8

Sequence, Progression and Series

8.01 Sequence :

Consider the following group of numbers :

(i) 2, 5, 8, 11, . . . (ii) 3, 6, 12, 24, . . .

(iii) 1, 4, 5, 9, 14, . . . (iv) 5, 6, 2, 9, 3, . . .

Clearly in (i) every term is 3 more than its previous term, in (ii) every term is twice its previous term, in (iii) every term is the sum of its previous two terms and in (iv) there is no pattern of numbers, so we can not find out the next numbers of group.

From the above groups first three are the examples of sequence.

Definition : If the numbers follow a particular pattern, logically rule. Then those group of numbers is called a sequence . Every number of the sequence is called its term.

Sequence in the form of Set : A function defined from a set of natural numbers, N to another set S, is called sequence, i.e. in any set N, sequence is a rule which relates every natural number to a unique element of S.

If $N_n = \{1, 2, 3..., n\}$ is a finite set of natural numbers and function $f : N_n \to S,$ is defined from N_n to another set S then set of images of natural numbers 1,2,3,...,n is $\{f(1), f(2), f(3),..., f(n)\}$ is called a finite sequence,similarly, if function $f: N \to S$ then set $\{f(1), f(2), f(3),...\}$ is called infinite sequence. This is denoted by $\{f(n)\}$ or $< f(n) >$ $f(1), f(2), f(3),..., f(n),...$ are called as first, second, third... n^{th} term of sequence respectively. The n^{th} term is called general term. General term is denoted as a_n, t_n or T_n .

Example : Sequence 1, 3, 5, 7, 9, 11 is a finite sequence where $T_n = 2n - 1, \quad n \in N_6$

Example : Sequence 2, 3, 5, 7, 11, 13, . . . is an infinite sequence of prime number, in which n^{th} term can not written in the form of any specific formula.

It is clear from the above examples that a sequence can be represented as follows :

(i) Some initial terms of sequence has to be writen, so that we can predict following terms of the same. Like general term of sequence 1, 8, 27, . . . is $T_n = n^3$

(ii) Specially of terms of sequence can be expressed by writing some initial terms. Like the terms of sequence 2, 3, 5, 7, 11, . . . are prime numbers, this is their property.

(iii) By writing the first two terms of sequence other terms can be expressed by using the previous terms. For example to represent sequence 1, 4, 5, 9, 14, . . . , $a_1 = 1, a_2 = 4$ and $a_{n+2} = a_n + a_{n+1}$ $(n = 1, 2, ...)$ can be written.

8.02 Series :

Let $a_1, a_2, a_3, ..., a_n, ...$ be a given sequence. Then, the expression $a_1 \pm a_2 \pm a_3 \pm \ldots \pm a_n \pm \ldots$ is called the series corresponding to the given sequence. The series is finite or infinite according as the number of terms in a given sequence is finite or infinite. In which between the terms are seprated by positive or negative signs. There is a corresponding sequence for every series.

8.03 Progression :

A sequence is said be a progression. If it follows a particular pattern and the numerical value of the terms increase or decrease with a certain pattern.

i.e. If the n^{th} term of the sequence is derived by a formula then it is called as a progression.

Remark : Difference between Sequence and Progression- In the sequence the next terms can be writen by using rule in it we write n^{th} term is not always possible but in progression n^{th} term is always written.

For example 2, 3, 5, 7, 11, 13 is an infinite sequence of prime numbers but its n^{th} term cannot represented by any mathematical formula. hence, it is not progression.

8.04 Arithmetical Progression :

An arithmetic progression is a progression whose each term can be obtained by adding or substracting a finite quantity in previous term. In other words arithmetic progression is a sequence in which the difference of one term and its previous term remains constant. This constant difference is called as common difference. In brief Arithmetic progression can be written as A.P.

Example : 2, 5, 8, 11,..... is an A.P. where the first term is 2 and the common difference is $5-2=8-5=11-8=3$

8.05 General term of arithmetical progression :

To find the nth term of A.P. whose first term is *a* and common difference is *d*

Let $T_1, T_2, \ldots, T_n, \ldots$ is an A.P. then

by defination $T_2 - T_1 = d$

$$T_3 - T_2 = d$$

$$T_4 - T_3 = d$$

....

....

$$T_n - T_{n-1} = d$$

adding all $(n-1)$ equations,

$$T_n - T_1 = (n-1)d$$

$\Rightarrow$ $T_n = T_1 + (n-1)d$ but $T_1 = a$

thus required term $T_n = a + (n-1)d$

Let us consider an A.P. (in its standrad form) with first term a and common difference d, then the general form is given by $a,\ a+d,\ a+2d, \ldots, a+(n-1)d$ or $a, a+d, a+2d, \ldots, a+(n-1)d, \ldots$ corresponding to finite or infinite sequence.

Then the n^{th} term (general term) of the A.P, is $\ell = a + (n-1)d$.

If the n^{th} term of the sequence is given then to know that the sequence is A.P. or not, check the following procedure :

(i) Write the n^{th} term as T_n

(ii) In T_n replace n with $n+1$ and find T_{n+1}

(iii) Find $T_{n+1} - T_n$. If the difference is free from n then the sequence is an A.P.

Illustrative Examples

Example 1 : Show that the sequence $\{T_n\}$, whose nth term is $T_n = 2n+7$ is an A.P.

Solution : Here $T_n = 2n+7$

substituting $n+1$ in place of n

$$T_{n+1} = 2(n+1)+7 = 2n+2+7 = 2n+9$$

$$\therefore \quad T_{n+1} - T_n = (2n+9)-(2n+7) = 2$$

the difference is free from n therefore $\{T_n\}$ is an A.P.

Example 2 : Show that $\{T_n\}$ is not A.P. where $T_n = n^2 - 2n$.

Solution : Here $T_n = n^2 - 2n$

substituting $n+1$ in place of n

$$T_{n+1} = (n+1)^2 - 2(n+1) = n^2 + 2n + 1 - 2n - 2 = n^2 - 1$$

$$\therefore \quad T_{n+1} - T_n = (n^2 - 2n) - (n^2 - 1) = -2n + 1$$

the difference is dependent of n therefore $\{T_n\}$ is not an A.P.

Note :

1. A sequence is not an A.P. if its nth term is not linear in n.
2. In an A.P. with first term a and common difference d, the pth term from the end will be the $(n-p+1)$ th term from the begining and can be derived using the formula $a+(n-p)d$:

Example 3 : Show that the sequence 2, 7, 12, 17, ... is an A.P. also find its general term.

Solution : Here the difference between the two consecutive terms is 5 thus, it is an A.P.
Here $a = 2$ and $d = 5$. Hence the general term of A.P.

$$T_n = 2+(n-1)5 = 2+5n-5 = 5n-3.$$

Example 4 : In an A.P. if the 5th term is 18 and 9th term is 10 then find the 20th term.

Solution : Here $T_5 = 18 \quad \Rightarrow \quad a+4d = 18 \qquad (1)$

$T_9 = 10 \quad \Rightarrow \quad a+8d = 10 \qquad (2)$

solving (1) and (2)

$$a = 26 \quad d = -2$$

$$\therefore \quad T_{20} = a+19d = 26 + 19(-2) = 26 - 38 = -12$$

Example 5 : Check whether 105 is a term of the A.P. 1 + 4 + 7 + 11 + ...?

Solution : Here $a = 1$ and $d = 3$ Let 105 be the nth term of A.P., then $T_n = 105$

or $\quad a + (n-1)\,d = 105$

$$1 + (n-1)3 = 105 \quad \Rightarrow \quad n = 35\frac{2}{3}$$

Here n is not a natural number hence 105 is not a term of an A.P.

Example 6 : In an A.P. if the common difference is 4 and the last term is 201 then find the 25th term from the end.

Solution : Here $d = 4$ and $\ell = 201$

$\because$ nth term from the end = $\ell - (n-1)d$

$\therefore$ 25th term from the end $= 201 - (25-1)\,4 = 201 - 96 = 105.$

Exercise 8.1

1. Which of the following sequence is an A.P. ?

   (i) 2, 6, 11, 17, ... (ii) 1, 1.4, 1.8, 2.2, ... (iii) –7, –5, –3, –1, ... (iv) 1, 8, 27, 64, ...

2. Find the first term, common difference and 5th term of an A.P. if the nth term is given below -

   (i) $3n + 7$ (ii) $a + (n-1)d$ (iii) $5 - 3n$

3. Show that the sequence of following given n th term is not an A.P. ?

   (i) $\dfrac{n}{n+1}$ (ii) $n^2 + 1$

4. Which term of the A.P. $2 + 5 + 8 + 11 + \ldots$ is 65 ?
5. Find the 13th term from the last of the A.P. $4 + 9 + 14 + 19 + \ldots + 124$
6. Determine the number of term of an A.P. $2 + 5 + 8 + 11 + \ldots$ whose last term is 95.
7. If the 9th term of an A.P. is zero then prove that its 29th term is twice that of the 19th term.
8. How many two digit natural numbers are divisible by 3 ?
9. If the pth term of an A.P. is q and qth term is p then find the $(p+q)$ th term
10. If the pth term of an A.P. is $1/q$ and qth term is $1/p$ then prove that pq th term is one.

8.06 Sum of first n terms of an A.P. :

We will now find the sum of the first n terms of an A.P. It is denoted by S_n with first a and common difference d and nth terms is l the general terms are $a, a+d, a+2d \ldots \ell-2d, \ell-d, \ell$ respectively.

$$\therefore\ S_n = a + (a+d) + (a+2d) + \ldots + (\ell-2d) + (\ell-d) + \ell \qquad (1)$$

Rewriting the terms in reverse order, we have

$$S_n = \ell + (\ell-d) + (\ell-2d) + \ldots + (a+2d) + (a+d) + a \qquad (2)$$

On adding (1) and (2) term wise. We get

$$2S_n = (a+\ell) + (a+\ell) + \ldots + (a+\ell) + (a+\ell) \qquad (n \text{ term})$$

$$= n(a+\ell)$$

$$\therefore\quad S_n = \frac{n}{2}(a+\ell)$$

$$\text{or}\quad S_n = \frac{n}{2}\left[a + a + (n-1)d\right] \qquad \left[\because \ell = T_n = a + (n-1)d\right]$$

$$\text{or}\quad S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Note :

1. There are four variables in the formula of sum of n terms of A.P.. if any three are known, the fourth can be calculated.
2. If the sum of first n terms is S_n then its nth term can be derived using the formula $T_n = S_n - S_{n-1}$
3. If the sum of an AP is given then the terms can be derived as

   odd terms $\begin{bmatrix} \text{3 terms : a} - \text{d, a, a} + \text{d} \\ \text{5 terms : a} - \text{2d, a} - \text{d, a, a} + \text{d, a} + \text{2d} \end{bmatrix}$

even terms $\begin{bmatrix} \text{4 terms : } a-3d,\ a-d,\ a+d,\ a+3d \\ \text{6 terms : } a-5d,\ a-3d,\ a-d,\ a+d,\ a+3d,\ a+5d \end{bmatrix}$ etc.

8.07 Arithmetic mean :

If three or more than three terms are in A.P. then the terms lying between the first and the last term are called Arithmatic Mean i.e. if $a, A_1, A_2, A_3, \ldots, A_n, b$ is an A.P. then $A_1, A_2, A_3, \ldots A_n$; are called the arithmetic means between a and b. Arithmetic mean is also written as A.M.

To Find A. M. between the two given numbers :

Given two numbers a and b. We can insert A between them so that a, A, b is an A.P.

$\therefore \quad A - a = b - A \quad$ or $\quad 2A = a + b$

or $\quad A = \dfrac{a+b}{2}$, A.M. between two numbers a and b

To Find n A. M. between the two given numbers :

Let there be two terms a and b and there are n arithmetic means $A_1, A_2, A_3 \ldots A_n$ between them then $a, A_1, A_2, A_3 \ldots . A_n, b$ are in A.P.

In this progression first term a last term is b and number of terms in $(n+2)$. Let common difference of this A.P. is d, then last term.

$$b = a + (n+2-1)d$$

or $\quad b = a + (n+1)d \quad \Rightarrow \quad d = (b-a)/(n+1)$

$$\Rightarrow \quad A_1 = a + d = a + (b-a)/(n+1)$$

$$\Rightarrow \quad A_2 = a + 2d = a + 2\left(\frac{b-a}{n+1}\right), \ldots, A_n = a + nd = a + n\left(\frac{b-a}{n+1}\right),$$

Which is the required A.M. between a and b.

8.08 Properties of A.P. :

1. If a definite number is added or substracted in each term of given A.P., then new A.P. have same common difference as that of given A.P.
2. If each term of given A.P. be multiplied or divided by a non zero definite number then new progression will also be an A.P.
3. In a finite A.P. the sum of equidistant terms from begining and end terms is constant and it is sum of first and last terms.
4. In a A.P. every term (leaving except first and last term) half of the sum of two equidistant terms.
5. If an A.P. numbers of terms are odd then sum of the series, is equal to product of its middle term and number of terms .
6. If $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, y_3, \ldots, y_n$; n are two A.P.'s then $(x_1 \pm y_1), (x_2 \pm y_2), (x_3 \pm y_3), \ldots, (x_n \pm y_n)$, will also be an A.P.

Illustrative Examples

Example 7 : Find the sum of 20 terms of an $7 + 12 + 17 + 22 + \ldots$

Solution : Here $a = 7$, $d = 5$ and $n = 20$ है।

$$\therefore \quad S_n = \frac{20}{2}[2 \times 7 + (20-1)5]$$

$= 10\ [14 + 95]$

$= 10 \times 109 = 1090.$

Example 8 : If the nth term of an A.P. is $2n + 7$ then find the sum of its 12 terms

Solution : Here $T_n = 2n + 7$

put $n = 1, 2, 3, \ldots$ Here $T_1 = 9,\ T_2 = 11,\ T_3 = 13$

$\therefore \quad a = 9,\ d = 11 - 9 = 2$

$$\therefore \quad S_{12} = \frac{12}{2}[2\times 9+(12-1)\times 2]$$

$= 6\ [18 + 22]$

$= 6 \times 40 = 240.$

Example 9 : In an A.P., if p term is $\frac{\mathbf{1}}{\boldsymbol{q}}$ and q term is $\frac{\mathbf{1}}{\boldsymbol{p}}$ prove that the sum $\boldsymbol{pq}$ term is $\frac{1}{2}(pq+1)$

Solution : Given

$$T_p = \frac{1}{q} \Rightarrow a+(p-1)d = \frac{1}{q} \qquad (1)$$

$$T_q = \frac{1}{p} \Rightarrow a+(q-1)d = \frac{1}{p} \qquad (2)$$

solving equation (1) and (2) we have

$$a = \frac{1}{pq},\ d = \frac{1}{pq}$$

thus sum of pq terms

$$S_{pq} = \frac{pq}{2}\left[2\times\frac{1}{pq}+(pq-1)\frac{1}{pq}\right]$$

$$= \frac{pq}{2}\left[\frac{2+pq-1}{pq}\right] = \frac{1}{2}(pq+1)$$

Example 10 : Determine the sum of 30 terms of the series $\mathbf{1 - 7 + 3 - 10 + 5 - 13} + \ldots$

Solution : Given series is a combination of two series, thus spliting the series into two parts we have

$$S = (1+3+5+\ldots 15\,terms)-(7+10+13+\ldots 15\,terms)$$

$$= \frac{15}{2}\left[2\times 1+(15-1)\times 2\right]-\frac{15}{2}\left[7\times 2+(15-1)\times 3\right]$$

$$= \frac{15}{2}[2+28]-\frac{15}{2}[14+42] = \frac{15}{2}\times 30-\frac{15}{2}\times 56 = -195$$

Example 11 : If x^2, y^2, z^2 are in A.P. then prove that

(i) $\frac{\mathbf{1}}{y+z}, \frac{\mathbf{1}}{z+x}, \frac{\mathbf{1}}{x+y}$ are in A.P. **(ii)** $\frac{x}{y+z}, \frac{y}{z+x}, \frac{z}{x+y}$ are in A.P.

Solution : (i) $\frac{1}{y+z}, \frac{1}{z+x}, \frac{1}{x+y}$ are in A.P. if

$$\frac{1}{z+x} - \frac{1}{y+z} = \frac{1}{x+y} - \frac{1}{z+x}$$

or $\quad \frac{y-x}{y+z} = \frac{z-y}{x+y}$

or $\quad y^2 - x^2 = z^2 - y^2$

or $\quad 2y^2 = x^2 + z^2$

or $\quad x^2, y^2, z^2$ are in A.P. which is given

therefore x^2, y^2, z^2 are in A.P. then $\frac{1}{y+z}, \frac{1}{z+x}, \frac{1}{x+y}$ are also in A.P.

(ii) $\frac{x}{y+z}, \frac{y}{z+x}, \frac{z}{x+y}$ are in A.P. if

$\frac{x}{y+z}+1, \frac{y}{z+x}+1, \frac{z}{x+y}+1$ are in A.P. [adding (1)]

or $\quad \frac{x+y+z}{y+z}, \frac{x+y+z}{z+x}, \frac{x+y+z}{x+y}$ are in A.P.

or $\quad \frac{1}{y+z}, \frac{1}{z+x}, \frac{1}{x+y}$ are in A.P. [dividing by $x+y+z$]

or $\quad 2y^2 = x^2 + z^2$, [by (i)]

which is given that x^2, y^2, z^2 are in A.P. $\frac{x}{y+z}, \frac{y}{z+x}, \frac{z}{x+y}$ are also in A.P.

Example 12 : If the sum of n terms of an A.P. is m and the sum of m terms of an A.P. is n then prove that the sum of $(m+n)$ terms is $-(m+n)$

Solution : Let the first term be a and difference be d then

$$S_m = n \Rightarrow \frac{m}{2}\{2a+(m-1)d\} = n \Rightarrow 2am + m(m-1)d = 2n \quad (1)$$

$$S_n = m \Rightarrow \frac{n}{2}\{2a+(n-1)d\} = m \Rightarrow 2an + n(n-1)d = 2m \quad (2)$$

substracting (2) from (1)

$$2a(m-n) + \{m(m-1) - n(n-1)\}d = 2n - 2m$$

or $\quad 2a(m-n) + \{(m^2-n^2) - (m-n)\}d = -2(m-n)$

or $\quad 2a + (m+n-1)d = -2 \quad$ [dividing by $(m-n)$] (3)

or $\quad S_{m+n} = \frac{m+n}{2}\{2a+(m+n-1)d\}$

$$= \frac{m+n}{2}(-2) = -(m+n) \quad \text{[using (3)]}$$

Example 13 : The sum of n terms of two arithmetic progressions are in the ratio $(3n+13):(5n+3)$. Find the ratio of their 17th terms.

Solution : Let a, A and d, D be the first terms and common differences of the first and second arithmetic progression respectively. According to the given condition, we have

$$\frac{\text{Sum of n terms of first A.P.}}{\text{Sum of n terms of second A.P.}}=\frac{3n+13}{5n+3}$$

or $$\frac{\frac{n}{2}\left[2a+(n-1)d\right]}{\frac{n}{2}\left[2A+(n-1)D\right]}=\frac{3n+13}{5n+3}$$

or $$\frac{2a+(n-1)d}{2A+(n-1)D}=\frac{3n+13}{5n+3}$$

or $$\frac{a+\left(\frac{n-1}{2}\right)d}{A+\left(\frac{n-1}{2}\right)D}=\frac{3n+13}{5n+3} \qquad (1)$$

Now $$\frac{17^{\text{th}}\text{ terms of first A.P.}}{17^{\text{th}}\text{ of second A.P.}}=\frac{a+16d}{A+16D} \qquad (2)$$

putting $\frac{n-1}{2}=16 \Rightarrow n=33$ in (1)

$$\frac{a+16d}{A+16D}=\frac{3\times 33+13}{5\times 33+3}=\frac{99+13}{165+3}$$

or $$\frac{a+16d}{A+16D}=\frac{112}{168}=\frac{2}{3}$$

Thus required ratio = 2 : 3

Example 14 : Find 3 arithmetic means between 18 and 30

Solution : Let A_1, A_2, A_3 be the three A.M.'s between 18 and 30 therefore

$18, A_1, A_2, A_3$ 30 are in A.P.

here $d=\frac{30-18}{3+1}=\frac{12}{4}=3$ $\qquad \left[\because d=\frac{b-a}{n+1}\right]$

A_1 = a + d = 18 + 3 = 21

A_2 = a + 2d = 18 + 6 = 24

A_3 = a + 3d = 18 + 9 = 27

Thus required A.M.'s are 21, 24 and 27

Example 15 : For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$, is the A.M. between a and b.

Solution : Since the A.M. between a and b is $\frac{a+b}{2}$ therefore

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\frac{a+b}{2}$$

or $\quad 2\left(a^{n+1}+b^{n+1}\right)=\left(a^n+b^n\right)(a+b)$

or $\quad 2a^{n+1}+2b^{n+1}=a^{n+1}+a^n b+b^n a+b^{n+1}$

or $\quad a^{n+1}+b^{n+1}=a^n b+b^n a$

or $\quad a^n(a-b)=b^n(a-b)$

or $\quad a^n=b^n \qquad [\because a\neq b]$

Which is only possible when $\quad n=0 \qquad \left[\because a^0=b^0=1\right]$

Exercise 8.2

1. Find the sum of the following series

   (i) $7+11+15+19+\ldots$ upto 20 terms

   (ii) $\frac{1}{3}+1+\frac{5}{3}+\frac{7}{3}+\ldots$ upto 10 terms

   (iii) $\frac{1}{\sqrt{2}+1}+\sqrt{2}+\frac{1}{\sqrt{2}-1}+\ldots$ upto 6 terms

2. Find the sum of odd integers from 1 to 101 which are divisible by 3.
3. Determine the sum of first n terms of an A.P. whose rth term is $2r+3$.
4. If for an A.P. $S_n=n^2+2n$, then find the first term and the common difference.
5. If the sum of n terms of an A.P. 1, 6, 11, ... is 148, then find the number of terms and the last term.
6. If the sum of p terms and the sum of q terms of an A.P. are equal then find the sum of $(p+q)$ terms.
7. If the sum of n, $2n$, $3n$ terms of an A.P. are S_1, S_2 and S_3 then Prove that $S_3=3\left(S_2-S_1\right)$.
8. If the sum of n terms of m A.P.'s are given by $S_1, S_2, S_3, \ldots, S_m$ with first terms are 1, 2, 3,....., m and the common differences are 1, 3, 5,......., $(2m-1)$ then prove that

$$S_1+S_2+S_3+\ldots+S_m=\frac{mn}{2}(mn+1)$$

9. If the sum of first p, q, r terms of an A.P. are a, b, c respectively then prove that

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0.$$

10. Find the three terms of an A.P. whose sum is 12 and the sum of whose cubes is 408.
11. If n A.M.'s are inserted between 1 and 51 in such a way that the ratio of fourth and seventh. A.M. is 3 : 5 then find n.
12. If x, y, z are in A.P. then Prove that :

    (i) $y+z,\ z+x,\ x+y$ are in A.P. **(ii)** $\frac{1}{yz}, \frac{1}{zx}, \frac{1}{xy}$ are in A.P.

(iii) $(x-y)(y-z)=\frac{(z-x)^2}{4}$ **(iv)** $(x-z)^2=4(y^2-xz)$

(v) $xy+yz+zx=\frac{x^2+z^2+4xz}{2}$

13. If $x^2(y+z), y^2(z+x), z^2(x+y)$ are in A.P., then prove that either x, y, z are in A.P. or $xy+yz+zx=0$
14. Find the sum of A.P. $a_1, a_2, a_3, \dots, a_{30}$ given that
$$a_1+a_7+a_{10}+a_{21}+a_{24}+a_{30}=540$$
15. The difference between any two consecutive interior angles of a polygon is 8^0. If the smallest angle is 52^0, find the number of the sides of the polygon.

8.09 Geometrical Progression :

In a non zero sequence of numbers if every term is obtained by multiplying the previous term with a constant number then that sequence of numbers is termed as geometric progression i.e. the ratio between the consective terms is always constant. The constant ratio is called as common ratio (r). Geometric Progressionis represented briefly as G.P.

Consider the following sequences :

(i) 2, 4, 8, 16, . . . (ii) 1, -3, 9, -27 ...

(iii) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ (iv) $a, ar, ar^2, ar^3, \dots$

In each of these sequences, we note that each term, is multiplied by a definite number e.g. in sequence (i) by 2 (ii) by -3 (iii) by $\frac{1}{2}$ (iv) by r, thus these all are examples of geometric progressions.

Note :

1. In geometric progression the first term and the common ratio are always non-zero.
2. If the terms of G.P. are altermate positive and negative, then the common ratio of progression is negative.

8.10 General term of G.P. :

To find the n th term of a G.P. whose first term is a and common ratio is r

Let $T_1, T_2, \dots, T_n$ are in G.P.

$$T_1 = \text{first term} = a = ar^{1-1},$$

$$\Rightarrow \quad \frac{T_2}{T_1}=r \quad \Rightarrow \quad T_2 = T_1 r = ar = ar^{2-1},$$

$$\frac{T_3}{T_2}=r \quad \Rightarrow \quad T_3 = T_2 r = ar.r = ar^2 = ar^{3-1},$$

similarly $T_4 = ar^{4-1}, \dots, T_n = ar^{n-1}$

Hence of any G.P. whose first term a and common ratio r, the nth term is given by $T_n = ar^{n-1}$

thus the last term is also written by ℓ i.e. $\ell = ar^{n-1}$

If the numbers of terms in a G.P. is n then pth term from the end will be $(n-p+1)$ term from the begining i.e. the pth term from the end will be $= ar^{n-p}$

If the last terms is ℓ then there will be a sequence from the last term to the first term whose common ratio will be $\frac{1}{r}$ and the nth term from the end will be $= \ell\left(\frac{1}{r}\right)^{n-1}$

Note :

1. If the product of the terms in G.P. is not given, then the consective terms of a G.P. are assumed as $a, ar, ar^2, \ldots$
2. If the product of the terms in G.P. is given, then to find the various terms can be determined in the following manner

Odd term	3 terms :	$\frac{a}{r}, a, ar$
	5 terms :	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$
Even term	4 terms :	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

6 terms $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$ etc.

8.11 Geometric mean :

If three or more than three terms are in G.P. then the terms between the first and the last is called as geometric means. It is also denoted as G.M. i.e. of $a, G_1, G_2, \ldots, G_n, b$ are in G.P. then $G_1, G_2, \ldots, G_n$ are termed as geometric means between a and b.

To find Geometric mean between the two numbers *a* and *b*

Let G be the geometric mean between a and b then a, G, b will be in G.P. and according to the definition

$$\frac{G}{a} = \frac{b}{G} \quad \text{or} \quad G^2 = ab$$

or $G = \pm\sqrt{ab}$, which is required geometric mean between a and b

Example 1 : If 2, 6, 18 are in G.P. then the G.M. between 2 and 18 is 6.

Example 2 : G.M. between 3 and 27 is $G = \sqrt{3 \times 27} = 9$

Example 3 : G.M. between -8 and -2 is. $G = \pm\sqrt{(-8)\times(-2)} = -4$

To find in G.M.'s between the two given numbers

Let $G_1, G_2, G_3, \ldots, G_n$ be n numbers between two positive numbers a and b such that $a, G_1, G_2, G_3, \ldots, G_n, b$ is a G.P. Thus b being the ($n+2$)th term, we have

$$\therefore \quad b = ar^{n+2-1} \quad \text{or} \quad r^{n+1} = b/a \quad \text{or} \quad r = (b/a)^{\frac{1}{n+1}}$$

hence, the required geometric means are as follows :

$$G_1 = ar = a(b/a)^{\frac{1}{n+1}}$$

$$G_2 = ar^2 = a(b/a)^{\frac{2}{n+1}}$$

$$G_n = ar^n = a(b/a)^{\frac{n}{n+1}}$$

Note : If two numbers a and b are of different signs then there will be no G.M. between terms.

8.12 Important properties of A.M. and GM between two quantities :

Property 1 : Let A and G be A.M. and G.M. of two given positive real numbers a and b respectively. Then $A > G$.

Proof : Here $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

now $A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} > 0$ $\quad \therefore \quad A > G$

Property 2 : Let A and G be A.M. and G.M. of two given positive real numbers a and b respectively. Then a quadratic equation having roots a, b is : $x^2 - 2Ax + G^2 = 0$

Proof : Here $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$ (1)

Quadratic equation whose roots are a and b is

$$x^2 - (a+b)x + ab = 0$$

or $x^2 - 2Ax + G^2 = 0$ [by (1)]

Property 3 : Let A and G be A.M. and G.M. of two given positive real numbers a and b respectively. Then the numbers are $A \pm \sqrt{A^2 - G^2}$

Proof : Quadratic equation whose roots are given

$$x^2 - 2Ax + G^2 = 0 \quad \text{[Property 2]}$$

or $x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2}$ [Shreedharacharya formula]

or $x = A \pm \sqrt{A^2 - G^2}$

Illustrative Examples

Example 16 : Find the 10th term and the general term of G.P. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

Solution : here $a = \frac{1}{2}$ and $r = \frac{1}{2}$ therefore

$$T_{10} = ar^{10-1} = ar^9 = \frac{1}{2} \times \left(\frac{1}{2}\right)^9 = \frac{1}{2^{10}} = \frac{1}{1024}$$

General term $T_n = ar^{n-1} = \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^n}$

Example 17 : Determine the G.P. if its second term is 10 and fifth term is 80.

Solution : Second term $T_2 = ar = 10$ (1)

fifth term $T_5 = ar^4 = 80$ (2)

Dividing (2) by (1)

$$\frac{ar^4}{ar} = \frac{80}{10}$$

or $r^3 = 8 \quad \Rightarrow \quad r = 2$

Put the value of r in (1)

$2a = 10 \quad \Rightarrow \quad a = 5$

required G.P. is 5, 10, 20, 40, . . .

Example 18 : Find the three numbers in G.P. whose products is 1000 and sum is 35.

Solution : Let the numbers be $\frac{a}{r}, a, ar$ that given that,

$$\frac{a}{r} \times a \times ar = 1000 \text{ से } a^3 = 1000$$

or $\quad a = 10 \quad$ (1)

and $\quad \frac{a}{r} + a + ar = 35$ or $\frac{10}{r} + 10 + 10r = 35 \quad$ [using (1)]

simplifying $\quad 2r^2 - 5r + 2 = 0$

or $\quad (r-2)(2r-1) = 0$

or $\quad r = 2, r = 1/2$

Required numbers are 5, 10, 20

Example 19 : If the pth, qth and rth terms of a G.P. are x, y, z respectively. Prove that

$$x^{q-r}\, y^{r-p}\, z^{p-q} = 1$$

Solution : let the first term be a and ratio be R then given is,

$$T_p = a\,R^{p-1} = x,$$

$$T_q = a\,R^{q-1} = y,$$

$$T_r = a\,R^{r-1} = z$$

$$\therefore \quad x^{q-r}\, y^{r-p}\, z^{p-q} = \left(aR^{p-1}\right)^{q-r}\left(aR^{q-1}\right)^{r-p}\left(aR^{r-1}\right)^{p-q}$$

$$= a^{(q-r)+(r-p)+(p-q)} R^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$$

$$= a^0 R^0 = 1\,.$$

Example 20 : If the A.M. between the two numbers a and b is n times the G.M., then prove that :

$$a/b = \left(2n^2 - 1\right) + 2n\sqrt{n^2 - 1}$$

Solution : A.M. between a and $b = \frac{a+b}{2}$ and G.M. $= \sqrt{ab}$

given

$$\frac{a+b}{2} = n\sqrt{ab} \qquad \text{or} \qquad a + b = 2n\sqrt{ab} \qquad (1)$$

now $\quad a - b = \sqrt{(a+b)^2 - 4ab} \qquad$ or $\qquad a - b = \sqrt{4n^2ab - 4ab}$

or $\quad a - b = 2\sqrt{ab}\sqrt{n^2 - 1} \qquad$ (2)

dividing (1) by (2) $\qquad \frac{a+b}{a-b} = \frac{n}{\sqrt{n^2 - 1}}$

or $\qquad \frac{a}{b} = \frac{n + \sqrt{n^2 - 1}}{n - \sqrt{n^2 - 1}}$

$$= \frac{\left(n+\sqrt{n^2-1}\right)^2}{n^2-\left(n^2-1\right)}$$ [By rationalisation]

$$= \left(2n^2-1\right)+2n\sqrt{n^2-1}$$

Exercise 8.3

1. (i) Find the 7th term of the series $1 + 3 + 9 + 27 + \ldots$

   (ii) Find the 10th term of the G.P. $\sqrt{2}+\frac{1}{\sqrt{2}}+\frac{1}{2\sqrt{2}}+\ldots$

2. (i) Which term of the G.P. $64 + 32 + 16 + 8 + \ldots$ is $1/64$?

   (ii) Which term of the G.P. $6 + 3 + 3/2 + 3/4 + \ldots$ is $3/256$?

3. Find the common ratio and the nth term of the G.P. $5 + 10 + 20 + 40 + \ldots$
4. Find the 5th term from the end of the G.P. 2, 6, 18, 54, ..., 118098
5. If the 3rd term of the G.P. is 32 and 7th term is 8192 then find its 10th term.
6. Find G.P. whose 3rd term is 1 and 7th term is 16.
7. (i) Find three G.M. 's between 3 and 48.

   (ii) Find six G.M. 's between 2 and 256
8. for what value of x, the numbers x, $x + 3$, $x + 9$ are in G.P. ?
9. Find four terms which are in G.P. whose third term is 4 more than the first term and second term is 36 more than the fourth term.
10. The 4th, 7th and 10th terms of a G.P. are p, q and r respectively. Show that $q^2 = pr$.
11. If in a G.P. $(p + q)$ th terms is x and $(p - q)$ th terms is y then find the pth term.
12. If a, b, c are in G.P. and $a^x = b^y = c^z$ then prove that $\frac{1}{x}+\frac{1}{z}=\frac{2}{y}$.
13. If n G.M.'s are inserted between a and b then prove that the product of all G.M.'s will be $\left(\sqrt{ab}\right)^n$.
14. x, y, z are in G.P. , if A.M. of x, y is A_1 and A.M. of y, z is A_2 then prove that :

    (i) $\frac{1}{A_1}+\frac{1}{A_2}=\frac{2}{y}$ **(ii)** $\frac{x}{A_1}+\frac{z}{A_2}=2$

8.13 Sum of first n terms of a G.P. :

Let the first term of a G.P. be a, the common ratio be r and S be the sum to first n terms of G.P. . Then

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \quad (1)$$

Multiplying both sides by r

$$r S_n = ar + ar^2 + ar^3 + \ldots + ar^n \quad (2)$$

substracting (2) from (1)

$$S_n - r S_n = a - ar^n$$

$$S_n\left(1-r\right) = a\left(1-r^n\right)$$

This gives

$$S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{a-lr}{1-r}$$

$$a\left(\frac{r^n-1}{r-1}\right) = \frac{lr-a}{r-1}, \text{ where } \ell = ar^{n-1} \qquad [r \neq 1]$$

Note : S_n can have the following two conditions

$$S_n = a\left(\frac{1-r^n}{1-r}\right), r<1$$

and
$$S_n = a\left(\frac{r^n-1}{r-1}\right), r>1$$

Actually both formule are equal. We need to keep in of mind that the above formula is not applicable for $r = 1$. For $r=1$, $S_n = na$.

Illustrative Examples

Example 21 : Find the sum of first ten terms of the G.P. $\sqrt{3}, 3, 3\sqrt{3}, 9, \ldots$

Solution : Here $a=\sqrt{3}, r=\sqrt{3}$ and $n=10$

$\therefore$ Sum of 10 terms $= S_{10} = \dfrac{\sqrt{3}\left[(\sqrt{3})^{10}-1\right]}{\sqrt{3}-1}$

$\Rightarrow$ $S_{10} = \dfrac{\sqrt{3}[243-1]}{\sqrt{3}-1} = \dfrac{\sqrt{3}(242)}{2}\times(\sqrt{3}+1)$

$\Rightarrow$ $S_{10} = 121(3+\sqrt{3})$.

Example 22 : How many terms of G.P. 3, 6, 12, ... are needed to give the sum 189 ?

Solution : Here $a = 3$, $r = 2$ and $S_n = 189$

$\therefore$ $S_n = \dfrac{a(r^n-1)}{r-1}$

$\Rightarrow$ $189 = \dfrac{3(2^n-1)}{2-1}$

$\Rightarrow$ $2^n - 1 = 189/3 = 63$

$\Rightarrow$ $2^n = 64 = 2^6 \quad \Rightarrow \quad n = 6$

Example 23 : In a G.P. if the first term is 7, last term is 567 and sum of the terms is 847 then find the common ratio.

Solution : Here $a = 7$, $T_n = 567$ and $S_n = 847$

$\therefore$ $T_n = ar^{n-1} = 567$ [or]

$\Rightarrow$ $r^{n-1} = \dfrac{567}{7} = 81$

$\Rightarrow$ $r^n = 81r$ [multiply both sides by r] (1)

$\Rightarrow \quad S_n = \dfrac{a(r^n - 1)}{r-1} \quad \Rightarrow \quad 847 = \dfrac{7(81r-1)}{r-1}$ [using (1)]

$\Rightarrow \quad 121(r-1) = 81r - 1$

$\Rightarrow \quad 40r = 120 \quad \Rightarrow \quad r = 3$

Example 24 : Find the sum to n terms of the series $5 + 55 + 555 + 5555 + \ldots$

Solution : $S_n = 5 + 55 + 555 + 5555 + \ldots .n$ terms

$$= 5[1 + 11 + 111 + 1111 + \ldots .n \text{ terms}]$$

$$= \frac{5}{9}[9 + 99 + 999 + \ldots .n \text{ terms}]$$

$$= \frac{5}{9}\left[(10-1) + (100-1) + \{1000-1\} + \ldots .n \text{ terms}\right]$$

$$= \frac{5}{9}\left[10 + 10^2 + 10^3 + \ldots .n \text{ terms- } (1+1+1+1\ldots\ldots \text{ n terms}\right]$$

$$= \frac{5}{9}\left[\frac{10(10^n - 1)}{10-1} - n\right] = \frac{50}{81}(10^n - 1) - \frac{5n}{9} \; .$$

Example 25 : If S_1, S_2, S_3 are the sum of n, $2n$ and $3n$ terms of the G.P., then prove that

$$S_1^2 + S_2^2 = S_1(S_2 + S_3)$$

Solution : Let the first term be a and common ratio be r, then

$$S_1 = \frac{a(r^n - 1)}{r-1}, S_2 = \frac{a(r^{2n} - 1)}{r-1}, S_3 = \frac{a(r^{3n} - 1)}{r-1}$$

$$\Rightarrow \quad S_1^2 + S_2^2 = \frac{a^2(r^n-1)^2}{(r-1)^2} + \frac{a^2(r^{2n}-1)^2}{(r-1)^2} = \frac{a^2}{(r-1)^2}\left\{(r^n-1)^2 + (r^{2n}-1)^2\right\}$$

$$= \frac{a^2}{(r-1)^2}\left\{(r^n-1)^2 + (r^n-1)^2(r^n+1)^2\right\} = \frac{a^2}{(r-1)^2}(r^n-1)^2\left\{1 + (r^n+1)^2\right\}$$

$$\Rightarrow \quad S_1^2 + S_2^2 = \frac{a^2}{(r-1)^2}(r^n-1)^2(r^{2n} + 2r^n + 2).$$

And $$S_1(S_2 + S_3) = \frac{a(r^n-1)}{r-1}\left[\frac{a(r^{2n}-1)}{r-1} + \frac{a(r^{3n}-1)}{r-1}\right] = \frac{a^2(r^n-1)}{(r-1)^2}\left[(r^{2n}-1) + (r^{3n}-1)\right]$$

$$= \frac{a^2(r^n-1)}{(r-1)^2}\left[(r^n-1)(r^n+1) + (r^n-1)(r^{2n}+r^n+1)\right] = \frac{a^2(r^n-1)^2}{(r-1)^2}\left[r^n + 1 + r^{2n} + r^n + 1\right]$$

$$= \frac{a^2 \left(r^n - 1\right)^2}{\left(r - 1\right)^2} \left(r^{2n} + 2r^n + 2\right) \qquad \ldots (2)$$

$\therefore \quad S_1^2 + S_2^2 = S_1 \left(S_2 + S_3\right)$.

Example 26 : Find the sum of the first n terms of the series : $0.2 + 0.22 + 0.222 + \ldots\ldots$

Solution : The sum of the first n terms of the series is

$$S_n = 0.2 + 0.22 + 0.222 + \ldots. n \text{ term}$$

$$= 2\left(0.1 + 0.11 + 0.111 + \ldots. n \text{ term}\right)$$

$$= \frac{2}{9}\left(0.9 + 0.99 + 0.999 + \ldots. n \text{ term}\right)$$

$$= \frac{2}{9}\left[(1 - 0.1) + (1 - 0.01) + (1 - 0.001) + \ldots. n \text{ term}\right]$$

$$= \frac{2}{9}\left[(1 + 1 + \ldots. n \text{ term}) - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots n \text{ term}\right)\right]$$

$$= \frac{2}{9}\left[n - \frac{1}{10}\left(\frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}}\right)\right]$$

$$= \frac{2}{9}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right] = \frac{2}{81}\left[9n - 1 + \frac{1}{10^n}\right]$$

Example 27 : Prove that the sum of n terms of the series $11 + 103 + 1005 + \ldots$ is $\frac{10}{9}\left(10^n - 1\right) + n^2$

Solution : Let us the sum of n terms of the series S_n . Then

$$S_n = 11 + 103 + 1005 + \ldots. n \text{ term}$$

$$= (10 + 1) + \left(10^2 + 3\right) + \left(10^3 + 5\right) + \ldots. + \left\{10^n + (2n - 1)\right\}$$

$$= \left(10 + 10^2 + \ldots.. + 10^n\right) + \left\{1 + 3 + \ldots.. + (2n - 1)\right\}$$

$$= \frac{10\left(10^n - 1\right)}{10 - 1} + \frac{n}{2}\left\{1 + 2n - 1\right\} = \frac{10}{9}\left(10^n - 1\right) + n^2$$ [Using the formula for sum for G.P. and A.P.]

8.14 Sum of an infinite G.P. :

Let the first term be a and common ratio be r of an infinite geometric progression where $-1 < r < 1$ i.e. $|r| < 1$. Then from section 8.13 in this G.P. sum of n terms is given of S_n

$$S_n = a\left(\frac{1 - r^n}{1 - r}\right) = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

$$\therefore \quad \lim_{n \to \infty} S_n = \lim_{n \to \infty}\left(\frac{a}{1 - r} - \frac{ar^n}{1 - r}\right)$$

$$S_\infty = \frac{a}{1-r} \qquad [\because |r| < 1 \text{ and } \lim_{n\to\infty} r^n = 0 \text{ if } |r| < 1]$$

therefore $S_\infty = \frac{a}{1-r}$, if $|r| < 1$.

Note:

1. If $|r| \geq 1$, then the sum of infinite G.P. tends to infinity. Thus sum of infinite terms to G.P. the appropriate formula is applicable only when numerical value of r is less then 1 i.e. $|r| < 1$
2. Using the above formula any rational number can be represented in fractional or by repeating decimal expansion. By representing any number in decimal form if any one or more than one group of digits repeat then it is called as repeating decimal expansion. For example $2\cdot454545$........, $0\cdot3565656$....... can be written in the form of $2\cdot\overline{45}$ and $0\cdot3\overline{56}$ respectively. mark a bar or dot on the group of digits that repeating in decimal expansion.

Illustrative Examples

Example 28 : Find the sum to infinite series :

(i) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$ (ii) $\frac{2}{3}-\frac{4}{9}-\frac{8}{27}+\ldots.$

Solution : (i) $a = 1$ and $r = \frac{1}{3}$ $\therefore$ $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

(ii) $a = \frac{2}{3}$ and $r = -\frac{2}{3}$ $\therefore$ $S_\infty = \frac{a}{1-r} = \frac{2/3}{1-(-2/3)} = \frac{2/3}{1+(2/3)} = \frac{2}{5}$.

Example 29 : The sum of the first two terms of an infinite (G.P.) series is 20 and every term is three times the sum of its succeding terms, find the G.P.

Solution : Let the general terms be $a, ar, ar^2, \ldots$ then sum of first two terms $= a + ar = 20$,

or $a(1+r) = 20$ (1)

also given each term is 3 times the sum of the succeding terms.

$$a = 3\left(\frac{ar}{1-r}\right) \quad \text{or} \quad 1-r = 3r \quad \Rightarrow \quad r = \tfrac{1}{4}$$

putting the value of r in

$$a = \frac{20}{1+\frac{1}{4}} \quad \Rightarrow \quad a = 16$$

therefore the required series is $16, 4, 1, \frac{1}{4}, \ldots$

Example 30 : Find the rational number whose decimal form is $0\cdot3\overline{75}$

Solution : $0\cdot3\overline{75} = \cdot375\ 75\ 75\ \ldots$

$$= \cdot3 + \cdot075 + \cdot00075 + \ldots$$

$$= \frac{3}{10} + \left(\frac{75}{1000} + \frac{75}{100000} + \frac{75}{10000000} + \ldots\right)$$

$$= \frac{3}{10} + \frac{75}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\right)$$

$$=\frac{3}{10}+\frac{75}{10^3}\left(\frac{1}{1-\frac{1}{10^2}}\right) \qquad \left[\text{Using formula } S_\infty = \frac{a}{1-r}\right]$$

$$=\frac{3}{10}+\frac{75}{1000}\times\frac{100}{99}=\frac{3}{10}+\frac{75}{990}=\frac{372}{990}=\frac{62}{165}.$$

Exercise 8.4

1. Determine the sum of geometric series :

(i) $2+6+18+51+\ldots$ upto 7 terms (ii) $\frac{2}{9}-\frac{1}{3}+\frac{1}{2}-\frac{3}{4}+\ldots$ upto 8 terms

(iii) $a^8-a^7b+a^6b^2-a^5b^3+\ldots$ upto 10 terms

2. Determine the sum of Geometric Series :

(i) $2+6+18+54+\ldots+486.$ (ii) $64+32+16+\ldots+\frac{1}{4}$

3. How many terms of G.P. 4, 12, 36, ... are needed to give the sum 484 ?

4. If the sum of the first five terms of the G.P. is 124 and common ratio is 2 then find the first term.

5. If the common ratio of the G.P. is 2 and last term is 160 and sum is 310, then find the first term.

6. Find the sum of the following series up to n terms :

(i) $7+77+777+\ldots$ (ii) $\cdot 5+\cdot 55+\cdot 555+\ldots$ (iii) $\cdot 9+\cdot 99+\cdot 999+\ldots$

7. Convert the following number whose expansion is not terminating repeating decimal expansion into a rational number :

(i) $2\cdot 3\overline{5}$ (ii) $\cdot 6\overline{25}$ (iii) $2\cdot\overline{752}$

8. If the first term of an infinite G.P. is 64 and every term is three times the sum of its successive terms, find the G.P.

9. If $y=x+x^2+x^3+\ldots\infty$, where $|x|<1$ then prove that $x=\frac{y}{1+y}$.

10. If $x=1+a+a^2+\ldots\infty$, where $|a|<1$ and

$y=1+b+b^2+\ldots\infty$, where $|b|<1$ then

prove that $$1+ab+a^2b^2+\ldots\infty=\frac{xy}{x+y-1}$$

11. Find the sum to infinity of the series :

$$\left(1+\frac{1}{2^2}\right)+\left(\frac{1}{2}+\frac{1}{2^4}\right)+\left(\frac{1}{2^2}+\frac{1}{2^6}\right)+\ldots\infty$$

8.15 Arithmetico Geometric Series :

The series obtained after multiplying the corresponding terms as A.P. and G.P. is known as arithmetico geometric series.

Example : $1+3x+5x^2+7x^3+\ldots$ is an arithmetico geometric series where 1, 3, 5, 7,... are in A.P. and $1, x, x^2, \ldots$ are in G.P. the nth term of A.P. will be $(2n-1)$ the nth term of G.P. is x^{n-1} therefore the nth term of arithmetico geometric series will be $(2n-1)x^{n-1}$

The General form will be $a, (a+d)r, (a+2d)r^2, \ldots$ and the nth term is $T_n = \{a+(n-1)d\}r^{n-1}$

To find the sum of n terms of Arithmetico Geometric Series :

Let the sum of first nth term be S_n then

$$S_n = a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \ldots + [a+(n-1)d]r^{n-1} \quad (1)$$

multiplying both sides by r

$$r S_n = ar + (a+d)r^2 + (a+2d)r^3 + \ldots + [a+(n-2)d]r^{n-1} + [a+(n-1)d]r^n \quad (2)$$

substracting (2) from (1)

$$(1-r)S_n = a + [dr + dr^2 + dr^3 + \ldots\ldots + dr^{n-1}] - [a+(n-1)d]r^n$$

$$= a + \frac{dr(1-r^{n-1})}{1-r} - [a+(n-1)d]r^n$$

$$\Rightarrow \quad S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r} \quad (3)$$

If $|r| < 1$ and number of terms are infinite then r^n, $r^n \to 0$

$$\Rightarrow \quad S_n \to \frac{a}{1-r} + \frac{dr}{(1-r)^2}, \text{ जब } n \to \infty$$

Thus when $|r| < 1$ If $S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$ (4)

Remark : The above result (3) and (4) should not use as a form of formula but find the sum using the given method in this.

8.16 Sum of series by difference method :

If in the series, the difference of consecutive ordered term is in G.P., then to find the sum of this type of series write the terms below the terms of same series but shifting one term, after substracting the terms of resultant series will be in G.P. We can find the nth term using this and placing $n = 1, 2, 3, \ldots$ obtaining each term we can find the sum of the series.

Illustrative Examples

Example 31 : Find the sum to n terms of the series $\frac{3}{4} + \frac{7}{4^2} + \frac{11}{4^3} + \ldots$

Solution : The given series is an Arithmetico Geometric series where

$3, 7, 11, \ldots$ is an A.P. and $\frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \ldots$ is a G.P.

nth term of an A.P. $= [3+(n-1)4] = 4n-1$ and

nth term of an G.P. $= \frac{1}{4}\left(\frac{1}{4}\right)^{n-1} = \frac{1}{4^n}$ है।

therefore the nth term will be $\frac{4n-1}{4^n}$

$\therefore$ $$S_n = \frac{3}{4} + \frac{7}{4^2} + \frac{11}{4^3} + \ldots + \frac{4n-5}{4^{n-1}} + \frac{4n-1}{4^n} \quad (1)$$

multiplying both sides by $\frac{1}{4}$

$$\frac{1}{4}S_n = \frac{3}{4^2} + \frac{7}{4^3} + \frac{11}{4^4} + \ldots + \frac{4n-5}{4^n} + \frac{4n-1}{4^{n+1}} \quad (2)$$

substracting (2) from (1)

$$S_n - \frac{1}{4}S_n = \frac{3}{4} + \left\{\frac{4}{4^2} + \frac{4}{4^3} + \frac{4}{4^4} + \ldots + \frac{4}{4^n}\right\} - \frac{4n-1}{4^{n+1}}$$

or $$S_n\left(1 - \frac{1}{4}\right) = \frac{3}{4} + \left\{\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^{n-1}}\right\} - \frac{4n-1}{4^{n+1}}$$

or $$S_n\left(\frac{3}{4}\right) = \frac{3}{4} + \frac{\frac{1}{4}\left[1 - \left(\frac{1}{4}\right)^{n-1}\right]}{1 - \frac{1}{4}} - \frac{4n-1}{4^{n+1}}$$

or $$S_n = 1 + \frac{4}{9}\left(1 - \frac{1}{4^{n-1}}\right) - \frac{4n-1}{3.4^n}$$

Example 32 : Find the sum to infinity of the series $1 + \frac{3}{4} + \frac{7}{4^2} + \frac{15}{4^3} + \ldots$

Solution : Let $S_\infty = 1 + \frac{3}{4} + \frac{7}{4^2} + \frac{15}{4^3} + \ldots$ (1)

multiplying both sides by $\frac{1}{4}$

$$\frac{1}{4}S_\infty = \frac{1}{4} + \frac{3}{4^2} + \frac{7}{4^3} + \frac{15}{4^4} + \ldots \quad (2)$$

substracting (2) from (1)

$$S_\infty - \frac{1}{4}S_\infty = 1 + \frac{2}{4} + \frac{4}{4^2} + \frac{8}{4^3} + \ldots = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

or $$S_\infty\left(1 - \frac{1}{4}\right) = \frac{1}{1 - \frac{1}{2}}$$ [Using formula $S_\infty = \frac{a}{1-r}$]

$\therefore$ $$S_\infty = 2.\frac{4}{3} = \frac{8}{3}$$

Example 33 : Find the sum to n terms and the nth term of the series $1 + 5 + 13 + 29 + 61 + \ldots$

Solution : The difference 4, 8, 16, . . . of two consecutive terms is in G.P. Using the difference method the n^{th} term be T_n and sum to n terms be S_n

$$S_n = 1 + 5 + 13 + 29 + 61 + \ldots + T_n \quad (1)$$

substracting (2) from (1)

$$S_n = 1+5+13+29+\ldots+T_{n-1}+T_n \qquad \ldots(2)$$

or $\quad 0 = 1+\{4+8+16+\ldots+(n-1)\text{ terms}\}-T_n$

or $\quad T_n = 1+\{4+8+16+\ldots+(n-1)\text{ terms}\} = 1+\dfrac{4(2^{n-1}-1)}{2-1} = 2^{n+1}-3$

putting $n = 1, 2, 3, \ldots$

$$T_1 = 2^2-3,\ T_2 = 2^3-3,\ T_3 = 2^4-3, \ldots,$$

$\therefore \quad S_n = T_1+T_2+\ldots+T_n = (2^2-3)+(2^3-3)+(2^4-3)+\ldots+(2^{n+1}-3)$

$$= (2^2+2^3+2^4+\ldots+2^{n+1})-3(1+1+\ldots n\text{ terms}) = \frac{2^2(2^n-1)}{2-1}-3n$$

or $S_n = 2^{n+2}-3n-4$

Exercise 8.5

1. Find the sum to n terms of the series :

(i) $1+1+\dfrac{3}{2^2}+\dfrac{4}{2^3}+\ldots$ **(ii)** $1+3x+5x^2+7x^3+\ldots$ **(iii)** $\dfrac{1}{5}-\dfrac{2}{5^2}+\dfrac{3}{5^3}-\dfrac{4}{5^4}+\ldots$

2. Find the sum to infinity of the series :

(i) $\dfrac{3}{7}+\dfrac{5}{21}+\dfrac{7}{63}+\dfrac{9}{189}+\ldots$ **(ii)** $\dfrac{1}{3}-\dfrac{2}{3^2}+\dfrac{3}{3^3}-\dfrac{4}{3^4}+\ldots$ **(iii)** $1-2x+3x^2-4x^3+\ldots$, $|x|<1$

3. Find the n^{th} term and sum to the n terms of the series :

(i) $2+5+14+41+122+\ldots$ **(ii)** $3.2+5.2^2+7.2^3+\ldots$ **(iii)** $1+4x+7x^2+10x^3+\ldots$

4. Find the sum of n terms of the series : $2+5x+8x^2+11x^3+\ldots$ also find the sum to infinity of the series : if $|x|<1$.

8.17 Sum to *n* terms of series of first natural numbers, their squares and cubes :

(i) Sum of first *n* natural numbers :

Let S_n $(or\ \Sigma n)$ denotes the sum of first n natural numbers then

$$S_n = 1+2+3+\ldots+n$$

here $a = 1$ and $d = 1$

$\therefore \quad S_n = \dfrac{n}{2}\{2a+(n-1)d\} = \dfrac{n}{2}\{2.1+(n-1)1\} = \dfrac{n(n+1)}{2}$

or $\sum n = \dfrac{n(n+1)}{2}$

(ii) Sum of squares of first *n* natural numbers

$$S_n = 1^2+2^2+3^2+\ldots n^2 = \sum n^2$$

Identity $(x+1)^3-x^3 = 3x^2+3x+1$ put $x = 1, 2, 3, \ldots, (n-1), n.$

or $$2^3-1^3=3.1^2+3.1+1$$
$$3^3-2^3=3.2^2+3.2+1$$
$$4^3-3^3=3.3^2+3.3+1$$
$$\ldots\ldots\qquad\qquad\ldots\ldots$$
$$n^3-(n-1)^3=3(n-1)^2+3(n-1)+1$$

and $$(n+1)^3-n^3=3n^2+3n+1$$

adding all column wise

$$(n+1)^3-1^3=3\left(1^2+2^2+3^2+\ldots+n^2\right)+3(1+2+\ldots+n)+(1+1+\ldots+1, n\text{ terms})$$

$$(n+1)^3-1^3=3S_n+3\sum n+n \quad \text{or} \quad n^3+3n^2+3n=3S_n+3\frac{n(n+1)}{2}+n$$

or $$3S_n=n^3+3n^2+3n-\frac{3n^2+3n}{2}-n=\frac{2n^3+3n^2+n}{2}=\frac{n(n+1)(2n+1)}{2}$$

$\therefore$ $$S_n=\sum n^2=\frac{n(n+1)(2n+1)}{6}$$

(iii) Sum of cubes of first n natural numbers :

Let $S_n=1^3+2^3+3^3+\ldots n^3=\sum n^3$

Identity $(x+1)^4-x^4=4x^3+6x^2+4x+1$ put $x=1,2,3,\ldots,(n-1),n$

$$2^4-1^4=4.1^3+6.1^2+4.1+1$$
$$3^4-2^4=4.2^3+6.2^2+4.2+1$$
$$4^4-3^4=4.3^3+6.3^2+4.3+1$$
$$\ldots\ldots\qquad\qquad\ldots\ldots$$
$$n^4-(n-1)^4=4(n-1)^3+6(n-1)^2+4(n-1)+1,$$

and $$(n+1)^4-n^4=4n^3+6n^2+4n+1$$

adding all column wise

$$(n+1)^4-1^4=4\left(1^3+2^3+3^3+\ldots+n^3\right)+6\left(1^2+2^2+3^2+\ldots+n^2\right)$$
$$+4(1+2+3+\ldots+n)+(1+1+\ldots+1, n\text{ terms})$$

or $$n^4+4n^3+6n^2+4n=4S_n+6.\frac{n(n+1)(2n+1)}{6}+4.\frac{n(n+1)}{2}+n$$

or $$4S_n=n^4+4n^3+6n^2+4n-n(n+1)(2n+1)-2n(n+1)-n$$

$$, 4S_n=n^2\left(n^2+2n+1\right)$$

or $$S_n=\frac{n^2(n+1)^2}{4} \qquad \text{or} \qquad \sum n^3=S_n=\left[\frac{n(n+1)}{2}\right]^2$$

8.18 Sum of series by difference method :

To find the sum of the series by difference method the terms of the given series are written below the series with one term ahead and then the terms are subtracted. Whose sum to n term can be found out then using the formula for $\sum n, \sum n^2$ and $\sum n^3$ the sum can be derived.

Example : Find the nth terms and sum to n terms of the series :

$1 + 6 + 13 + 22 + ...$

Solution : In the given series, the difference of consecutive terms 5, 7, 9, ...is in A.P. so find its nth term and sum of n term by using difference method.

Let nth term of series it T_n and sum of n terms is S_n then,

$$S_n = 1 + 6 + 13 + 22 + \ldots + T_n \quad (1)$$

Write after shifting one place

$$S_n = 1 + 6 + 13 + \ldots + T_{n-1} + T_n \quad (2)$$

On subtraction, we get

$$0 = 1 + \{5 + 7 + 9 + ... + (n-1) \text{ terms}\} - T_n$$

or $$T_n = 1 + \{5 + 7 + 9 + ... + (n-1) \text{ terms}\} = 1 + \frac{(n-1)}{2}\{2 \cdot 5 + (n-2)2\} = 1 + (n-1)(n+3)$$

Simplyfying

$$T_n = n^2 + 2n - 2$$

$$\therefore \quad S_n = \sum T_n = \Sigma(n^2 + 2n - 2) = \Sigma n^2 + 2\Sigma n - 2\Sigma 1$$

$$= \frac{n(n+1)(2n+1)}{6} + 2.\frac{n(n+1)}{2} - 2n = \frac{2n^3 + 3n^2 + n + 6n^2 + 6n - 12n}{6}$$

$$= \frac{2n^3 + 9n^2 - 5n}{6}.$$

Illustrative Examples

Example 34 : Find the sum to n terms of the series whose nth terms is $n(n+1)(3n-1)$

Solution : Here $T_n = n(n+1)(3n-1) = 3n^3 + 2n^2 - n$

$$\therefore \quad S_n = \Sigma T_n$$

or $$S_n = \Sigma(3n^3 + 2n^2 - n) = 3\Sigma n^3 + 2\Sigma n^2 - \Sigma n$$

$$= 3.\left(\frac{n(n+1)}{2}\right)^2 + \frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{12}[9n(n+1) + 4(2n+1) - 6]$$

$$= \frac{n(n+1)}{12}[9n^2 + 17n - 2] = \frac{n(n+1)(n+2)(9n-1)}{12}.$$

Example 35 : Find the sum to n terms of the series :

$$1.3.5 + 3.5.7 + 5.7.9 + \ldots$$

Solution : The given series is a combination of three sequences 1, 3, 5, . . . ; 3, 5, 7, . . . and 5, 7, 9, . . . thus the nth term of the sequence is

$$\therefore \quad T_n = (2n-1)(2n+1)(2n+3) = 8n^3 + 12n^2 - 2n - 3$$

$$\therefore \quad S_n = \sum T_n$$

$$\text{or} \quad S_n = \sum\left(8n^3 + 12n^2 - 2n - 3\right)$$

$$= 8\sum n^3 + 12\sum n^2 - 2\sum n - 3\sum 1$$

$$= 8\left(\frac{n(n+1)}{2}\right)^2 + 12\frac{n(n+1)(2n+1)}{6} - \frac{2n(n+1)}{2} - 3n$$

$$= n(n+1)\left[2n(n+1) + 2(2n+1) - 1\right] - 3n = n(n+1)\left(2n^2 + 6n + 1\right) - 3n.$$

Exercise 8.6

1. Find the sum to n terms of the series whose nth terms is -

   (i) $3n^2 + 2n + 5$ **(ii)** $4n^3 + 7n + 1$ **(iii)** $n(n+1)(n+2)$

2. Find the sum to n terms of the series :

   (i) $3^2 + 7^2 + 11^2 + 15^2 + \ldots$ **(ii)** $2^3 + 5^3 + 8^3 + 11^3 + \ldots$ **(iii)** $1.2^2 + 2.3^2 + 3.4^2 + \ldots$

3. Find the n^{th} term and sum to n terms of the series

   (i) $1.3 + 3.5 + 5.7 + \ldots$ **(ii)** $1.2.4 + 2.3.7 + 3.4.10 + \ldots$

4. Find the n^{th} term and sum to n terms of the series.

   (i) $3 + 8 + 15 + 24 + \ldots$ **(ii)** $1 + 6 + 13 + 22 + \ldots$

5. Find the n^{th} term and sum to n terms of the series.

   (i) $1 + (1+2) + (1+2+3) + \ldots$ **(ii)** $1^2 + \left(1^2 + 2^2\right) + \left(1^2 + 2^2 + 3^2\right) + \ldots$

8.19 Harmonic Progression :

If the reciprocal of the terms form an arithmetic progression then that sequence is termed as harmonic progression.

Consider the following sequences -

(i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$ **(ii)** $\frac{1}{20}, \frac{1}{17}, \frac{1}{14}, \frac{1}{11} \ldots$

Above sequences are harmonic progression as its reciprocal 3,5,7,9, . . .; 20,17,14,11, . . . from an A.P.

General term : Standard form of harmonic progression -

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots, \frac{1}{a+(n-1)d}, \ldots$$

corresponding A.P. will be

$$a, a+d, a+2d, \ldots, a+(n-1)d, \ldots$$

The nth term of A.P. is $a+(n-1)d$, so nth term of H.P. is,

$$T_n = \frac{1}{a+(n-1)d}$$

Note :

1. To solve the problems of H.P. the terms are reciprocated and an A.P. is formed, and use the formula of A.P. and solve.
2. There is no specific formula to find the sum of nth terms of H.P.

8.20 Harmonic Mean :

If three or more than three terms are in H.P. then the terms between the first and the last term are known as Harmonic Means.

Example : If $a, H_1, H_2, H_3, \ldots\ldots, H_n, b$ are in H.P. then $H_1, H_2, H_3, \ldots\ldots, H_n$; are the n harmonic mean between a and b.

To find the H.M. between the two given numbers :

Let the numbers be a and b and let H be the H.M. between them thus a, H, b are in H.P. i.e.

$\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P. $\quad \therefore \quad \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$

or $\frac{2}{H} = \frac{1}{a} + \frac{1}{b}$ $\quad$ or $\frac{2}{H} = \frac{a+b}{ab}$ $\quad$ or $H = \frac{2ab}{a+b}$

To find the n H.M.'s between the two given numbers :

Let the numbers be a and b and there be n H.M.'s $H_1, H_2, H_3, \ldots, H_n$ between them

$\therefore \quad a, H_1, H_2, H_3, \ldots, H_n, b$ are in H.P. i.e.

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \ldots, \frac{1}{H_n}, \frac{1}{b}$ is A.P.

let term $1/b, (n+2)$ th term then

$\frac{1}{b} = \frac{1}{a} + (n+2-1)d$ $\quad$ or $\quad \frac{1}{b} = \frac{1}{a} + (n+1)d$

Simplifying, $\quad d = \frac{a-b}{ab(n+1)}$

$\therefore$ there are following n A.M.'s between $\frac{1}{a}$ and $\frac{1}{b}$

or $\quad \frac{1}{a}+d, \frac{1}{a}+2d, \frac{1}{a}+3d, \ldots, \frac{1}{a}+nd$

or $\quad \frac{1+ad}{a}, \frac{1+2ad}{a}, \frac{1+3ad}{a}, \ldots, \frac{1+nad}{a}$

Hence, there are following n H.M.'s between a and b.

$\therefore \quad \frac{a}{1+ad}, \frac{a}{1+2ad}, \frac{a}{1+3ad}, \ldots, \frac{a}{1+nad},$ $\quad$ where $d = \frac{a-b}{ab(n+1)}$.

Remark : It is clear from above that to find H.M. first we find A.M. using corresponding A.P. So the reciprocal of obtained A.M.'s will be required H.M.'s

Illustrative Examples

Example 36 : Find the terms of the following H.P.'s

(i) $\frac{1}{7},\frac{1}{11},\frac{1}{15},\frac{1}{19},\ldots 10$ th term (ii) $\frac{1}{5},\frac{2}{9},\frac{1}{4},\frac{2}{7},\ldots 15$ th term

Solution : (i) The A.P. corresponding to the given H.P. is

7, 11, 15, 19, ...

here $a = 7, d = 11 - 7 = 4$

$\therefore \quad T_{10} = a + 9d = 7 + 9 \times 4 = 43$

Thus the 10 term $= 1/43$

(ii) The A.P. corresponding to the given H.P. is

$5,\frac{9}{2},4,\frac{7}{2},\ldots..$

here $a = 5, d = 9/2 - 5 = -1/2$

$\therefore \quad T_{15} = a + 14d = 5 + 14 \times \left(-\frac{1}{2}\right) = -2$

Thus the 15 term $= -1/2$.

Example 37 : Find H.P. whose fourth is $1/2$ and 10th term is $1/4$.

Solution : The fourth term is 2 and 10 term is 4 of corresponding an A.P.

$\therefore \quad T_4 = 2 \Rightarrow a + 3d = 2$ (1)

and $T_{10} = 4 \Rightarrow a + 9d = 4$ (2)

solving (1) and (2)

$a = 1$ and $d = 1/3$

$\therefore \quad 1,\left(1+\frac{1}{3}\right),\left(1+\frac{2}{3}\right),\ldots$

Thus H.P. is $1,\frac{3}{4},\frac{3}{5},\ldots$

Example 38 : Which term of the series $\frac{\sqrt{5}}{3}+\frac{\sqrt{5}}{4}+\frac{1}{\sqrt{5}}+\ldots$ is $\frac{\sqrt{5}}{13}$?

Solution : Let $\frac{\sqrt{5}}{13}$ be the nth term of the series

The series obtained by reciprocating the terms is $\frac{3}{\sqrt{5}}+\frac{4}{\sqrt{5}}+\frac{\sqrt{5}}{1}+\ldots$ which is an A.P.

common difference $= \frac{4}{\sqrt{5}}-\frac{3}{\sqrt{5}} = \sqrt{5}-\frac{4}{\sqrt{5}} = \frac{1}{\sqrt{5}}$, a constant

The given terms of series are in H.P. thus the nth term of corresponding A.P. will be $\frac{13}{\sqrt{5}}$

$\therefore \quad \frac{3}{\sqrt{5}}+(n-1)\frac{1}{\sqrt{5}} = \frac{13}{\sqrt{5}}$ or $\frac{n-1}{\sqrt{5}} = \frac{13}{\sqrt{5}}-\frac{3}{\sqrt{5}}$

or $\quad n-1=10 \qquad \Rightarrow \qquad n=11$

$\therefore \quad \dfrac{\sqrt{5}}{13}$ is the 11th term of the given series.

Example 39 : Find the four H.M.'s between 1/ 2 and 3 .

Solution : Let there be H_1, H_2, H_3, H_4 H.M.'s between 1/ 2 and 3

$\therefore \quad 1/2, H_1, H_2, H_3, H_4, 3$

$a=2$ and sixth term is $1/3$

$\therefore \quad a+5d=1/3 \qquad [\because T_6 = a+5d]$

or $\quad 5d=1/3-2 \quad \Rightarrow \quad d=-1/3 \qquad [\because a=2]$

$\therefore$ Four H.M.'s between 1/ 2 and 3

$2+d, 2+2d, 2+3d, 2+4d$

or $\quad 2-\dfrac{1}{3}, 2-\dfrac{2}{3}, 2-\dfrac{3}{3}, 2-\dfrac{4}{3} \qquad$ or $\qquad \dfrac{5}{3}, \dfrac{4}{3}, 1, \dfrac{2}{3}$

Thus H.M.s are $\dfrac{3}{5}, \dfrac{3}{4}, 1, \dfrac{3}{2}$

Example 40 : If the A.M. between the two terms is 4 and H.M. is 3 then find the numbers.

Solution : Let the two numbers be a and b

A.M. $\quad = \dfrac{a+b}{2} = 4 \quad$ (given)

or $\quad a+b=8 \qquad (1)$

and H.M. $\quad \Rightarrow \dfrac{2ab}{a+b}=3 \qquad$ (given)

or $\quad 2ab = 3(a+b)$

or $\quad 2ab = 3\times 8 = 24 \qquad$ [by (1)] (2)

or $\quad a-b = \pm\sqrt{(a+b)^2 - 4ab}$

or $\quad a-b = \pm\sqrt{64-48} \qquad$ [by (1) and (2)]

or $\quad a-b = \pm 4$

taking positive sign

$a-b=4 \qquad (3)$

solving (1) and (2)

$a=6, b=2$

taking negative sign

$a-b=-4 \qquad (4)$

solving (1) and (2)

$a=2, b=6$

thus required terms 6, 2, or 2, 6

Example 41 : If ***a, b, c*** are in H.P., then prove that $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in H.P.

Solution : $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in H.P. if $\dfrac{b+c}{a}, \dfrac{c+a}{b}, \dfrac{a+b}{c}$ are in A.P.

i.e. $\frac{c+a}{b}-\frac{b+c}{a}=\frac{a+b}{c}-\frac{c+a}{b}$

or $\frac{ac+a^2-b^2-bc}{ab}=\frac{ab+b^2-c^2-ac}{bc}$

or $\frac{(a-b)(a+b+c)}{a}=\frac{(b-c)(a+b+c)}{c}$ [or]

simplifying $2ac=b(a+c)$

or $b=\frac{2ac}{a+c}$

or a, b, c are in H.P., given

$\therefore$ $\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}$ are in H.P.

Example 42 : If a car travels a distance of 200 km with a speed of 40 kmph and next 200 km with a speed of 50 kmph then find the average speed of the car. Check your result.

Solution : Here a = 40 kmph and b = 50 kmph

$\therefore$ Harmonic Mean $H=\frac{2ab}{a+b}$

$=\frac{2.40.50}{40+50}=\frac{4000}{90}=44.4$ kmph

Verification : Time taken for first 200 km $=\frac{200}{40}=5$ Hrs

Time taken for next 200 km $=\frac{200}{50}=4$ Hrs

$\therefore$ Total time taken to travel 400 km is 9 Hrs

$\therefore$ Average speed$=\frac{400}{9}=44.4$ kmph

Note : Here for average speed taking $=\frac{40+50}{2}=45$ is wrong because in 9 hrs car will travel a distance of $45\times 9=405$ km which is wrong, thus if the speed is given in different intervals then to find the average speed harmonic mean is taken.

Example 43 : If in a H.P. pth term is q and qth term is p then prove that rth term is $\frac{pq}{r}$

Solution : pth term is q and qth term is p

$\therefore$ then the corresponding A.P. of pth term is $1/q$ and q th term is $1/p$ thus let first term of A.P. is 'a' and common difference is d. So,

$a+(p-1)d=1/q$ (1) and $a+(q-1)d=1/p$ (2)

substracting (2) from (1)

$(p-q)d=\frac{1}{q}-\frac{1}{p}$ or $(p-q)d=\frac{p-q}{pq}$ or $d=1/pq$

putting the value of d in (1)

$$a = \frac{1}{q} - \frac{(p-1)}{pq} \qquad \text{or} \qquad a = 1/pq$$

$$\therefore \quad r \text{ th term} = a + (r-1)d = \frac{1}{pq} + (r-1)\frac{1}{pq} = \frac{r}{pq}$$

thus the r th term of H.P. is pq/r

Exercise 8.7

1. Determine the term of the following H.P. :

(i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \ldots 6$ th term. **(ii)** $\frac{1}{9}, \frac{1}{19}, \frac{1}{29}, \frac{1}{39}, \ldots 18$ th term. **(iii)** $\frac{1}{14}, \frac{2}{29}, \frac{1}{15}, \frac{2}{31}, \ldots 10$ th term.

2. Determine the nth term of the following H.P. :

(i) $\frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{2}{7}, \ldots$ **(ii)** $\frac{2}{a+b}, \frac{1}{a}, \frac{2}{3a-b}, \ldots$

3. Determine the H.P. whose second term is $\frac{2}{5}$ and seventh term is $\frac{4}{25}$

4. If the 7th term of an H.P. is $17/2$ and 11th terms is $13/2$ then find its 20th term :

5. Find :

(i) 4 H.M.'s between 1 and $1/16$

(ii) 5 H.M.'s between $1/19$ and $1/7$

(iii) 4 H.M.'s between $-2/5$ and $4/25$

6. If the pth, qth and rth term of H.P. are a, b, c respectively, then prove that

$$bc(q-r) + ca(r-p) + ab(p-q) = 0$$

7. If a, b, c are in H.P. then prove that $a, a-c, a-b$ are also in H.P.

8. If a, b, c are in H.P. then prove that $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c} = \frac{2}{b}$

9. Find the H.M. of roots of equation $ax^2 + bx + c = 0$

10. In a H.P. if the pth term is q and qth term is p then prove that $(p+q)$ th term is $pq/(p+q)$

11. If the roots of equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal then prove that a, b, c are in H.P.

12. If a child goes to the school with a speed of 8 kmph and comes back with a speed of 6 kmph then find the average speed of the child if the distance from the school to the house is 6 km. Verify the answer.

8.21 Relation between A.M., G.M. and H.M. :

Let A, G and H are the A.P. G.P. and H.P. between the two numbers a and b . Then

$$\therefore \quad A = \frac{a+b}{2}, G = \sqrt{ab} \text{ and } H = \frac{2ab}{a+b}$$

$$\therefore \quad AH = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab = G^2$$

$$\therefore \quad G^2 = AH$$

i.e. G is the G.M. between A and H.

again $A-G=\frac{a+b}{2}-\sqrt{ab}=\frac{a+b-2\sqrt{a}\sqrt{b}}{2}=\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}>0$

$\therefore \quad A>G$ (1)

and $G-H=\sqrt{ab}-\frac{2ab}{a+b}=\frac{\sqrt{ab}}{a+b}\left(a+b-2\sqrt{ab}\right)=\frac{\sqrt{ab}}{(a+b)}\left(\sqrt{a}-\sqrt{b}\right)^2>0$

$\therefore \quad G>H$

$\therefore \quad A>G>H$ [(1) and (2)]

8.22 Three terms *a, b, c* are in A.P. , G.P. and H.P. if :

(i) $\frac{a-b}{b-c}=\frac{a}{a}$ **(ii)** $\frac{a-b}{b-c}=\frac{a}{b}$ **(iii)** $\frac{a-b}{b-c}=\frac{a}{c}$

(i) $\frac{a-b}{b-c}=\frac{a}{a}\Rightarrow a-b=b-c \quad \Rightarrow \quad b=\frac{a+c}{2}$

$\therefore$ *a, b, c* are in A.P.

(ii) $\frac{a-b}{b-c}=\frac{a}{b}\Rightarrow ab-b^2=ab-ac \quad \Rightarrow \quad b^2=ac$

$\therefore$ *a, b, c* are in G..P.

(iii) $\frac{a-b}{b-c}=\frac{a}{c}\Rightarrow ac-bc=ab-ac \quad \Rightarrow \quad b=\frac{2ac}{a+c}$

$\therefore$ *a, b, c* are in H.P.

Illustrative Examples

Example 44 : If the A.M. of two numbers is two more than its G.M. and the ratio between them is 4 : 1 , then find the numbers.

Solution : Let the two numbers be *a* and *b* then given is,

$$\frac{a+b}{2}=\sqrt{ab}+2 \qquad (1)$$

and $\frac{a}{b}=\frac{4}{1}$ or $a=4b$ (2)

substituting the value of *a* from (2) in (1)

$\frac{4b+b}{2}=\sqrt{4b^2}+2$ or $5b=4b+4$ or $b=4$

$a=4b=4\times 4=16$

thus the numbers are 16, 4

Example 45 : If the H.M. of two numbers *a* and *b* is $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ then find the value of *n*.

Solution : Let the H.M. of two numbers be *a* and *b* is, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ then

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\frac{2ab}{a+b}$$

or $\quad a\,a^{n+1}+a\,b^{n+1}+b\,a^{n+1}+b\,b^{n+1}=2a^{n+1}b+2b^{n+1}a$

or $\quad a\,a^{n+1}+b\,b^{n+1}=a^{n+1}b+b^{n+1}a$

or $\quad a^{n+1}(a-b)=b^{n+1}(a-b)$

or $\quad a^{n+1}=b^{n+1}$ $\quad [\because a\neq b]$

or $\quad \left(\frac{a}{b}\right)^{n+1}=1$

It is only possible when $n+1=0$ or $n=-1$.

Example 46 : If three terms *a, b, c* are in H.P., then prove that $a(b+c),b(c+a),c(a+b)$ are in A.P.

Solution : $a(b+c),b(c+a),c(a+b)$ are in A.P. If

or $\quad b(c+a)-a(b+c)=c(a+b)-b(c+a)$

or $\quad bc+ba-ab-ac=ca+cb-bc-ba$

or $\quad bc+ab=2ac$

or $\quad b=\frac{2ac}{a+c}$, which is true because *a, b, c* are in H.P.

$\Rightarrow \quad a(b+c),b(c+a),c(a+b)$ are in A.P.

Exercise 8.8

1. If A.M. of two numbers is 50 and their H.M. is 18, then find the numbers.
2. If the ratio of H.M. and G.M. of two numbers are 12 : 13, then prove that the numbers are in the ratio 4 : 9.
3. If the difference of A.M. and G.M. of two numbers is 2 and the difference of G.M. and H.M. is 12 then find the numbers.
4. Three terms *a, b, c* are in G.P. and if $a^x=b^y=c^z$, then prove that *x, y, z* are in H.P.
5. Three terms *a, b, c* are in H..P. then prove that $2a-b$, b, $2c-b$ are in G.P.
6. If *a, b, c* are in A.P., *x, y, z* are in H.P. *ax, by, cz* are in H.P. then prove that

   $$\frac{x}{z}+\frac{z}{x}=\frac{a}{c}+\frac{c}{a}$$
7. If there are two A.M.'s A_1, A_2; two G.M.'s G_1, G_2; and two H.M.'s H_1, H_2; between the two numbers *a* and *b* then prove that

   (i) $A_1H_2=A_2H_1=G_1G_2=ab$

   (ii) $G_1G_2:H_1H_2=(A_1+A_2):(H_1+H_2)$
8. If *a, b, c* are in A.P. *b, c, d* are in G.P. and *c, d, e* are in H.P. then prove that *a, c, e* are in G.P.
9. If three terms *a, b, c* are in A.P. and H.P. then prove that they are in G.P. also.

Miscellaneous Exercise 8

1. The 10th term of the sequence $-4,-1,+2,+5,\ldots..$ is

   (A) 23 (B) –23 (C) 32 (D) –32

2. If the 9th term of an A.P. is 35 and 19th term is 75 then its 20th term is :
(A) 78 (B) 79 (C) 80 (D) 81

3. Find the sum to n terms of the sequences 1, 3, 5, ...
(A) $(n-1)^2$ (B) $(n+1)^2$ (C) $(2n-1)^2$ (D) n^2

4. If the first term of an A.P. is 5 and the last term is 45 and the sum of the terms is 400 then the number of terms.
(A) 8 (B) 10 (C) 16 (D) 20

5. If the 3rd term of an A.P. is 18 and 7th term is 30 then the sum of its first 17 terms is
(A) 600 (B) 612 (C) 624 (D) 636

6. If $(x+1), 3x, (4x+2)$ are in A.P. then its 5th term is,
(A) 14 (B) 19 (C) 24 (D) 28

7. a, b, c are in A.P., the A.M. of a and b is x, the A.M. of b and c is y then the A.M. of x and y is
(A) a (B) b (C) c (D) $a + c$

8. If $S_n = 3n^2 + 5n$ then its 27th terms is :
(A) 160 (B) 162 (C) 164 (D) 166

9. The sum of 50 A.M.'s between 20 and 30 is :
(A) 1255 (B) 1205 (C) 1250 (D) 1225

10. The common ratio of G.P. $\sqrt{3}, \frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}}, \ldots$ is :
(A) $\frac{1}{3}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\sqrt{3}$ (D) 3

11. The number of terms in G.P. 96, 48, 24, 12, . . ., $\frac{3}{16}$ is :
(A) 8 (B) 10 (C) 12 (D) 15

12. The value of $9^{1/3} \times 9^{1/9} \times 9^{1/27} \times \ldots\infty$ is :
(A) 1 (B) 3 (C) 9 (D) 27

13. The sum of infinity of the series $\sqrt{3} + \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} + \cdots$ is :
(A) $\frac{\sqrt{3}}{2}$ (B) $3\sqrt{3}$ (C) $\frac{3\sqrt{3}}{2}$ (D) $\frac{3}{2}$

14. The sum to infinity of the series $\frac{1}{7} + \frac{2}{7^2} + \frac{1}{7^3} + \frac{2}{7^4} + \ldots$ is :
(A) $\frac{1}{6}$ (B) $\frac{3}{16}$ (C) $\frac{1}{16}$ (D) $\frac{7}{16}$

15. If the 3rd term of the G.P. is 2 then the product of its first five terms is :
(A) 4 (B) 16 (C) 32 (D) 64

16. For what value of n does the expression $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$, is a G.M. between a and b
(A) 1 (B) 2 (C) 0 (D) $-\frac{1}{2}$

17. If G_1 and G_2 are the two G.M.'s between a and b then the value of G_1G_2 is :

(A) $\sqrt{ab}$ (B) ab (C) $(ab)^2$ (D) $(ab)^3$

18. The G.M. between -9 and -4 is :

(A) –36 (B) 6 (C) –6 (D) 36

19. The series $\frac{1}{2}, \frac{5}{13}, \frac{5}{16}, \ldots$ is :

(A) A.P. (B) G.P. (C) H.P. (D) other series

20. The 6th term of the sequence $1, \frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \ldots$ is :

(A) $\frac{1}{13}$ (B) $\frac{1}{16}$ (C) $\frac{1}{15}$ (D) none of these

21. If a, b, c, d are in H.P. then which among the following is true :

(A) $ab > cd$ (B) $ac > bd$ (C) $ad > bc$ (D) none of these

22. Two numbers have H.M. 4, A.M. A and G.M. G, if $2A + G^2 = 27$ then the numbers are :

(A) 6, 4 (B) 8, 2 (C) 8, 6 (D) 6, 3

23. If the ratio of H.M. and G.M. of two numbers is 12 : 13 then the ratio of the numbers are :

(A) 1 : 2 (B) 2 : 3 (C) 3 : 5 (D) 4 : 9

24. If the A.M., G.M. and H.M. between the two numbers a and b are A, G and H then A, G, H will be in,

(A) H.P. (B) G.P. (C) A.P. (D) none of these

25. If the H.M. between a and b is H then the value of $\frac{H}{a} + \frac{H}{b}$ is :

(A) 2 (B) $\frac{a+b}{ab}$ (C) $\frac{ab}{a+b}$ (D) none of these

26. If a, b, c are in H.P. then which of the following is true :

(A) $ac = b^2$ (B) $\sqrt{ac} < b$ (C) $a + c = 2b$ (D) $\sqrt{ac} > b$

27. If the n th term of a sequence is $\frac{n^2}{3^n}$ then find its first three terms.

28. Which term of the sequence 72, 70, 68, 66, ... is 40 ?

29. If the ratio of sum of m and n terms of an A.P. is $m^2 : n^2$ then prove that the ratio of m and n th term is $(2m-1):(2n-1)$

30. If the sides of a right angled triangle are in A.P. then find the ratio of length of its sides.

[Hint : $(a+d)^2 = (a-d)^2 + a^2 \Rightarrow \frac{a}{d} = 4$]

31. If $-\frac{2}{7}, a, -\frac{7}{2}$ are in G.P. then find the value of a:

32. Find the sum of n terms of the series $1 - 1 + 1 - 1 + \ldots$

33. Find the value of $2^{1/2}.4^{1/8}.16^{1/32}\ldots\infty$

34. For what value of n does the expression $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$, is the G.M. of a and b ?

35. If the A.M. and H.M. of a and b are A and H then prove that :

$$\frac{a-A}{a-H} \times \frac{b-A}{b-H} = \frac{A}{H}$$

36. If a, b, c are in A.P. and b, c, d are in H.P. then prove that $ad = bc$.

[Hint $b = \frac{a+c}{2}$ and $c = \frac{2bd}{b+d}$ $\therefore bc = \frac{a+c}{2} \cdot \frac{2bd}{b+d}$ or $c(b+d) = (a+c)d$ or $bc = ad$

37. If $a + b + ... + \ell$ are in G.P. then prove that its sum is $= \frac{b\ell - a^2}{b-a}$

38. Find the sum of n terms of sequence 3, 33, 333, ...

39. Find the sum of the products of the corresponding terms of the sequences 1, 2, 4, 8, 16, 32 and 32, 8, 2, 1 / 2, 1 / 8, 1 / 32

40. Find the value of n so that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$, may be the geometric mean between a and b.

41. If G_1 and G_2, are the G.M.'s between a and b then prove that $G_1 G_2 = a\,b$.

42. The ratio of the A.M. and G.M. of two positive numbers a and b is $m : n$. Show that

$a : b = m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$

43. If the A.M. and H.M. of two numbers are 50 and 18 respectively, then find the numbers.

44. If the difference of A.M. and G.M. of two numbers is 2 and the difference of G.M. and H.M. is 1.2, then find the numbers.

45. If a, b, c are in A.P., x, y, z are in H.P. and ax, by, cz are in G.P., then prove that

$$\frac{x}{z} + \frac{z}{x} = \frac{a}{c} + \frac{c}{a}$$

46. If there are two A.M.'s A_1, A_2, two G.M.'s G_1, G_2 and two H.M.'s H_1, H_2; between the two numbers a and b, then prove that $A_1 H_2 = A_2 H_1 = G_1 G_2 = a\,b$

Important Points

1. By a sequence, we mean an arrangement of numbers in a definite order according to some logical rule. Also we define a sequence as a function whose domain is the set of natural numbers or some subsets of the type $N_k = \{1, 2, 3 ..., k\}$. The number of sequence is called its terms. A sequence containing a finite number of terms is called a finite sequence. A sequence is called infinite if it is not a finite sequence.
2. Let $a_1, a_2, a_3, ...$ be the sequence, then the sum expressed as $a_1 + a_2 + a_3 + ...$ is known as series. A series is called finite series if it has got finite number of terms.
3. Progression : A sequence is called as progression if the number value of its terms is increasing or decreasing in a specific rule. Thus, if nth term of any sequence is expressed by using a finite formula so that sequence is called as progression.
4. An arithmetic progression (A.P.) is a sequence in which terms increase or decrease regularly by the same constant. This constant is called common difference of the A.P. Usually, we denote the first term of A.P. by a, the common difference by d and the last term by l. The general term or the nth term of the A.P. is given by $T_n = a + (n-1)d$

   The sum S_n of the first n terms of an A.P. is given by

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}[a + l]$$

5. The arithmetic mean A of any two numbers a and b is given by $A = \frac{a+b}{2}$ i.e., the sequence a, A, b is an A.P.
6. A sequence is said to be a geometric progression or G.P., if the ratio of any term to its preceding term is same throughout. This constant factor is called the common ratio. Usually, we denote the first term of a G.P. by a and its common ratio by r. The general or the nth term of G.P. is given by $T_n = ar^{n-1}$
7. The sum S_n of the first n terms of G.P. is given by $S_n = \frac{a(r^n - 1)}{r-1}$ or $\frac{a(1-r^n)}{1-r}$ if $r \neq 1$
8. The geometric mean G of any two numbers a and b is given by $G = \sqrt{ab}$ i.e. the sequence a, G, b is G.P.
9. If the reciprocal of a sequence is an A.P. then that sequence is called as harmonic progression (H.P.)
10. The nth term of H.P. is $T_n = \frac{1}{a+(n-1)d}$ where a, d are first term and common difference of corresponding A.P.
11. The harmonic mean H of any two numbers a and b is given by $H = \frac{2ab}{a+b}$
12. If A.M., G.M. and H.M. between the two numbers is A, G, H then
    (i) $G^2 = AH$ (B) $A \geq G \geq H$

Answers

Exercise 8.1

1. (i), (ii), (iii) A.P. (iv) not an A.P.
2. (i) a = 10: d = 3 $T_5 = 22$
   (ii) a = a : d = d $T_5 = a + 4d$
   (iii) a = 2 : d = 3 $T_5 = -10$

4. 22 5. 64 6. 32 8. 30

Exercise 8.2

1. (i) 900 (ii) $\frac{100}{3}$ (iii) $\frac{3(7+5\sqrt{2})}{\sqrt{2}+1}$ 2. 33, 1683 3. n (n + 4)

4. 3,2 5. 8, 36 6. zero 10. 1,4,7 11. 24 14. 2700 15. 3

Exercise 8.3

1. (i) 29 (ii) $\frac{\sqrt{2}}{512}$ 2. (i) n = 13 (ii) n = 0 3. 5.2^{n-1} 4. 1458 5. r = 4 6. $\frac{1}{4}, \frac{1}{2}, 1, 2.$

7. (i) 6,12, 24 (ii) 4, 8, 16, 32, 64, 128 8. $x = 3,$ 9. $\frac{1}{20}, \frac{-9}{20}, \frac{81}{20}, \frac{-729}{20}$ 11. $\sqrt{xy}$

Exercise 8.4

1. (i) 2186 (ii) $-\frac{6305}{2880}$ (iii) $\frac{a^{10}-b^{10}}{a(a+b)}$ 2. (i) 728 (ii) $\frac{511}{4}$

3. 5 terms 4. 4 5. 10

6. (i) $\frac{70}{81}\left(10^n-1\right)-\frac{7n}{9}$ (ii) $\frac{5}{81}\left(9n-1+\frac{1}{10^n}\right)$ (iii) $n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)$

7. (i) $\frac{106}{45}$ (ii) $\frac{619}{990}$ (iii) $\frac{2750}{999}$ 8. 64, 16, 4, 1, $\frac{1}{4}$,... 11. $\frac{7}{3}$

Exercise 8.5

1. (i) $4-\frac{2+n}{2^{n-1}}$ **(ii)** $\frac{1-(2n-1)x^n}{1-x}+\frac{2x\left(1-x^{n-1}\right)}{(1-x)^2}$ **(iii)** $\frac{5}{36}+(-1)^{n-1}\frac{5+6n}{6^2\,5^n}$

2. (i) $\frac{6}{7}$ **(ii)** $\frac{3}{16}$ **(iii)** $\frac{1}{(1+x)^2}$

3. (i) $\frac{3^n+1}{2},\frac{3^{n+1}+2n-3}{4}$ **(ii)** $(2n+1)2^n,(2n-1)2^{n+1}+2$

(iii) $(3n-2)x^{n-1},\left[\frac{1}{1-x}+\frac{3x\left(1-x^{n-1}\right)}{(1-x)^2}-\frac{(3n-2)x^n}{1-x}\right]$

4. $\frac{2+x}{(1-x)^2}-\frac{3x^n}{(1-x)^2}-\frac{(3n-1)x^n}{1-x},\frac{2+x}{(1+x)^2}$

Exercise 8.6

1. (i) $\frac{n(n+1)(2n+3)}{2}+5n$ **(ii)** $\frac{n(n+1)\left(2n^2+2n+7\right)}{2}+n$ **(iii)** $\frac{n(n+1)(n+2)(n+3)}{4}$

2. (i) $\frac{4}{3}n(n+1)(4n-1)+n$ **(ii)** $\frac{9}{4}n^2(n+1)(3n-1)-n$ **(iii)** $\frac{n}{12}(n+1)(n+2)(3n+5)$

3. (i) $(2n-1)(2n+1)\frac{n}{3}\left(4n^2+6n-1\right)$ **(ii)** $n(n+1)(3n+1)\frac{1}{12}n(n+1)(n+2)(9n+7)$

4. (i) $n(n+2),\frac{n(n+1)(2n+7)}{6}$ **(ii)** $n^2+2n-2,\frac{n(n+1)(2n+7)}{6}-2n$

5. (i) $\frac{n(n+1)}{2},\frac{n}{6}(n+1)(n+2)$ **(ii)** $\frac{n(n+1)(2n+1)}{6},\frac{n}{12}(n+1)^2(n+2)$

Exercise 8.7

1. (i) $\frac{1}{17}$ **(ii)** $\frac{1}{179}$ **(iii)** $\frac{2}{37}$ **2. (i)** $\frac{2}{11-n}$ **(ii)** $\frac{2}{na+(2-n)b}$

3. $\frac{4}{7}, \frac{4}{10}, \frac{4}{13} \ldots\ldots$ **4.** $\frac{17}{4}$

5. (i) $\frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}$ **(ii)** $\frac{1}{17}, \frac{1}{15}, \frac{1}{13}, \frac{1}{11}, \frac{1}{9}$ **(iii)** $\frac{8}{15}$

9. $-\frac{2c}{b}$ **12.** $6\frac{6}{7}$ kmph

Exercise 8.8

1. 90, 10 3. 1, 9

Miscellaneous Answer 8

1. (A) **2.** (B) **3.** (D) **4.** (C) **5.** (B) **6.** (C) **7.** (B)

8. (C) **9.** (C) **10.** (A) **11.** (B) **12.** (B) **13.** (C) **14.** (B)

15. (C) **16.** (D) **17.** (B) **18.** (C) **19.** (C) **20.** (B) **21.** (C)

22. (D) **23.** (D) **24.** (B) **25.** (A) **26.** (D) **27.** $\frac{1}{3}, \frac{4}{9}, \frac{1}{3}$ **28.** 17 th

30. 3 : 4 : 5 **31.** ± 1 **32.** $\frac{1-(-1)^n}{2}$ **33.** 2 **34.** $n = -1$ **39.** $\frac{30}{81}\left[10^n - 1\right] - \frac{n}{3}$