Complex Numbers

Theory →

$iota = i = \underbrace{\sqrt{-1}}_{\text{Not real}}$

$x^2 = -1$

$x = \pm\sqrt{-1} \Rightarrow x = \pm i$

$i^1 = i$	$i^5 = i$	$i^9 = i$	$\boxed{i^{4n+1} = i}$ ☆
$i^2 = -1$	$i^6 = -1$	$i^{10} = -1$	$\boxed{i^{4n+2} = -1}$ ☆
$i^3 = -i$	$i^7 = -i$	$i^{11} = -i$	$\boxed{i^{4n+3} = -i}$ ☆
$\boxed{i^4 = 1}$	$\boxed{i^8 = 1}$	$\boxed{i^{12} = 1}$	$\boxed{i^{4n} = 1}$ ☆

$\pm 1 \quad \pm i$

$i^{63} = i^{\underline{4\times 15+3}}$

$= -i$ ✓

Special Case :→

$$\boxed{\sqrt{a}\cdot\sqrt{b} = \sqrt{a\cdot b}}$$

Condition:

atleast one of a & b must be non-negative.

$a = -1$ ✓

$b = -1$ ✓

$\sqrt{-1}\cdot\sqrt{-1} = \sqrt{(-1)\cdot(-1)}$

$i\cdot i = \sqrt{+1}$

$i^2 = 1$

$\boxed{-1 = 1}$

False

a	b	
+	+	$\sqrt{a}\sqrt{b} = \sqrt{ab}$ ✓
+	−	$\sqrt{a}\sqrt{b} = \sqrt{ab}$ ✓
−	+	$\sqrt{a}\sqrt{b} = \sqrt{ab}$ ✓
−	−	$\sqrt{a}\sqrt{b} \neq \sqrt{ab}$ ✗

☆

Complex No.

2 → Real No.

5i → imaginary

2+3i → imaginary No.

2 → Ratⁿ.

$5\sqrt{3}$ → Irr.

$2+5\sqrt{3}$ → Irr.

Complex No.

- Real No.
  - $2, 3, \sqrt{3}, \pi$, $-\frac{1}{2}$
- Imaginary No.
  - $2+3i$, $5i$, $\sqrt{-7} = \sqrt{-1}\sqrt{7} = i\sqrt{7}$

Note

\# Complex No.

$\boxed{z = x+iy}$

- Purely Real No.
  - $z = x$, $y = 0$
  - $x \in R$
- Purely Imaginary No.
  - $z = iy$
  - $x = 0$

Complex No. $\boxed{z = x+iy}$ $x \in R$, $y \in R$

- Real Part of z $= Re(z) = x$
- Imaginary Part of z $= Im(z) = y$

Algebra of complex Numbers.

=	≷	±	×	÷

$i^2 = -1$

$$\frac{2+\sqrt{3}}{5-\sqrt{3}} \times \frac{5+\sqrt{3}}{5+\sqrt{3}}$$

Equality

$z_1 = z_2$

$x_1 + iy_1 = x_2 + iy_2$

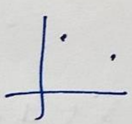
$\boxed{x_1 = x_2}$

$\boxed{y_1 = y_2}$

Inequality in complex numbers does not make any SENSE

$2+i3$ & $100+i1000$

Point Point



$z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2)$

$= (x_1 \pm x_2) + i(y_1 \pm y_2)$

$z_1 . z_2$

$= (x_1 + iy_1).(x_2 + iy_2)$

$= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2 y_1y_2$

$= \underline{(x_1x_2 - y_1y_2)} + i\,\underline{(x_1y_2 + y_1x_2)}$

$i^2 = -1$

Note: Multiplicative inverse of

$z = \frac{1}{z}$

e.g. MI of $2+i3 = \frac{1}{2+i3}$

e.g. Simplify $\frac{1}{2+i3}$ into $a+ib$.

$$\frac{1}{2+i3} = \frac{1}{2+i3} \times \frac{2-i3}{2-i3}$$

$(a+b)\cdot(a-b) = a^2 - b^2$

$$= \frac{2-3i}{2^2 - (i3)^2}$$

$$= \frac{2-3i}{4-(-1)9}$$

$$= \frac{2-3i}{13}$$

$$= \frac{2}{13} - \frac{3i}{13} = a+ib$$

$$a = \frac{2}{13}, \quad b = -\frac{3}{13}$$

Tools, Modulus + ~~Argument~~ + Conjugate

Modulus of a complex No ($z = x+iy$)

$$= |z| = |x+iy| = \sqrt{x^2+y^2}$$

$$|z| = \sqrt{[Re(z)]^2 + (Im(z))^2}$$

Conjugete of a complex No. ($z = x+iy$)

$$\bar{z} = \overline{x+yi} = x - yi$$

e.g. $z = 5-4i$

$$|z| = \sqrt{5^2+(-4)^2} = \sqrt{25+16} = \sqrt{41}$$

$$\bar{z} = \overline{5-4i} = 5+4i$$

Properties of modulus & Conjugate

① $\boxed{z.\bar{z} = |z|^2}$ ☆

② ☆ $|z_1.z_2| = |z_1|.|z_2|$, $\overline{(z_1.z_2)} = \bar{z}_1.\bar{z}_2$

$\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

$\checkmark$ $\overline{(z_1+z_2)} = \bar{z}_1 + \bar{z}_2$

$\times$ $\overline{z_1-z_2} = \bar{z}_1 - \bar{z}_2$

Note:

Mult. Inverse of z

$= \dfrac{1}{z}$

$= \dfrac{1}{z} \times \dfrac{\bar{z}}{\bar{z}}$

$= \dfrac{\bar{z}}{|z|^2}$ ☆

Proof: $z.\bar{z} = |z|^2$

$\Rightarrow (x+iy).(x-iy) = \left(\sqrt{x^2+y^2}\right)^2$

$\Rightarrow (x)^2 - (iy)^2 = x^2+y^2$

$\Rightarrow x^2 + y^2 = x^2+y^2$

Exercise 4.1

$i =$ imaginary

$$\left.\begin{aligned} i &= \sqrt{-1} \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \end{aligned}\right\} \quad \left.\begin{aligned} i^{4n+1} &= i \\ i^{4n+2} &= -1 \\ i^{4n+3} &= -i \\ i^{4n} &= 1 \end{aligned}\right\} \; n \in I$$

Q.1.

$$(\cancel{8}i).\left(-\frac{3}{\cancel{8}}i\right) = -3.i^2 = -3.(-1) = 3 = 3+i0$$

Q.2

$$i + i^{19} = i - i = 0 = 0 + i0$$

$a+ib$

$$i^{19} = i^{16+3} = i^{4\times4+3} = -i$$

Q.3

$$i^{-39} = \frac{1}{i^{39}} \times \frac{i}{i} = \frac{i}{i^{40}} = \frac{i}{1} = i$$

$= 0 + i.1$

Q.4

$$3(7+i7) + i(7+i7)$$

$$= 21 + \underline{i.21 + 7i} + i^2.7$$

$$= 21 + 28.i + (-1).7$$

$$= 14 + 28i = a+ib$$

Q.5 $(1-i) - (-1+i6)$

$$= 1 - i + 1 - i.6$$

$$= 2 - i.7$$

$$= 2 - 7i$$

Q.6 $\left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right)$

$$= \frac{1}{5} - 4 + i\left(\frac{2}{5} - \frac{5}{2}\right)$$

$$= \frac{-19}{4} + i\left(\frac{-21}{10}\right)$$

(7) $\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$

$= \underbrace{\left(\frac{1}{3}+4+\frac{4}{3}\right)}_{\text{Real part}} + i\,\underbrace{\left(\frac{7}{3}+\frac{1}{3}-1\right)}_{\text{Im. Part}}$

$= \frac{1+12+4}{3} + i\left(\frac{7+1-3}{3}\right)$

$= \frac{17}{3} + i\left(\frac{5}{3}\right) = a+ib$

$a=\frac{17}{3}, \quad b=\frac{5}{3}$

(8) $(1-i)^4$

$(a-b)^4 = (a-b)^2.(a-b)^2$

$= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

$= (1-i)^2.(1-i)^2$

$(a-b)^2 = a^2+b^2-2ab$

$= (1^2+i^2-2.1.i).(1-i)^2$

$= (\cancel{1}-\cancel{1}-2i).(1-i)^2 = (-2i).(-2i) = (+4)(i^2) = -4$

(9) $\left(\frac{1}{3}+3i\right)^3$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$\left(\frac{1}{3}+3i\right)^3 = \left(\frac{1}{3}\right)^3 + \cancel{3}\left(\frac{1}{\cancel{3}}\right)^2.(\cancel{3}i) + \cancel{3}.\frac{1}{\cancel{3}}.(3i)^2 + (3i)^3$

$= \frac{1}{27} + i - 9 - 27i$

$= \frac{1-243}{27} - 26i = -\frac{242}{27} - 26i$

(10) $\left(-2-\frac{1}{3}i\right)^3 = -\left(2+\frac{1}{3}i\right)^3$

$= -\left[2^3 + \cancel{3}.2^2.\frac{i}{\cancel{3}} + 3.2 * \left(\frac{i}{3}\right)^2 + \left(\frac{i}{3}\right)^3\right]$

$= -\left[8 + 4i - \frac{2}{3} - \frac{i}{27}\right]$

$= -\left(\frac{22}{3} + \frac{108-1}{27}i\right)$

$= -\frac{22}{3} - \frac{107}{27}i$

Q.11 multiplicative inverse of $4-3i = \frac{1}{4-3i}$

$\boxed{\text{multiplicative inverse of } z = \frac{1}{z}}$

$z = x + iy$

$i^2 = -1$

$i \times i = -1$ Real

$2+\sqrt{3} \rightarrow 2-\sqrt{3}$

$$= \frac{1}{4-3i} \times \frac{4+3i}{4+3i}$$

$(a-b)(a+b) = a^2 - b^2$

$$= \frac{4+3i}{(4)^2-(3i)^2} = \frac{4+3i}{16-(9(-1))}$$

$$= \frac{4+3i}{16+9} = \frac{4+3i}{25} = \frac{4}{25} + \frac{3i}{25}$$

(12) multiplicative Inverse of $\sqrt{5}+3i$

$$= \frac{1}{\sqrt{5}+3i} \times \frac{\sqrt{5}-3i}{\sqrt{5}-3i}$$

$$= \frac{\sqrt{5}-3i}{(\sqrt{5})^2-(3i)^2}$$

$$= \frac{\sqrt{5}-3i}{5-9i^2}$$

$$= \frac{\sqrt{5}-3i}{5-9(-1)} = \frac{\sqrt{5}-3i}{5+9}$$

$$= \frac{\sqrt{5}-3i}{14}$$

(13) multiplicative inverse of $(-i)$ $= \frac{1}{-i} \times \frac{i}{i}$

$$= \frac{i}{-i^2} = \frac{i}{-(-1)}$$

$$= \frac{i}{+1} = i$$

(14)

$$a+ib = \frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}$$

$(x+y)(x-y) = x^2-y^2$

$$= \frac{(3)^2-(i\sqrt{5})^2}{\cancel{\sqrt{3}}+\sqrt{2}i-\cancel{\sqrt{3}}+i\sqrt{2}}$$

$$= \frac{9-i^2 \cdot 5}{2\sqrt{2}i} \quad i^2 \rightarrow -1$$

$$= \frac{9+5}{2\sqrt{2}i} \times \frac{\sqrt{2}i}{\sqrt{2}i}$$

$$= \frac{\overset{7}{\cancel{14}}\sqrt{2}i}{\cancel{2}\cdot 2\cdot(-1)}$$

$$= \frac{-7\sqrt{2}i}{2} = 0 - \frac{7\sqrt{2}}{2}i$$

Argand Plane

Modulus

Conjugate

Argument

Polar form

+ EXTRA

Complex Number (Geometry Part)

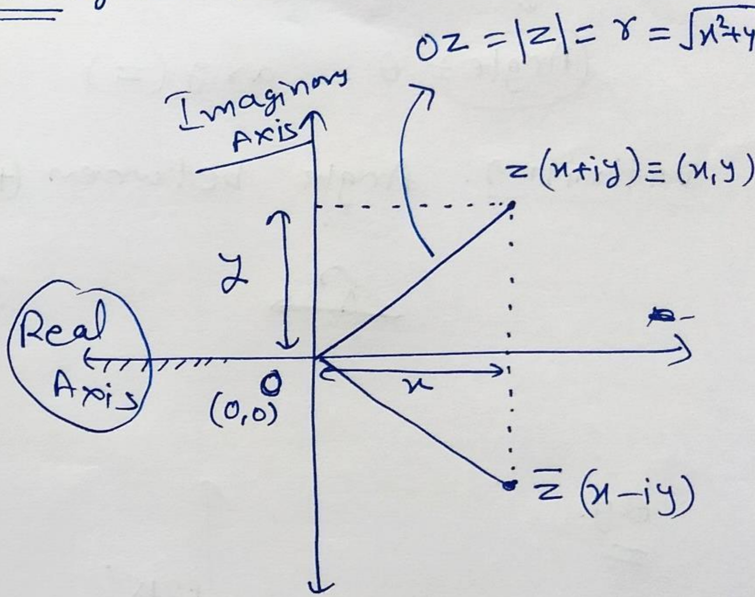
$z = x + iy \rightarrow (x, y)$

Number → point

$|z| = \sqrt{x^2 + y^2}$ = distance between $z(x+iy)$ and $O(0+i0)$

$\bar{z} = x - iy \rightarrow (x, -y)$

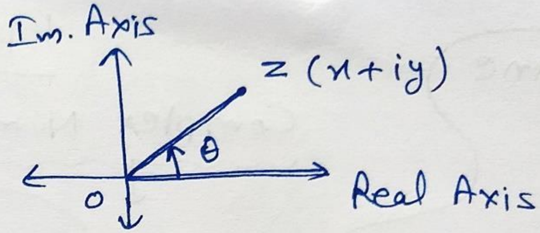
$\bar{z}$ = mirror image of 'z' in the mirror "Real Axis".



Argand Plane
(Complex Plane)
(Gaussian Plane)

Argument of a Complex Number $= \theta$

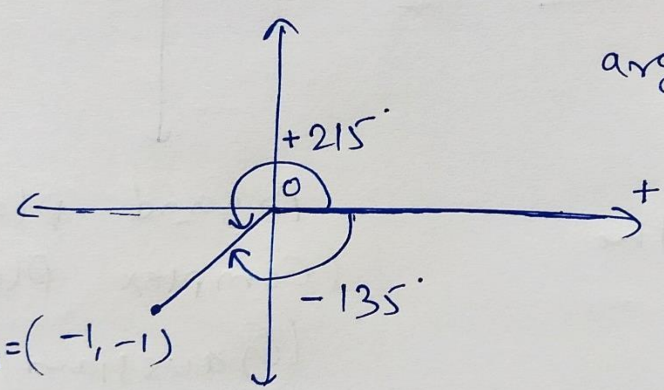
(Angle) $= \theta = \arg(z)$



Definition: Angle between (+) Real Axis & line segment joining $O(0+i0)$ and $z\ (x+iy)$

$\underbrace{\text{(+) Real Axis}}_{\text{Initial Ray}}$ $\underbrace{\text{line segment joining } O(0+i0) \text{ and } z\ (x+iy)}_{\text{Final Ray}}$

e.g.



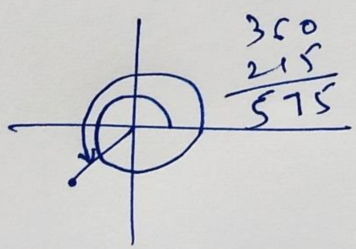
$-1-i = z_1 = (-1,-1)$

$\arg(z_1) = \arg(-1-i)$

$= 215°, -135°, +575°$

$\times$ ✓ $\times$

P.A

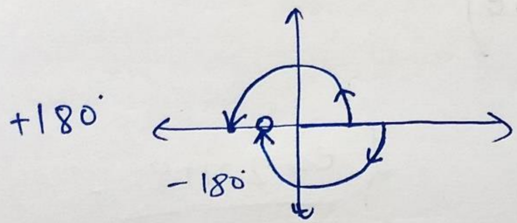


Concept of Principal Argument (θ)

Amplitude

$$-\pi < \theta \le \pi$$

$$-180^\circ < \theta \le +180^\circ$$

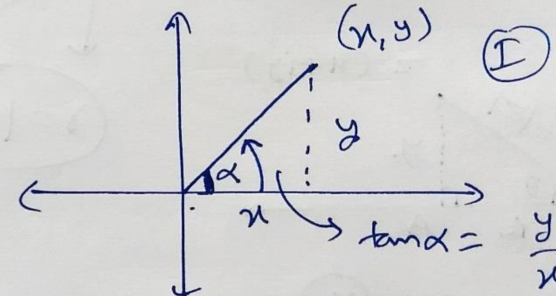


θ

Steps to find Principal Argument :→

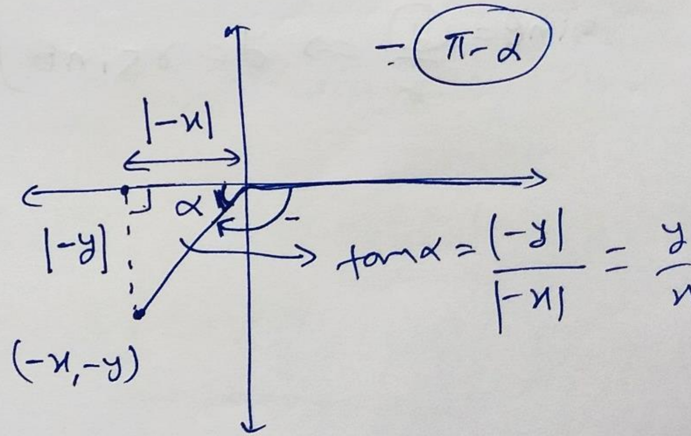
$$z = x + iy$$

(I) $\alpha = \tan^{-1}\left|\dfrac{y}{x}\right|$



(II)

Quadrant	θ = P.A.
I	$\theta = \alpha$
II	$\theta = \pi - \alpha$
III	$\theta = \alpha - \pi$
IV	$\theta = -\alpha$



Cartesian Form $\longrightarrow$ Polar Form

Extra

Euler's Form

$$z = x + iy \;=\; r(\cos\theta + i\sin\theta) \;=\; r.(e^{i\theta})$$

$$\hookrightarrow r(\text{cis}\,\theta) \longrightarrow$$

$(x, y) \leftarrow$ Cartesian Coordinates

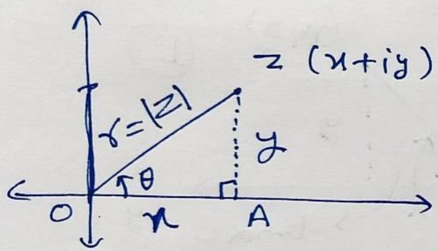
$(r, \theta) \leftarrow$ Polar Coordinates

$r = |z|$

$\theta = \arg(z)$

$\{ e = 2.718..$

$\{ e^x =$ Exponential fn^n.



$$\left.\begin{aligned} \cos\theta = \frac{x}{r} &\Rightarrow x = r\cos\theta \\ \sin\theta = \frac{y}{r} &\Rightarrow y = r\sin\theta \end{aligned}\right\}$$

(r, θ)

e.g. Convert $-1 + i\sqrt{3}$ into Polar form.

z $\qquad$ $r(\cos\theta + i\sin\theta)$

$$r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2}$$

$$= \sqrt{1+3} = 2 = r$$

$\alpha = \tan^{-1}\sqrt{3}$ $\qquad$ $\tan\alpha = \sqrt{3}$

Argument (P.A).

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| = \tan^{-1}|\sqrt{3}| = \tan^{-1}\sqrt{3} = 60^\circ = \frac{\pi}{3}$$

Quadrant. $\quad (-1 + i\sqrt{3}) \equiv (-1, \sqrt{3})$ II

$(-, +)$ $\qquad$ $-\,+$

$r.e^{i\theta}$

Argument $\theta = \pi - \alpha$

$$= \pi - \frac{\pi}{3}$$

$$\theta = \frac{2\pi}{3}$$

Polar Form: $r(\cos\theta + i\sin\theta) = e$

$$-1 + i\sqrt{3} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2.e^{i\frac{2\pi}{3}}$$

Exercise 4.2

Revision $\boxed{z = x + iy}$

Modulus

$$|z| = r = \sqrt{x^2 + y^2}$$

Argument θ

I) $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

II)

I	$\theta = \alpha$
II	$\theta = \pi - \alpha$
III	$\theta = \alpha - \pi$
IV	$\theta = -\alpha$

$\boxed{Q.1}$ $z = -1 - i\sqrt{3}$ $\rightarrow x = -1$, $y = -\sqrt{3}$

$$|z| = r = \sqrt{(-1)^2 + (-\sqrt{3})^2}$$

$$= \sqrt{1 + 3}$$

$$= 2$$

$-+$	$++$
$--$	$+-$

III

$\arg(z) = \theta = ?$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-\sqrt{3}}{-1}\right|$$

$$\alpha = \tan^{-1}\left|\sqrt{3}\right|$$

$\downarrow$

Angle

$\downarrow$

$60^\circ = \frac{\pi}{3}$

$\tan\alpha = \sqrt{3}$

$\downarrow$

?? 60°

$z = -1 - \sqrt{3}\,i \longrightarrow$ III - quadrant.

$$\arg(z) = \theta = \alpha - \pi$$

$$= \frac{\pi}{3} - \pi$$

$$= -\frac{2\pi}{3}$$

$|z| = 2$

$\arg(z) = -\frac{2\pi}{3}$

Q.2 $\quad z_1 = -\sqrt{3} + i$ $\quad x = -\sqrt{3}$, $\quad y = 1$

$|z| = r = \sqrt{x^2+y^2}$

$= \sqrt{(-\sqrt{3})^2 + 1^2}$

$= \sqrt{4}$

$\boxed{|z| = 2}$

$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$\alpha = \tan^{-1}\left|\frac{1}{-\sqrt{3}}\right|$

$\alpha = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right|$

Angle. $\to 30° = \frac{\pi}{6}$

Quadrant. $(-\sqrt{3}, 1)$ $(-,+)$ II

$\arg(z) = \theta = \pi - \alpha$

$\theta = \pi - \frac{\pi}{6}$

$\boxed{\theta = \frac{5\pi}{6}}$

modulus $= 2$

arg. $= \frac{5\pi}{6}$

Polar Form: $r(\cos\theta + i\sin\theta) \leftarrow x+iy$

$r \to$ modulus $= |z| = \sqrt{x^2+y^2}$

$\theta \to$ argument $=$ $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$; Table (Quadrant)

$I \to \theta = \alpha$

$II \to \theta = \pi - \alpha$

$III \to \theta = \alpha - \pi$

$IV \to \theta = -\alpha$

$\boxed{Q.3}$ $z = 1-i$; $x = 1$, $y = -1$; $(x, y) \equiv (1, -1)$

$$\boxed{r = |z| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}}$$

$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-1}{1}\right| = \tan^{-1}|1|$

$\downarrow$

Angle

$\alpha = \tan^{-1} 1$

$\alpha = \frac{\pi}{4} = 45^\circ$

Quadrant: $(1, -1)$ $(+, -)$

$(+, -)$ IV

Argument $\theta = -\alpha$

$$\boxed{\theta = -\frac{\pi}{4}}$$

Polar form: (r, θ)

$r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

④ $z = -1+i$ $\begin{cases} x = -1 \\ y = 1 \end{cases}$ $(-1,1)$

Quadrant (II)

$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{1}{-1}\right|$

$\alpha = \tan^{-1}|1|$

$\alpha = \frac{\pi}{4} = 45^\circ$

$Arg. = \theta = \pi - \alpha$
$= \pi - \frac{\pi}{4}$
$\theta = \frac{3\pi}{4}$ ✓

$|z| = r = \sqrt{x^2+y^2}$
$= \sqrt{1+1} = \sqrt{2}$

Polar form $= r(\cos\theta + i\sin\theta)$
$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ ✓

⑤ $z = -1-i$ $\begin{cases} x = -1 \\ y = -1 \end{cases}$

$|z| = r = \sqrt{(-1)^2+(-1)^2}$
$= \sqrt{2}$ ✓

θ = argument

$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-1}{-1}\right| = \tan^{-1}|1|$

$\alpha = \frac{\pi}{4}$

Quadrant: $(x,y) \equiv (-1,-1)$
$\Rightarrow$ (III)

$\theta = \alpha - \pi$

$\theta = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$ ✓

Polar form:
$r(\cos\theta + i\sin\theta) \equiv \sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$

Q.6 $\boxed{-3}$ $\xrightarrow{\text{Convert}}$ $\boxed{\text{Polar form}}$ (r, θ)

$z = \underline{-3} = \underline{-3 + i.0}$ $\rightarrow x = -3$, $y = 0$

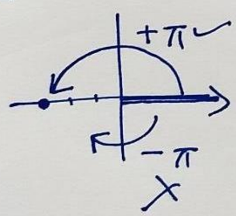
$|z| = r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + (0)^2}$

$= \sqrt{9} = 3$

$\boxed{r = 3}$

Argument (θ) $\rightarrow (-\pi, \pi]$

Point $(-3, 0)$



$\cancel{\alpha = \tan^{-1}\left(\frac{y}{x}\right)}$

$\boxed{\theta = +\pi}$

$r = 3$

$\theta = \pi$

Polar Form:

$r(\cos\theta + i\sin\theta)$

$= 3(\cos\pi + i\sin\pi)$

(7) $z = \sqrt{3} + i = (x + iy)$ $(x, y) \equiv (\sqrt{3}, 1)$

$r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3 + 1} = 2$

Arg. $\boxed{\theta = \alpha}$ $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

(I)

$\alpha = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = \frac{\pi}{6}$

$\boxed{\theta = \frac{\pi}{6}}$

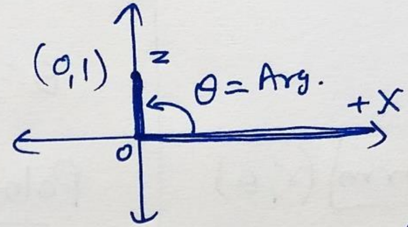
Polar Form: $r(\cos\theta + i\sin\theta)$

$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(8) $z = i = \underset{\substack{\uparrow \\ x}}{0} + i.\underset{\substack{\nearrow \\ y}}{1}$ $\qquad \underline{x = 0,\ y = 1}$

$x + i.y$

Point $(x, y) \equiv (0, 1)$



+ AC

$$\text{Argument} = \theta = +\frac{\pi}{2} = 90^\circ$$

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$$

Polar Form: $r(\cos\theta + i\sin\theta)$

$$= 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

Exercise $\boxed{4.3}$

$ax^2+bx+c=0 \longrightarrow x = \frac{-b \pm \sqrt{D}}{2a}$ $\sqrt{D} \geq 0$ $\sqrt{+}$

Roots. $\sqrt{-}$

D = Discriminant $= b^2 - 4ac$

$\boxed{D \geq 0}$ ✓

Theory: Fundamental Theorem of Algebra

"A polynomial Equation has at least one root"

"A polynomial equation of degree n has n roots"

Note:

① A Quadratic equation has maximum 2 real roots.

② A quadratic equation has exactly 2 roots.

$\boxed{\text{Solve}}$

$x = ?$ roots $= ?$

(1) $\boxed{x^2 + 3 = 0}$

$\Rightarrow x^2 = -3$

$\sqrt{-1} = i$

$\Rightarrow x = \pm\sqrt{-3}$

$x = \pm i\sqrt{3}$

$\left.\begin{aligned} x_1 &= i\sqrt{3} \\ x_2 &= -i\sqrt{3} \end{aligned}\right\}$ roots;

(2) $2x^2 + x + 1 = 0$ $a = 2$, $b = 1$, $c = 1$

Quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1-8}}{2(2)}$$

$$x = \frac{-1 \pm \sqrt{-7}}{4} = \frac{-1 \pm \sqrt{7}\,i}{4}$$

$\boxed{3}$ $x^2 + 3x + 9 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{9 - 4.1.9}}{2.1}$$

$$x = \frac{-3 \pm \sqrt{-27}}{2}$$

$27 = 3 \times 3 \times 3$

$$\boxed{x = \frac{-3 \pm i.3.\sqrt{3}}{2}}$$

$\boxed{4}$ $-x^2 + x - 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(-1)(-2)}}{2(-1)}$$

$$x = \frac{-1 \pm \sqrt{1 - 8}}{-2}$$

$$x = \frac{-1 \pm \sqrt{-7}}{-2}$$

$$\boxed{x = \frac{-1 \pm i\sqrt{7}}{-2}}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

Q.5 $x^2 + 3x + 5 = 0$

$$x = \frac{-3 \pm \sqrt{3^2 - 4.1.5}}{2 \cdot 1}$$

$$x = \frac{-3 \pm \sqrt{9-20}}{2}$$

$$x = \frac{-3 \pm \sqrt{-11}}{2}$$

$$x = \frac{-3 \pm i\sqrt{11}}{2}$$

Q.6 $x^2 - x + 2 = 0$

$1 \quad -1 \quad 2$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 8}}{2}$$

$$x = \frac{1 \pm \sqrt{-7}}{2}$$

$$x = \frac{1 \pm i\sqrt{7}}{2}$$

Q.7 $\sqrt{2}\,x^2 + x + \sqrt{2} = 0$

$a = \sqrt{2} \quad b = 1 \quad c = \sqrt{2}$

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot \sqrt{2} \cdot \sqrt{2}}}{2 \cdot \sqrt{2}}$$

$$x = \frac{-1 \pm \sqrt{1-8}}{2 \cdot \sqrt{2}}$$

$$x = \frac{-1 \pm \sqrt{-7}}{2\sqrt{2}}$$

$$x = \frac{-1 \pm i\sqrt{7}}{2\sqrt{2}}$$

Q.8

$$\sqrt{3}\,x^2 - \sqrt{2}\,x + 3\sqrt{3} = 0$$

$$x = \frac{-(-\sqrt{2}) \pm \sqrt{(-\sqrt{2})^2 - 4.\sqrt{3}.3\sqrt{3}}}{2.\sqrt{3}}$$

$$x = \frac{\sqrt{2} \pm \sqrt{2 - 36}}{2\sqrt{3}}$$

$$x = \frac{\sqrt{2} \pm \sqrt{-34}}{2\sqrt{3}}$$

$$x = \frac{\sqrt{2} \pm i\sqrt{34}}{2\sqrt{3}}$$

$$x = \frac{-1 \pm \sqrt{1 - 2\sqrt{2}}}{2}$$

$$\boxed{\begin{aligned} 4 &= 2 \times 2 \\ &= 2 \times \sqrt{2}.\sqrt{2} \end{aligned}}$$

Q.9

$$x^2 + x + \frac{1}{\sqrt{2}} = 0$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4.1.\frac{1}{\sqrt{2}}}}{2.1}$$

$$x = \frac{-1 \pm \sqrt{1 - \frac{4}{\sqrt{2}}}}{2}$$

$$x = \frac{-1 \pm \sqrt{\frac{\sqrt{2} - 4}{\sqrt{2}}}}{2}$$

$$\Rightarrow x = \frac{-1 \pm i\sqrt{\frac{4 - \sqrt{2}}{\sqrt{2}}}}{2}$$

$$\Rightarrow \boxed{x = \frac{-1 \pm \frac{i\sqrt{4 - \sqrt{2}}}{\sqrt{\sqrt{2}}}}{2}}$$

Q.10

$$x^2 + \frac{x}{\sqrt{2}} + 1 = 0$$

$$x = \frac{-\frac{1}{\sqrt{2}} \pm \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 - 4.1.1}}{2.1}$$

$$x = \frac{-\frac{1}{\sqrt{2}} \pm \sqrt{\frac{1}{2} - 4}}{2}$$

$$x = \frac{-\frac{1}{\sqrt{2}} \pm \sqrt{\frac{-7}{2}}}{2}$$

$$x = \frac{\left(\frac{-1 \pm i\sqrt{7}}{\sqrt{2}}\right)}{(2)}$$

$$\boxed{x = \frac{-1 \pm i\sqrt{7}}{2\sqrt{2}}}$$

Miscellaneous Exercise 4.4

$\boxed{Q.1}$ Evaluate $\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^{4n} = 1$	$i^{4n+2} = -1$
$i^{4n+1} = i$	$i^{4n+3} = -i$

$i^{18} = i^{16+2} = i^{4.4+2} = -1$

$\left(\frac{1}{i}\right)^{25} = \frac{1}{i^{25}} = \frac{1}{i^{24+1}} = \frac{1}{i} \times \frac{i}{i}$

$= \frac{i}{i^2} = \frac{i}{-1} = -i$

$\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3 = (-1-i)^3 = -(1+i)^3$

$(a+b)^3$

$(a+b)^3 = \left[a^3 + b^3 + 3a^2b + 3ab^2\right]$

$-(1+i)^3 = -\left[1 + i^3 + 3.1^2.i + 3.1.i^2\right]$

$= -\left[1 - i + 3i - 3\right]$

$= -\left[-2 + 2i\right]$

$= 2 - 2i$

$\boxed{Q.2}$ Complex Numbers z_1, z_2

To Prove:

$Re(z_1 z_2) = \underset{x_1}{Re\, z_1} \cdot \underset{x_2}{Re\, z_2} - \underset{y_1}{Im\, z_1} \cdot \underset{y_2}{Im\, z_2}$

let $\boxed{z_1 = x_1 + iy_1}$

$\boxed{z_2 = x_2 + iy_2}$

$RHS = x_1 \cdot x_2 - y_1 \cdot y_2$

$LHS = Re(z_1 . z_2)$

$= Re\left[(x_1 + iy_1).(x_2 + iy_2)\right]$

$$LHS = Re(x_1(x_2+iy_2) + iy_1(x_2+iy_2))$$

$$= Re[x_1x_2 + x_1 \cdot i \cdot y_2 + iy_1x_2 + i^2 \cdot y_1y_2]$$

$$= Re[x_1x_2 - y_1 \cdot y_2 + i(x_1y_2 + y_1x_2)] \quad i^2 = -1$$

$$= x_1x_2 - y_1y_2$$

$$= Re\,z_1 \cdot Re\,z_2 - Im\,z_1 \cdot Im\,z_2$$

$$= RHS.$$

③ Reduce $\left[\left(\frac{1}{1-4i}\right) - \left(\frac{2}{1+i}\right)\right] \cdot \left[\frac{3-4i}{5+i}\right]$ into standard form. $a+ib$

$$\frac{1}{1-4i} \times \frac{1+4i}{1+4i} = \frac{1+4i}{1^2-(4i)^2} = \frac{1+4i}{1+16} = \frac{1+4i}{17} \quad z \cdot \bar{z} = |z|^2$$

$$\frac{2}{1+i} \times \frac{1-i}{1-i} = \frac{2-2i}{1^2-i^2} = \frac{2-2i}{1+1} = 1-i$$

$$\frac{3-4i}{5+i} \times \frac{5-i}{5-i} = \frac{15-3i-20i-4}{5^2-i^2}$$

$$= \frac{11-23i}{26}$$

$$\left(\frac{1}{1-4i} - \frac{2}{1+i}\right) \cdot \left(\frac{3-4i}{5+i}\right)$$

$$= \left(\frac{1+4i}{17} - 1 + i\right) \cdot \left(\frac{11-23i}{26}\right)$$

$$= \left(\frac{1+4i-17+17i}{17}\right) \cdot \left(\frac{11-23i}{26}\right)$$

$$= \frac{-16+21i}{17} \times \frac{11-23i}{26}$$

$$= \frac{-176 + \overset{368}{\cancel{408}}i + 231i + 483}{442}$$

$$= \frac{307 + \overset{599}{\cancel{639}}i}{442}$$

④ $x - iy = \sqrt{\frac{a-ib}{c-id}}$ (Given)

To Prove: $\boxed{(x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}}$

Concept: $|x+iy| = \sqrt{x^2+y^2}$

Given: $x - iy = \sqrt{\frac{a-ib}{c-id}}$ $\boxed{|z|^2 = |z^2|}$

Square:

$$(x-iy)^2 = \left(\frac{a-ib}{c-id}\right)$$

Modulus

$$|(x-iy)^2| = \left|\frac{a-ib}{c-id}\right|$$

Properties,

$$|z_1 . z_2| = |z_1| . |z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$\Rightarrow |(x-iy).(x-iy)| = \left|\frac{a-ib}{c-id}\right|$

$\Rightarrow |x-iy|.|x-iy| = \frac{|a-ib|}{|c-id|}$

$\Rightarrow \sqrt{x^2+y^2} . \sqrt{x^2+y^2} = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$

$\Rightarrow$ Square

$$(x^2+y^2)^2 = \left(\frac{a^2+b^2}{c^2+d^2}\right)$$

(5) Polar Form: $r(\cos\theta + i\sin\theta)$

(i) $\dfrac{1+7i}{(2-i)^2} \longrightarrow$ Simpified form (a+ib) $\longrightarrow r=|z|$, $\theta=\arg.$

$i^2=-1$

$$= \frac{1+7i}{2^2+i^2-4i} = \frac{1+7i}{3-4i}$$

$$= \frac{1+7i}{3-4i}\times\frac{3+4i}{3+4i} = \frac{3+4i+21i-28}{3^2-(4i)^2}$$

$$= \frac{-25+25i}{9+16} = \frac{-25+25i}{25}$$

$z = \boxed{-1+i} = x+iy \qquad (x,y)\equiv(-1,1)$

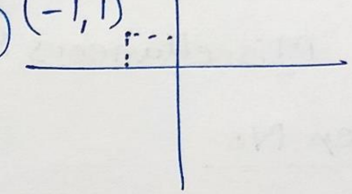
$r=|z|=\sqrt{(-1)^2+(1)^2}=\sqrt{2}$

$\theta=\arg(z)=$ Principal Argument.

$\alpha=\tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{1}{-1}\right| = \tan^{-1}(1)$

$\alpha = 45^\circ = \frac{\pi}{4}$

II



$\arg(z)=\theta=\pi-\alpha$

$\theta=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

Polar Form =

$= r(\cos\theta+i\sin\theta)$

$=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

Q.6 $\left(3x^2 - 4x + \frac{20}{3} = 0\right) \times 3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$9x^2 - 12x + 20 = 0$

$$x = \frac{-(-12) \pm \sqrt{144 - 4.9.20}}{2 \times 9}$$

$$x = \frac{12 \pm \sqrt{144 - 720}}{18}$$

$$x = \frac{12 \pm \sqrt{-576}}{18}$$

$$x = \frac{12 \pm i.24}{18}$$

$$x = \frac{2}{3} \pm i\frac{4}{3}$$

Q.7 $1.x^2 - 2x + \frac{3}{2} = 0$

$$x = \frac{-(-2) \pm \sqrt{4 - 4.1.3/2}}{2 \times 1}$$

$$x = \frac{2 \pm \sqrt{4-6}}{2}$$

$$x = \frac{2 \pm \sqrt{-2}}{2}$$

$$x = \frac{2 \pm i\sqrt{2}}{2}$$

$$x = 1 \pm \frac{i}{\sqrt{2}}$$

Q.8 $27x^2 - 10x + 1 = 0$

$$x = \frac{-(-10) \pm \sqrt{100 - 4.27.1}}{2(27)}$$

$$x = \frac{10 \pm \sqrt{100-108}}{54}$$

$$x = \frac{10 \pm \sqrt{-8}}{54}$$

$$x = \frac{10 \pm i2\sqrt{2}}{54}$$

$$x = \frac{5 \pm i\sqrt{2}}{27}$$

Q.9 $21x^2 - 28x + 10 = 0$

$$x = \frac{-(-28) \pm \sqrt{(-28)^2 - 4.21.10}}{2(21)}$$

$$x = \frac{28 \pm \sqrt{784 - 840}}{42}$$

$$x = \frac{28 \pm \sqrt{-56}}{42}$$

$$x = \frac{28 \pm i\sqrt{4 \times 14}}{42}$$

$$x = \frac{28 \pm 2\sqrt{14}.i}{42}$$

$$x = \frac{14 \pm \sqrt{14}\,i}{21}$$

(10) If $z_1 = 2 - i$, $z_2 = 1 + i$,

Find $\left| \dfrac{z_1 + z_2 + 1}{z_1 - z_2 + i} \right|$. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ ☆

$$\left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + i} \right| = \frac{|2 - \cancel{i} + 1 + \cancel{i} + 1|}{|2 - i - 1 - \cancel{i} + \cancel{i}|}$$

$$= \frac{|4|}{|1 - i|}$$

$\sqrt{4^2 + 0^2}$, $|4|$
$\downarrow$ $\downarrow$
4 4

$$= \frac{4}{\sqrt{1^2 + (-1)^2}} = \frac{4}{\sqrt{2}} = \frac{(2 \times \sqrt{2} \cdot \cancel{\sqrt{2}})}{\cancel{\sqrt{2}}}$$

$$= 2\sqrt{2}$$

(11) $a + ib = \dfrac{(x + i)^2}{2x^2 + 1}$ $\boxed{\text{Given}}$

To Prove: $a^2 + b^2 = \dfrac{(x^2 + 1)^2}{(2x^2 + 1)^2}$

$$a + ib = \frac{(x + i)^2}{2x^2 + 1}$$

By taking modulus Both Sides.

$$|a + ib| = \left| \frac{(x + i)^2}{2x^2 + 1} \right|$$

$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

$$\Rightarrow \sqrt{a^2 + b^2} = \frac{|x + i|^2}{|2x^2 + 1 + 0 \cdot i|}$$

$$\Rightarrow \sqrt{a^2 + b^2} = \frac{\left(\sqrt{x^2 + 1^2}\right)^2}{\sqrt{(2x^2 + 1)^2 + 0^2}}$$

$$\Rightarrow \sqrt{a^2+b^2} = \frac{\left(\sqrt{x^2+1^2}\right)^2}{\sqrt{(2x^2+1)^2+0^2}}$$

By squaring both sides.

$$\Rightarrow \left(\sqrt{a^2+b^2}\right)^2 = \left[\frac{x^2+1}{2x^2+1}\right]^2$$

$$\Rightarrow a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$$

$\boxed{12}$ $\underbrace{z_1 = 2-i}$ $\underbrace{z_2 = -2+i}$

(i) $Re\left(\frac{z_1 z_2}{\overline{z_1}}\right)$

$$= Re\left(\frac{(2-i)\cdot(-2+i)}{\overline{2-i}}\right)$$

$$= Re\left(\frac{-4+2i+2i-\overset{-1}{i^2}}{2+i}\right) \quad i^2=-1$$

$$= Re\left(\frac{-4+1+4i}{2+i} \times \frac{2-i}{2-i}\right)$$

$$= Re\left(\frac{(-3+4i)\cdot(2-i)}{2^2-(i)^2}\right)$$

$$= Re\left(\frac{-6+3i+8i-4\overset{-1}{i^2}}{4+1}\right)$$

$$= Re\left(\frac{-2+11i}{5}\right)$$

$$= \underline{\underline{Re}}\left(\underbrace{\frac{-2}{5}+\frac{11i}{5}}\right)$$

$$= -\frac{2}{5}$$

(12) (ii) $\text{Im}\left(\frac{1}{z_1 . \bar{z}_1}\right)$

$= \text{Im}\left(\frac{1}{(2-i).\overline{(2-i)}}\right)$

$= \text{Im}\left(\frac{1}{(2-i).(2+i)}\right)$

$= \text{Im}\left(\frac{1}{2^2-(i)^2}\right)$

$= \text{Im}\left(\frac{1}{4-(-1)}\right)$

$= \text{Im}\left(\frac{1}{5}\right)$

$= \text{Im}\left(\frac{1}{5} + 0.i\right)$

$= 0$

(13) $z = \frac{1+2i}{1-3i}$ $\qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Modulus $= |z| = \left|\frac{1+2i}{1-3i}\right| = \frac{|1+2i|}{|1-3i|}$

$= \frac{\sqrt{1^2+2^2}}{\sqrt{1^2+(-3)^2}}$

$= \frac{\sqrt{5}}{\sqrt{10}} = \sqrt{\frac{\cancel{5}}{\cancel{10}\,2}} = \frac{1}{\sqrt{2}}$ ✓

Argument (P.A.)

$z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{1+3i+2i+6i^2}{1^2-(3i)^2}$ $\quad i^2 \to -1$

$= \frac{-5+5i}{1+9} = \frac{-5+5i}{10} = -\frac{5}{10} + \frac{5i}{10}$

$= \frac{-1}{2} + \frac{i}{2}$

$$z = \frac{1+2i}{1-3i} = -\frac{1}{2} + \frac{i}{2} = x + iy$$

argument of $z = \theta = ?$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{\frac{1}{2}}{-\frac{1}{2}}\right|$$

$$\alpha = \tan^{-1}|-1| = \tan^{-1}(1) = 45^\circ = \frac{\pi}{4}$$

$$(x, y) \equiv \left(-\frac{1}{2}, \frac{1}{2}\right)$$

$-$ $\quad$ $+$

II $(-,+)$

II- quadrant $\quad$ $\arg=(\theta) = \pi - \alpha$

$$= \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

$$\underbrace{r = |z| = \frac{1}{\sqrt{2}}} \qquad \underbrace{\arg(z) = \frac{3\pi}{4}} = \theta$$

14

$(x-iy)(3+5i) =$ Conjugate of $-6-24i$

$\Rightarrow (x-iy)(3+5i) = \overline{(-6-24i)}$

$\Rightarrow 3x + 5x.i - 3y.i - 5.y.i^2 = -6+24i$ ($i^2 = -1$)

$\Rightarrow (3x+5y) + i(5x-3y) = -6+24i$

By Comparision

Real Part

$3x+5y=-6$ — (1) $\times 3 \Rightarrow 9x+15y=-18$

Im. Part

$5x-3y=24$ — (2) $\times 5 \Rightarrow 25x-15y=120$

$34x = 102$

$x = \frac{\cancel{102}\ \cancel{51}\ 3}{\cancel{34}\ \cancel{17}}$

$x=3$

eqn (1) :→

$3x+5y=-6$

$\Rightarrow 9+5y=-6$

$\Rightarrow 5y=-15$

$\boxed{y=-3}$

$\boxed{x=3}$
$\boxed{y=-3}$

$\boxed{15}$ modulus of $\left(\frac{1+i}{1-i} - \frac{1-i}{1+i}\right)$

$$= \left|\frac{1+i}{(1-i)} - \frac{1-i}{(1+i)}\right|$$

$|z_1 - z_2| = |z_1| - |z_2|$ ✗

$$= \left|\frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)}\right|$$

$$= \left|\frac{(1+i^2+2.1.i) - (1+i^2-2.1.i)}{(1-i)(1+i)}\right|$$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$|z_1 z_2| = |z_1|.|z_2|$

$$= \frac{|\cancel{1}+\cancel{i^2}+2i-\cancel{1}-\cancel{i^2}+2i|}{|1-i|.|1+i|}$$

$4i = 0+4i$

$$= \frac{|4i|}{\sqrt{1^2+(-1)^2}.\sqrt{1^2+1^2}} = \frac{\sqrt{0^2+4^2}}{\sqrt{2}.\sqrt{2}} = \frac{4}{2} = 2$$ ✓

(16) $(x+iy)^3 = u+iv$

To Prove $\boxed{\frac{u}{x} + \frac{v}{y} = 4(x^2-y^2)}$

$\Rightarrow \underbrace{(a+b)^3 = a^3+b^3+3a^2b+3ab^2}$

$\Rightarrow x^3+(iy)^3+3.x^2.iy+3.x.(iy)^2 = u+iv$

$\underbrace{i^2=-1}, \underbrace{i^3=-i}$

$\Rightarrow x^3+(-i)y^3+3x^2.y.i+3xy^2(-1) = u+iv$

$\Rightarrow (x^3-3xy^2)+i(3x^2y-y^3) = u+i.v$

By Comparison

$u = x^3-3xy^2$

$v = 3x^2y-y^3$

$LHS = \frac{u}{x} + \frac{v}{y}$

$= \frac{x^3-3xy^2}{x} + \frac{3x^2y-y^3}{y}$

$= x^2 - 3y^2 + 3x^2 - y^2$

$= 4x^2 - 4y^2$

$= 4(x^2-y^2)$

$= RHS.$

H.P.

$\boxed{Q.17}$ ☆

$\alpha, \beta \to$ Complex No.

$|\beta| = 1$

$\left|\dfrac{\beta - \alpha}{1 - \bar{\alpha}.\beta}\right| = ?$

Theory.

☆ $\boxed{z.\bar{z} = |z|^2}$

Results.

$\beta.\bar{\beta} = |\beta|^2$

$\boxed{\beta.\bar{\beta} = 1}$

Properties

$\overline{z_1.z_2} = \bar{z}_1.\bar{z}_2$

$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

$\bar{\bar{\alpha}} = \alpha$

$\alpha = x + iy$

$\bar{\alpha} = x - iy$

$\bar{\bar{\alpha}} = x + iy$

$|z|^2 = z.\bar{z}$

put $z = \dfrac{\beta - \alpha}{1 - \bar{\alpha}.\beta}$

$\Rightarrow \left|\dfrac{\beta - \alpha}{1 - \bar{\alpha}.\beta}\right|^2 = \left(\dfrac{\beta - \alpha}{1 - \bar{\alpha}.\beta}\right) . \overline{\left(\dfrac{\beta - \alpha}{1 - \bar{\alpha}.\beta}\right)}$

$= \left(\dfrac{\beta - \alpha}{1 - \bar{\alpha}.\beta}\right) . \left(\dfrac{\bar{\beta} - \bar{\alpha}}{1 - \alpha.\bar{\beta}}\right)$

$\downarrow$

$$\Rightarrow \left|\frac{\beta-\alpha}{1-\bar{\alpha}.\beta}\right|^2 = \left(\frac{\beta-\alpha}{1-\bar{\alpha}.\beta}\right)\cdot\left(\frac{\bar{\beta}-\bar{\alpha}}{1-\alpha.\bar{\beta}}\right)$$

Put $\underline{\beta.\bar{\beta}=1}$

$\downarrow$

1

$$= \frac{\beta.\bar{\beta} - \beta.\bar{\alpha} - \alpha.\bar{\beta} + \alpha.\bar{\alpha}}{1-\alpha.\bar{\beta} - \bar{\alpha}.\beta + \alpha.\bar{\alpha}.\beta.\bar{\beta}}$$

$$= \frac{\cancel{(1-\beta.\bar{\alpha}-\alpha.\bar{\beta}+\alpha.\bar{\alpha})}}{\cancel{(1-\alpha.\bar{\beta}-\bar{\alpha}.\beta+\alpha.\bar{\alpha})}}$$

$$\Rightarrow \left|\frac{\beta-\alpha}{1-\bar{\alpha}.\beta}\right|^2 = 1$$

$x^2 = 1$

$x = \pm 1$

$$\Rightarrow \left|\frac{\beta-\alpha}{1-\bar{\alpha}.\beta}\right| = \pm 1 = +1$$

$|z| = \cancel{\ominus}$

$$\boxed{\left|\frac{\beta-\alpha}{1-\bar{\alpha}.\beta}\right| = 1}$$

$\boxed{Q.18}$ Find the no. of non zero integral solutions of the equation $|1-i|^x = 2^x$.

Integers

$\ldots -3, -2, -1, \cancel{0}, 1, 2, 3 \ldots$

$$|1-i|^x = 2^x$$

$$\Rightarrow \left(\sqrt{1^2+(-1)^2}\right)^x = 2^x$$

$$\sqrt{2} = (2)^{\frac{1}{2}}$$

$$\Rightarrow \left(\sqrt{2}\right)^x = 2^x$$

$$(a^m)^n = a^{m \cdot n}$$

$$\Rightarrow \left(2^{\frac{1}{2}}\right)^x = 2^x$$

$$\Rightarrow 2^{x/2} = 2^{x}$$

$$\Rightarrow \frac{x}{2} = x$$

$$\Rightarrow x = 2x$$

$$\Rightarrow \boxed{0 = x}$$ only.

Non-zero sol^n := No sol^n,

No. of non. zero solutions. = 0

(19) To Prove: $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$

Given: $(a+ib).(c+id).(e+if).(g+ih) = A+iB$

By taking Modulus Both sides: $|z_1 . z_2| = |z_1||z_2|$

$\Rightarrow \left|(a+ib).(c+id).(e+if).(g+ih)\right| = |A+iB|$

$\Rightarrow |a+ib|.|c+id|.|e+if|.|g+ih| = |A+iB|$

$\Rightarrow \sqrt{a^2+b^2}.\sqrt{c^2+d^2}.\sqrt{e^2+f^2}.\sqrt{g^2+h^2} = \sqrt{A^2+B^2}$

Square

$\Rightarrow$

$(a^2+b^2).(c^2+d^2).(e^2+f^2).(g^2+h^2) = (A^2+B^2)$

20 If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least integral value of m.

Natural

$\{ \ldots -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \ldots \}$

$$\left(\frac{1+i}{1-i}\right)^m = 1$$

$$\Rightarrow \left[\frac{(1+i) \times (1+i)}{(1-i) \times (1+i)}\right]^m = 1$$

$$\Rightarrow \left[\frac{(1+i)^2}{1^2 - i^2}\right]^m = 1$$

$$\Rightarrow \left[\frac{1^2 + i^2 + 2.1.i}{1+1}\right]^m = 1$$

$$\Rightarrow \left(\frac{\cancel{1} - \cancel{1} + 2i}{2}\right)^m = 1$$

$$\Rightarrow \boxed{(i)^m = 1}$$

$\vdots$

$i^{-12} = 1$ ✗

$i^{-8} = 1$ ✗

$i^{-4} = 1$ ✗

$i^0 = 1$ ✗

$i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$

$i^4 = 1$

$i^8 = 1$

$i^{12} = 1$

$\vdots$

Naturals

Ans. $= 4$