Bayes Theorem

Tip 1

- Probability is a key topic in the CAT. Bayes Theorem (conditional probability) is not a very important topic.

- You don't have to go too deep into this topic, but ensure that you learn the basics well.

- So look through this formula list a few times and understand the formulae.

Tip 2(i)

Bayes Theorem (Conditional Probability) for CAT:

Conditional probability is used in case of events which are not independent. In the discussion of probabilities all events can be classified into 2 categories: Dependent and Independent.

Independent events are those where the happening of one event does not affect the happening of the other. For example, if an unbiased coin is thrown 'n' times then the probability of head turning up in any of the attempts will be 1/2. It will not be dependent on the results of the previous outcomes.

Dependent events, on the other hand, are the events in which the outcome of the second event is dependent on the outcome of the first event.

For example, if you have to draw two cards from a deck one after the other, then the probability of second card being of a particular suit will depend on the which card was drawn in the first attempt.

Tip 2(ii)

Bayes Theorem (Conditional Probability) for CAT:

Let us first discuss the definition of conditional probability.
Let 'A' and 'B' be two events which are not independent then the probability of occurrence of B given that A has already occurred is given by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Here, $P(A \cap B)$ is nothing but the probability of occurrence of both A and B.
We often use Bayes theorem to solve problems on conditional probability.
Bayes theorem is defined as follows

$$P(B|A) = \frac{P(A|B) * P(B)}{P(A)}$$

Here, P(A|B) is the probability of occurrence of A given that B has already occurred.
P(A) is the probability of occurrence of A
P(B) is the probability of occurrence of B

Examples

Example-1

Let us try to understand the application of the conditional probability and Bayes theorem with the help of few examples.

Ravi draws two cards from a deck of 52 cards one after another. If it is known that the first card was king then what is the probability of second card being 'spades'?

Let us use the conditional probability concept which we discussed above.

Let 'A' be the event of getting a king. Then P(A) = 4/52 = 1/13

Let 'B' be the event of getting a spade. Then P(B) = 13/52 = 1/4

Now we know that one of the spade cards is also a king. Hence, the event $P(A \cap B)$ contains 1 element.

Thus, $P(A \cap B) = 1/52$

Hence, by using the formula for conditional probability, we get

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{52}}{\frac{1}{13}} = 1/4$$

Examples

Example-2

Let us consider another problem to get a better understanding of conditional probability.

Ram plays a game of Russian roulette. He loads 2 bullets in the adjacent slots of a six slot revolver. He revolves the cylinder and then pulls the trigger. Luckily, it is an empty slot. Ram has an option either to pull the trigger again or to spin the cylinder first and then pull the trigger. What must Ram choose to maximize his chances of survival?

Let us number the slots as 1, 2, 3, 4, 5 and 6. Let us assume that slots 1 and 2 contain the bullets. The various combinations when the trigger is pressed continuously are (1,2), (2,3), (3,4), (4,5), (5,6) and (6,1).

We know that $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

Here, $P(A \cap B)$ represents the probability in which both A and B are empty slots and $p(A)$ represents the probability that A is an empty slot. Among the 6 combinations mentioned above, 3 combinations ((3,4), (4,5) and (5,6)) have both the slots empty. Therefore, $n(A \cap B) = 3$ and $p(A \cap B) = 3/6$.

P(A) represents the probability of the first slot being empty. The empty slot can be one among 3, 4, 5 or 6. Therefore, $n(A) = 4$ and $p(A \cap B) = 4/6$.

$P(B|A) = \dfrac{3/6}{4/6} = \dfrac{3}{4} = 75\%.$

If Ram prefers to spin the cylinder, he has P(A) chances of survival (Choosing an empty slot among the given slots).

=> $P(A) = 4/6 = 66.66\%$. Hence, Ram must prefer to press the trigger immediately without revolving the cylinder as chances of survival will be more.

Examples

Example-3

Let us now have a look at very famous problem on conditional probability. This is known as the Monty Hall problem.

There is a game show in which there are three doors. There is a car behind one door and there is nothing behind the other two doors. After you pick a door, the host opens one of the other two doors and shows you that it is empty. Now, he gives you two options – either stick with your initial selection or switch to the other door. What is the optimal strategy that should be followed? Will you switch or remain with the same door?
(The host knows which door has a car behind it.)

To find the optimal strategy let's compute the probability of wining in the both events.

Without the loss of generality, let's assume that the contestant has picked the door one.

Let W1, W2, W3 be the events that the car is behind door 1, 2, 3 respectively.
Hence, $p(W1) = p(W2) = p(W3) = \frac{1}{3}$

Let A, B and C be the events that the host opens doors 1, 2 and 3 respectively.
Now,
The probability that the host opens the third door provided the car is in the second door.
$p(C|W2) = 1$

The probability that the host opens the third door provided the car is in the third door is
$p(C|W3) = 0$

Examples

Example-3

The probability that the host opens the third door provided the car is in the first door.

P(c|W1) = ½

Now let's use Baye's theorem

Chances of winning by not switching

$$P(W1|C) = \frac{P(C|W1) * P(W1)}{P(C)} = \frac{\left(\frac{1}{2}\right) * \left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{3}$$

Chances of winning by switching

$$P(W2|C) = \frac{p(C|W2) * P(W2)}{P(C)} = \frac{1 * \left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)} = \frac{2}{3}$$

Therefore, it is always optimal to switch.