Application of Derivatives

Short Answer Type Questions

1. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then how fast is the slope of curve changing when $x = 3$?

Sol. Slope of curve $= \frac{dy}{dx} = 5 - 6x^2$

$$\Rightarrow \frac{d}{dt}\left(\frac{dy}{dx}\right) = -12x.\frac{dx}{dt}$$

$$= -12.(3).(2)$$

$$= -72\, units/\sec.$$

Thus, slope of curve is decreasing at the rate of $72\ units/sec$ when x is increasing at the rate of $2\ units/sec$.

2. Water is dripping out from a conical funnel of semi-vertical angle $\frac{\pi}{4}$ at the uniform rate of $2\ cm^2/sec$ in the surface area, through a tiny hole at the vertex of the bottom. When the slant height of cone is 4 cm, find the rate of decrease of the slant height of water.

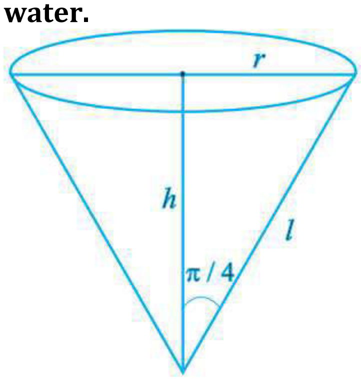


Fig. 6.1

Sol. If s represents the surface area, then

$$\frac{ds}{dt} = 2cm^2/\sec$$

$$s = \pi r.l = \pi l.\sin\frac{\pi}{4} l = \frac{\pi}{\sqrt{2}} l^2$$

Therefore, $\frac{ds}{dt} = \frac{2\pi}{\sqrt{2}} l.\frac{dl}{dt} = \sqrt{2}\pi l.\frac{dl}{dt}$

when $l = 4\,cm, \frac{dl}{dt} = \frac{1}{\sqrt{2}\pi.4}.2 = \frac{1}{2\sqrt{2}\pi} = \frac{\sqrt{2}}{4\pi} cm/s$

3. Find the angle of intersection of the curves $y^2 = x$ and $x^2 = y$.

Sol. Solving the given equations, we have $y^2 = x$ and $x^2 = y \Rightarrow x^4 = x$ or $x^4 - x = 0$

$\Rightarrow x\left(x^3-1\right)=0 \;\Rightarrow x=0, x=1$

Therefore, $y=0,\; y=1$

i.e. points of intersection are $(0,\; 0)$ *and* $(1,1)$

Further $y^2 = x \Rightarrow 2y\dfrac{dy}{dx}=1 \;\Rightarrow \dfrac{dy}{dx}=\dfrac{1}{2y}$

and $x^2 = y \Rightarrow \dfrac{dy}{dx}=2x.$

At $(0,\; 0)$, the slope of the tangent to the curve $y^2=x$ is parallel to y-axis and the tangent to the curve $x^2=y$ is parallel to x-axis.

$\Rightarrow$ angle of intersection $=\dfrac{\pi}{2}$

At (1, 1) slope of the tangent to the curve $y_2=x$ is equal to $\dfrac{1}{2}$ and that of $x^2=y$ is 2.

$$\tan\theta=\left|\frac{2-\frac{1}{2}}{1+1}\right|=\frac{3}{4} \;\Rightarrow \theta=\tan^{-1}\left(\frac{3}{4}\right)$$

4. Prove that the function $f\left(x\right)=tan\,x-4x$ is strictly decreasing on $\left(\dfrac{-\pi}{3},\dfrac{\pi}{3}\right)$.

Sol. $f(x)=\tan x-4x \Rightarrow f\,'(x)=\sec^2 x-4$

When $\dfrac{-\pi}{3}<x<\dfrac{\pi}{3}, 1<\sec<2$

Therefore, $1<\sec^2 x<4 \Rightarrow -3<(\sec^2 x-4)<0$

Thus for $\dfrac{-\pi}{3}<x<\dfrac{\pi}{3}, f\,'(x)<0$

Hence f is strictly decreasing on $\left(\dfrac{-\pi}{3},\dfrac{\pi}{3}\right)$.

5. Determine for which values of x, the function $y=x^4-\dfrac{4x^3}{3}$ is increasing and for which values, it is decreasing.

Sol. $y=x^4-\dfrac{4x^3}{3} \;\Rightarrow \dfrac{dy}{dx}=4x^3-4x^2=4x^2(x-1)$

Now, $\dfrac{dy}{dx}=0 \Rightarrow x=0, x=1.$

Since $f\,'(x)<0\;\forall x\in(-\infty,0)\cup(0,\,1)$ and f is continuous in $(-\infty,\;0]$ and $[0,\,1]$.

Therefore, f is decreasing in $(-\infty,\,1]$ and f is increasing in $[1,\;\infty)$.

Note: Here f is strictly decreasing in $(-\infty,0)\cup(0,\,1)$ and is strictly increasing in $(1,\,\infty)$.

6. **Show that the function $f(x)=4x^3-18x^2+27x-7$ has neither maxima nor minima.**

Sol. $f(x)=4x^3-18x^2+27x-7$

$f'(x)=12x^2-36x+27=3(4x^2-12x+9)=3(2x-3)^2$

$f'(x)=0 \Rightarrow x=\frac{3}{2}(critical\ point)$

Since $f'(x)>0$ for all $x<\frac{3}{2}$ and for all $x>\frac{3}{2}$

Hence $x=\frac{3}{2}$ is a point of inflexion i.e., neither a point of maxima nor a point of minima.

$x=\frac{3}{2}$ is the only critical point, and f has neither maxima nor minima.

7. **Using differentials, find the approximate value of $\sqrt{0.082}$**

Sol. Let $f(x)=\sqrt{x}$

Using $f(x+\Delta x); f(x)+\Delta x.f'(x), taking\ x=.09\ and\ \Delta x=-0.008,$

we get $f(0.09)-0.008)=f(0.09)+(-0.008)f'(0.09)$

$\Rightarrow \sqrt{0.082}=\sqrt{0.09}-0.008.\left(\frac{1}{2\sqrt{0.09}}\right)=0.3-\frac{0.008}{0.6}$

= 0.3 – 0.0133 = 0.2867.

8. **Find the condition for the curves $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1; xy=c^2$ to intersect orthogonally.**

Sol. Let the curves intersect at (x_1, y_1). Therefore,

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{b^2x}{a^2y}$

$\Rightarrow$ slope of tangent at the point intersection $(m_1)=\frac{b^2x_1}{a^2y_1}$

Again $xy=c^2 \Rightarrow x\frac{dy}{dx}+y=0 \Rightarrow \frac{dy}{dx}=\frac{-y}{x} \Rightarrow m_2=\frac{-y_1}{x_1}.$

For orthoganality, $m_1\times m_2=-1 \Rightarrow \frac{b^2}{a^2}=1\, or\, a^2-b^2=0.$

9. **Find all the point of local maxima and local minima of the function**

$f(x)=-\frac{3}{4}x^4-8x^3-\frac{45}{2}x^2+105.$

Sol. $f'(x)=-3x^2-24x^2-45x$

$=-3x(x^2+8x+15)=-3x(x+5)(x+3)$

$f'(x)=0 \Rightarrow x=-5,\ x=-3,\ x=0$

$f''(x) = -9x^2 - 48x - 45$

$= -3(3x^2 + 16x + 15)$

$f''(0) = -45 < 0$. Therefore, $x = 0$ is point of local maxima

$f''(-3) = 18 > 0$. Therefore, $x = -3$ is point of local minima

$f''(-5) = -30 < 0$. Therefore, $x = -5$ is point of local maxima.

10. Show that the local maximum value of $x + \frac{1}{x}$ is less than local minimum value.

Sol. Let $y = x + \frac{1}{x} \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{x^2}$

$\frac{dy}{dx} = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$

$\frac{d^2y}{dx^2} = +\frac{2}{x^3}, therefore \frac{d^2y}{dx^2}(at\, x = 1) > 0\, and\, \frac{d^2y}{dx^2}(at\, x = -1) < 0.$

Hence local maximum value of y is at $x = -1$ and the local maximum value $= -2$.

Local minimum value of y is at $x = 1$ and local minimum value $= 2$.

Therefore, local maximum value (-2) is less than local minimum value 2.

Long Answer Type Questions

11. Water is dripping out at a steady rate of $1\ cu\ cm/sec$ through a tiny hole at the vertex of the conical vessel, whose axis is vertical. When the slant height of water in the vessel is 4 cm, find the rate of decrease of slant height, where the vertical angle of the conical vessel is $\frac{\pi}{6}$.

Sol. Given that $\frac{dv}{dt} = 1 cm^3/s$ where v is the volume of water in the conical vessel.

From the Fig.6.2, $l = 4cm, h = l\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}l\ and\ r = l\sin\frac{\pi}{6} = \frac{l}{2}.$

Therefore, $v = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}\frac{l^2}{4}\frac{\sqrt{3}}{2}l = \frac{\sqrt{3}}{24}l^3.$

$\frac{dv}{dt} = \frac{\sqrt{3}\pi}{8}l^2\frac{dl}{dt}$

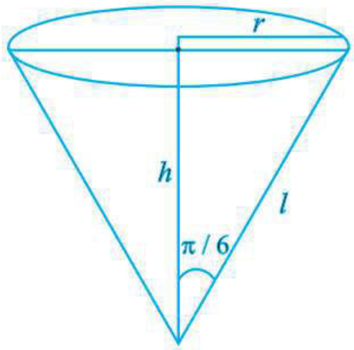


Fig. 6.2

Therefore, $1=\frac{\sqrt{3}\pi}{8}16.\frac{dl}{dt}$

$\Rightarrow \frac{dl}{dt}=\frac{1}{2\sqrt{3}\pi}cm/s$

Therefore, the rate of decrease of slant height $=\frac{1}{2\sqrt{3}\pi}cm/s$.

12. Find the equation of all the tangents to the curve y = cos (x + y), $-2\pi \le x \le 2\pi$ **that are parallel to the line** $x+2y=0$.

Sol. Given that $y=\cos(x+y)\Rightarrow \frac{dy}{dx}=-\sin(x+y)\left[1+\frac{dy}{dx}\right]$ ...(i)

$or\,\frac{dy}{dx}=-\frac{\sin(x+y)}{1+\sin(x+y)}$

Since tangent is parallel to $x+2y=0$, therefore slope of tangent $=-\frac{1}{2}$

Therefore, $-\frac{\sin(x+y)}{1+\sin(x+y)}=-\frac{1}{2}\Rightarrow \sin(x+y)=1$ ...(ii)

Since $cos(x+y)=y$ and $sin(x+y)=1\Rightarrow cos^2(x+y)+sin^2(x+y)=y^2+1$

$\Rightarrow 1=y^2+1\ or\ y=0.$

Therefore, $cos\,x=0$

Therefore, $x=(2n+1)\frac{\pi}{2},\ n=0,\pm 1,\pm 2...$

Thus, $x=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\ but\ x=\frac{\pi}{2}, x=\frac{-3\pi}{2}\ satisfy\ equation\ (ii)$

Hence, the points are $\left(\frac{\pi}{2},0\right),\left(\frac{-3\pi}{2},0\right)$.

Therefore, equation of tangent at $\left(\frac{\pi}{2},0\right) is\ y=-\frac{1}{2}\left(x-\frac{\pi}{2}\right) or\ 2x+4y-\pi=0$, and equation of tangent at $\left(\frac{-3\pi}{2},0\right) is\ y=-\frac{1}{2}\left(x+\frac{3\pi}{2}\right)$ or $2x+4y+3\pi=0$.

13. Find the angle of intersection of the curves $y^2=4ax$ and $x^2=4by$.

Sol. Given that $y^2=4ax\ \ldots(i)$ and $x^2=4by\ \ldots(ii)$. Solving (i) and (ii), we get

$$\left(\frac{x^2}{4b}\right)^2=4ax\Rightarrow x^4=64ab^2x$$

Or $x\left(x^3-64ab^2\right)=0\Rightarrow x=0,\ x=4a^{\frac{1}{3}}b^{\frac{2}{3}}$

Therefore, the points of intersection are (0, 0) and $\left(4a^{\frac{1}{3}}b^{\frac{2}{3}},4a^{\frac{2}{3}}b^{\frac{1}{3}}\right)$.

Again, $y^2=4ax\Rightarrow\frac{dy}{dx}=\frac{4a}{2y}=\frac{2a}{y}\ and\ x^2=4by\Rightarrow\frac{dy}{dx}=\frac{2x}{4b}=\frac{x}{2b}$

Therefore, at (0, 0) the tangent to the curve $y^2=4ax$ is parallel to y-axis and tangent to the curve $x^2=4by$ is parallel to x-axis.

$\Rightarrow$ Angle between curves $=\frac{\pi}{2}$

At $\left(4a^{\frac{1}{3}}b^{\frac{2}{3}},4a^{\frac{2}{3}}b^{\frac{1}{3}}\right),m_1$ (slope of the tangent to the curve (i)) $=2\left(\frac{a}{b}\right)^{\frac{1}{3}}$

$=\frac{2a}{4a^{\frac{2}{3}}b^{\frac{1}{3}}}=\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{3}},m_2$ (slope of the tangent to the curve (ii)) $=\frac{4a^{\frac{1}{3}}b^{\frac{2}{3}}}{2b}=2\left(\frac{a}{b}\right)^{\frac{1}{3}}$

Therefore, $\tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right|=\left|\frac{2\left(\frac{a}{b}\right)^{\frac{1}{3}}-\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{3}}}{1+2\left(\frac{a}{b}\right)^{\frac{1}{3}}\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{3}}}\right|=\frac{3a^{\frac{1}{3}}.b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)}$

Hence, $\theta=\tan^{-1}\left(\frac{3a^{\frac{1}{3}}.b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)}\right)$

14. Show that the equation of normal at any point on the curve $x=3\cos\theta-\cos^3\theta,\ y=3\sin\theta-\sin^3\theta$ is $4(y\cos^3\theta-x\sin^3\theta)=3\sin4\theta$.

Sol. We have $x=3\cos\theta-\cos^3\theta$

Therefore, $\frac{dy}{d\theta}=-3\sin\theta+3\cos^2\theta\sin\theta=-3\sin\theta(1-\cos^2\theta)=-3\sin^3\theta$

$$\frac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta = 3\cos\theta(1-\sin^2\theta) = 3\cos^3\theta$$

$$\frac{dy}{dx} = -\frac{\cos^3\theta}{\sin^3\theta}.\text{Therefore, slope of normal} = +\frac{\sin^3\theta}{\cos^3\theta}$$

Hence the equation of normal is

$$y-(3\sin\theta-\sin^3\theta) = \frac{\sin^3\theta}{\cos^3\theta}[x-(3\cos\theta-\cos^3\theta)]$$

$$\Rightarrow y\cos^3\theta - 3\sin\theta\cos^3\theta + \sin^3\theta\cos^3\theta = x\sin^3\theta - 3\sin^3\theta\cos\theta + \sin^3\theta\cos^3\theta$$

$$\Rightarrow y\cos^3\theta - x\sin^3\theta = 3\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$$

$$= \frac{3}{2}\sin 2\theta.\cos 2\theta$$

$$= \frac{3}{4}\sin 4\theta$$

Or $4(y\cos^3\theta - x\sin^3\theta) = 3\sin 4\theta$

15. Find the maximum and minimum values of

$f(x) = secx + log\ cos^2 x,\ 0 < x < 2$

Sol. $f(x) = sec\,x + 2log\ cos\,x$

Therefore, $f'(x) = sec\,x\ tan\,x - 2tan\,x = tan\,x(sec\,x - 2)$

$f'(x) = 0 \Rightarrow tanx = 0\ or\ sec\,x = 2\ or \cos x = \frac{1}{2}$

Therefore, possible values of x are $x = 0\,or\,x = \pi\,and$

$x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$

Again, $f''(x) = sec^2 x(sec\,x - 2) + tan\,x(sec\,x\ tan\,x)$

$= sec^3 x + sec\,x\ tan^2 x - 2sec^2 x$

$= sec\,x(sec^2 x + tan^2 x - 2sec\,x)$. We not that

$f''(0) = 1(1+0-2) = -1 < 0.$ Therefore, $x = 0$ is a point of maxima.

$f''(\pi) = 1(1+0+2) = -3 < 0$. Therefore, $x = \pi$ is a point of maxima.

$f''\left(\frac{\pi}{3}\right) = 2(4+3-4) = 6 > 0$. Therefore, $x = \frac{\pi}{3}$ is a point of minima.

$f''\left(\frac{5\pi}{3}\right) = 2(4+3-4) = 6 > 0$. Therefore, $x = \frac{5\pi}{3}$ is a point of minima.

Maximum Value of y at x = 0 is 1 + 0 = 1

Maximum Value of y at x = π is -1 + 0 = -1

Minimum Value of y at $x = \frac{\pi}{3}$ is $2 + 2log\frac{1}{2} = 2(1 - log 2)$

Minimum Value of y at $x=\frac{5\pi}{3}$ is $2+2log\frac{1}{2}=2(1-log2)$

16. Find the area of greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol. Let ABCD be the rectangle of maximum area with sides $AB=2x$ *and* $BC=2y$, where $C(x,\ y)$ is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as shown in the Fig. 6.3.

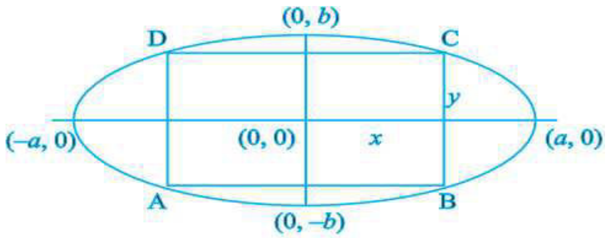


Fig. 6.3

The area A of the rectangle is $4xy$ *i.e.* $A=4xy$ which gives $A^2=16x^2y^2=s\ (say)$

Therefore, $s=16x^2\left(1-\frac{x^2}{a^2}\right).b^2=\frac{16b^2}{a^2}(a^2x^2-x^4)$

$\Rightarrow\frac{ds}{dx}=\frac{16b^2}{a^2}.[2a^2x-4x^3].$

Again, $\frac{ds}{dx}=0\Rightarrow x=\frac{a}{\sqrt{2}}\ and\ y=\frac{b}{\sqrt{2}}$

Now, $\frac{d^2s}{dx^2}=\frac{16b^2}{a^2}[2a^2-12x^2]$

At $x=\frac{a}{\sqrt{2}},\frac{d^2s}{dx^2}=\frac{16b^2}{a^2}[2a^2-6a^2]=\frac{16b^2}{a^2}(-4a^2)<0$

Thus, at $x=\frac{a}{\sqrt{2}},y=\frac{b}{\sqrt{2}},s$ is maximum and hence the area A is maximum.

Maximum area $=4.x.y=4.\frac{a}{\sqrt{2}}.\frac{b}{\sqrt{2}}=2ab\,sq\,units.$

17. Find the differential between the greatest and least values of the function $f(x)=\sin 2x-x, on\left[-\frac{\pi}{2},\frac{\pi}{2}\right].$

Sol. $f(x)=sin2x-x$

$\Rightarrow f'(x)=2cos2x-1$

Therefore, $f'(x)=0\Rightarrow\cos 2x=\frac{1}{2}\Rightarrow 2x\ is\ \frac{-\pi}{3}\ or\ \frac{\pi}{3}\Rightarrow x=-\frac{\pi}{6}or\frac{\pi}{6}$

$$f\left(-\frac{\pi}{2}\right)=\sin(-\pi)+\frac{\pi}{2}=\frac{\pi}{2}$$

$$f\left(-\frac{\pi}{6}\right)=\sin\left(-\frac{2\pi}{6}\right)+\frac{\pi}{6}=-\frac{\sqrt{3}}{2}+\frac{\pi}{6}$$

$$f\left(\frac{\pi}{6}\right)=\sin\left(\frac{2\pi}{6}\right)-\frac{\pi}{6}=\frac{\sqrt{3}}{2}-\frac{\pi}{6}$$

$$f\left(\frac{\pi}{2}\right)=\sin(\pi)-\frac{\pi}{2}=-\frac{\pi}{2}$$

Clearly, $\frac{\pi}{2}$ is the greatest value and $-\frac{\pi}{2}$ is the least.

Therefore, difference $=\frac{\pi}{2}+\frac{\pi}{2}=\pi$

18. An isosceles triangle of vertical angle 2 θ is inscribed in a circle of radius a. Show that the area of triangle is maximum when $\theta=\frac{\pi}{6}$.

Sol. Let ABC be an isosceles triangle inscribed in the circle with radius a such that AB = AC.
$AD=AO+OD=a+a\ cos2\theta$ and $BC=2BD=2a\,sin2\theta$ (see fig. 16.4)

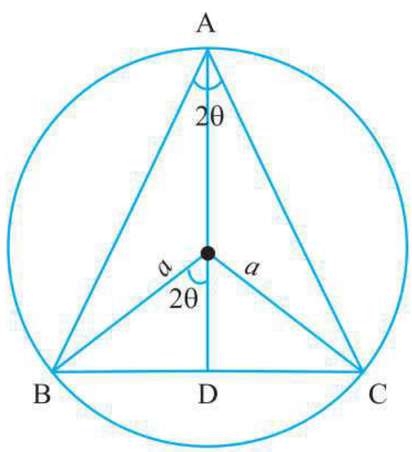


Fig. 6.4

Therefore, area of the triangle ABC i.e. $\Delta=\frac{1}{2}BC.AD$

$$=\frac{1}{2}2a\sin 2\theta.(a+a\cos 2\theta)$$

$$=a^2\sin 2\theta(1+\cos 2\theta)$$

$$\Rightarrow \Delta=a^2\sin 2\theta+\frac{1}{2}a^2\sin 4\theta$$

Therefore, $\frac{d\Delta}{d\theta}=2a^2\cos 2\theta+2a^2\cos 4\theta$

$= 2a^2 (\cos 2\theta + \cos 4\theta)$

$\frac{d\Delta}{d\theta} = 0 \Rightarrow \cos 2\theta = -\cos 4\theta = \cos(\pi - 4\theta)$

Therefore, $2\theta = \pi - 4\theta \Rightarrow \theta = \frac{\pi}{6}$

$\frac{d^2\Delta}{d\theta^2} = 2a^2(-2\sin 2\theta - 4\sin 4\theta) < 0 (at\, \theta = \frac{\pi}{6}).$

Therefore, Area of triangle is maximum when $\theta = \frac{\pi}{6}$.

Objective Type Questions

Choose the correct answer from the given four options in each of the following Examples 19 to 23.

19. The abscissa of the point on the curve $3y = 6x - 5x^3$, the normal at which passes through origin is:

(A) 1

(B) $\frac{1}{3}$

(C) 2

(D) $\frac{1}{2}$

Sol. Let (x_1, y_1) be the point on the given curve $3y = 6x - 5x^3$ at which the normal passes through the origin. Then we have $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2 - 5x_1^2$. Again the equation of the normal at (x_1, y_1) passing through the origin gives $2 - 5x_1^2 = \frac{-x_1}{y_1} = \frac{-3}{6 - 5x_1^2}$.

Since $x_1 = 1$ satisfies the equation, therefore, Correct answer is (A).

20. The two curves $x^3 - 3xy^2 + 2 = 0$ *and* $3x^2y - y^3 = 2$

(A) touch each other

(B) cut at right angle

(C) cut at an angle $\frac{\pi}{3}$

(D) cut at an angle $\frac{\pi}{4}$

Sol. From first equation of the curve, we have $3x^2 - 3y^2 - 6xy\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = (m_1)$ say and second equation of the curve gives

$6xy + 3x^2 \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2xy}{x^2 - y^2} = (m_2)$ say

Since, $m_1 . m_2 = -1$. Therefore, correct answer is (B).

21. The tangent to the curve given by $x = e^t .\cos t, y = e^t .\sin t \; at \; t = \frac{\pi}{4}$ make with x-axis an angle:

(A) 0

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

Sol. $\frac{dx}{dt} = -e^t \; . \sin t + e^t \cos t, \frac{dy}{dt} = e^t \cos + e^t \sin t$

Therefore, $\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{\cos t + \sin t}{\cos t - \sin t} = \frac{\sqrt{2}}{0}$ and hence the correct answer is (D).

22. The equation of the normal to the curve y = sinx at (0, 0) is:

(A) $x = 0$

(B) $y = 0$

(C) $x + y = 0$

(D) $x - y = 0$

Sol. $\frac{dy}{dx} = \cos x$. Therefore, slope of normal $= \left(\frac{-1}{\cos x}\right)_{x=0} = -1$. Hence the equation of normal is

$y - 0 = -1(x - 0) \; or \; x + y = 0$

Therefore, correct answer is (C).

23. The point on the curve $y^2 = x$, where the tangent makes an angle of $\frac{\pi}{4}$ with x-axis is

(A) $\left(\frac{1}{2}, \frac{1}{4}\right)$

(B) $\left(\frac{1}{4}, \frac{1}{2}\right)$

(C) (4, 2)

(D) (1, 1)

Sol. $\frac{dy}{dx} = \frac{1}{2y} = \tan\frac{\pi}{4} = 1 \Rightarrow y = \frac{1}{2} \Rightarrow x = \frac{1}{4}$

Therefore, correct answer is B.

Fill in the blanks in each of the following Examples 24 to 29.

24. The values of a for which $y = x^2 + ax + 25$ touches the axis of x are ____________ .

Sol. $\frac{dy}{dx} = 0 \Rightarrow 2x + a = 0$ *i.e.* $x = -\frac{a}{2}$,

Therefore, $\frac{a^2}{4} + a\left(-\frac{a}{2}\right) + 25 = 0 \Rightarrow a = \pm 10$

Hence, the values of a are ± 10

25. If $f(x) = \frac{1}{4x^2 + 2x + 1}$, then its maximum value is __________.

Sol. For f to be maximum, $4x^2 + 2x + 1$ should be minimum i.e.

$4x^2 + 2x + 1 = 4(x + \frac{1}{4})^2 + \left(1 - \frac{1}{4}\right)$ giving the minimum value of $4x^2 + 2x + 1 = \frac{3}{4}$.

Hence, maximum value of $f = \frac{4}{3}$.

26. Let f have second derivative at c such that $f'(c) = 0$ *and* $f''(c) > 0$, then c is a point of ________.

Sol. Local minima.

27. Minimum value of f if $f(x) = \sin x$ *in* $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ _________.

Sol. -1

28. The maximum value of $\sin x + \cos x$ is _________.

Sol. $\sqrt{2}$

29. The rate of change of volume of a sphere with respect to its surface area, when the radius is 2 cm, is _________.

Sol. $1\ cm^3 / cm^2$

$$v = \frac{4}{3}\pi r^3 \Rightarrow \frac{dv}{dr} = 4\pi r^2, s = 4\pi r^2 \Rightarrow \frac{ds}{dr} = 8\pi r \Rightarrow \frac{dv}{ds} = \frac{r}{2} = 1 \text{ at } r = 2.$$

Application of Derivatives
Objective Type Questions

Choose the correct answer from the given four options in each of the following questions 35 to 39.

35. **The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate at which the area increase, when side is 10 cm is:**

(A) $10\ cm^2/s$

(B) $\sqrt{3}\ cm^2/s$

(C) $10\sqrt{3}\ cm^2/s$

(D) $\frac{10}{3} cm^2/s$

Sol. (C) Let the side of an equilateral triangle be $x\ cm$.

$\therefore$ Area of equilateral triangle, $A = \frac{\sqrt{3}}{4}x^2 \quad ...(i)$

Also, $\frac{dx}{dt} = 2cm/s$

On differentiating Eq. (i) w.r.t. t, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}.2x.\frac{dx}{dt}$$

$$= \frac{\sqrt{3}}{4}.2.10.2 \left[\because x = 10 \ and \ \frac{dx}{dt} = 2\right]$$

$$= 10\sqrt{3}cm^2/s$$

36. **A ladder, 5 meter long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 meters from the wall is:**

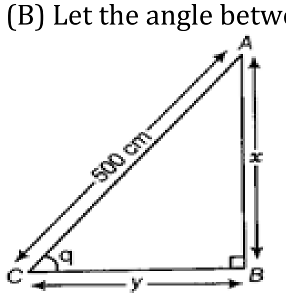
(A) $\frac{1}{10} radian/sec$

(B) $\frac{1}{20} radian/sec$

(C) $20\ radian/sec$

(D) $10\ radian/sec$

Sol. (B) Let the angle between floor and the ladder be θ



Let $AB = x\ cm\ and\ BC = y\,cm$

$\therefore \sin\theta = \frac{x}{500}\, and \cos\theta = \frac{y}{500}$

$\Rightarrow x = 500\sin\theta\, and\ y = 500\cos\theta$

Also, $\frac{dx}{dt} = 10\,cm/s$

$\Rightarrow 500.\cos\theta.\frac{d\theta}{dt} = 10\,cm/s$

$\Rightarrow \frac{d\theta}{dt} = \frac{10}{500\cos\theta} = \frac{1}{50\cos\theta}$

For $y = 2\ m = 200\ cm$,

$\frac{d\theta}{dt} = \frac{1}{50.\frac{y}{500}} = \frac{10}{y}$

$= \frac{10}{200} = \frac{1}{20}\,rad/s$

37. The curve $y = x^{\frac{1}{5}}$ has at (0, 0)

(A) a vertical tangent (parallel to y-axis)
(B) a horizontal tangent (parallel to x-axis)
(C) an oblique tangent
(D) no tangent

Sol. (A) We have, $y = x^{1/5}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{5}x^{\frac{1}{5}-1} = \frac{1}{5}x^{-4/5}$

$\therefore \left(\frac{dy}{dx}\right)_{(0,0)} = \frac{1}{5}\times(0)^{-4/5} = \infty$ v

So, the curve $y = x^{1/5}$ has vertical tangent at (0, 0), which is parallel to Y-axis.

38. The equation of normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is

(A) $3x - y = 8$
(B) $3x + y + 8 = 0$
(C) $x + 3y \pm 8 = 0$
(D) $x + 3y = 0$

Sol. (C) we have, the equation of the curve is $3x^2 - y^2 = 8\ \ldots(i)$

Also, the given equation of the line is $x + 3y = 8$

$\Rightarrow 3y = 8 - x$

$\Rightarrow y = -\frac{x}{3} + \frac{8}{3}$

Thus, slope of the line is $-\frac{1}{3}$ which should be equal to slope of the equation of normal to the curve.

On differentiating Eq. (i) w.r.t. x, we get

$6x - 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{6x}{2y} = \frac{3x}{y} =$ Slope of the curve

Now, slope of normal to the curve $= -\frac{1}{\left(\frac{dy}{dx}\right)}$

$= -\frac{1}{\left(\frac{3x}{y}\right)} = -\frac{y}{3x}$

$\therefore -\left(\frac{y}{3x}\right) = -\frac{1}{3}$

$\Rightarrow -3y = -3x$

$\Rightarrow y = x$

On substituting the value of the given equation of the curve, we get

$3x^2 - x^2 = 8$

$\Rightarrow x^2 = \frac{8}{2}$

$\Rightarrow x = \pm 2$

$For\, x = 2,\ 3(2)^2 - y^2 = 8$

$\Rightarrow y^2 = 4$

$\Rightarrow y = \pm 2$

and $for\, x = -2,\ 3(-2)^2 - y^2 = 8$

$\Rightarrow y = \pm 2$

So, the points at which normal are parallel to the given line are $(\pm\, 2, \pm\, 2)$.

Hence, the equation of normal at $(\pm\, 2, \pm\, 2)$ is

$y - (\pm 2) = -\frac{1}{3}[x - (\pm\, 2)]$

$\Rightarrow 3[y - (\pm\, 2)] = -[x - (\pm\, 2)]$

$\Rightarrow x + 3y \pm 8 = 0$

39. **If the curve** $ay + x^2 = 7\ and\ x^3 = y,$ **cut orthogonally at** $(1,\ 1),$ **then the value of a is:**

(A) 1

(B) 0

(C) -6
(D) 6

Sol. (D) We have, $ay + x^2 = 7 \, and \; x^3 = y$

On differentiating w.r.t. x. in both equations, we get

$$a.\frac{dy}{dx} + 2x = 0 \, and \; 3x^2 = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x}{a} \, and \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{-2}{a} = m_1$$

and $\left(\frac{dy}{dx}\right)_{(1,1)} = 3.1 = 3 = m_2$

Since, the curves cut orthogonally at (1, 1)

$$\therefore m_1 . m_2 = -1$$

$$\Rightarrow \left(\frac{-2}{a}\right).3 = -1$$

$$\therefore \; a = 6$$

40. If $y = x^4 - 10$ and if x changes from 2 to 1.99, what is the change in y

(A) .32
(B) .032
(C) 5.68
(D) 5.968

Sol. (A) We have, $y = x^4 - 10 \Rightarrow \frac{dy}{dx} = 4x^3$

and $\Delta x = 2.00 - 1.99 = 0.01$

$$\therefore \Delta y = \frac{dy}{dx} \times \Delta x$$

$$= 4x^3 \times \Delta x$$

$$= 4 \times 2^3 \times 0.01$$

$$= 32 \times 0.01 = 0.32$$

So, the approximate change in y is 0.32.

41. The equation of tangent to the curve $y(1 + x^2) = 2 - x$, where it crosses x-axis is:

(A) $x + 5y = 2$
(B) $x - 5y = 2$
(C) $5x - y = 2$
(D) $5x + y = 2$

Sol. (A) We have, equation of the curve $y(1 + x^2) = 2 - x \; ...(i)$

$$\therefore \; y.(0 + 2x) + (1 + x^2).\frac{dy}{dx} = 0 - 1 \;\; [on \, differentiating \; w.r.t. \, x]$$

$\Rightarrow 2xy+(1+x^2)\dfrac{dy}{dx}=-1$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{-1-2xy}{1+x^2}\ \ ...(ii)$

Since, the given curve passes through X-axis i.e., y = 0.

$\therefore 0(1+x^2)=2-x$ [using $Eq.(i)$]

$\Rightarrow x=2$

So, the curve passes through the point (2, 0).

$\therefore \left(\dfrac{dy}{dx}\right)_{(2,0)}=\dfrac{-1-2\times 0}{1+2^2}=-\dfrac{1}{5}=$ Slope of the curve

$\therefore$ Slope of tangent to the curve $=-\dfrac{1}{5}$

$\therefore$ Equation of tangent of the curve passing through (2, 0) is

$y-0=-\dfrac{1}{5}(x-2)$

$\Rightarrow y+\dfrac{x}{5}+\dfrac{2}{5}$

$\Rightarrow 5y+x=2$

42. The points at which the tangents to the curve $y=x^3-12x+18$ are parallel to x-axis are:

(A) $(2,-2),(-2,-34)$

(B) $(2,\ 34),(-2,\ 0)$

(C) $(0,34),(-2,0)$

(D) $(2,2),(-2,34)$

Sol. (D) The given equation of curve is

$y=x^3-12x+18$

$\therefore \dfrac{dy}{dx}=3x^2-12$ [on differentiating $w.r.t.\ x$]

So, the slope of line parallel to the X-axis.

$\therefore \left(\dfrac{dy}{dx}\right)=0$

$\Rightarrow 3x^2-12=0$

$\Rightarrow x^2=\dfrac{12}{3}=4$

$\therefore x=\pm 2$

$For\ x=2,\ \ y=2^3-12\times 2+18=2$

$and\ for\ x=-2, y=(-2)^3-12(-2)+18=34$

So, the points are $(2, 2)$ *and* $(-2, 34)$.

43. **The tangent to the curve $y = e^{2x}$ at the point $(0, 1)$ meets x-axis at:**

(A) $(0, 1)$

(B) $\left(-\frac{1}{2}, 0\right)$

(C) $(2, 0)$

(D) $(0, 2)$

Sol. (B) The equation of curve is $y = e^{2x}$

Since, it passes through the point (0, 1).

$\therefore \frac{dy}{dx} = e^{2x}.2 = 2.e^{2x}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = 2.e^{2.0} = 2 =$ Slope of tangent to the curve

$\therefore$ Equation of tangent is $y - 1 = 2(x - 0)$

$\Rightarrow y = 2x + 1$

Since, tangent to curve $y = e^{2x}$ at the point (0, 1) meets X-axis i.e., y = 0.

$\therefore 0 = 2x + 1 \Rightarrow x = -\frac{1}{2}$

So, the required points is $\left(\frac{-1}{2}, 0\right)$.

44. **The slope of tangent to the curve $x = t^2 + 3t - 8,\ y = 2t^2 - 2t - 5$ at the point $(2, -1)$ is:**

(A) $\frac{22}{7}$

(B) $\frac{6}{7}$

(C) $\frac{-6}{7}$

(D) – 6

Sol. (B) Equation of curve is given is given by

$x = t^2 + 3t - 8$ *and* $y = 2t^2 - 2t - 5$.

$\therefore \frac{dx}{dt} = 2t + 3$ *and* $\frac{dy}{dx} = 4t - 2$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t - 2}{2t + 3} \quad ...(i)$$

Since, the curve passes through the point $(2, -1)$.

$\therefore\ 2 = t^2 + 3t - 8$

and $-1 = 2t^2 - 2t - 5$

$\Rightarrow t^2 + 3t - 10 = 0$

and $2t^2 - 2t - 4 = 0$

$\Rightarrow\ t^2 + 5t - 2t - 10 = 0$

and $2t^2 + 2t - 4t - 4 = 0$

$\Rightarrow t(t+5) - 2(t+5) = 0$

and $2t(t+1) - 4(t+1) = 0$

$\Rightarrow\ (t-2)(t+5) = 0$

and $(2t-4)(t+1) = 0$

$\Rightarrow t = 2, -5\ and\ t = -1, 2$

$\Rightarrow\ t = 2$

$\therefore$ Slope of tangent,

$$\left(\frac{dy}{dx}\right)_{at\, t=2} = \frac{4\times2-2}{2\times2+3} = \frac{6}{7}\ \text{[using } Eq.(i)]$$

45. **The two curve** $x^3 - 3xy^2 + 2 = 0\ and\ 3x^2y - y^3 - 2 = 0$ **intersect at an angle of**

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$

(D) $\frac{\pi}{6}$

Sol. (C) Equation of two curves are given by

$x^3 - 3xy^2 + 2 = 0$

and $3x^2y - y^3 - 2 = 0$ [on differentiating $w.r.t.\,x$]

$$\Rightarrow 3x^2 - 3\left[x.2y\frac{dy}{dx} + y^2.1\right] + 0 = 0$$

$$and\ 3\left[x^2\frac{dy}{dx} + y.2x\right] - 3y^2\frac{dy}{dx} - 0 = 0$$

$$\Rightarrow 3x.2y\frac{dy}{dx} + 3y^2 = 3x^2$$

and $3y^2\frac{dy}{dx} = 3x^2\frac{dy}{dx} + 6xy$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2 - 3y^2}{6xy}$$

Application of Derivatives
<u>Short Answer Type Questions</u>

1. **A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface. Prove that the radius is decreasing at a constant rate.**

Sol. We have, rate of decrease of the volume of spherical ball of salt at any instant is α surface.
Let the radius of the spherical ball of the salt be r.

$$\therefore \text{ } Volume\ of\ the\ ball\,(V) = \frac{4}{3}\pi r^3$$

$$and\ \ surface\ area\,(S) = 4\pi r^2$$

$$\because \frac{dV}{dt}\alpha S \Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)\alpha\, 4\pi r^2$$

$$\Rightarrow \frac{4}{3}\pi.3r^2.\frac{dr}{dt}\alpha 4\pi r^2 \Rightarrow \frac{dr}{dt}\alpha\frac{4\pi r^2}{4\pi r^2}$$

$$\Rightarrow \frac{dt}{dr} = k.1 \text{ [where, k is the proportionality constant]}$$

$$\Rightarrow \frac{dr}{dt} = k$$

Hence, the radius of ball is decreasing at a constant rate.

2. **If the area of a circle increase at a uniform rate, then prove that perimeter varies inversely as the radius.**

Sol. Let the radius of circle = r And area of the circle, $A = \pi r^2$

$$\therefore \frac{d}{dt}A = \frac{d}{dt}\pi r^2$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r.\frac{dr}{dt} \quad ...(i)$$

Since, the area of a circle increases at a uniform rate, then

$$\frac{dA}{dt} = k \quad ...(ii)$$

Where, k is a constant.

From Eqs. (i) and (ii), $2\pi r.\frac{dr}{dt} = k$

$$\Rightarrow \frac{dr}{dt} = \frac{k}{2\pi r} = \frac{k}{2\pi}.\left(\frac{1}{r}\right) \quad ...(iii)$$

Let the perimeter, $P = 2\pi r$

$$\therefore \frac{dP}{dt} = \frac{d}{dt}.2\pi r \quad \Rightarrow \quad \frac{dP}{dt} = 2\pi.\frac{dr}{dt}$$

$$= 2\pi.\frac{k}{2\pi}.\frac{1}{r} = \frac{k}{r} \quad [\text{using } Eq.(iii)]$$

$\Rightarrow \frac{dP}{dt} \alpha \frac{1}{r}$ *Hence proved.*

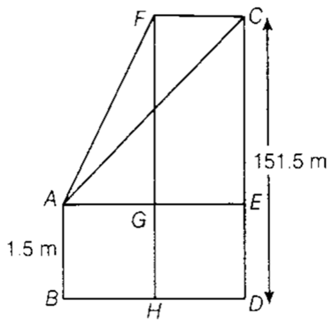
3. A kite is moving horizontally at a height of 151.5 meters. If the speed of kite is $10\ m/s$, how fast is the string being let out; when the kite is 250 m away from the boy who is flying the kite? The height of boy is 1.5 m.

Sol. We have, height $(h) = 151.5\ m$, speed of kite $(v) = 10\ m/s$

Let CD be the height of kite and AB be the height of boy.

Let $DB = x\ m = EA\ and\ AC = 250\ m$

$\therefore\ \frac{dx}{dt} = 10\ m/s$



From the figure, we see that

$EC = 151.5 - 1.5 = 150m$

And $AE = x$

Also, $AC = 250m$

In right angles ΔCEA,

$AE^2 + EC^2 = AC^2$

$\Rightarrow\ x^2 + (150)^2 = y^2\ \ ...(i)$

$\Rightarrow\ x^2 + (150)^2 = (250)^2$

$\Rightarrow\ x^2 = (250)^2 - (150)^2$

$= (250+150)(250-150)$

$= 400 \times 100$

$\therefore\ x = 20 \times 10 = 200$

From Eq. (i), on differentiating w.r.t, we get

$2x.\frac{dx}{dt} + 0 = 2y\frac{dy}{dt}$

$\Rightarrow\ 2y\frac{dy}{dt} = 2x\frac{dx}{dt}$

$\therefore\ \frac{dy}{dt} = \frac{x}{y}.\frac{dx}{dt}$

$= \frac{200}{250}.10 = 8m/s\ \left[\because \frac{dx}{dt} = 10m/s\right]$

So, the required rate at which the string is being let out is 8 m/s.

4. Two men A and B start with velocities v at the same time from the junction of two roads inclined at 45^0 to each other. If they travel by different roads, find the rate at which they are being separated.

Sol. Let two men start from the point C with velocity v each at the same time.

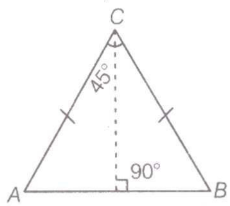
Also, $\angle BCA = 45^\circ$

Since, A and B are moving with same velocity v, so they will cover same distance in same time.

Therefore, ΔABC is an isosceles triangle with AC = BC.

Now, draw $CD \perp AB$

Let at any instant t, the distance between them is AB



Let $AC = BC = x \, and \, AB = y$

In ΔACD and ΔDCB,

$\angle CAD = \angle CBD$ $\quad [\because AC = BC]$

$\angle CDA = \angle CDB = 90^\circ$

$\therefore \angle ACD = \angle DCB$

$$or \quad \angle ACD = \frac{1}{2} \times \angle ACB$$

$$\Rightarrow \angle ACD = \frac{1}{2} \times 45^\circ$$

$$\Rightarrow \angle ACD = \frac{\pi}{8}$$

$$\therefore \sin\frac{\pi}{8} = \frac{AD}{AC}$$

$$\Rightarrow \sin\frac{\pi}{8} = \frac{y/2}{x} \quad [\because AD = y/2]$$

$$\Rightarrow \frac{y}{2} = x\sin\frac{\pi}{8}$$

$$\Rightarrow y = 2x.\sin\frac{\pi}{8}$$

Now, differentiating both sides w.r.t, we get

$$\frac{dy}{dt} = 2.\sin\frac{\pi}{8}.\frac{dx}{dt}$$

$$= 2.\sin\frac{\pi}{8}.v \quad \left[\because v = \frac{dx}{dt}\right]$$

$$= 2v.\frac{\sqrt{2-\sqrt{2}}}{2} \quad \left[\because \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}\right]$$

$$= \sqrt{2-\sqrt{2}}\ v\,unit/s$$

which is the rate at which A and B are being separated.

5. Find an angle $\theta, where\ 0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine.

Sol. Let θ increases twice as fast as its sine.

$\Rightarrow\ \theta = 2\sin\theta$

Now, on differentiating both sides w.r.t. t, we get

$$\frac{d\theta}{dt} = 2.\cos\ \theta.\frac{d\theta}{dt} \ \Rightarrow\ 1 = 2\cos\ \theta$$

$$\Rightarrow\ \frac{1}{2} = \cos\theta \ \Rightarrow\ \cos\theta = \cos\frac{\pi}{3}$$

$$\therefore\ \theta = \frac{\pi}{3}$$

So, the required angle is $\frac{\pi}{3}$.

6. Find the approximate value of (1.999)^5.

Sol. $Let\ x = 2$

$and\ \Delta x = -0.001\ [\because 2 - 0.001 = 1.999]$

$Let\ y = x^5$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 5x^4$$

Now, $\Delta y = \frac{dy}{dx}.\Delta x = 5x^4 \times \Delta x$

$= 5 \times 2^4 \times [-0.001]$

$= -80 \times 0.001 = -0.080$

$\therefore\ (1.999)^5 = y + \Delta y$

$= 2^5 + (-0.080)$

$= 32 - 0.080 = 31.920$

7. Find the approximate volume of metal in a hollow spherical shell whose internal and external radii are 3 cm and 3.0005 cm, respectively.

Sol. Let internal radius = r and external radius = R

$\therefore$ Volume of hollow spherical shell, $V = \frac{4}{3}\pi(R^3 - r^3)$

$\Rightarrow V = \frac{4}{3}\pi[(3.0005)^3 - (3)^3]$ ...(i)

Now, we shell use differentiation to get approximate value of $(3.0005)^3$.

$Let\ (3.0005)^3 = y + \Delta y$

$and\ x = 3, \Delta x = 0.0005$

$Also,\ let\ y = x^3$

On differentiating both sides w.r.t. x, we get

$\frac{dy}{dx} = 3x^2$

$\therefore\ \Delta y = \frac{dy}{dx} \times \Delta x = 3x^2 \times 0.0005$

$= 3 \times 3^2 \times 0.0005$

$= 27 \times 0.0005 = 0.0135$

Also, $(3.0005)^3 = y + \Delta y$

$= 3^3 + 0.0135 = 27.0135$

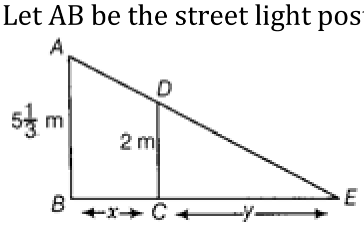
$\therefore\ V = \frac{4}{3}\pi[27.0135 - 27.000]$ [using $Eq.(i)$]

$= \frac{4}{3}\pi[0.0135] = 4\pi \times (0.0045)$

$= 0.0180\pi\, cm^3$

8. A man, 2 m tall, walks at the rate of $1\frac{2}{3} m/s$ towards a street light which is $5\frac{1}{3} m$ above the ground. At what rate is the tip of his shadow moving? At what rate is the length of the shadow changing when he is $3\frac{1}{3} m$ from the base of the light?

Sol. Let AB be the street light post and CD be the height of man i.e., CD = 2 m.



$Let\ BC = x\, m, CE = y\, m\, and\ \frac{dx}{dt} = \frac{-5}{3} m/s$

From $\Delta ABE\ and\ \Delta DCE$, we see that

$\Delta ABE \sim \Delta DCE$ [by AAA similarity]

$$\therefore \frac{AB}{DC}=\frac{BE}{CE}\Rightarrow\frac{\frac{16}{3}}{2}=\frac{x+y}{y}$$

$$\Rightarrow\frac{16}{6}=\frac{x+y}{y}$$

$$\Rightarrow 16y=6x+6y\Rightarrow 10y=6x$$

$$\Rightarrow y=\frac{3}{5}x$$

On differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt}=\frac{3}{5}.\frac{dx}{dt}=\frac{3}{5}.\left(-1\frac{2}{3}\right)$$

[since, *man is moving towards the light post*]

$$=\frac{3}{5}.\left(\frac{-5}{3}\right)=-1\,m/s$$

$Let\ \ z=x+y$

Now, differentiating both sides w.r.t. t, we get

$$\frac{dz}{dt}=\frac{dx}{dt}+\frac{dy}{dt}=-\left(\frac{5}{3}+1\right)$$

$$=-\frac{8}{3}=--2\frac{2}{3}m/s$$

Hence, the tip of shadow is moving at the rate of $2\frac{2}{3}m/s$ towards the light source and length of the shadow is decreasing at the rate of 1 m/s.

9. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L=200(10-t)^2$. How fast is the water running out at the end of 5 seconds? What is the average rate at which the water flows out during the first 5 seconds?

Sol. Let L represents the number of litres of water in the pool t seconds after the pool been plugged off to drain, then

$$L=200(10-t)^2$$

$\therefore$ Rate at which the water is running out $=-\frac{dL}{dt}$

$$\frac{dL}{dt}=-200.2(10-t).(-1)$$

$$=400(10-t)$$

Rate at which the water is running out at the end of 5 s

$$=400(10-5)$$

$$=2000L/s=Final\ rate$$

Since, $initial\ rate = -\left(\frac{dL}{dt}\right)_{t=0} = 4000L/s$

$\therefore\ Average\ rate\ during\ 5\,s = \frac{Initial\ rate + Final\ rate}{2}$

$= \frac{4000+2000}{2}$

$= 3000L/s$

10. The volume of a cube increases at a constant rate. Prove that the increase in its surface area varies inversely as the length of the side.

Sol. Let the side of a cube be x unit.

$\therefore\ Volume\ of\ cube\,(V) = x^3$

On differentiating both side w.r.t. t, we get

$\frac{dV}{dt} = 3x^2\frac{dx}{dt} = k$ [constant]

$\Rightarrow\ \frac{dx}{dt} = \frac{k}{3x^2}\ \ ...(i)$

$Also, surface\ area\ of\ cube,\ S = 6x^2$

On differentiating w.r.t. t, we get

$\frac{dS}{dt} = 12x.\frac{dx}{dt}$

$\Rightarrow \frac{dS}{dt} = 12\,x.\frac{k}{3x^2}$ [using $Eq.(i)$]

$\Rightarrow \frac{dS}{dt} = \frac{12k}{3x} = 4\left(\frac{k}{x}\right)$

$\Rightarrow \frac{dS}{dt}\,\alpha\,\frac{1}{x}$

Hence, the surface area of the cube varies inversely as the length of the side.

11. $x\ and\ y$ **are the sides of two square such that** $y = x - x^2$. **Find the rate of change of the area of second square with respect to the area of first square.**

Sol. Since, $x\ and\ y$ are the sides of two squares such that $y = x - x^2$.

$\therefore$ Area of the first square $(A_1) = x^2$

And area of the second square $(A_2) = y^2 = (x - x^2)^2$

$\therefore\ \frac{dA_2}{dt} = \frac{d}{dt}(x-x^2)^2 = 2\,(x-x^2)\left(\frac{dx}{dt} - 2x.\frac{dx}{dt}\right)$

$= \frac{dx}{dt}(1-2x)\,2\,(x-x^2)$

and $\frac{dA_1}{dt} = \frac{d}{dt}x^2 = 2x.\frac{dx}{dt}$

$$\therefore \frac{dA_2}{dA_1} = \frac{dA_2/dt}{dA_1/dt} = \frac{\frac{dx}{dt}.(1-2x)(2x-2x^2)}{2x.\frac{dx}{dt}}$$

$$= \frac{(1-2x)\,2x(1-x)}{2x}$$

$$= (1-2x)(1-x)$$

$$= 1 - x - 2x + 2x^2$$

$$= 2x^2 - 3x + 1$$

12. Find the condition that the curves $2x = y^2$ *and* $2xy = k$ intersect orthogonally.

Sol. Given, equation of curves are $2x = y^2 \ \ldots(i)$

and $2xy = k \ \ldots(ii)$

$$\Rightarrow y = \frac{k}{2x} \quad [\text{from } Eq.(ii)]$$

$$From\, Eq.(i),\ 2x = \left(\frac{k}{2x}\right)^2$$

$$\Rightarrow 8x^3 = k^2$$

$$\Rightarrow x^3 = \frac{1}{8}k^2$$

$$\Rightarrow x = \frac{1}{2}k^{2/3}$$

$$\therefore\ y = \frac{k}{2x} = \frac{k}{2.\frac{1}{2}k^{2/3}} = k^{1/3}$$

Thus, we get point of intersection of curves which is $\left(\frac{1}{2}k^{2/3}, k^{1/3}\right)$

From Eqs. (i) and (ii),

$$2 = 2y\frac{dy}{dx}$$

and $2\left[x.\frac{dy}{dx} + y.1\right] = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{y}$$

$$\left(\frac{dy}{dx}\right) = \frac{-2y}{2x} = -\frac{y}{x}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}k^{2/3},\, k^{1/3}\right)} = \frac{1}{k^{1/3}} \quad [say\, m_1]$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}k^{2/3},k^{1/3}\right)} = \frac{-k^{1/3}}{\frac{1}{2}k^{2/3}} = 2k^{-1/3} \ [say\, m_2]$$

Since, the curves intersect orthogonally.

$i.e., \quad m_1 . m_2 = -1$

$\Rightarrow \frac{1}{k^{1/3}}.(-2k^{-1/3}) = -1$

$\Rightarrow -2k^{-2/3} = -1$

$\Rightarrow \frac{2}{k^{2/3}} = 1$

$\Rightarrow \ k^{2/3} = 2$

$\therefore \ k^2 = 8$

Which is the required condition.

13. Prove that the curves $xy = 4$ *and* $x^2 + y^2 = 8$ touch each other.

Sol. Given equation of curves are

$xy = 4 \ \ldots(i)$

and $x^2 + y^2 = 8 \ \ldots(ii)$

$\Rightarrow x.\frac{dy}{dx} + y = 0$

and $2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$ and $\frac{dy}{dx} = \frac{-2x}{2y}$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{x} m_1 \ [say]$ and $\frac{dy}{dx} = \frac{-x}{y} = m_2 \ [say]$

Since, both the curves should have same slope.

$\therefore \frac{-y}{x} = \frac{-x}{y} \Rightarrow -y^2 = -x^2$

$\Rightarrow x^2 = y^2 \ \ldots(iii)$

Using the value of x^2 in Eq. (ii), we get

$y^2 + y^2 = 8$

$\Rightarrow y^2 = 4 \Rightarrow y = \pm 2$

$For \ y = 2, x = \frac{4}{2} = 2$

and for $y = -2, x = \frac{4}{-2} = -2$

Thus, the required points of intersection are $(2,\ 2)$ *and* $(-2,\ -2)$.

For $(2,2)$, $m_1 = \frac{-y}{x} = \frac{-2}{2} = -1$

and $m_2 = \frac{-x}{y} = \frac{-2}{2} = -1$

$\because m_1 = m_2$

For $(-2,2)$, $m_1 = \frac{-y}{x} = \frac{-(-2)}{-2} = -1$

and $m_2 = \frac{-x}{y} = \frac{-(-2)}{-2} = -1$

Thus, for both the intersection points, we see that slope of both the curves are same.
Hence, the curves touch each other.

14. Find the co-ordinates of the point on the curve $\sqrt{x}+\sqrt{y}=4$ at which tangent is equally inclined to the axes.

Sol. We have, $\sqrt{x}+\sqrt{y}=4$ ...(i)

$\Rightarrow x^{1/2}+y^{1/2}=4$

$\Rightarrow \frac{1}{2}.\frac{1}{x^{1/2}}+\frac{1}{2}.\frac{1}{y^{1/2}}.\frac{dy}{dx}=0$

$\therefore \frac{dy}{dx}=-\frac{1}{2}.x^{-1/2}\ 2.y^{1/2}$

$=-\sqrt{\frac{y}{x}}$

Since, tangent is equally inclined to the axes.

$\therefore \frac{dy}{dx}=\pm 1$

$\Rightarrow -\sqrt{\frac{y}{x}}=\pm 1$

$\Rightarrow \frac{y}{x}=1 \Rightarrow y=x$

$From\,Eq.(i),\ \sqrt{y}+\sqrt{y}=4$

$\Rightarrow 2\sqrt{y}=4$

$\Rightarrow 4y=16$

$\therefore\ y=4\, and\ x=4$

When $y=4\, and\ x=4$

So, the required coordinates are (4, 4).

15. Find the angle of intersection of the curves $y=4-x^2$ *and* $y=x^2$.

Sol. We have, $y=4-x^2$ ...(i)

and $y=x^2$ ...(ii)

$\Rightarrow \dfrac{dy}{dx} = -2x$ and $\dfrac{dy}{dx} = 2x$

$\Rightarrow m_1 = -2x$ and $m_2 = 2x$

$From\, Eqs. (i)\, and\, (ii),\; x^2 = 4 - x^2$

$\Rightarrow 2x^2 = 4$

$\Rightarrow x^2 = 2$

$\Rightarrow x = \pm\sqrt{2}$

$\therefore\; y = x^2 = (\pm\sqrt{2})^2 = 2$

So, the points of intersection are $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$.

For point $(+\sqrt{2}, 2)$ $m_1 = -2x = -2.\sqrt{2} = -2\sqrt{2}$

and $m_2 = 2x = 2\sqrt{2}$

and for point $(\sqrt{2}, 2)$ $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\dfrac{-2\sqrt{2} - 2\sqrt{2}}{1 - 2\sqrt{2}.2\sqrt{2}}\right| = \left|\dfrac{-4\sqrt{2}}{-7}\right|$

$\therefore \theta = \tan^{-1}\left(\dfrac{4\sqrt{2}}{7}\right)$

16. Prove that the curves $y^2 = 4x\; and\; x^2 + y^2 - 6x + 1 = 0$ touch each other at the point (1,2).

Sol. We have, $y^2 = 4x\; and\; x^2 + y^2 - 6x + 1 = 0$

Since, both the curves touch each other at (1, 2) i.e., curves are passing through (1, 2).

$\therefore 2y.\dfrac{dy}{dx} = 4$ and $2x + 2y\dfrac{dy}{dx} = 6$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{4}{2y}$ and $\dfrac{dy}{dx} = \dfrac{6 - 2x}{2y}$

$\Rightarrow \left(\dfrac{dy}{dx}\right)_{(1,2)} = \dfrac{4}{4} = 1$

and $\left(\dfrac{dy}{dx}\right)_{(1,2)} = \dfrac{6 - 2.1}{2.2} = \dfrac{4}{4} = 1$

$\Rightarrow m_1 = 1\, and\, m_2 = 1$

Thus, we see that slope of both the curves are equal to each other i.e., $m_1 = m_2 = 1$ at the point (1, 2).

Hence, both the curves touch each other.

17. Find the equation of the normal lines to the curves $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.

Sol. Given equation of the curve is

$3x^2 - y^2 = 8 \;\ldots(i)$

On differentiating both sides w.r.t. x, we get

$$6x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{6x}{2y} = \frac{3x}{y}$$

$$\Rightarrow m_1 = \frac{3x}{y} \ [say]$$

and $slop\, of\ normal\, (m_2) = \frac{-1}{m_1} = \frac{-y}{3x}$ ...(ii)

Since, slope of normal to the curve should be equal to the slope of line $x + 3y = 4,$ which is parallel to curve.

For line, $y = \frac{4-x}{3} = \frac{-x}{3} + \frac{4}{3}$

$$\Rightarrow Slope\, of\ the\, line\ (m_3) = \frac{-1}{3}$$

$$\therefore m_2 = m_3$$

$$\Rightarrow \frac{-y}{3x} = -\frac{1}{3}$$

$$\Rightarrow -3y = -3x$$

$$\Rightarrow y = x \ \ ...(iii)$$

On substituting the value of y in Eq. (i), we get

$$3x^2 - x^2 = 8$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2$$

$For\ \ x = 2,\ y = 2$ [using $Eq.(iii)$]

$and\ \ for\ x = -2,\ y = -2$ [using $Eq.(iii)$]

Thus, the points at which normal to curve are parallel to the line $x + 3y = 4,$ are $(2,\ 2)\ and\ (-2,\ -2).$

Required equations of normal are

$$y - 2 = m_2(x-2)\ and\ \ y + 2 = m_2(x+2)$$

$$\Rightarrow y - 2 = \frac{-2}{6}(x-2)\ and\ \ y + 2 = \frac{-2}{6}(x+2)$$

$$\Rightarrow 3y - 6 = -x + 2\ and\ 3y + 6 = -x - 2$$

$$\Rightarrow 3y + x = +8\ and\ 3y + x = -8$$

So, the required equations are $3y + x = \pm 8.$

18. **At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the y-axis?**

Sol. Given, equation of curve which is $x^2 + y^2 - 2x - 4y + 1 = 0$ ...(i)

$$\Rightarrow\ 2x+2y\frac{dy}{dx}-2-4\frac{dy}{dx}=0$$

$$\Rightarrow\ \frac{dy}{dx}(2y-4)=2-2x$$

$$\Rightarrow\ \frac{dy}{dx}=\frac{2(1-x)}{2(y-2)}$$

Since, the tangents are parallel to the Y-axis i.e., $\tan\theta=\tan 90°=\frac{dy}{dx}$.

$$\frac{1-x}{y-2}=\frac{1}{0}$$

$$\Rightarrow y-2=0\ \Rightarrow y=2$$

For y = 2 from Eq. (i), we get

$$x^2+2^2-2x-4\times 2+1=0$$

$$\Rightarrow x^2-2x-3=0$$

$$\Rightarrow x^2-3x+x-3=0$$

$$\Rightarrow x(x-3)+1(x-3)=0$$

$$\Rightarrow (x+1)(x-3)=0$$

$$\therefore\ x=-1, x=3$$

So, the required points are $(-1,\ 2)$ *and* $(3,\ 2)$.

19. Show that the line $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y=b\,.e^{-x/a}$ at the point where the curve intersects the axis of y?

Sol. We have the equation of line given by $\frac{x}{a}+\frac{y}{b}=1$, which touches the curve $y=b\,.e^{-x/a}$ at the point, where the curve intersects the axis of Y i.e., x = 0.

$$\therefore\ y=b.e^{-0/a}=b\ \ [\because e^0=1]$$

So, slope point of intersection on the curve with Y-axis is (0, b).

Now, slope of the given line at (0, b) is given by

$$\frac{1}{a}.1+\frac{1}{b}.\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-1}{a}.b$$

$$\Rightarrow \frac{dy}{dx}-\frac{1}{a}.b=\frac{-b}{a}=m_1\ \ [say]$$

Also, the slope of the curve (0, b) is

$$\frac{dy}{dx}=b.e^{-x/a}.\frac{-1}{a}$$

$$\frac{dy}{dx}=\frac{-b}{a}e^{-x/a}$$

$$\left(\frac{dy}{dx}\right)_{(0,b)} = \frac{-b}{a}e^{-0} = \frac{-b}{a} = m_2 \ \ [say]$$

Since, $m_1 = m_2 = \dfrac{-b}{a}$

Hence, the line touches the curve at the point, where the curve intersects the axis of Y.

20. **Show that** $f(x) = 2x + \cot^{-1} x + \log\left(\sqrt{1+x^2} - x\right)$ **is increasing in R.**

Sol. We have, $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$

$$\therefore \ f'(x) = 2 + \left(\frac{-1}{1+x^2}\right) + \frac{1}{(\sqrt{1+x^2} - x)}\left(\frac{1}{2\sqrt{1+x^2}}.2x - 1\right)$$

$$= 2 - \frac{1}{1+x^2} + \frac{1}{(\sqrt{1+x^2} - x)}.\frac{(x - \sqrt{1+x^2})}{\sqrt{1+x^2}}$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{2 + 2x^2 - 1 - \sqrt{1+x^2}}{1+x^2} = \frac{1 + 2x^2 - \sqrt{1+x^2}}{1+x^2}$$

For increasing function, $f'(x) \geq 0$

$$\Rightarrow \frac{1 + 2x^2 - \sqrt{1+x^2}}{1+x^2} \geq 0$$

$$\Rightarrow 1 + 2x^2 \geq \sqrt{1+x^2}$$

$$\Rightarrow (1 + 2x^2)^2 \geq 1 + x^2$$

$$\Rightarrow 1 + 4x^4 + 4x^2 \geq 1 + x^2$$

$$\Rightarrow 4x^4 + 3x^2 \geq 0$$

$$\Rightarrow x^2(4x^2 + 3) \geq 0$$

which is true for any real value of x.

Hence, $f(x)$ increasing in R.

21. **Show that for** $a \geq 1, f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ **is decreasing in R.**

Sol. We have, $a \geq 1, f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$

$$\therefore \ f'(x) = \sqrt{3}\cos x - (-\sin x) - 2a$$

$$= \sqrt{3}\cos x + \sin x - 2a$$

$$= 2\left[\frac{\sqrt{3}}{2}.\cos x + \frac{1}{2}.\sin x\right] - 2a$$

$$= 2\left[\cos\frac{\pi}{6}.\cos x + \sin\frac{\pi}{6}.\sin x\right] - 2a$$

$= 2\left(\cos\frac{\pi}{6} - x\right) - 2a$

$[\because \cos(A-B) = \cos A.\cos B + +\sin A.\sin B]$

$= 2\left[\left(\cos\frac{\pi}{6} - x\right) - a\right]$

We know that, $\cos x \in [-1, 1]$ and $a \geq 1$

So, $2\left[\cos\left(\cos\frac{\pi}{6} - x\right) - a\right] \leq 0$

$\therefore \; f'(x) \leq 0$

Hence, $f(x)$ is a decreasing function in R.

22. Show that $f(x) = tan^{-1}(sin\,x + cos\,x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

Sol. We have, $f(x) = \tan^{-1}(\sin x + \cos x)$

$\therefore \; f'(x) = \frac{1}{1 + (\sin x + \cos x)^2}.(\cos x - \sin x)$

$= \frac{1}{1 + \sin^2 x + \cos^2 x + 2\sin x.\cos x}(\cos x - \sin x)$

$= \frac{1}{(2 + \sin 2x)}(\cos x - \sin)$

$[\because \sin 2x = 2\sin x \cos x \; and \; \sin^2 x + \cos^2 x = 1]$

For $f'(x) \geq 0$.

$\frac{1}{(2 + \sin 2x)}.(\cos x - \sin x) \geq 0$

$\Rightarrow \; \cos x - \sin x \geq 0 \; \left[\because (2 + \sin 2x) \geq 0 \; in \left(0, \frac{\pi}{4}\right)\right]$

$\Rightarrow \cos x \geq \sin x$

Which is true, if $x \in \left(0, \frac{\pi}{4}\right)$

Hence, $f(x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

23. At what point, the slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is maximum? Also, find the maximum slope.

Sol. We have, $y = -x^3 + 3x^2 + 9x - 27$

$\therefore \; \frac{dy}{dx} = -3x^2 + 6x + 9 =$ Slope of tangent to the curve

Now, $\frac{d^2y}{dx^2} = -6x + 6$

For $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0,$

$-6x + 6 = 0$

$\Rightarrow x = \frac{-6}{-6} = 1$

$\therefore \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = -6 < 0$

So, the slope of tangent to the curve is maximum, when x = 1.

For x = 1, $\left(\frac{dy}{dx}\right)_{(x=1)} = -3.1^2 + 6.1 + 9 = 12,$

Which is maximum slope.

$Also, for\ x = 1, y = -1^3 + 3.1^2 + 9.1 - 27$

$= -1 + 3 + 9 - 27 \quad = -16$

So, the required point is (1, - 16).

24. **Prove that $f(x) = \sin x + \sqrt{3}\cos x$ has maximum value at $x = \frac{\pi}{6}$.**

Sol. We have, $f(x) = \sin x + \sqrt{3}\cos x$

$\therefore f'(x) = \cos x + \sqrt{3}(-\sin x)$

$= \cos x - \sqrt{3}\sin x$

For $f'(x) = 0$, $\cos x = \sqrt{3}\sin x$

$\Rightarrow \tan x = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$

$\Rightarrow x = \frac{\pi}{6}$

Again, differentiating $f'(x)$ we get

$f''(x) = -\sin x - \sqrt{3}\cos x$

$At\ x = \frac{\pi}{6},\ f''(x) = -\sin\frac{\pi}{6} - \sqrt{3}\cos\frac{\pi}{6}$

$= -\frac{1}{2} - \sqrt{3}.\frac{\sqrt{3}}{2}$

$= -\frac{1}{2} - \frac{3}{2} = -2 < 0$

Hence, at $x = \frac{\pi}{6}$, $f(x)$ has maximum value at $\frac{\pi}{6}$ is the point of local maxima.

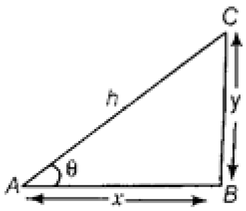
Application of Derivatives
<u>Long Answer Type Questions</u>

25. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$

Sol. Let ABC be a triangle with $AC = h,\ AB = x\ and\ BC = y.$

Also, $\angle CAB = \theta$

Let $h + x = k\ \ ...(i)$



$\therefore\ \cos\theta = \frac{x}{h}$

$\Rightarrow x = h\cos\theta$

$\Rightarrow h + h\cos\theta = k\ \ [\text{using } Eq.(i)]$

$\Rightarrow h(1+\cos\theta) = k$

$\Rightarrow h = \frac{k}{(1+\cos\theta)}\ \ ...(ii)$

Also, $area\,of\ \Delta ABC = \frac{1}{2}(AB.BC)$

$A = \frac{1}{2}.x.y$

$= \frac{1}{2}h\cos\theta.h\sin\theta\ \left[\because \sin\theta = \frac{y}{h}\right]$

$= \frac{1}{2}h^2\sin\theta.\cos\theta$

$= \frac{2h^2}{4}\sin\theta.\cos\theta$

$= \frac{1}{4}h^2\sin\theta.2\theta\ \ ...(iii)$

Since, $h = \frac{k}{1+\cos\theta}$

$\therefore\ A = \frac{1}{4}\left(\frac{k}{1+\cos\theta}\right)^2.\sin 2\theta$

$$\Rightarrow A=\frac{k^2}{4}.\frac{\sin 2\theta}{(1+\cos\theta)^2} \quad ...(iv)$$

$$\therefore \frac{dA}{d\theta}=\frac{k^2}{4}\left[\frac{(1+\cos\theta)^2.\cos 2\,\theta.2-\sin 2\theta.2(1+\cos\theta).(0-\sin\theta)}{(1+\cos\theta)^4}\right]$$

$$=\frac{k^2}{4}.\left\{\frac{2(1+\cos\theta)[(1+\cos\theta).\cos 2\theta+\sin 2\theta(\sin\theta)}{(1+\cos\theta)^4}\right\}$$

$$=\frac{k^2}{4}.\frac{2}{(1+\cos\theta)^3}\left[(1+\cos\theta).\cos 2\theta+2\sin^2\theta.\cos\theta\right]$$

$$=\frac{k^2}{2(1+\cos\theta)^3}\left[(1+\cos\theta)(1-2\sin^2\theta)+2\sin^2\theta.\cos\theta\right]$$

$$=\frac{k^2}{2(1+\cos\theta)^3}\left[1+\cos\theta-2\sin^2\theta-2\sin^2\theta.\cos\theta+2\sin^2\theta.\cos\theta\right]$$

$$=\frac{k^2}{2(1+\cos\theta)^3}\left[(1+\cos\theta)-2\sin^2\theta\right]$$

$$=\frac{k^2}{2(1+\cos\theta)^3}[1+\cos\theta-2+2\cos^2\theta]$$

$$=\frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta+\cos\theta-1) \quad ...(v)$$

For $\frac{dA}{d\theta}=0,$

$$\frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta+\cos\theta-1)=0$$

$$\Rightarrow 2\cos^2\theta+\cos\theta-1=0$$

$$\Rightarrow 2\cos^2\theta+2\cos\theta-\cos\theta-1=0$$

$$\Rightarrow 2\cos\theta(\cos\theta+1)-1(\cos\theta+1)=0$$

$$\Rightarrow (2\cos\theta-1)(\cos\theta+1)=0$$

$$\Rightarrow \cos\theta=\frac{1}{2}\, or \cos\theta=-1$$

$$\Rightarrow \theta=\frac{\pi}{3} \ [possible]$$

$$or \ \theta=2n\pi\pm\pi \ [not\ possible]$$

$$\therefore \ \theta=\frac{\pi}{3}$$

Again, differentiating w.r.t. θ in Eq. (v), we get

$$\frac{d}{d\theta}\left(\frac{dA}{d\theta}\right)=\frac{d}{d\theta}\left[\frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta+\cos\theta-1)\right]$$

$$\therefore\ \frac{d^2A}{d\theta}=\frac{d}{d\theta}\left[\frac{k^2(2\cos\theta-1)(1+\cos\theta)}{2(1+\cos\theta)^3}\right]=\frac{d}{d\theta}\left[\frac{k^2}{2}.\frac{(2\cos\theta-1)}{(1+\cos\theta)^2}\right]$$

$$=\frac{k^2}{2}\left[\frac{(1+\cos\theta)^2.(-2\sin\theta)-2(1+\cos\theta).(-\sin\theta)(2\cos\theta-1)}{(1+\cos\theta)^4}\right]$$

$$=\frac{k^2}{2}\left[\frac{(1+\cos\theta).[1+\cos\theta](-2\sin\theta)+2\sin\theta(2\cos\theta-1)}{(1+\cos\theta)^4}\right]$$

$$=\frac{k^2}{2}\left[\frac{-2\sin\theta-2\sin\theta.\cos\theta+4\sin\theta.\cos\theta-2\sin\theta}{(1+\cos\theta)^3}\right]$$

$$=\frac{k^2}{2}\left[\frac{-4\sin\theta-\sin2\theta+2\sin2\theta}{(1+\cos\theta)^3}\right]=\frac{k^2}{2}\left[\frac{\sin2\theta-4\sin\theta}{(1+\cos\theta)^3}\right]$$

$$\therefore\ \left(\frac{d^2A}{d\theta}\right)_{at\,\theta=\frac{\pi}{3}}=\frac{k^2}{2}\left[\frac{\sin\frac{2\pi}{3}-4\sin\frac{\pi}{3}}{\left(1+\cos\frac{\pi}{3}\right)^3}\right]=\frac{k^2}{2}\left[\frac{\frac{\sqrt{3}}{2}-\frac{4\sqrt{3}}{2}}{\left(1+\frac{1}{2}\right)^3}\right]$$

$$=\frac{k^2}{2}\left[\frac{-3\sqrt{3}.8}{2.27}\right]=-k^2\left(\frac{2\sqrt{3}}{9}\right)$$

which is less than zero.

Hence, area of the right angled triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

26. Find the points of local maxima, local minima and the points of inflection of the function $f(x)=x^5-5x^4+5x^3-1$. Also, find the corresponding local maximum and local minimum values.

Sol. Given that, $f(x)=x^5-5x^4+5x^3-1$

On differentiating w.r.t. x, we get

$f'(x)=5x^4-20x^3+15x^2$

For maxima or minima $f'(x)=0$

$\Rightarrow 5x^4-20x^3+15x^2=0$

$\Rightarrow 5x^2(x^2-4x+3)=0$

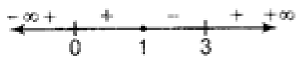
$\Rightarrow 5x^2(x^2-3x-x+3)=0$

$\Rightarrow 5x^2[x(x-3)-1(x-3)]=0$

$\Rightarrow 5x^2[(x-1)(x-3)]=0$

$\therefore\ x=0,1,3$

Sing scheme for $\frac{dy}{dx}=5x^2(x-1)(x-3)$



So, y has maximum value at x = 1 and minimum value at x = 3.
At x = 0, y has neither maximum nor minimum value.
$\therefore$ *Maximum value of* $y=1-5+5-1=0$

and Minimum value $=(3)^5-5(3)^4+5(3)^3-1$

$=243-81\times5-27\times5-1=-298$

27. A telephone company in a town has 500 subscribers on its list and collects fixed charges of Rs. 300/- per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of Re 1/- on subscriber will discontinue the service. Find what increase will bring maximum profit?

Sol. Consider that company increases the annual subscription by Rs. x
So, x subscribes will discontinue the service.
$\therefore$ Total revenue of company after the increment is given by
$R(x)=(500-x)(300+x)$

$=15\times10^4+500x-300x-x^2$

$=-x^2+200x+150000$

On differentiating both sides w.r.t. x, we get
$R'(x)=-2x+200$

Now, $R'(x)=0$

$\Rightarrow 2x=200 \Rightarrow x=100$

$\therefore\ R''(x)=-2<0$

So, $R(x)$ is maximum when $x=100$

Hence, the company should increase the subscription fee by $Rs.\ 100$, so that it has maximum profit.

28. If the straight line $x\ cos\alpha+y\ sin\alpha=p$ touches the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then prove that

$a^2cos^2\alpha+b^2sin^2\alpha=p^2$.

Sol. Given, *line is* $x\cos\alpha+y\sin\alpha=p\ \ ...(i)$

and *curve is* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\Rightarrow b^2x^2+a^2y^2=a^2b^2\ \ ...(ii)$

Now, differentiating Eq. (ii) w.r.t. x, we get

$b^2.2x+a^2.2y.\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=\frac{-2b^2x}{2a^2y}=\frac{-xb^2}{ya^2}\ \ ...(iii)$

From Eq.(i), $y\sin\alpha=p-x\cos\alpha$

$\Rightarrow y = -x\cot\alpha + \dfrac{p}{\sin\alpha}$

Thus, slope of the line is $(-\cot\alpha)$

So, the given equation of line will be tangent to the Eq. (ii), if $\left(-\dfrac{x}{y}.\dfrac{b^2}{a^2}\right) = (-\cot\alpha)$

$\Rightarrow \dfrac{x}{a^2\cos\alpha} = \dfrac{y}{b^2\sin\alpha} = k\ \ [say]$

$\Rightarrow x = ka^2\cos\alpha$

and $y = b^2\ k\sin\alpha$

So, the line $x\cos a + y\sin\alpha = p$ will touch the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ at point

$(ka^2\cos\alpha, kb^2\sin\alpha).$

$From\, Eq.(i),\ ka^2\cos^2\alpha + kb^2\sin^2\alpha = p$

$\Rightarrow a^2\cos^2\alpha + b^2\sin^2\alpha = \dfrac{p}{k}$

$\Rightarrow (a^2\cos^2\alpha + b^2\sin\alpha)^2 = \dfrac{p^2}{k^2}\ \ ...(iv)$

$From\, Eq.(ii),\ b^2k^2a^4\cos^2\alpha + a^2k^2b^4\sin^2\alpha = a^2b^2$

$\Rightarrow k^2(a^2\cos^2\alpha + b^2\sin^2\alpha) = 1$

$\Rightarrow (a^2\cos^2\alpha + b^2\sin^2\alpha) = \dfrac{1}{k^2}\ \ ...(v)$

On dividing Eq. (iv) by Eq. (v), we get

$a^2\cos^2\alpha + b^2\sin^2\alpha = p^2\ \ \ Hence\ proved.$

Alternate Method

We know that, if a line $y = mx + c$ touches ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then

the required condition is $c^2 = a^2m^2 + b^2$

Here, given equation of the line is

$x\cos\alpha + y\sin\alpha = p$

$\Rightarrow y = \dfrac{p - x\cos\alpha}{\sin\alpha}$

$= -x\cot\alpha + \dfrac{p}{\sin\alpha}$

$\Rightarrow c = \dfrac{p}{\sin\alpha}$

and $m = \cot\alpha$

$\therefore \left(\frac{p}{\sin\alpha}\right)^2 = a^2(-\cot\alpha)^2 + b^2$

$\Rightarrow \frac{p^2}{\sin^2\alpha} = a^2\frac{\cot^2\alpha}{\sin^2\alpha} + b$

$\Rightarrow p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$ *Hence proved.*

29. An open box with square base is to be made of a given quantity of card board of area c^2. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

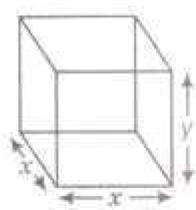
Sol. Let the length of side of the square base of open box be x *units* and its height be y *units*.

$\therefore$ *Area of the metal used* $= x^2 + 4xy$

$\Rightarrow x^2 + 4xy = c^2$ [*given*]

$\Rightarrow y = \frac{c^2 - x^2}{4x}$ ...(*i*)

Now, *Volume of the box* $(V) = x^2 y$



$\Rightarrow V = x^2.\left(\frac{c^2 - x^2}{4x}\right)$

$= \frac{1}{4}x(c^2 - x^2)$

$= \frac{1}{4}(c^2x - x^3)$

On differentiating both sides w.r.t. x, we get

$\frac{dV}{dx} = \frac{1}{4}(c^2 - 3x^2)$ ...(*ii*)

Now, $\frac{dV}{dx} = 0 \Rightarrow c^2 = 3x^2$

$\Rightarrow x^2 = \frac{c^2}{3}$

$\Rightarrow x = \frac{c}{\sqrt{3}}$ [using positive sign]

Again, differentiating Eq. (ii) w.r.t. x, we get

$\frac{d^2V}{dx^2} = \frac{1}{4}(-6x) = \frac{-3}{2}x < 0$

$$\therefore \left(\frac{d^2v}{dx^2}\right)_{at\, x=\frac{c}{\sqrt{3}}} = -\frac{3}{2}.\left(\frac{c}{\sqrt{3}}\right) < 0$$

Thus, we see that volume (V) is maximum at $x = \frac{c}{\sqrt{3}}$.

$\therefore$ Maximum volume of the box, $(V)_{x=\frac{c}{\sqrt{3}}} = \frac{1}{4}\left(c^2.\frac{c}{\sqrt{3}} - \frac{c^3}{3\sqrt{3}}\right)$

$$= \frac{1}{4}.\frac{(3c^3 - c^3)}{3\sqrt{3}} = \frac{1}{4}.\frac{2c^3}{3\sqrt{3}}$$

$$= \frac{c^3}{6\sqrt{3}}\ cu\ units$$

30. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also, find the maximum volume.

Sol. Let breadth and length of the rectangle be x and y, respectively.

$\therefore$ Perimeter of the rectangle $= 36\,cm$

$\Rightarrow 2x + 2y = 36$

$\Rightarrow x + y = 18$

$\Rightarrow y = 18 - x \ \ ...(i)$

Let the rectangle is being revolved about its length y.

Then, volume (V) of resultant cylinder $= \pi x^2.y$

$\Rightarrow V = \pi x^2.(18 - x)\ \ [\because V = \pi r^2 h][\text{using } Eq.(i)]$

$= 18\pi x^2 - \pi x^3 = \pi[18x^2 - x^3]$

On differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \pi(36x - 3x^2)$$

Now, $\frac{dV}{dx} = 0$

$\Rightarrow 36x = 3x^2$

$\Rightarrow 3x^2 - 36x = 0$

$\Rightarrow 3(x^2 - 12x) = 0$

$\Rightarrow 3x(x - 12) = 0$

$\Rightarrow x = 0, x - 12$

$\therefore\ x = 12\ \ [\because, x \neq 0]$

Again, differentiating w.r.t. x, we get

$$\frac{d^2V}{dx^2} = \pi(36-6x)$$

$$\Rightarrow \left(\frac{d^2V}{dx^2}\right)_{x=12} = \pi(36-6\times12) = -36\pi < 0$$

At x = 12, volume of the resultant cylinder is the maximum.

So, the dimensions of rectangle are 12 cm and 6 cm, respectively. [using Eq. (i)]

$\therefore$ Maximum volume of resultant cylinder,

$$(V)_{x=12} = \pi[18.(12)^2-(12)^3]$$

$$= \pi[12^2(18-12)]$$

$$= \pi\times144\times6$$

$$= 864\pi\, cm^3$$

31. If the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum?

Sol. Let length of one cube be x units and radius of sphere be r units.

$\therefore$ Surface area of cube $= 6x^2$

and Surface area of sphere $= 4\pi r^2$

Also, $6x^2 + 4\pi r^2 = k$ [constant, *given*]

$$\Rightarrow 6x^2 = k - 4\pi r^2$$

$$\Rightarrow x^2 = \frac{k-4\pi r^2}{6}$$

$$\Rightarrow x = \left[\frac{k-4\pi r^2}{6}\right]^{1/2} \quad ...(i)$$

Now, $volume\ of\ cube = x^3$

and $volume\ of\ sphere = \frac{4}{3}\pi r^3$

Let sum of volume of the cube and volume of the sphere be given by

$$S = x^3 + \frac{4}{3}\pi r^3 = \left[\frac{k-4\pi r^2}{6}\right]^{3/2} + \frac{4}{3}\pi r^3$$

On differentiating both sides w.r.t. x, we get]

$$\frac{dS}{dr} = \frac{3}{2}\left[\frac{k-4\pi r^2}{6}\right]^{1/2}.\left(\frac{-8\pi r}{6}\right) + \frac{12}{3}\pi r^2$$

$$= -2\pi r\left[\frac{k-4\pi r^2}{6}\right]^{1/2} + 4\pi r^2 \quad ...(ii)$$

$$= -2\pi r\left[\left\{\frac{k-4\pi r^2}{6}\right\}^{1/2} - 2r\right]$$

Now, $\frac{dS}{dr} = 0$

$$\Rightarrow r = 0\, or\, 2r = \left(\frac{k-4\pi r^2}{6}\right)^{1/2}$$

$$\Rightarrow 4r^2 = \frac{k-4\pi r^2}{6} \Rightarrow 24r^2 = k - 4\pi r^2$$

$$\Rightarrow 24r^2 + 4\pi r^2 = k \Rightarrow\ r^2[24+4\pi] = k$$

$$\therefore\ r = 0\ or\quad r = \sqrt{\frac{k}{24+4\pi}} = \frac{1}{2}\sqrt{\frac{k}{6+\pi}}$$

We know that, $r \neq 0$

$$\therefore\ r = \frac{1}{2}\sqrt{\frac{k}{6+\pi}}$$

Again, differentiating w.r.t. r in Eq. (ii), we get

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[-2\pi r\left\{\left(\frac{k-4\pi r^2}{6}\right)^{1/2} + 4\pi r^2\right\}\right]$$

$$= 2\pi\left[r.\frac{1}{2}\left(\frac{k-4\pi r^2}{6}\right)^{-1/2}.\left(\frac{-8\pi r}{6}\right) + \left(\frac{k-4\pi r^2}{6}\right)^{1/2}.1\right] + 4\pi.2r$$

$$= -2\pi\left[r.\frac{1}{2\sqrt{\frac{k-4\pi r^2}{6}}}.\left(\frac{-8\pi r}{6}\right) + \sqrt{\frac{k-4\pi r^2}{6}}\right] + 8\pi r$$

$$= 2\pi\left[\frac{-8\pi r^2 + 12\left(k - \frac{4\pi r^2}{6}\right)}{12\sqrt{\frac{k-4\pi r^2}{6}}}\right] + 8\pi r$$

$$= -2\pi\left[\frac{-48\pi r^2 + 72k - 48\pi r^2}{72\sqrt{\frac{k-4\pi r^2}{6}}}\right] + 8\pi r = -2\pi\left[\frac{-96\pi r^2 + 72k}{72\sqrt{\frac{k-4\pi r^2}{6}}}\right] + 8\pi r > 0$$

For $r = \frac{1}{2}\sqrt{\frac{k}{6+\pi}}$ then the sum of their volume is minimum.

For $r=\frac{1}{2}\sqrt{\frac{k}{6+\pi}}$, $x=\left[\frac{k-4\pi.\frac{1}{4}\frac{k}{(6+\pi)}}{6}\right]^{1/2}$

$$=\left[\frac{(6+\pi)k-\pi k}{6(6+\pi)}\right]^{1/2}=\left[\frac{k}{6+\pi}\right]^{1/2}=2r$$

Since, the sum of their volume is minimum when $x=2r$.

Hence, the ratio of an edge of cube to the diameter of the sphere is $1:1$.

32. AB is a diameter of a circle and C is any point on the circle. Show that the area of Δ ABC is maximum, when its isosceles.

Sol. We have, $AB=2r$

and $\angle ACB=90^0$ [since, angle in the semi-circle is alsways 90^0]

Let $AC=x \, and \, BC=y$

$\therefore (2r)^2=x^2+y^2$

$\Rightarrow y^2=4r^2-x^2$

$\Rightarrow y=\sqrt{4r^2-x^2}$ ...(i)

Now, $area\, of\ \Delta ABC, A=\frac{1}{2}\times x\times y$

$=\frac{1}{2}\times x\times(4r^2-x^2)^{1/2}$ [using $Eq.(i)$]

Now, differentiating both sides w.r.t. x, we get

$$\frac{dA}{dx}=\frac{1}{2}\left[x.\frac{1}{2}(4r^2-x^2)^{-1/2}.(0-2x)+(4r^2-x^2)^{1/2}.1\right]$$

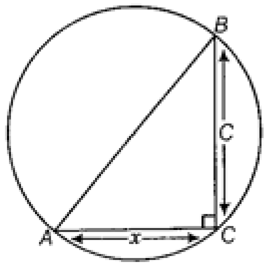
$$=\frac{1}{2}\left[\frac{-2x^2}{2\sqrt{4r^2-x^2}}+(4r^2-x^2)^{1/2}\right]$$

$$=\frac{1}{2}\left[\frac{-x^2}{\sqrt{4r^2-x^2}}+\sqrt{4r^2-x^2}\right]$$

$$=\frac{1}{2}\left[\frac{-x^2+4r^2-x^2}{\sqrt{4r^2-x^2}}\right]=\frac{1}{2}\left[\frac{-2x^2+4r^2}{\sqrt{4r^2-x^2}}\right]$$

$$\Rightarrow \frac{dA}{dx}=\left[\frac{(-x^2+2r^2)}{\sqrt{4r^2-x^2}}\right]$$

Now, $\frac{dA}{dx}=0$



$\Rightarrow -x^2+2r^2=0$

$\Rightarrow r^2=\frac{1}{2}x^2$

$\Rightarrow r=\frac{1}{\sqrt{2}}x$

$\Rightarrow x=r\sqrt{2}$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2A}{dx^2}=\frac{\sqrt{4r^2-x^2}.(-2x)+(2r^2-x^2).\frac{1}{2}(4r^2-x^2)^{-1/2}(-2x)}{(\sqrt{4r^2-x^2})^2}$$

$$=\frac{-2x\left[\sqrt{4r^2-x^2}+(2r^2-x^2).\frac{1}{2\sqrt{4r^2-x^2}}\right]}{(\sqrt{4r^2-x^2})^2}$$

$$=\frac{-4x.\left(\sqrt{4r^2-x^2}\right)^2+(2r^2-x^2)(-2x)}{2.(4r^2-x^2)^{3/2}}$$

$$=\frac{-4x(4r^2-x^2)+(2r^2-x^2).(-2x)}{2.(4r^2-x^2)^{3/2}}$$

$$=\frac{-16xr^2+4x^3+(2r^2-x^2)(-2x)}{2.(4r^2-x^2)^{3/2}}$$

$$\left(\frac{d^2A}{dx^2}\right)_{x=r\sqrt{2}}=\frac{-16r\sqrt{2}.r^2+4.(r\sqrt{2})^3+[2r^2-(r\sqrt{2})^2].(-2.r\sqrt{2})}{2.(4r^2-2r^2)^{3/2}}\ [\because x=r\sqrt{2}]$$

$$=\frac{-16\sqrt{2}.r^3+8\sqrt{2}r^3}{2(2r^2)^{3/2}}=\frac{8\sqrt{2}\ r^2[r-2r]}{4r^3}$$

$$=\frac{-8\sqrt{2}\ r^3}{4r^3}=-2\sqrt{2}<0$$

For $x=r\sqrt{2}$, the area of triangle is maximum.

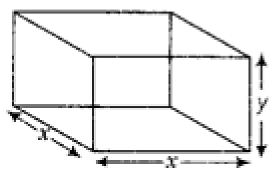
$For\ x = r\sqrt{2},\ \ y = \sqrt{4r^2 - (r\sqrt{2})^2} = \sqrt{2r^2} = r\sqrt{2}$

Since, $x = r\sqrt{2} = y$

Hence, the triangle is isosceles.

33. **A metal box with a square base and vertical sides is to contain** $1024\ cm^3$. **The material for the top and bottom costs** $Rs.\ 5/cm^2$ **and the material for the sides costs** $Rs.\ 2.50/cm^2$. **Find the least cost of the box.**

Sol. Since, volume of the box = $1024\ cm^3$.
Let length of the side of square base be $x\ cm$ and height of the box be $y\ cm$.



$\therefore Volume\ of\ the\ box\ (V) = x^2 . y = 1024$

Since, $x^2 y = 1024 \Rightarrow y = \dfrac{1024}{x^2}$

Let C denotes the cost of the box.

$\therefore C = 2x^2 \times 5 + 4xy \times 2.50$

$= 10x^2 + 10xy = 10x(x + y)$

$= 10x\left(x + \dfrac{1024}{x^2}\right)$

$= \dfrac{10x}{x^2}(x^3 + 1024)$

$\Rightarrow C = 10x^2 + \dfrac{10240}{x}\ \ ...(i)$

On differentiating both sides w.r.t. x, we get

$\dfrac{dC}{dx} = 20x + 10240(-x)^{-2}$

$= 20x - \dfrac{10240}{x^2}\ \ ...(ii)$

Now, $\dfrac{dC}{dx} = 0$

$\Rightarrow 20x = \dfrac{10240}{x^2}$

$\Rightarrow 20x^3 = 10240$

$\Rightarrow x^3 = 512 = 8^3 \Rightarrow\ x = 8$

Again, differentiating Eq. (ii) w.r.t. x, we get

$$\frac{d^2C}{dx^2} = 20 - 10240(-2).\frac{1}{x^3}$$

$$= 20 + \frac{20480}{x^3} > 0$$

$$\therefore \left(\frac{d^2C}{dx^2}\right)_{x=8} = 20 + \frac{20480}{512} = 60 > 0$$

For $x = 8,$ cost is minimum and the corresponding least cost of the box

$$C(8) = 10.8^2 + \frac{10240}{8}$$

$$= 640 + 1280 = 1920$$

$$\therefore \; Least \cos t = Rs.1920$$

34. The sum of the surface areas of a rectangular parallelopiped with sides $x,\ 2x\ and\ \frac{x}{3}$ and a sphere is given to constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of the sphere. Also, find the minimum value of the sum of their volumes.

Sol. We have given that, the sum of the surface areas of a rectangular parallelepiped with sides $x,\ 2x\ and\ \frac{x}{3}$ and a sphere is constant.

Let S be the sum of both the surface area.

$$\therefore \; S = 2\left(x.2x + 2x.\frac{x}{3} + \frac{x}{3}.x\right) + 4\pi r^2 = k$$

$$k = 2\left[2x^2 + \frac{2x^2}{3} + \frac{x^2}{3}\right] + 4\pi r^2$$

$$= 2[3x^2] + 4\pi r^2 = 6x^2 + 4\pi r^2$$

$$\Rightarrow 4\pi r^2 = k - 6x^2$$

$$\Rightarrow r^2 = \frac{k - 6x^2}{4\pi}$$

$$\Rightarrow r = \sqrt{\frac{k - 6x^2}{4\pi}} \quad ...(i)$$

Let V denotes the volume of both the parallelepiped and the sphere.

Then, $V = 2x.x.\frac{x}{3} + \frac{4}{3}\pi r^3 = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi\left(\frac{k - 6x^2}{4\pi}\right)^{3/2}$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi.\frac{1}{8\pi^{3/2}}(k - 6x^2)^{3/2}$$

$$= \frac{2}{3}x^3 + \frac{1}{6\sqrt{\pi}}(k-6x^2)^{3/2} \quad ...(ii)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \frac{2}{3}.3x^2 + \frac{1}{6\sqrt{\pi}}.\frac{3}{2}(k-6x^2)^{1/2}.(-12x)$$

$$= 2x^2 - \frac{12x}{4\sqrt{\pi}}\sqrt{k-6x^2}$$

$$= 2x^2 - \frac{3x}{\sqrt{\pi}}(k-6x^2)^{1/2} \quad ...(iii)$$

$$\because \frac{dV}{dx} = 0$$

$$\Rightarrow 2x^2 = \frac{3x}{\sqrt{\pi}}(k-6x^2)^{1/2}$$

$$\Rightarrow 4x^4 = \frac{9x^2}{\pi}(k-6x^2)$$

$$\Rightarrow 4\pi x^4 = 9k\,x^2 - 54x^4$$

$$\Rightarrow 4\pi x^4 + 54x^4 = 9\,k\,x^2$$

$$\Rightarrow x^4[4\pi+54] = 9.k.x^2$$

$$\Rightarrow x^2 = \frac{9\,k}{4\pi+54}$$

$$\Rightarrow x = 3.\sqrt{\frac{k}{4\pi+54}} \quad ...(iv)$$

Again, differentiating Eq. (iii) w.r.t. x, we get

$$\frac{d^2V}{dx^2} = 4x - \frac{3}{\sqrt{\pi}}\left[x.\frac{1}{2}(k-6x^2)^{-1/2}.(-12x) + (k-6x^2)^{1/2}.1\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}[-6x^2.(k-6x^2)^{-1/2} + (k-6x^2)^{1/2}]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2+k-6x^2}{\sqrt{k-6x^2}}\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{k-12x^2}{\sqrt{k-6x^2}}\right]$$

$$Now, \left(\frac{d^2V}{dx^2}\right)_{x=3.\sqrt{\frac{k}{4\pi+54}}} = 4.3\sqrt{\frac{k}{4\pi+54}} - \frac{3\pi}{\sqrt{\pi}}\left[\frac{k-12.9.\frac{k}{4\pi+54}}{\sqrt{k-\frac{6.9.k}{4\pi+54}}}\right]$$

$$=12\sqrt{\frac{k}{4\pi+54}}-\frac{3}{\sqrt{\pi}}\left[\frac{k-\frac{108k}{4\pi+54}}{\sqrt{k-\frac{54k}{4\pi+54}}}\right]$$

$$=12\sqrt{\frac{k}{4\pi+54}}-\frac{3}{\sqrt{\pi}}\left[\frac{4k\pi+54k-108k/4\pi+54}{\sqrt{4\pi+54k-54k/4\pi+54}}\right]$$

$$=12\sqrt{\frac{k}{4\pi+54}}-\frac{3}{\sqrt{\pi}}\left[\frac{4k\pi-54k}{\sqrt{4k\pi}\sqrt{4\pi+54}}\right]$$

$$=12\sqrt{\frac{k}{4\pi+54}}-\frac{6}{\sqrt{\pi}}\left[\frac{k(2\pi-27)}{\sqrt{k}\sqrt{16\pi^2+216\pi}}\right]$$

$$\left[\text{since}, (2\pi-27)<0 \Rightarrow \frac{d^2V}{dx^2}>0; k>0\right]$$

For $x=3\sqrt{\frac{k}{4\pi+54}}$, the sum of volumes is minimum.

For $x=3\sqrt{\frac{k}{4\pi+54}}$, then $r=\sqrt{\frac{k-6x^2}{4\pi}}$ [using $Eq.(i)$]

$$=\frac{1}{2\sqrt{\pi}}\sqrt{k-6.\frac{9k}{4\pi+54}}$$

$$=\frac{1}{2\sqrt{\pi}}\sqrt{\frac{4k\pi+54k-54k}{4\pi+54}}$$

$$=\frac{1}{2\sqrt{\pi}}\sqrt{\frac{4k\pi}{4\pi+54}}=\frac{\sqrt{k}}{\sqrt{4\pi+54}}=\frac{1}{3}x$$

$\Rightarrow x=3r$ *Hence proved.*

$\therefore$ Minimum sum of volume,

$$V_{x=3.\sqrt{\frac{k}{4\pi+54}}}=\frac{2}{3}x^3+\frac{4}{3}\pi r^3=\frac{2}{3}x^3+\frac{4}{3}\pi.\left(\frac{1}{3}x\right)^3$$

$$=\frac{2}{3}x^3+\frac{4}{3}\pi.\frac{x^3}{27}=\frac{2}{3}x^3\left(1+\frac{2\pi}{27}\right)$$