UNIT–III : CALCULUS

CHAPTER 5

Term-I

CONTINUITY & DIFFERENTIABILITY

Syllabus

- ***Continuity and differentiability, derivative of composite functions, chain rule, derivative of inverse trigonometric functions, derivative of implicit functions. Concept of exponential and logarithmic functions.***
- ***Derivatives of logarithmic and exponential functions. Logarithmic differentiation, derivative of functions expressed in parametric forms. Second order derivatives.***

STAND ALONE MCQs (1 Mark each)

Q. 1. If $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$ then which of the following can be a discontinuous function?

(A) $f(x) + g(x)$ **(B)** $f(x) - g(x)$

(C) $f(x).g(x)$ **(D)** $\frac{g(x)}{f(x)}$

Ans. Option (D) is correct.

> *Explanation:* Since $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$ are continuous functions, then by using the algebra of continuous functions , the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x).g(x)$ are also continuous functions but $\frac{g(x)}{f(x)}$ is discontinuous function at $x = 0$.

Q. 2. The function $f(x) = \frac{4 - x^2}{4x - x^3}$

(A) discontinuous at only one point

(B) discontinuous at exactly two points

(C) discontinuous at exactly three points

(D) none of these

Ans. Option (C) is correct.

> *Explanation:* Given that,
>
> $$f(x) = \frac{4 - x^2}{4x - x^3},$$
>
> then it is discontinuous if
>
> $\Rightarrow \quad 4x - x^3 = 0$
>
> $\Rightarrow \quad x(4 - x^2) = 0$
>
> $\Rightarrow x(2 + x)(2 - x) = 0$
>
> $\Rightarrow \quad x = 0, -2, 2$
>
> Thus, the given function is discontinuous at exactly three points.

Q. 3. The function $f(x) = \cot x$ is discontinuous on the set

(A) $\{x = n\pi; n \in Z\}$

(B) $\{x = 2n\pi; n \in Z\}$

(C) $\left\{x = (2n + 1)\frac{\pi}{2}; n \in Z\right\}$

(D) $\left\{x = \frac{n\pi}{2}; n \in Z\right\}$

Ans. Option (A) is correct.

Explanation: Given that,

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

It is discontinuous at

$$\sin x = 0$$

$$\Rightarrow \quad x = n\pi, n \in Z$$

Thus, the given function is discontinuous at $\{x = n\pi : n \in Z\}$.

Q. 4. If $f(x) = \begin{cases} mx+1 \text{ if } x \le \frac{\pi}{2} \\ \sin x + n, \text{ if } x > \frac{\pi}{2} \end{cases}$, is continuous at $x = \frac{\pi}{2}$

then

(A) $m = 1, n = 0$ **(B)** $m = \frac{n\pi}{2} + 1$

(C) $n = \frac{m\pi}{2}$ **(D)** $m = n = \frac{\pi}{2}$

Ans. Option (C) is correct.

Explanation: Given that,

$$f(x) = \begin{cases} mx+1 \text{ if } x \le \frac{\pi}{2} \\ \sin x + n, \text{ if } x > \frac{\pi}{2} \end{cases}$$

is continuous function at $x = \frac{\pi}{2}$, then

$$\text{LHL} = \text{RHL}$$

$$\Rightarrow \lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^+} f(x)$$

$$\Rightarrow \lim_{h \to 0} f\left(\frac{\pi}{2} - h\right) = \lim_{h \to 0} f\left(\frac{\pi}{2} + h\right)$$

$$\Rightarrow \lim_{h \to 0} m\left(\frac{\pi}{2} - h\right) + 1 = \lim_{h \to 0} \sin\left(\frac{\pi}{2} + h\right) + n$$

$$\Rightarrow \lim_{h \to 0} m\left(\frac{\pi}{2} - h\right) + 1 = \lim_{h \to 0} \cos h + n$$

$$\Rightarrow m\left(\frac{\pi}{2}\right) + 1 = 1 + n$$

$$\Rightarrow n = \frac{m\pi}{2}$$

Q. 5. If $y = Ae^{5x} + Be^{-5x}$, then $\frac{d^2y}{dx^2}$ is equal to

(A) $25y$ **(B)** $5y$

(C) $-25y$ **(D)** $15y$

[CBSE Delhi Set-I 2020]

Ans. Option (A) is correct.

Explanation:

$$y = Ae^{5x} + Be^{-5x}$$

$$\Rightarrow \frac{dy}{dx} = 5Ae^{5x} - 5Be^{-5x}$$

$$\Rightarrow \frac{d^2y}{d^2x} = 25Ae^{5x} + 25Be^{-5x}$$

$$= 25y$$

Q. 6. If $y = \log_e\left(\frac{x^2}{e^2}\right)$, then $\frac{d^2y}{dx^2}$ equals

(A) $-\frac{1}{x}$ **(B)** $-\frac{1}{x^2}$

(C) $\frac{2}{x^2}$ **(D)** $-\frac{2}{x^2}$

[CBSE Delhi Set-III 2020]

Ans. Option (D) is correct.

Explanation:

Given, $$y = \log_e\left(\frac{x^2}{e^2}\right)$$

$$\Rightarrow y = 2\log_e x - \log_e e^2$$

$$\Rightarrow y = 2\log_e x - 2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2}{x^2}$$

Q. 7. The set of points where the function f given by $f(x) = |2x - 1| \sin x$ is differentiable is

(A) R **(B)** $R - \left\{\frac{1}{2}\right\}$

(C) $(0, \infty)$ **(D)** none of these

Ans. Option (B) is correct.

Explanation: Given that,

$$f(x) = |2x - 1| \sin x$$

The function $\sin x$ is differentiable.

The function $|2x - 1|$ is differentiable, except

$$2x - 1 = 0$$

$$\Rightarrow x = \frac{1}{2}$$

Thus, the given function is differentiable $R - \left\{\frac{1}{2}\right\}$.

Q. 8. The function $f(x) = e^{|x|}$ is

(A) continuous everywhere but not differentiable at $x = 0$

(B) continuous and differentiable everywhere

(C) not continuous at $x = 0$

(D) none of these

Ans. Option (A) is correct.

Explanation: Given that,

$$f(x) = e^{|x|}$$

The functions e^x and $|x|$ are continuous functions for all real value of x.
Since e^x is differentiable everywhere but $|x|$ is non-differentiable at $x = 0$.
Thus, the given functions $f(x) = e^{|x|}$ is continuous everywhere but not differentiable at $x = 0$.

Q. 9. Let $f(x) = |\sin x|$, then

(A) f is everywhere differentiable

(B) f is everywhere continuous but not differentiable at $x = n\pi$, $n \in Z$.

(C) f is everywhere continuous but not differentiable at $x = (2n+1)\frac{\pi}{2}$, $n \in Z$.

(D) none of these

Ans. Option (B) is correct.

Explanation: Given that,

$$f(x) = |\sin x|$$

The functions $|x|$ and $\sin x$ are continuous function for all real value of x.
Thus, the function $f(x) = |\sin x|$ is continuous function everywhere.
Now, $|x|$ is non-differentiable function at $x = 0$.
Since $f(x) = |\sin x|$ is non-differentiable function at $\sin x = 0$
Thus, f is everywhere continuous but not differentiable at $x = n\pi$, $n \in Z$.

Q. 10. If $y = \log\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to

(A) $\frac{4x^3}{1-x^4}$ **(B)** $\frac{-4x}{1-x^4}$

(C) $\frac{1}{4-x^4}$ **(D)** $\frac{-4x^3}{1-x^4}$

Ans. Option (B) is correct.

Explanation: Given that,

$$y = \log\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \quad y = \log(1-x^2) - \log(1+x^2).$$

Differentiate with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(1-x^2)\right] - \frac{d}{dx}\left[\log(1+x^2)\right]$$

$$= \frac{-2x}{1-x^2} - \frac{2x}{1+x^2}$$

$$= -2x\left(\frac{2}{(1-x^2)(1+x^2)}\right)$$

$$= -\frac{4x}{1-x^4}$$

Q. 11. If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ is equal to

(A) $\frac{\cos x}{2y-1}$ **(B)** $\frac{\cos x}{1-2y}$

(C) $\frac{\sin x}{1-2y}$ **(D)** $\frac{\sin x}{2y-1}$

Ans. Option (A) is correct.

Explanation: Given that,

$$y = \sqrt{\sin x + y}$$

$$\Rightarrow y^2 = \sin x + y$$

Differentiate with respect to x, we have

$$\Rightarrow \quad 2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\Rightarrow (2y-1)\frac{dy}{dx} = \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos x}{2y \quad 1}$$

Q. 12. The derivative of $\cos^{-1}(2x^2-1)$ w.r.t. $\cos^{-1} x$ is

(A) 2 **(B)** $\frac{-1}{2\sqrt{1-x^2}}$

(C) $\frac{2}{x}$ **(D)** $1-x^2$

Ans. Option (A) is correct.

Explanation: Let

$$u = \cos^{-1}(2x^2-1)$$

$$\Rightarrow \frac{du}{dx} = -\frac{4x}{\sqrt{1-(2x^2-1)^2}}$$

$$\Rightarrow \frac{du}{dx} = -\frac{4x}{\sqrt{1-4x^4+4x^2-1}}$$

$$\Rightarrow \frac{du}{dx} = -\frac{4x}{\sqrt{-4x^4+4x^2}}$$

$$\Rightarrow \frac{du}{dx} = -\frac{2}{\sqrt{1-x^2}}$$

And, $v = \cos^{-1} x$

$$\frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Thus, $\frac{du}{dv} = 2$

Q. 13. If $x = t^2$ and $y = t^3$ then $\frac{d^2y}{dx^2}$ is

(A) $\frac{3}{2}$ **(B)** $\frac{3}{4t}$

(C) $\frac{3}{2t}$ **(D)** $\frac{3}{4}$

Ans. Option (A) is correct.

Explanation: Given that,

$x = t^2$ and $y = t^3$

Then, $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2$

Thus,

$$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{3}{2}$$

ASSERTION AND REASON BASED MCQs (1 Mark each)

Directions: In the following questions, A statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as.

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is NOT the correct explanation of A

(C) A is true but R is false

(D) A is false and R is True

Q. 1. Assertion (A): $|\sin x|$ is continuous for all $x \in R$.

Reason (R): $\sin x$ and $|x|$ are continuous in R.

Ans. Option (A) is correct.

Explanation: $\sin x$ and $|x|$ are continuous in R. hence R is true.

Consider the functions $f(x) = \sin x$ and $g(x) = |x|$ both of which are continuous in R.

$$gof(x) = g(f(x)) = g(\sin x) = |\sin x|.$$

Since $f(x)$ and $g(x)$ are continuous in R, $gof(x)$ is also continuous in R.

Hence A is true.

R is the correct explanation of A.

Q. 2. Assertion (A): $f(x) = \tan^2 x$ is continuous at $x = \frac{\pi}{2}$.

Reason (R): $g(x) = x^2$ is continuous at $x = \frac{\pi}{2}$.

Ans. Option (D) is correct.

Explanation: $g(x) = x^2$ is a polynomial function. It is continuous for all $x \in R$.

Hence R is true.

$f(x) = \tan^2 x$ is not defined when $x = \frac{\pi}{2}$.

Therefore $f\left(\frac{\pi}{2}\right)$ does not exist and hence $f(x)$ is not continuous at $x = \frac{\pi}{2}$.

A is false.

Q. 3. Consider the function $f(x) = \begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$

which is continuous at $x = 0$.

Assertion (A): The value of k is – 3.

Reason (R): $|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$

Ans. Option (A) is correct.

Explanation:

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}.$$

This is the definition for modulus function and hence true.

Hence R is true.

Since f is continuous at $x = 0$,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

Here $f(0) = 3$,

$$\text{LHL} = \lim_{x \to 0^-} f(x)$$

$$= \lim_{x \to 0^-} \frac{kx}{|x|} = \lim_{x \to 0^-} \frac{kx}{-x} = -k$$

$\therefore$ $-k = 3$ or $k = -3$.

Hence A is true.

R is the correct explanation of A.

Q. 4. Consider the function

$$f(x) = \begin{cases} \frac{x^2 + 3x - 10}{x - 2}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$$

which is continuous at $x = 2$.

Assertion (A): The value of k is 0.

Reason (R): $f(x)$ is continuous at $x = a$, if

$$\lim_{x \to a} f(x) = f(a).$$

Ans. Option (D) is correct.

Explanation:

$f(x)$ is continuous at $x = a$, if $\lim_{x \to a} f(x) = f(a)$.

$\therefore$ R is true.

$$\lim_{x \to 2} f(x) = f(2) = k$$

$$\lim_{x \to 2} \frac{(x+5)(x-2)}{x-2} = k$$

$\therefore$ $k = 7$

Hence A is false.

Q. 5. Assertion (A): $|\sin x|$ is continuous at $x = 0$.

Reason (R): $|\sin x|$ is differentiable at $x = 0$.

Ans. Option (C) is correct.

Explanation: Since sin x and $|x|$ are continuous functions in R, $|\sin x|$ is continuous at $x = 0$.

Hence A is true.

$$|\sin x| = \begin{cases} -\sin x, & \text{if } x < 0 \\ \sin x, & \text{if } x \geq 0 \end{cases}$$

$$f(0) = |\sin 0| = 0$$

$$\text{LHD} = f'(0^-) = \lim_{x \to 0} \frac{-\sin x - 0}{x} = -1$$

$$\text{RHD} = f'(0^+) = \lim_{x \to 0} \frac{\sin x - 0}{x} = 1$$

At $x = 0$, LHD ≠ RHD.

So $f(x)$ is not differentiable at $x = 0$.

Hence R is false.

Q. 6. Assertion (A): $f(x) = [x]$ is not differentiable at $x = 2$.

Reason (R): $f(x) = [x]$ is not continuous at $x = 2$.

Ans. Option (A) is correct.

Explanation: $f(x) = [x]$ is not continuous when x is an integer.

So $f(x)$ is not continuous at $x = 2$. Hence R is true.

A differentiable function is always continuous.

Since $f(x) = [x]$ is not continuous at $x = 2$, it is also not differentiable at $x = 2$.

Hence A is true.

R is the correct explanation of A.

Q. 7. Assertion (A): A continuous function is always differentiable.

Reason (R): A differentiable function is always continuous.

Ans. Option (D) is correct.

Explanation: The function $f(x)$ is differentiable at $x = a$, if it is continuous at $x = a$ and

LHD = RHD at $x = a$.

A differentiable function is always continuous.

Hence R is true.

A continuous function need not be always differentiable.

For example, $|x|$ is continuous at $x = 0$, but not differentiable at $x = 0$.

Hence A is false.

Q. 8. Assertion (A): If $y = \sin^{-1}(6x\sqrt{1-9x^2})$, then $\frac{dy}{dx} = \frac{6}{\sqrt{1-9x^2}}$

Reason (R): $\sin^{-1}(6x\sqrt{1-9x^2}) = 3\sin^{-1}(2x)$

Ans. Option (C) is correct.

Explanation:

put $3x = \sin\theta$ or $\theta = \sin^{-1} 3x$

$$y = \sin^{-1}(6x\sqrt{1-9x^2}) = \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$= 2\sin^{-1} 3x$$

$$\therefore \quad \frac{dy}{dx} = \frac{6}{\sqrt{1-9x^2}}$$

A is true. R is false.

CASE-BASED MCQs

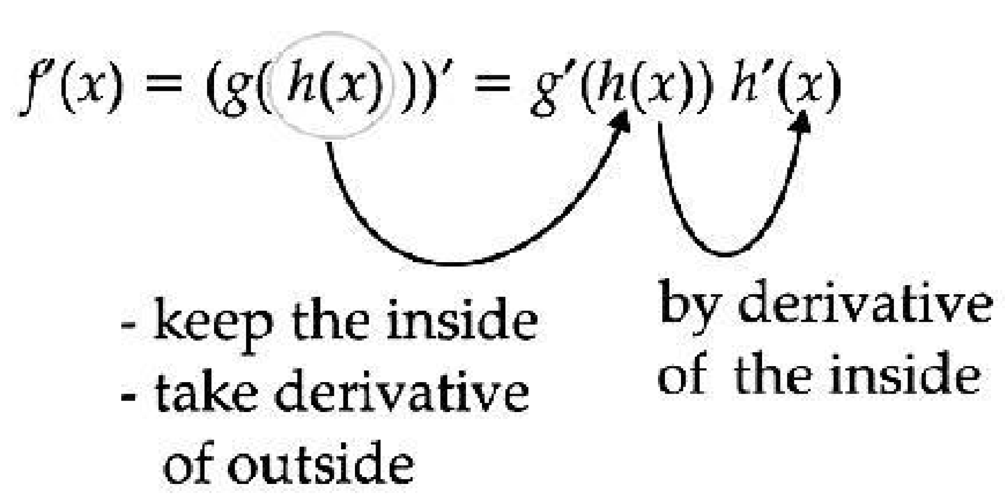
Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same:

Ms. Remka of city school is teaching chain rule to her students with the help of a flow-chart

The chain rule says that if h and g are functions and $f(x) = g(h(x))$, then

$$f'(x) = (g(h(x)))' = g'(h(x))\, h'(x)$$



Let $f(x) = \sin x$ and $g(x) = x^3$

Q. 1. $fog(x) =$ _______.

(A) $\sin x^3$ (B) $\sin^3 x$

(C) $\sin 3x$ (D) $3\sin x$

Ans. Option (A) is correct.

Explanation:

$$fog(x) = f(g(x)) = f(x^3) = \sin(x^3)$$

Q. 2. $gof(x) =$ _______.

(A) $\sin x^3$ (B) $\sin^3 x$

(C) $\sin 3x$ (D) $3 \sin x$

Ans. Option (B) is correct.

Explanation:

$$gof(x) = g(f(x)) = g(\sin x) = \sin^3 x$$

Q. 3. $\frac{d}{dx}(\sin^3 x) =$ ______.

(A) $\cos^3 x$ **(B)** $3\sin x\cos x$

(C) $3\sin^2 x \cos x$ **(D)** $-\cos^3 x$

Ans. Option (C) is correct.

> ***Explanation:***
> $$\frac{d}{dx}(\sin^3 x) = 3\sin^2 x \frac{d}{dx}(\sin x)$$
> $$= 3\sin^2 x \cos x$$

Q. 4. $\frac{d}{dx}\sin x^3$ ______.

(A) $\cos (x^3)$ **(B)** $-\cos (x^3)$

(C) $3x^2 \sin (x^3)$ **(D)** $3x^2 \cos (x^3)$

Ans. Option (D) is correct.

> ***Explanation:***
> $$\frac{d}{dx}(\sin x^3) = \cos x^3 \frac{d}{dx}(x^3)$$
> $$= 3x^2 \cos x^3$$

Q. 5. $\frac{d}{dx}(\sin 2x)$ at $x = \frac{\pi}{2}$ is ______.

(A) 0 **(B)** 1

(C) 2 **(D)** -2

Ans. Option (D) is correct.

> ***Explanation:***
> $$\frac{d}{dx}(\sin 2x) = \cos 2x \frac{d}{dx}(2x)$$
> $$= 2\cos 2x$$
> $$\left.\frac{d}{dx}(\sin 2x)\right|_{x=\frac{\pi}{4}} = 2\cos 2 \times \frac{\pi}{2} = 2\cos \pi$$
> $$= 2(-1)$$
> $$= -2$$

II. Read the following text and answer the following questions on the basis of the same:

A potter made a mud vessel, where the shape of the pot is based on $f(x) = |x - 3| + |x - 2|$, where $f(x)$ represents the height of the pot.

[CBSE QB 2021]

Q. 1. When $x > 4$ what will be the height in terms of x?

(A) $x - 2$ **(B)** $x - 3$

(C) $2x - 5$ **(D)** $5 - 2x$

Ans. Option (C) is correct.

> ***Explanation:*** The given function can be written as
> $$f(x) = \begin{cases} 5 - 2x, & \text{if } x < 2 \\ 1, & \text{if } 2 \le x < 3 \\ 2x - 5, & \text{if } x \ge 3 \end{cases}$$
> When $x > 4$, $f(x) = 2x - 5$

Q. 2. Will the slope vary with x value?

(A) Yes **(B)** No

(C) Can't say **(D)** In complete data

Ans. Option (A) is correct.

> ***Explanation:***
> $$f'(x) = \begin{cases} -2, & \text{if } x < 2 \\ 0, & \text{if } 2 \le x < 3 \\ 2, & \text{if } x \ge 3 \end{cases}$$

Q. 3. What is $\frac{dy}{dx}$ at $x = 3$

(A) 2

(B) -2

(C) Function is not differentiable

(D) 1

Ans. Option (C) is correct.

> ***Explanation:*** $f(x)$ is not differentiable at $x = 2$ and $x = 3$.

Q. 4. When the value of x lies between (2, 3) then the function is

(A) $2x - 5$ **(B)** $5 - 2x$

(C) 1 **(D)** 5

Ans. Option (C) is correct.

> ***Explanation:*** In (2, 3), $f(x) = 1$

Q. 5. If the potter is trying to make a pot using the function $f(x) = [x]$, will he get a pot or not? Why?

(A) Yes, because it is a continuous function

(B) Yes, because it is not continuous

(C) No, because it is a continuous function

(D) No, because it is not continuous

Ans. Option (D) is correct.

> ***Explanation:*** $[x]$ is not continuous at integral values of x.