} - \frac{\sin 0}{2}\right] dr\end{aligned}$$

$$= \frac{1}{2}\int_0^2 r[\pi]\, dr - \frac{1}{2}\int_0^2 r[0-0]\, dr$$

$$= \frac{1}{2}\pi\left[\frac{r^2}{2}\right]_0^2$$

$$= \frac{1}{2}\pi\left[\frac{4}{2}\right]$$

$$= \pi$$

Therefore, $\int_0^2 \int_0^\pi r\sin^2\theta \, d\theta \, dr = \boxed{\pi}$.

Chapter 15 Multiple Integrals 15.2 14E

$$\int_0^1 \int_0^1 \sqrt{s+t}\, \mathrm{d}s\, \mathrm{d}t = \int_0^1 \frac{2}{3}(s+t)^{\frac{3}{2}}\, \mathrm{d}t$$

$$= \frac{2}{3}\int_0^1 (1+t)^{\frac{3}{2}} - t^{\frac{3}{2}}\, \mathrm{d}t$$

$$= \frac{2}{3}\left\{\left[\frac{(1+t)^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1 - \frac{t^{\frac{5}{2}}}{\frac{5}{2}}\right\}$$

$$= \frac{4}{15}\left[2^{\frac{5}{2}} - 1^{\frac{5}{2}} - \left(1^{\frac{5}{2}} - 0\right)\right]$$

$$= \frac{4}{15}\left[\sqrt{32} - 1 - 1\right]$$

$$= \frac{4}{15}\left[4\sqrt{2} - 2\right]$$

Chapter 15 Multiple Integrals 15.2 15E

Consider the double integral

$$\iint_R \sin(x-y)\, dA, \quad R = \left\{(x,y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{2}\right\}$$

Note that, if f is continuous on the rectangle $R = \left\{(x, y) \mid a \le x \le b, c \le y \le d\right\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Therefore,

$$\iint_R \sin(x-y)\,dA = \int_0^{\pi/2}\int_0^{\pi/2} \sin(x-y)\,dy\,dx$$

$$= \int_{x=0}^{\pi/2}\int_{u=x}^{x-\frac{\pi}{2}} \sin u\,(-du)\,dx$$

$$= \int_{x=0}^{\pi/2}\left(\int_{u=x-\frac{\pi}{2}}^{x} \sin u\,du\right)dx$$

$$= \int_{x=0}^{\pi/2} (-\cos u)_{x-\frac{\pi}{2}}^{x}\,dx$$

$$= \int_{x=0}^{\pi/2}\left(-\cos x+\cos\left(x-\frac{\pi}{2}\right)\right)dx$$

$$= \int_{x=0}^{\pi/2} (-\cos x+\cos x)\,dx$$

$$= \int_{x=0}^{\pi/2} (0)\,dx$$

$$= \boxed{0}$$

Chapter 15 Multiple Integrals 15.2 16E

Consider the integral:

$$\iint_R \left(y + xy^{-2}\right)dA\,,$$

The region is $R=\{(x,y):\ 0\le x\le 2,\ 1\le y\le 2\}$

Here $0\le x\le 2$ which means that $x=0$ and $x=2$ are the lower and upper limits of integration of *x* respectively, and $1\le y\le 2$, which means that $y=1$ and $y=2$ are the lower and upper limits of integration of *y* respectively.

Now, the integral can be evaluated as follows:

$$\iint_R \left(y + xy^{-2}\right) dA = \int_0^2 \int_1^2 \left(y + xy^{-2}\right) dy\, dx$$

$$= \int_0^2 \left[\frac{y^2}{2} - \frac{x}{y}\right]_1^2 dx$$

$$= \int_0^2 \left[\frac{(2)^2}{2} - \frac{x}{2} - \left(\frac{(1)^2}{2} - \frac{x}{1}\right)\right] dx$$

$$= \int_0^2 \left(\frac{3}{2} + \frac{x}{2}\right) dx$$

$$= \left[\frac{3x}{2} + \frac{x^2}{4}\right]_0^2$$

$$= \frac{3(2)}{2} + \frac{(2)^2}{4} - 0$$

$$= 3 + 1$$

$$= 4$$

Therefore, the value of the integral is $\boxed{\int_0^2 \int_1^2 \left(y + xy^{-2}\right) dy\, dx = 4}$.

Chapter 15 Multiple Integrals 15.2 17E

$$\iint_R \frac{xy^2}{x^2+1} dA,\ R = \{(x, y) : 0 \le x \le 1,\ -3 \le y \le 3\}$$

$$= \int_0^1 \int_{-3}^{3} \frac{xy^2}{x^2+1}\, dy\, dx$$

$$= \frac{1}{3}\int_0^1 \left[\frac{xy^3}{x^2+1}\right]_{y=-3}^{y=3} dx$$

$$= \frac{1}{3}\int_0^1 \frac{x}{x^2+1}\left[27-(-27)\right] dx$$

i.e. $\iint_R \frac{xy^2}{x^2+1}\, dA = \frac{1}{3}\int_0^1 \frac{x}{x^2+1}(54)\, dx$

$$= 18\int_0^1 \frac{x\,dx}{x^2+1}$$

$$= 18 \times \frac{1}{2}\left[\ln\left|x^2+1\right|\right]_0^1$$

$$= 9\left[\ln(2) - \ln(1)\right]$$

$$= \boxed{9\ln 2}$$

(As $\ln 1 = 0$)

Chapter 15 Multiple Integrals 15.2 18E

$\iint_R \frac{1+x^2}{1+y^2}\, dA$, $R = \{(x, y):\ 0 \le x \le 1,\ \ 0 \le y \le 1\}$

$$\int_0^1\int_0^1 \frac{1+x^2}{1+y^2}\,dy\,dx$$

$$=\int_0^1 (1+x^2)\left[\tan^{-1} y\right]_0^1 dx$$

$$=\int_0^1 (1+x^2)\left[\tan^{-1} 1-\tan^{-1} 0\right] dx$$

$$=\int_0^1 (1+x^2)\left[\frac{\pi}{4}-0\right] dx$$

i.e. $\iint_R \frac{1+x^2}{1+y^2}\,dA=\frac{\pi}{4}\int_0^1 (1+x^2)\,dx$

$$=\frac{\pi}{4}\left[x+\frac{x^3}{3}\right]_0^1$$

$$=\frac{\pi}{4}\left[1+\frac{1}{3}-0\right]$$

$$=\frac{\pi}{4}\left(\frac{4}{3}\right)$$

$$=\boxed{\frac{\pi}{3}}$$

Chapter 15 Multiple Integrals 15.2 19E

Consider the following double integral:

$$\iint_R x\sin(x+y)\,dA,\quad R=\left[0,\frac{\pi}{6}\right]\times\left[0,\frac{\pi}{3}\right]$$

So the double integral becomes,

$$\iint_R x\sin(x+y)\,dA=\int_0^{\frac{\pi}{6}}\int_0^{\frac{\pi}{3}} x\sin(x+y)\,dy\,dx$$ Since $dA=dxdy=dydx$

First integrate with respect to y.

$$\iint_R x\sin(x+y)\,dA=\int_0^{\frac{\pi}{6}}\left[-x\cos(x+y)\right]_{y=0}^{y=\frac{\pi}{3}} dx$$

$$=-\int_0^{\frac{\pi}{6}}\left[x\cos\left(x+\frac{\pi}{3}\right)-x\cos(x+0)\right]dx$$

$$=-\int_0^{\frac{\pi}{6}} x\cos\left(x+\frac{\pi}{3}\right)dx+\int_0^{\frac{\pi}{6}} x\cos x\,dx$$

Apply integrating by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Consider the integral $\int x\cos\left(x+\frac{\pi}{3}\right)dx$.

Use integrating by parts:

$$u = x \quad dv = \cos\left(x+\frac{\pi}{3}\right)$$

Then $du = dx \quad v = \sin\left(x+\frac{\pi}{3}\right)$

$$\int x\cos\left(x+\frac{\pi}{3}\right)dx = x\sin\left(x+\frac{\pi}{3}\right) - \int \sin\left(x+\frac{\pi}{3}\right)dx$$

$$= x\sin\left(x+\frac{\pi}{3}\right) + \cos\left(x+\frac{\pi}{3}\right) + C$$

Now consider the integral $\int x\cos x\,dx$.

Use integrating by parts:

$$u = x \quad dv = \cos x$$

Then $du = dx \quad v = \sin x$

Then integral becomes,

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

Therefore, the integral is as follows:

$$\iint_R x\sin(x+y)\,dA = -\left[x\sin\left(x+\frac{\pi}{3}\right)+\cos\left(x+\frac{\pi}{3}\right)\right]_0^{\frac{\pi}{6}} + \left[x\sin x+\cos x\right]_0^{\frac{\pi}{6}}$$

$$= -\left[\frac{\pi}{6}\sin\left(\frac{\pi}{6}+\frac{\pi}{3}\right)+\cos\left(\frac{\pi}{6}+\frac{\pi}{3}\right)-0-\cos\left(\frac{\pi}{3}\right)\right]$$
$$+\left[\frac{\pi}{6}\sin\left(\frac{\pi}{6}\right)+\cos\left(\frac{\pi}{6}\right)-0-\cos(0)\right]$$

$$= -\left[\frac{\pi}{6}\sin\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{\pi}{3}\right)\right]+\left[\frac{\pi}{6}\sin\left(\frac{\pi}{6}\right)+\cos\left(\frac{\pi}{6}\right)-1\right]$$

$$= -\left(\frac{\pi}{6}(1)+0-\frac{1}{2}\right)+\left(\frac{\pi}{6}\left(\frac{1}{2}\right)+\frac{\sqrt{3}}{2}-1\right)$$

$$= -\frac{\pi}{6}+\frac{1}{2}+\frac{\pi}{6}\times\frac{1}{2}+\frac{\sqrt{3}}{2}-1$$

$$= -\frac{\pi}{12}+\frac{\sqrt{3}}{2}-\frac{1}{2}$$

$$= \frac{1}{2}\left(\sqrt{3}-1\right)-\frac{\pi}{12}$$

Thus, the integral of $\iint_R x\sin(x+y)\,dA$ is $\boxed{\frac{1}{2}\left(\sqrt{3}-1\right)-\frac{\pi}{12}}$.

Chapter 15 Multiple Integrals 15.2 20E

Consider the following integral:

$$\iint_R \frac{x}{1+xy}\,dA,\quad R=[0,1]\times[0,1]$$

The objective is to evaluate the double integral for the region R.

Evaluate the integral is as follows:

$$\iint_R \frac{x}{1+xy}\,dA=\int_0^1\int_0^1 \frac{x}{1+xy}\,dy\ dx$$

$$=\int_0^1 \left[\ln(1+xy)\right]_{y=0}^{y=1} dx \qquad \frac{d}{dy}(1+xy)=x \quad \int\frac{f'}{f}=\ln f$$

$$=\int_0^1 \left[\ln(1+x)-\ln(1+0)\right]dx$$

$$=\int_0^1 \ln(1+x)\,dx$$

$$=x\ln(1+x)\Big|_0^1-\int_0^1 \frac{1}{1+x}.x\ dx \qquad \text{integration by parts}$$

Continuation on above:

$$\iint_R \frac{x}{1+xy}\,dA = x\ln(1+x)\Big|_0^1-\int_0^1\frac{1+x-1}{1+x}dx$$

$$=\ln 2-\left(\int_0^1 dx-\int_0^1\frac{1}{1+x}dx\right)$$

$$=\ln 2-\left[x-\ln(1+x)\right]_0^1$$

$$=\ln(2)-0-\left[1-\ln 2-0+\ln 1\right]$$

$$=\ln 2-1+\ln 2 \text{ (Because ln1 = 0)}$$

$$=\boxed{2\ln 2-1}$$

Therefore, the value of the integral is $\boxed{2\ln 2-1}$.

Chapter 15 Multiple Integrals 15.2 21E

Consider the double integral,

$$\iint_R ye^{-xy}\,dA,$$

Here R is the rectangle given by $R=[0,2]\times[0,3]$

The objective is to determine the value of the double integral over the rectangle $R=[0,2]\times[0,3]$.

Note that, if f is continuous on the rectangle $R = \{(x, y) \mid a \le x \le b, c \le y \le d\}$, then

$$\iint_R f(x, y)\,dA = \int_a^b\int_c^d f(x, y)\,dy\,dx = \int_c^d\int_a^b f(x, y)\,dx\,dy.$$

From the given intervals, observe that x varies from 0 to 2, and y varies from 0 to 3.

$$\iint_R \left(ye^{-xy}\right)dA = \int_0^3\int_0^2 ye^{-xy}\,dx\,dy$$

First, integrate the function $f(x,y)=ye^{-xy}$ with respect to x from 0 to 2 and holding y as a constant.

$$\begin{aligned}\iint_R ye^{-xy}\,dA &= \int_0^3\int_0^2 ye^{-xy}\,dx\,dy\\ &= \int_0^3\left[-e^{-xy}\right]_0^2 dy\\ &= \int_0^3\left(-e^{-2y}+1\right)dy\end{aligned}$$

Now, integrate the function $f(y)=1-e^{-2y}$ with respect to y from 0 to 3.

$$\begin{aligned}\int_0^3\left(-e^{-2y}+1\right)dy &= \left[\frac{1}{2}e^{-2y}+y\right]_0^3\\ &= \left(\frac{1}{2}e^{-6}+3\right)-\left(\frac{1}{2}\right)\\ &= \frac{1}{2}e^{-6}+\frac{5}{2}\end{aligned}$$

Therefore, the value of the double integral $\iint_R ye^{-xy}\,dA$ over the rectangle $R=[0,2]\times[0,3]$ is

$$\boxed{\int_0^3\int_0^2 ye^{-xy}\,dx\,dy=\frac{1}{2}e^{-6}+\frac{5}{2}}.$$

Chapter 15 Multiple Integrals 15.2 22E

Consider the double integral,

$$\iint_R \frac{1}{1+x+y}\,dA, R=[1,3]\times[1,2].$$

The object is to calculate the double integral.

Recall that, if f is continuous on the rectangle $R=\{(x,y)\mid a\le x\le b, c\le y\le d\}$, then the double integral can be calculated as follows:

$$\iint_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dydx=\int_c^d\int_a^b f(x,y)\,dxdy.$$

Observe the rectangle, it confirms that $a=1, b=3, c=1,$ and $d=2$.

So the integral can be written as follows:

$$\iint_R \frac{1}{1+x+y}\,dA=\int_1^2\int_1^3 \frac{1}{1+x+y}\,dxdy$$

Use substitution method: Let $1+x+y=u$.

Lower limit: If $x=1$ then $u=2+y$

Upper limit: If $x=3$ then $u=4+y$.

Then,

$$dx=du.$$

So the integral can be written as follows:

$$\iint_R \frac{1}{1+x+y}\,dA=\int_1^2\int_1^3 \frac{1}{1+x+y}\,dxdy$$

$$=\int_1^2\int_{u=2+y}^{u=4+y}\frac{1}{u}\,dudy$$

$$=\int_1^2\left[\ln(u)\right]_{u=2+y}^{u=4+y}dy$$

$$=\int_1^2\left[\ln(4+y)-\ln(2+y)\right]dy$$

$$=\int_1^2 \ln(4+y)\,dy-\int_1^2\ln(2+y)\,dy \quad \text{...... (1)}$$

Recall that, $\int \ln(x)\,dx = x\ln x - x + C$.

Assume $4+y=t_1$, then $dy = dt_1$.

Limits for the first integral: If $y=1$ then $t_1=5$.

If $y=2$ then, $t_1=6$.

For the second integral assume $2+y=t_2$, then $dy=dt_2$.

Limits for the first integral: If $y=1$ then $t_2=3$.

If $y=2$ then, $t_2=4$.

So the integral (1) can be calculated as follows:

$$\iint_R \frac{1}{1+x+y}\,dA = \int_1^2\int_1^3 \frac{1}{1+x+y}\,dxdy$$

$$= \int_1^2 \ln(4+y)\,dy - \int_1^2 \ln(2+y)\,dy$$

$$= \int_5^6 \ln(t_1)\,dt_1 - \int_3^4 \ln(t_2)\,dt_2$$

$$= \left[t_1\ln(t_1)-t_1\right]_5^6 - \left[t_2\ln(t_2)-t_2\right]_3^4$$

$$= \left[(6\ln(6)-6)-(5\ln(5)-5)\right] - \left[(4\ln(4)-4)-(3\ln(3)-3)\right]$$

$$= 6\ln(6)-5\ln(5)-1-4\ln(4)+4+3\ln(3)-3$$

$$= 6\ln(6)-5\ln(5)-4\ln(4)+3\ln(3)$$

$$= 6\ln(2\times 3)-5\ln(5)-4\ln(2^2)+3\ln(3)$$

$$= 6(\ln(2)+\ln(3))-5\ln(5)-8\ln(2)+3\ln(3)$$

$$= 9\ln(3)-2\ln(2)-5\ln(5)$$

Therefore, the double integral value is,

$$\int_1^2\int_1^3 \frac{1}{1+x+y}\,dxdy = \boxed{9\ln(3)-2\ln(2)-5\ln(5)}.$$

Chapter 15 Multiple Integrals 15.2 23E

Consider the following iterated integral.

$$\int_0^1\int_0^1 (4-x-2y)\,dx\,dy$$

The objective is to sketch the volume of the solid.

The iterated integral says that the solid is bounded by the plane and the rectangle $R=[0,1]\times[0,1]$.

The solid lies under the plane $z=4-x-2y$ and above the rectangle $R=[0,1]\times[0,1]$

First integrate with respect to x.

So, x lies between 0 and 1

$$0\le x\le 1$$

Therefore, $x=0$ that is $yz-$ plane, $x=1$ plane.

Now, integrate with respect to y

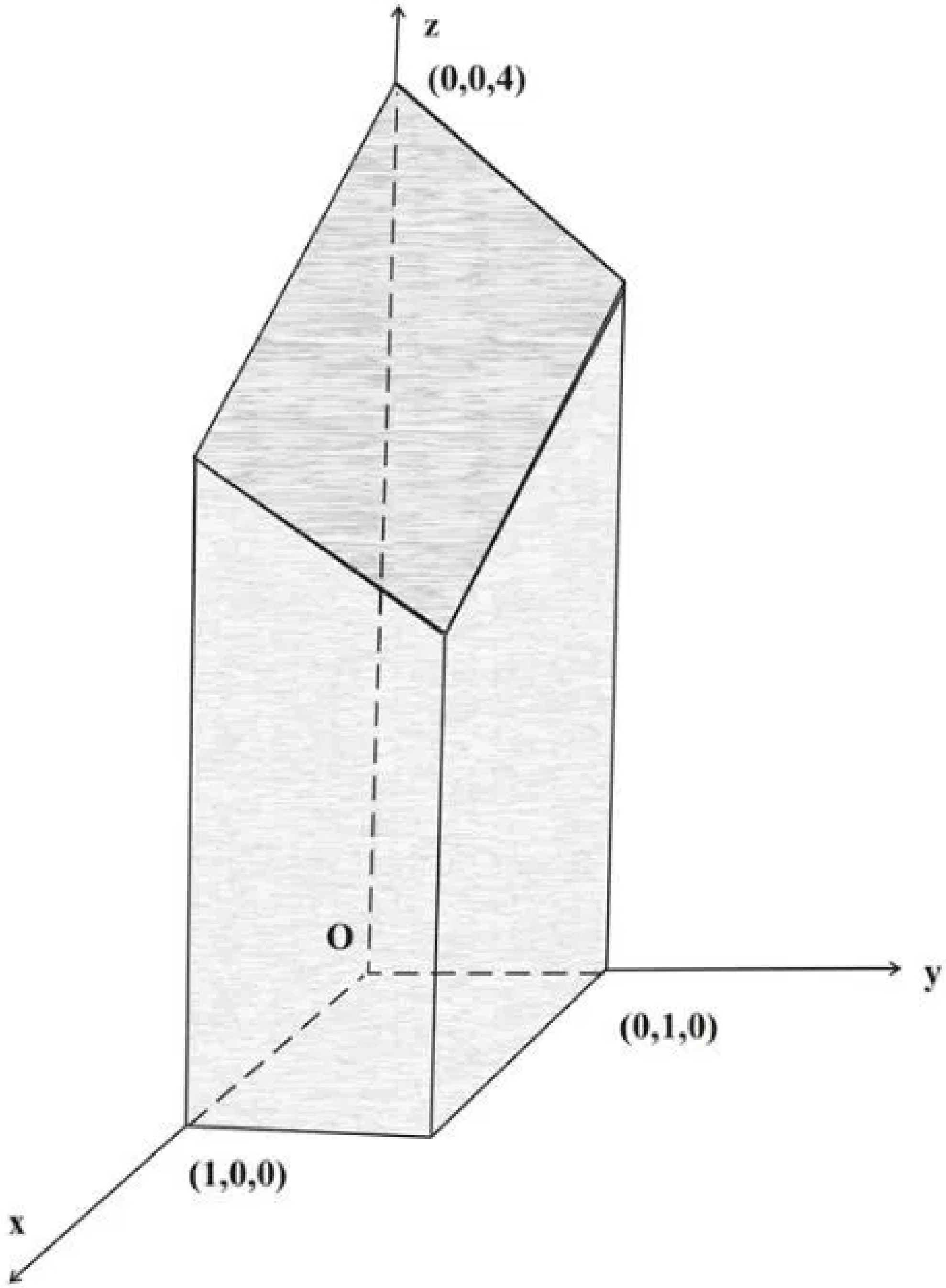
Here y lies between 0 and 1

Therefore, $y=0$ that is $xz-$ plane, and $y=1$ plane are obtain

Then $z=0$ that is $xy-$ plane and $z=4-x-2y$ planes are obtain

Therefore, there are six planes obtain

The graph of iterated integral $\int_0^1\int_0^1 (4-x-2y)\,dx\,dy$ is



The Volume of the iterated integral is

$$V = \int_0^1\int_0^1 (4-x-2y)\,dx\,dy$$

$$= \int_0^1 \left[4x - \frac{x^2}{2} - 2xy\right]_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(4 - \frac{1}{2} - 2y\right) dy$$

$$= \int_0^1 \left(\frac{7}{2} - 2y\right) dy$$

$$= \left[\frac{7}{2}y - y^2\right]_{y=0}^{y=1}$$

$$= \frac{7}{2} - 1$$

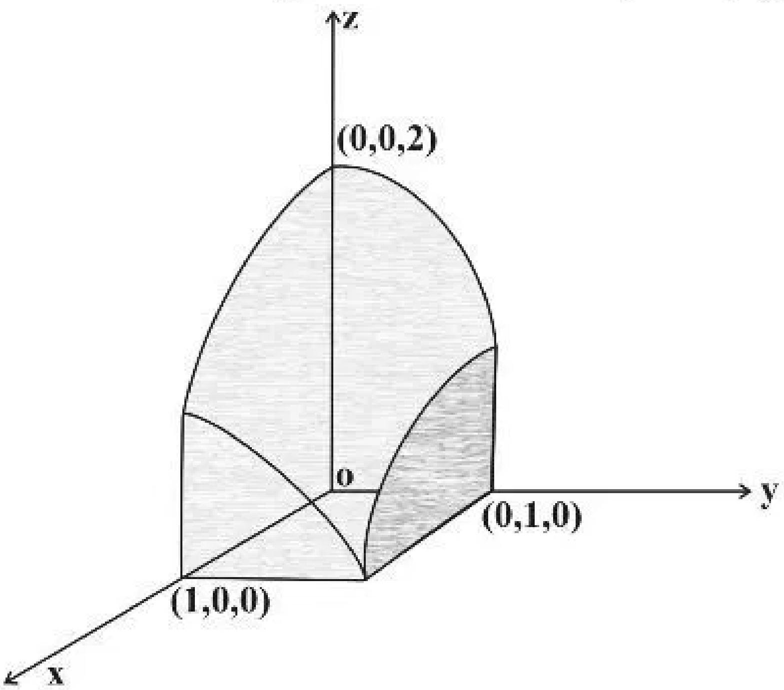
$$= \frac{5}{2}$$

Therefore the volume is $V = \boxed{\frac{5}{2}}$.

Chapter 15 Multiple Integrals 15.2 24E

$$\int_0^1\int_0^1 \left(2 - x^2 - y^2\right) dy\;dx$$

The solid is the region in the first octant which lies below the circular parabolic $z = 2 - x^2 - y^2$ and above the rectangle $R = [0,1]\times[0,1]$



Chapter 15 Multiple Integrals 15.2 25E

Given $4x+6y-2z+15=0$

The volume of the region bounded above by a surface $z=f(x,y)$ and below by a rectangle $[a\times b]$ and $[c\times d]$ is given by $\int_c^d\int_a^b f(x,y)\,dx\,dy$.

From the equation of the plane, we get

$$4x+6y-2z+15=0$$

$$\Rightarrow\ 4x+6y+15=2z$$

$$\Rightarrow z=\frac{4x+6y+15}{2}$$

$$=2x+3y+\frac{15}{2}$$

Now $f(x,y)$ is $2x+3y+\frac{15}{2}$.

We have the limits for x as $(-1, 2)$ and the limits for y as $(-1, 1)$.

Now, evaluating the integral we have

$$\int_{-1}^{1}\int_{-1}^{2}\left(2x+3y+\frac{15}{2}\right)dx\,dy=\int_{-1}^{1}\left(2\frac{x^2}{2}+3xy+\frac{15x}{2}\right)_{-1}^{2}dy$$

$$=\int_{-1}^{1}\left(x^2+3xy+\frac{15x}{2}\right)_{-1}^{2}dy$$

$$=\int_{-1}^{1}\left(\left(2^2+3(2)y+\frac{15(2)}{2}\right)-\left((-1)^2+3(-1)y+\frac{15}{2}(-1)\right)\right)dy$$

$$=\int_{-1}^{1}\left[(4+6y+15)-\left(1-3y-\frac{15}{2}\right)\right]dy$$

$$=\int_{-1}^{1}\left[\left(18+\frac{15}{2}+6y+3y\right)\right]dy$$

$$=\int_{-1}^{1}\left(\frac{51}{2}+9y\right)dy$$

$$=\left(\frac{51}{2}y+\frac{9y^2}{2}\right)_{-1}^{1}$$

$$=\left(\frac{51}{2}+\frac{9}{2}\right)-\left(\frac{-51}{2}+\frac{9}{2}\right)$$

$$=\frac{51+51}{2}$$

$$=\frac{102}{2}$$

$$=51$$

Therefore, the volume of the solid is $\boxed{51}$ cubic units.

Given $z = 3y^2 - x^2 + 2$ and $R = [-1,1] \times [1,2]$

The volume of the region bounded above by a surface $z = f(x, y)$ and below by a rectangle $[a \times b]$ and $[c \times d]$ is given by $\int_c^d \int_a^b f(x, y)\, dx dy$.

From the equation of the plane, we get $f(x, y)$ as $3y^2 - x^2 + 2$. We have the limits for x as $(-1, 1)$ and the limits for y as $(1, 2)$.

Now, evaluating the integral we have

$$\int_1^2 \int_{-1}^1 (3y^2 - x^2 + 2)\, dx dy = \int_1^2 \left(3y^2 x - \frac{x^3}{3} + 2x\right)_{-1}^1 dy$$

$$= \int_1^2 \left(3y^2 - \frac{1^3}{3} + 2 - \left(-3y^2 - \frac{(-1)^3}{3} + 2(-1)\right)\right) dy$$

$$= \int_1^2 \left(3y^2 - \frac{1}{3} + 2 + 3y^2 - \frac{1}{3} + 2\right) dy$$

$$= \int_1^2 \left(6y^2 - \frac{2}{3} + 4\right) dy$$

$$= \int_1^2 \left(6y^2 + \frac{10}{3}\right) dy$$

$$= \left(6\frac{y^3}{3} + \frac{10}{3}y\right)_1^2$$

$$= \left(2y^3 + \frac{10}{3}y\right)_1^2$$

$$= \left(\left(2(8) + \frac{10}{3}(2)\right) - \left(2(1) + \frac{10}{3}(1)\right)\right)$$

$$= \left(16 + \frac{20}{3} - 2 - \frac{10}{3}\right)$$

$$= \left(14 + \frac{10}{3}\right)$$

$$= \frac{52}{3}$$

Therefore, the volume of the solid is $\boxed{\frac{52}{3}}$ cubic units.

Chapter 15 Multiple Integrals 15.2 27E

The solid lies under the surface $z = 1 - \frac{x^2}{4} - \frac{y^2}{9}$

And above the rectangle $R = [-1, 1] \times [-2, 2]$

Then the volume of the solid is

$$v = \iint_R \left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right) dA$$

$$= \int_{-1}^{1}\int_{-2}^{2} \left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right) dy\,dx$$

$$= \int_{-1}^{1} \left[y - \frac{x^2 y}{4} - \frac{y^3}{27}\right]_{y=-2}^{y=2} dx$$

$$= \int_{-1}^{1} \left[2 - \frac{x^2}{2} - \frac{8}{27} + 2 - \frac{x^2}{2} - \frac{8}{27}\right] dx$$

$$= 2\int_{-1}^{1} \left(\frac{46}{27} - \frac{x^2}{2}\right) dx$$

i.e. v $= 2\left[\frac{46}{27}x - \frac{x^3}{6}\right]_{-1}^{1}$

$$= 2\left[\frac{46}{27} - \frac{1}{6} + \frac{46}{27} - \frac{1}{6}\right]$$

$$= 4\left[\frac{46}{27} - \frac{1}{6}\right]$$

$$= 4 \times \frac{83}{54}$$

$$= \boxed{\frac{166}{27}}$$

Chapter 15 Multiple Integrals 15.2 28E

The solid lies under the surface $z = 1 + e^x \sin y$

And above the rectangle $R = [-1,1] \times [0,\pi]$

Then the volume of the solid is

$$v = \iint_R \left(1 + e^x \sin y\right) dA$$

$$= \int_{-1}^{1} \int_{0}^{\pi} \left(1 + e^x \sin y\right) dy\, dx$$

$$= \int_{-1}^{1} \left[y - e^x \cos y\right]_{y=0}^{y=\pi} dx$$

$$= \int_{-1}^{1} \left[\pi - e^x \cos \pi - 0 + e^x \cos 0\right] dx$$

$$= \int_{-1}^{1} \left[\pi + e^x + e^x\right] dx$$

i.e. $v = \int_{-1}^{1} \left[\pi + 2e^x\right] dx$

$$= \left[\pi x + 2e^x\right]_{-1}^{1}$$

$$= \pi + 2e + \pi - 2e^{-1}$$

$$= 2\pi + 2\left(e - e^{-1}\right)$$

$$= \boxed{2\left(\pi + e - e^{-1}\right)}$$

Chapter 15 Multiple Integrals 15.2 29E

Volume of the solid is given by $\int_0^{\frac{\pi}{4}} \int_0^2 x \sec^2 y \, dx \, dy$

$$= \int_0^{\frac{\pi}{4}} \sec^2 y \, dy . \int_0^2 x \, dx$$

$$= [\tan y]_0^{\frac{\pi}{4}} . \left[\frac{x^2}{2}\right]_0^2$$

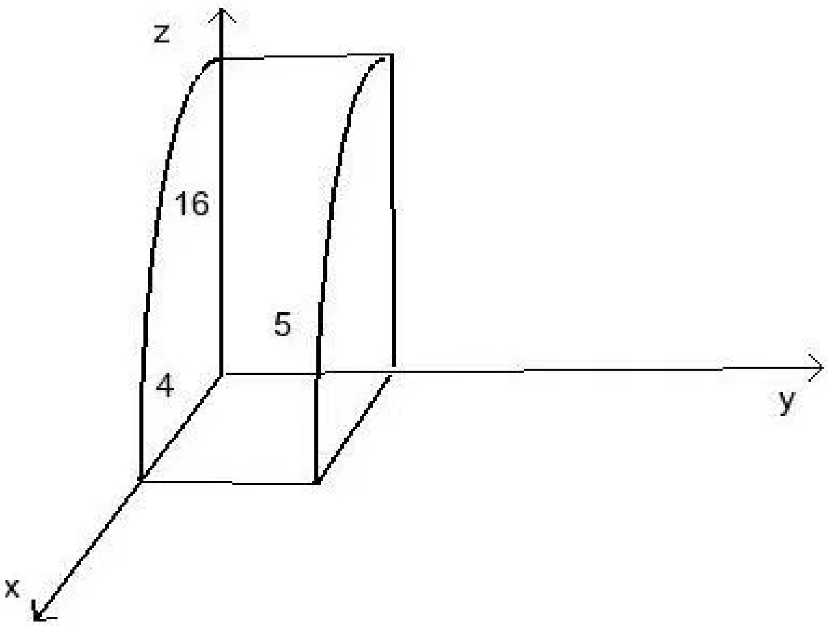
$$= [1 - 0] . \frac{4}{2}$$

$$= 2$$

Chapter 15 Multiple Integrals 15.2 30E

Find the volume of the solid in the first octant bounded by the cylinder

$z = 16 - x^2$ and the plane y = 5



R = [0,4] X [0,5]

$$\int_0^4 \int_0^5 \left(16 - x^2\right) \, dy \, dx$$

$$= \int_0^4 \left[\left(16y - x^2 y\right)\right]_0^5 \, dx$$

$$= \int_0^4 \left(80 - 5x^2\right) dx$$

$$= 80x - \frac{5}{3}x^3 \Big|_0^4$$

$$= \left(320 - \frac{320}{3}\right) - 0 = \frac{640}{3}$$

Chapter 15 Multiple Integrals 15.2 31E

Find the volume of the solid enclosed by paraboloid z=2+x2+(y-2)2 & plane z=1, x=1, z=-1, y=0, y=4

Thus the region is R= [-1,1] x [0,4] and the volume between z=2+x2+(y-2)2 and z=1

$$\iint_R \left[\left(2+x^2+(y-2)^2\right)-1\right]dA$$

$$\iint_R \left[1+x^2+(y-2)^2\right]dA = \int_{-1}^{1}\int_0^4 \left(1+x^2+(y-2)^2\right)dy\,dx$$

$$= \int_{-1}^{1}\left(y+x^2y+\frac{(y-2)^3}{3}\Big|_0^4\right)dx$$

$$= \int_{-1}^{1}\left[\left(4+4x^2+\frac{(4-2)^3}{3}\right)-\left(\frac{(-2)^2}{3}\right)\right]dx$$

$$= \int_{-1}^{1}\left(4+4x^2+\frac{16}{3}\right)dx$$

$$= \int_{-1}^{1}\left(4x^2+\frac{28}{3}\right)dx$$

$$= 2\int_0^1\left(4x^2+\frac{28}{3}\right)dx \quad \text{[symmetric about y-axis]}$$

$$= 2\cdot\left[\frac{4x^3}{3}+\frac{28}{3}x\Big]_0^1\right]$$

$$= 2\left[\frac{4}{3}+\frac{28}{3}\right]$$

$$= \frac{64}{3}$$

Chapter 15 Multiple Integrals 15.2 32E

Consider the surface $z = \dfrac{2xy}{x^2+1}$ and the plane $z = x + 2y$.

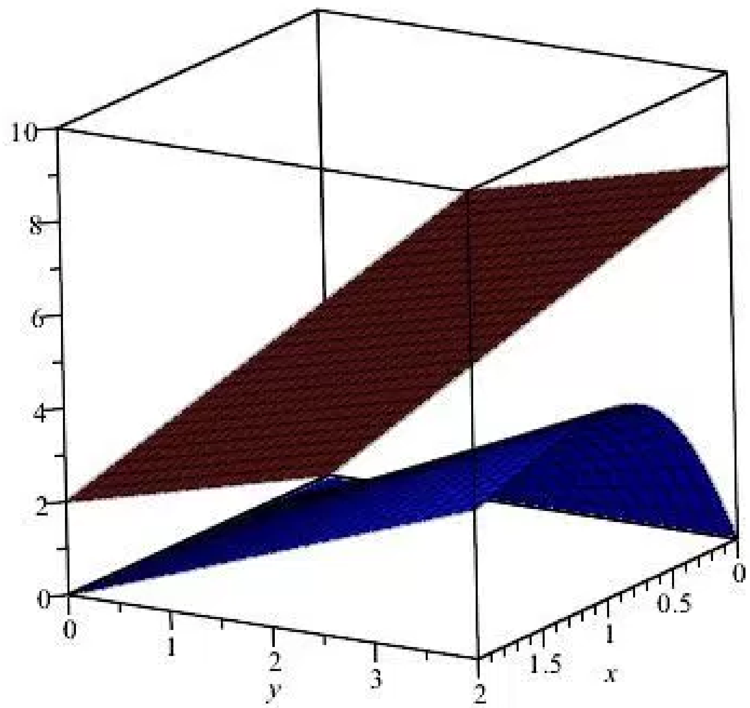
Use Maple to graph these surfaces bounded by the planes $x = 0$, $x = 2$, $y = 0$, and $y = 4$.

Keystrokes:

```
plot3d({2*x*y/(x^2+1), x+2*y}, x = 0 .. 2, y = 0 .. 4, color = [blue, orange]);
```

Maple result:

$$plot3d\left(\left\{\frac{2 \cdot x \cdot y}{x^2+1}, x + 2 \cdot y\right\}, x = 0 .. 2, y = 0 .. 4, color = [blue, orange]\right)$$



Let $g(x,y) = z = \frac{2xy}{x^2+1}$ and $h(x,y) = z = x + 2y.$

From the graph notice that $h(x,y) > g(x,y)$, so take

$$f(x,y) = h(x,y) - g(x,y)$$
$$= x + 2y - \frac{2xy}{x^2+1}$$

And $R = [0,2] \times [0,4].$

Volume of the solid lies between the given surfaces on the interval $0 \le x \le 2, 0 \le y \le 4$ is

$$V = \iint_R f(x,y)\,dA$$
$$= \int_0^2 \int_0^4 \left[x + 2y - \frac{2xy}{x^2+1} \right] dy\,dx$$

Use Maple to evaluate this integral.

Maple keystrokes:

```
evalf(int(int(x+2*y-2*x*y/(x^2+1), x = 0 .. 2), y = 0 .. 4));
```

Maple result:

$$> > \ evalf\left(int\left(int\left(\left(x + 2 \cdot y - \frac{2 \cdot x \cdot y}{x^2 + 1} \right), x = 0..2 \right), y = 0..4 \right) \right);$$

27.12449670

Hence the required volume of the solid is $\boxed{27.12449670}$.

Chapter 15 Multiple Integrals 15.2 33E

Consider the integral $\iint_R x^5 y^3 e^{xy}\,dA.$

Here use Maple to find the exact value of the given integral.

Keystrokes:

with(student);

f:=(x,y)->x^5*y^3*exp(y*x) ;

evalf(Doubleint(f(x, y), x = 0 .. 1, y = 0 .. 1));

Maple result:

> *with*(*student*);

[D, *Diff*, *Doubleint*, *Int*, *Limit*, *Lineint*, *Product*, *Sum*, *Tripleint*, *changevar*, *completesquare*, *distance*, *equate*, *integrand*, *intercept*, *intparts*, *leftbox*, *leftsum*, *makeproc*, *middlebox*, *middlesum*, *midpoint*, *powsubs*, *rightbox*, *rightsum*, *showtangent*, *simpson*, *slope*, *summand*, *trapezoid*]

> $f := (x, y) \rightarrow (x^5) * (y^3) * (e^{x*y});$

$f := (x, y) \rightarrow x^5\, y^3\, e^{y\,x}$

> $evalf(Doubleint(f(x, y), x = 0..1, y = 0..1));$

0.08391839764

Hence the required value of the given integral is $\boxed{0.08391839764}$.

(b)

Solid is shown in the below figure:

Use Maple to sketch the solid.

Keystrokes:

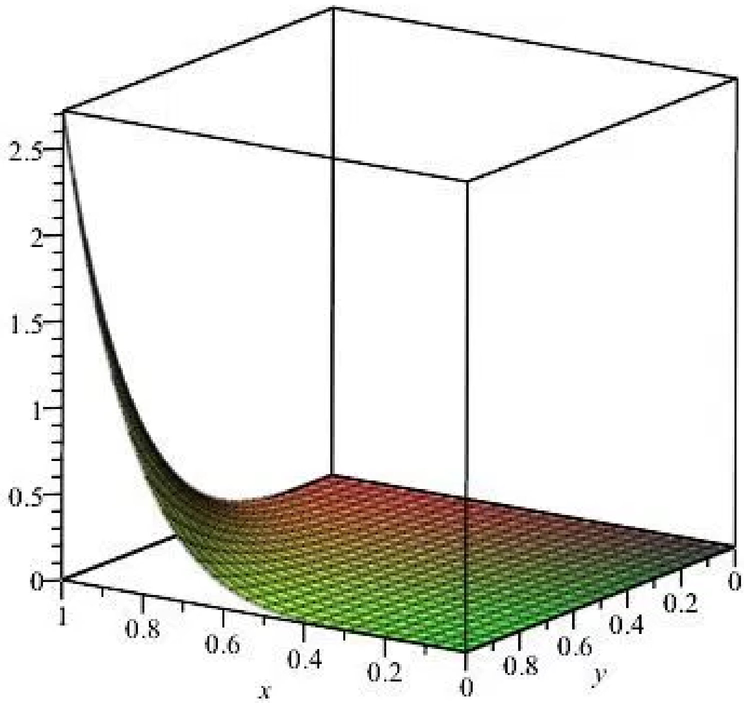
with(plots);

plot3d(x^5*y^3*exp(y*x), x = 0 .. 1, y = 0 .. 1);

Maple result:

with(*plots*) :

plot3d(x^5*y^3*exp(y*x), *x* = 0 ..1, *y* = 0 ..1);



Chapter 15 Multiple Integrals 15.2 34E

Consider the surfaces $z = e^{-x^2}\cos\left(x^2 + y^2\right)$ and $z = 2 - x^2 - y^2$.

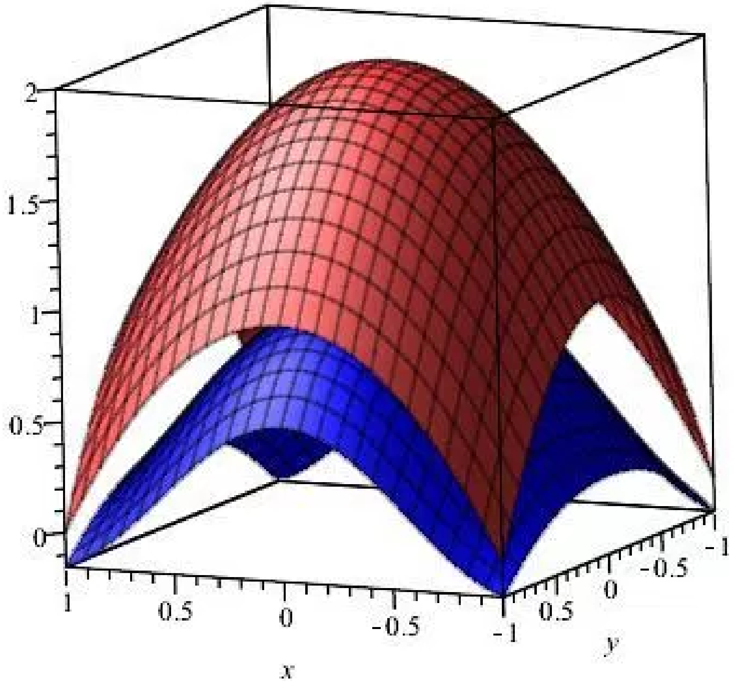
Use Maple to graph these surfaces on the interval $|x| \le 1, |y| \le 1$.

Keystrokes:

plot3d({exp(-x^2)*cos(x^2+y^2), -x^2-y^2+2}, x = -1 .. 1, y = -1 .. 1, color = [blue, orange])

Maple result:

$plot3d\left(\left\{e^{-x^2}\cdot\cos\left(x^2+y^2\right), 2-x^2-y^2\right\}, x=-1..1, y=-1..1, color=[blue, orange]\right);$



Let $g(x,y) = z = e^{-x^2}\cos(x^2+y^2)$ and $h(x,y) = z = 2 - x^2 - y^2$.

From the graph notice that $h(x,y) > g(x,y)$, so take

$$f(x,y) = h(x,y) - g(x,y)$$
$$= 2 - x^2 - y^2 - e^{-x^2}\cos(x^2+y^2)$$

And $R = [-1,1]\times[-1,1]$.

Volume of the solid lies between the given surfaces on the interval $-1 \le x \le 1, -1 \le y \le 1$ is

$$V = \iint_R f(x,y)\,dA$$
$$= \int_{-1}^{1}\int_{-1}^{1}\left[2 - x^2 - y^2 - e^{-x^2}\cos(x^2+y^2)\right]dx\,dy$$

Use Maple to evaluate this integral.

Maple keystrokes:

```
evalf(int(int(2-x^2-y^2-exp(-x^2)*cos(x^2+y^2), x = -1 .. 1), y = -1 .. 1));
```

Maple result:

```
> evalf(int(int((2 - x^2 - y^2 - (e^(-x^2)·cos(x^2 + y^2))), x = -1 ..1), y = -1 ..1));
```

3.027069069 − 1.104478312 10^-9 I

Hence the required volume of the solid is $\boxed{3.027069069 - 1.104478312\times10^{-9}\,\mathrm{I}}$.

Chapter 15 Multiple Integrals 15.2 35E

Consider the function $f(x,y) = x^2y$

The given region is $[-1,1]\times[0,5]$

Then,

$$A(R) = (1+1)\times(5-0)$$

$$= 10$$

The formula for the average value of a function is:

$$f_{ave} = \frac{1}{A(R)}\iint_R f(x,y)\,dA$$

Here the *x* limits are from -1 to 1 and *y* limits are from 0 to 5.

Now the average value of a function is calculated as follows:

$$f_{\text{ave}} = \frac{1}{A}\int_0^5\int_{-1}^1 f(x,y)\,dx\,dy$$

$$f_{\text{ave}} = \frac{1}{10}\int_0^5\int_{-1}^1 x^2 y\,dx\,dy$$

Integrate on *x*.

$$f_{\text{ave}} = \frac{1}{10}\int_0^5 \frac{1}{3}x^3 y\Big|_{-1}^{1}\,dy$$

$$= \frac{1}{10}\int_0^5\left(\frac{1}{3}y + \frac{1}{3}y\right)dy$$

$$= \frac{1}{15}\int_0^5 y\,dy$$

Integrate on *y*.

$$f_{\text{ave}} = \frac{1}{15}\left(\frac{1}{2}y^2\Big|_0^5\right)$$

$$= \frac{1}{30}(25-0)$$

$$= \boxed{\frac{5}{6}}$$

Chapter 15 Multiple Integrals 15.2 36E

Consider the function $f(x,y) = e^y\sqrt{x+e^y}$

The given region is $R = [0,4]\times[0,1]$

The rectangle has width of 1 and length of 4, so its area is 4.

The formula for the average value of a function is:

$$f_{ave} = \frac{1}{A(R)}\iint_R f(x,y)\,dA$$

Here the *x* limits are from 0 to 4 and *y* limits are from 0 to 1.

Now the average value of a function is calculated as follows:

$$f_{ave} = \frac{1}{A}\int_0^4\int_0^1 f(x,y)\,dxdy$$

$$f_{ave} = \frac{1}{4}\int_0^4\int_0^1 e^y\sqrt{x+e^y}\,dydx$$

Use the substitution $u = x + e^y$ and $du = e^y dy$.

And the limits are if $y = 0$, then $u = x+1$

If $y = 1$, then $u = x + e$

$$f_{ave} = \frac{1}{4}\int_0^4\int_{x+1}^{x+e} u^{\frac{1}{2}}\,dudx$$

$$= \frac{1}{4}\int_0^4\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_{x+1}^{x+e} dx$$

$$= \frac{1}{6}\int_0^4 (x+e)^{\frac{3}{2}} - (x+1)^{\frac{3}{2}}\,dx$$

$$= \frac{1}{6}\left(\frac{2}{5}(x+e)^{\frac{5}{2}} - \frac{2}{5}(x+1)^{\frac{5}{2}}\Bigg|_0^4\right.$$

$$= \frac{1}{15}\left[(4+e)^{\frac{5}{2}} - e^{\frac{5}{2}} - 5^{\frac{5}{2}} + 1^{\frac{5}{2}}\right]$$

$$\approx \boxed{3.327}$$

Chapter 15 Multiple Integrals 15.2 37E

Given $\iint_R \frac{xy}{1+x^4}\,dA$, $R = \{(x,y)\,|\,-1 \le x \le 1, 0 \le y \le 1\}$

Let $g(x)$ and $h(y)$ be two functions. Then, it is known that

$$\int_c^d\int_a^b g(x)\,h(y)\,dxdy = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

Thus, we can rewrite the given integral as $\iint_R \frac{xy}{1+x^4}\,dA = \left(\int_{-1}^1 \frac{x}{1+x^4}\,dx\right)\left(\int_0^1 y\,dy\right)$.

We note that $\frac{x}{1+x^4}$ is an odd function.

By the properties of definite integrals, we know that $\int_{-a}^{a} f(x)\,dx = 0$, provided $f(x)$ is an odd function.

Then, $\int_{-1}^{1} \frac{x}{1+x^4}\,dx = 0$ (Since the integral is zero).

Thus, we get

$$\iint_R \frac{xy}{1+x^4}\,dA = (0)\left(\int_0^1 y\,dy\right).$$
$$= 0$$

Chapter 15 Multiple Integrals 15.2 38E

Given $\iint_R \left(1+x^2\sin y+y^2\sin x\right)dA,\quad R=[-\pi,\pi]\text{x}\,[-\pi,\pi]$

Let $g(x)$ and $h(y)$ be two functions.

Then, it is known that

$$\int_c^d\int_a^b f(x,y)+g(x,y)\,dx\,dy = \int_c^d\int_a^b f(x,y)\,dx\,dy + \int_c^d\int_a^b g(x,y)\,dx\,dy.$$

Thus, we can rewrite the given integral as

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(1+x^2\sin y+y^2\sin x\right)dx\,dy = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} 1\,dx\,dy + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^2\sin y\,dx\,dy + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} y^2\sin x\,dx\,dy$$
$$= \int_{-\pi}^{\pi}\left(\pi-(-\pi)\right)dy+(0)+(0)$$
$$= (2\pi)(2\pi)+0$$
$$= 4\pi^2$$

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^2\sin y\,dx\,dy = 0 \text{ (Since they are odd functions whose integral is zero)}$$

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} y^2\sin x\,dx\,dy = 0 \text{ (Since they are odd functions whose integral is zero)}$$

Thus, the integral evaluates to $\boxed{4\pi^2}$.

Chapter 15 Multiple Integrals 15.2 39E

Consider the integrals $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx$ and $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy.$

Here use Maple to find the exact values of the given integrals.

Keystrokes:

```
with(student);
evalf(int(int((x-y)/(x+y)^3, x = 0 .. 1), y = 0 .. 1));
evalf(int(int((x-y)/(x+y)^3, y = 0 .. 1), x = 0 .. 1));
```

Maple result:

$> evalf\left(int\left(int\left(\frac{x-y}{(x+y)^3}, x=0..1\right), y=0..1\right)\right);$

-0.5000000000

$> evalf\left(int\left(int\left(\frac{x-y}{(x+y)^3}, y=0..1\right), x=0..1\right)\right);$

0.5000000000

So $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx = 0.5$ and $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy = -0.5.$

Therefore $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx \neq \int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy.$

Sketch the graph of the function $f(x,y)=\dfrac{x-y}{(x+y)^3}$ on $R=[0,1]\times[0,1]$.

Use Maple to sketch the function:

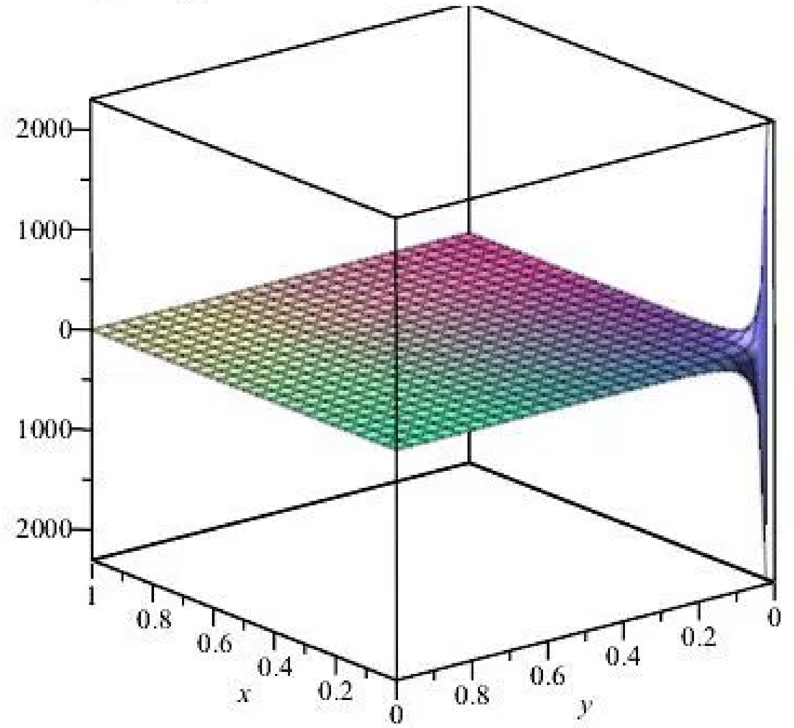
with(plots);

plot3d((x-y)/(x+y)^3, x = 0 .. 1, y = 0 .. 1);

Maple result:

> *with*(*plots*) :

> $plot3d\left(\dfrac{x-y}{(x+y)^3}, x=0..1, y=0..1\right)$;



According to Fubini's Theorem, if the integrand $f(x,y)=\dfrac{x-y}{(x+y)^3}$ is continuous on $R=[0,1]\times[0,1]$ then only we can write $\int_0^1\int_0^1 \dfrac{x-y}{(x+y)^3}\,dy\,dx=\int_0^1\int_0^1 \dfrac{x-y}{(x+y)^3}\,dx\,dy.$

But from the graph notice that the function $f(x,y)=\dfrac{x-y}{(x+y)^3}$ is not defined at $x=0$ and $y=0$, so the integrand $f(x,y)=\dfrac{x-y}{(x+y)^3}$ has an infinite discontinuity at the origin.

So Fubini's Theorem does not apply here.

Chapter 15 Multiple Integrals 15.2 40E

(a)

Fubini's Theorem states that if a function $f(x,y)$ is continuous on the rectangle $R=\{(x,y)\,|\,a\le x\le b, c\le y\le d\}$, the order of integration of the function $f(x,y)$ does not affect the value of the double integral $\iint\limits_R f(x,y)\,dA.$

And the Fubini's Theorem is applicable to the function such that f is bounded on R and discontinuous on a finite number of smooth curves and the iterated integrals $\int_a^b\int_c^d f(x,y)dydx$ and $\int_c^d\int_a^b f(x,y)dxdy$ exists.

Clairaut's Theorem states that if a function $f(x,y)$ is continuous on a disk D and the mixed partial derivatives f_{xy} and f_{yx} of the function $f(x,y)$ are also continuous on the disk D then they are equal at a point $(a,b)\in D.$

The similar thing is the function $f(x,y)$ must be continuous on respective domains in both the theorems.

(b)

Let $f(x,y)$ is continuous on a rectangle $[a,b]\times[c,d].$

Consider $g(x,y)=\int_a^x\int_c^y f(s,t)dtds,$ where $a<x<b, c<y<d.$

Differentiate $g(x,y)$ with respect to x on both sides, get

$$g_x(x,y)=\frac{d}{dx}\left[\int_a^x\left(\int_c^y f(s,t)dt\right)ds\right]$$

$$=\int_c^y f(x,t)dt \qquad \left[\begin{array}{l}\text{Fundamental Theorem of Calculus:}\\ g(x)=\int_a^x f(t)dt \text{ then } g'(x)=f(x).\end{array}\right]$$

Differentiate $g_x(x,y)$ with respect to y on both sides, get

$$g_{xy}(x,y)=\frac{d}{dy}\left[\int_c^y f(x,t)dt\right]$$

$$=f(x,y) \qquad [\text{Again use Fundamental Theorem of Calculus}]$$

Therefore $g_{xy}=f(x,y).$

Similarly we get $g_{yx}=f(x,y).$

Hence $g_{xy}=g_{yx}=f(x,y).$