CHAPTER

7.8

SPREAD SPECTRUM

Statement for Question 1-3 :

A pseudo-noise (PN) sequance is generated using a feedback shift register of length $m = 4$. The chip rate is 10^7chips per second

1. The PN sequance length is

(A) 10 (B) 12

(C) 15 (D) 18

2. The chip duration is

(A) $1\mu s$ (B) $0.1\ \mu s$

(C) 0.1 ms (D) 1 ms

3. The period of PN sequance is

(A) $1.5\ \mu s$ (B) $15\ \mu s$

(C) 6.67 ns (D) 0.67 ns

Statement for Question 4-5:

A direct sequence spread binary phase-shift-keying system uses a feedback shift register of Length 19 for the generation of PN sequence . The system is required to have an average probability of symbol error due to externally generated interfering signals that does not exceed 10^{-5}

4. The processing gain of system is

(A) 37 dB (B) 43 dB

(C) 57 dB (D) 93 dB

5. The Antijam margin is

(A) 47.5 dB (B) 93.8 dB

(C) 86.9 dB (D) 12.6 dB

6. A slow FH/MFSK system has the following parameters.

Number of bits per MFSK symbol = 4

Number of MFSK symbol per hop = 5

The processing gain of the system is

(A) 13.4 dB (B) 37.8 dB

(C) 6 dB (D) 26 dB

7. A fast FH/MFSK system has the following parameters.

Number of bits per MFSK symbol = 4

Number of pops per MFSK symbol = 4

The processing gain of the system is

(A) 0 dB (B) 7 dB

(C) 9 dB (D) 12 dB

Statement for Question 8-9:

A rate 1/2 convolution code with dfrec = 10 is used to encode a data requeence occurring at a rate of 1 kbps. The modulation is binary PSK. The DS spread spectrum sequence has a chip rate of 10 MHz

8. The coding gain is

(A) 7 dB (B) 12 dB

(C) 14 dB (D) 24 dB

9. The processing gain is

(A) 14 dB (B) 37 dB

(C) 58 dB (D) 104 dB

10. A total of 30 equal-power users are to share a common communication channel by CDM. Each user transmit information at a rate of 10 kbps via DS spread spectrum and binary PSK. The minimum chip rate to obtain a bit error probability of 10^{-5}

(A) 1.3×10^6 chips/sec (B) 2.9×10^5 chips/sec

(C) 1.9×10^6 chips/sec (D) 1.3×10^5 chips/sec

11. A CDMA system is designed based on DS spread spectrum with a processing gain of 1000 and BPSK modulation scheme. If user has equal power and the desired level of performance of an error probability of 10^{-6}, the number of user will be

(A) 89 (B) 117

(C) 147 (D) 216

12. In previous question if processing gain is changed to 500, then number of users will be

(A) 27 users (B) 38 users

(C) 42 users (D) 45 users

Statement for Question 13-15 :

A DS spread spectrum system transmit at a rate of 1 kbps in the presets of a tone jammer. The jammer power is 20 dB greater then the desired signal, and the required $\in_b / J_0$ to achieve satisfactory performance is 10 dB.

13. The spreading bandwidth required to meet the specifications is

(A) 10^7 Hz (B) 10^3 Hz

(C) 10^5 Hz (D) 10^6 Hz

14. If the jammer is a pulse jammer, then pulse duty cycle that results in worst case jamming is

(A) 0.14 (B) 0.05

(C) 0.07 (D) 0.10

15. The correspond probability of error is

(A) 4.9×10^{-3} (B) 6.3×10^{-3}

(C) 9.4×10^{-4} (D) 8.3×10^{-3}

Statement for question 16-18 :

A CDMA system consist of 15 equal power user that transmit information at a rate of 10 kbps, each using a DS spread spectrum signal operating at chip rate of 1 MHz. The modulation scheme is BPSK.

16. The Processing gain is

(A) 0.01 (B) 100

(C) 0.1 (D) 10

17. The value of ε_b / J_0 is

(A) 8.54 dB (B) 7.14 dB

(C) 17.08 dB (D) 14.28 dB

18. How much should the processing gain be increased to allow for doubling the number of users with affecting the autopad SNR

(A) 1.46 MHz (B) 2.07 MHz

(C) 4.93 MHz (D) 2.92 MHz

19. A DS/BPSK spread spectrum signal has a processing gain of 500. If the desired error probability is 10^{-5} and (ε_b / J_0) required to obtain an error probability of 10^{-5} for binary PSK is 9.5 dB, then the Jamming margin against a containers tone jammer is

(A) 23.6 dB (B) 17.5 dB

(C) 117.4 dB (D) 109.0 dB

Statement for Question 20-21 :

An $m = 10$ ML shift register is used to generate the pre hdarandlm sequence in a DS spread spectrum system. The chip duration is $T_c = l\,\mu s$ and the bit duration is $T_b = NT_c$, where N is the length (period of the m sequence).

20. The processing gain of the system is

(A) 10 dB (B) 20 dB

(C) 30 dB (D) 40 dB

21. If the required ε_b / J_0 is 10 and the jammer is a tone jammer with an average power J_{av}, then jamming margin is.

(A) 10 dB (B) 20 dB

(C) 30 dB (D) 40 dB

Statement for Question 22-23 :

An FH binary orthogonal FSK system employs an $m = 15$ stage liner feedback shift register that generates an ML sequence. Each state of the shift register selects one of L non over lapping frequency bands in the hopping pattern. The bit rate is 100 bits/s. The demodulator employ non coherent detection.

22. If the hop rate is one per bit, the hopping bandwidth for this channel is

(A) 6.5534 MHz (B) 9.4369 MHz

(C) 2.6943 MHz (D) None of the above

23. Suppose the hop rate is increased to 2 hops/bit and the receiver uses square law combining the signal over two hops. The hopping bandwidth for this channel is

(A) 3.2767 MHz (B) 13.1068 MHz

(C) 26.2136 MHz (D) 1.6384 MHz

Statement for Qquestion 24-25 :

In a fast FH spread spectrum system, the information is transmitted via FSK with non coherent detection. Suppose there are $N = 3$ hops/bit with hard decision decoding of the signal in each hop. The channel is AWGN with power spectral density $\frac{1}{2}\mathcal{N}_0$ and an SNR 20-13 dB (total SNR over the three hops)

24. The probability of error for this system is

(A) 0.013 (B) 0.0013

(C) 0.049 (D) 0.0049

25. In case of one hop per bit the probability of error is

(A) 1.96×10^{-5} (B) 1.96×10^{-7}

(C) 2.27×10^{-5} (D) 2.27×10^{-7}

Statement for Question 26-29 :

A slow FH binary FSK system with non coherent detection operates at $\varepsilon_b / J_0 = 10$, with hopping bandwidth of 2 GHz, and a bit rate of 10 kbps.

26. The processing gain of this system is

(A) 23 dB (B) 43 dB

(C) 43 dB (D) 53 dB

27. If the jammer operates as a partial band jammar, the bandwidth occupancy for worst case jamming is

(A) 0.4 GHz (B) 0.6 GHz

(C) 0.7 GHz (D) 0.9 GHz

28. The probability of error for the worst-case partial band jammer is

(A) 0.2996 (B) 0.1496

(C) 0.0368 (D) 0.0298

29. The minimum hop rate for a FH spread spectrum system that will prevent a jammer from operating five onives away from the receiver is

(A) 3.2 bHz (B) 3.2 MHz

(C) 18.6 MHz (D) 18.6 kHz

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SOLUTION

1. (C) The PN sequence length is

$N = 2^m - 1 = 2^4 - 1 = 15$

2. (B) The chip duration is

$T_C = \frac{1}{10^7} s = 0.1$ ms

3. (A) The period of the PN sequence is

$T = NT_C = 15 \times 0.1 = 1.5\ \mu s$

4. (C) $m = 19$

$n = 2^m - 1 = 2^{19} - 1 = 2^{19}$

The processing gain is $10\log_{10} N = 10\log_{10} 2^{19}$

$= 190 \times 0.3$ or 57 dB

5. (A) Antijam margin = (Processing gain) - $10\log_{10}\left(\frac{Eb}{N_0}\right)$

The probability of error is

$P_e = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$

With $P_e = 10^{-5}$, we have $E_b / N_0 = 9$.

Hence, Antijam margin $= 57 - 10\log_{10} 9 = 57 - 9.5$

or $= 47.5$ dB

6. (D) The precessing gain (PG) is

$\text{PG} = \frac{\text{FH Bandwidth}}{\text{Symbol Rate}} = \frac{W_c}{R_s} = 5 \times 4 = 20$

Hence, expressed in decibels, PG $= 10\log_{10} 20 = 26$ db

7. (D) The processing gain is

$\text{PG} = 4 \times 4 = 16$

Hence, in decibels,

$\text{PG} = 10\log_{10} 16 = 12$ dB

8. (A) The coding gain is $R_{cd\,min} = \frac{1}{2} \times 10 = 5$ or 7 dB

9. (B) The processing gain

$\frac{W}{R} = \frac{10^7}{2 \times 10^3} = 5 \times 10^3$ or 37 dB

10. (C) We assume that the interference is characterized as a zero-mean AWGN process with power spectral density J_0. To achieve an error probability of 10^{-5}, the required $\epsilon_b / J_0 = 10$ we have

$$\frac{W/R}{J_{av}/P_{av}} = \frac{W/R}{N_u - 1} = \frac{\epsilon_b}{J_0}$$

$$W/R = \left(\frac{\epsilon_b}{J_0}\right)(N_u - 1)$$

$$W = R\left(\frac{\epsilon_b}{J_0}\right)(N_u - 1)$$

where $R = 10^4$ bps, $N_u = 30$ and $\epsilon_b / J_0 = 10$

Therefore, $W = 2.9 \times 10^6$ Hz

The minimum chip rate is $1 / T_c = W = 2.9 \times 10^6$ chips/sec

11. (D) To achieve an error probability of 10^{-6}, we required $\left(\frac{\epsilon_b}{J_0}\right)_{dB} = 10.5$ dB

Then, the number of users of the CDMA system is

$N_u = \frac{W/R}{\epsilon_b / J_0} + 1 = \frac{1000}{11.3} + 1 = 89$ users

12. (D) If the processing gain is reduced to W/R = 500, then

$N_u = \frac{500}{11.3} + 1$ = 45 users

13. (D) We have a system where $(J_{av}/P_{av})_{dB} = 20$ dB, $R = 1000$ bps and $(\epsilon_b / J_0)_{dB} = 10$ dB

Hence, we obtain $\left(\frac{W}{R}\right)_{dB} = \left(\frac{J_{av}}{P_{av}}\right)_{dB} + \left(\frac{\epsilon_b}{J_0}\right)_{dB} = 30$ dB

$\frac{W}{R} = 1000$

$W = 1000R = 10^6$ Hz

14. (C) The duty cycle of a pulse jammer of worst-case jamming is $\alpha = \frac{0.71}{\epsilon_b / J_0} = \frac{0.7}{10} = 0.07$

15. (D) The corresponding probability of error for this worst-case jamming is

$P_2 = \frac{0.083}{\varepsilon_b / J_0} = \frac{0.083}{10} = 8.3 \times 10^{-3}$

16. (B) Precessing gain is $\frac{W}{R} = \frac{10^6}{10^4} = 100$

17. (A) We have $N_u = 15$ users transmitting at a rate of 10,000 bps each, in a bandwidth of $W = 1MHz$.

The ε_b/J_0 is. $\frac{\varepsilon_b}{J_0} = \frac{W/R}{N_u - 1} = \frac{10^6 / 10^4}{14} = \frac{100}{14}$

$= 7.14$ or 8.54 dB

18. (B) With $N_u = 30$ and $\varepsilon_b/J_0 = 7.14$, the processing gain should be increased to

$W/R = (7.14)(29) = 207$

$W = 207 \times 104 = 2.07$ MHz

Hence the bandwidth must be increased to 2.07 MHz

19. (B) The processing gain is given as

$\frac{W}{R} = 500$ or 27 dB

The (ε_b/J_0) required to obtain an error probability of 10^{-5} for binary PSK is 9.5 dB. Hence, the jamming margin is

$$\left(\frac{J_{av}}{P_{av}}\right)_{dB} = \left(\frac{W}{R}\right)_{dB} - \left(\frac{\varepsilon_b}{J_0}\right)_{dB} = 27.95 \text{ or } 17.5 \text{ dB}$$

20. (C) The period of the maximum length shift register sequence is

$N = 2^{10} - 1 = 1023$

Since $T_b = NT_c$ then the processing gain is

$N\frac{T_b}{T_c} = 1023$ or 30 dB

21. (B) A Jamming margin is

$$\left(\frac{J_{av}}{P_{av}}\right)_{dB} = \left(\frac{W}{R_b}\right)_{dB} - \left(\frac{\varepsilon_b}{J_0}\right)_{dB} = 30 - 10 = 20 \text{ dB}$$

where $J_{av} = J_0 W \approx J_0/T_c = J_0 \times 10^6$

22. (A) The length of the shift-register sequence is

$L = 2^m - 12^{15} - 1 = 32767$ bits

For binary FSK modulation, the minimum frequency separation is $2/T$, where $1/T$ is the symbol (bit) rate. The hop rate is 100 hops/sec. Since the shift register has $L = 32767$ states and each state utilizes a bandwidth of $2/T = 200$ Hz, then the total bandwidth for the FH signal is 6.5534 MHz.

23. If the hopping rate is 2 hops/bit and the bit rate is 100 bits/sec, then, the hop rate is 200 hops/sec. The minimum frequency separation for orthogonality $2/T = 400$ Hz. Since there are $N = 32767$ states of the shift register and for each state we select one of two frequencies separated by 400 Hz, the hopping bandwidth is 13.1068 MHz.

24. (B) The total SNR for three hops is 20 ~ 13 dB. Therefore the SNR per hop is 20/3. The probability of a chip error with non-coherent detection is

$$P = \frac{1}{2}e^{-\frac{\varepsilon_c}{2N_0}}$$

where $\varepsilon_c / N_0 = 20/3$. The probability of a bit error is

$$P_b = 1 - (1-p)^2 = 1 - (1 - 2p + p^2) = 2p - p^2$$

$$= e^{-\frac{\varepsilon_c}{2N_0}} - \frac{1}{2}e^{-\frac{\varepsilon_c}{2N_0}} = 0.0013$$

25. (C) In the case of one hop per bit, the SNR per bit is 20, Hence, $P_b = \frac{1}{2}e^{-\frac{\varepsilon_c}{2N_0}} = \frac{1}{2}e^{-10} = 2.27 \times 10^{-5}$

26. (D) We are given a hopping bandwidth of 2 GHz and a bit rate of 10 kbs.

Hence, $\frac{W}{R} = \frac{2 \times 10^9}{10^4} = 2 \times 10^5$ or 53 dB

27. (A) The bandwidth of the worst partial-band jammer is $\alpha * W$, where

$\alpha * W = 2/(\varepsilon_b/J_0) = 0.2$

Hence $\alpha * W = 0.4$ GHz

28. (C) The probability of error with worst-case partial-band jamming is $P_2 = \frac{e^{-1}}{(\varepsilon_b/J_0)} = \frac{e^{-1}}{10} = 3.68 \times 10^{-2}$

29. (D) $d = 5$ miles $= 8050$ meters

$\Delta d = 2 \times 8050 = 16100$

$\Delta d = x \times t$ or $t = \frac{\Delta d}{t}$

$\Rightarrow t = \frac{\Delta d}{x} = \frac{16100}{3 \times 10^8} = 5.367 \times 10^5$

$f = \frac{1}{t} = \frac{1}{5.367 \times 10^{-5}} = 18.63$ kHz

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10. A field is given as

$$\mathbf{G} = \frac{13}{x^2 + y^2}(y\mathbf{u}_x + 3\mathbf{u}_y + x\mathbf{u}_z)$$

The field at point (–2, 3, 4) is

(A) $13(-2\mathbf{u}_x + 3\mathbf{u}_y + 4\mathbf{u}_z)$ (B) $-2\mathbf{u}_x + 3\mathbf{u}_y + 4\mathbf{u}_z$

(C) $13(3\mathbf{u}_x + 4\mathbf{u}_y - 2\mathbf{u}_z)$ (D) $3\mathbf{u}_x + 4\mathbf{u}_y - 2\mathbf{u}_z$

11. A field is given as $\mathbf{F} = y\mathbf{u}_x + z\mathbf{u}_y + x\mathbf{u}_z$ The angle between $\mathbf{G}$ and $\mathbf{u}_x$ at point (2, 2, 0) is

(A) 45° (B) 30°

(C) 60° (D) 90°

12. A vector field is given as

$$\mathbf{G} = 12xy\mathbf{u}_x + 6(x^2 + 2)\mathbf{u}_y + 18z^2\mathbf{u}_z$$

The equation of the surface $\mathbf{M}$ on which $|\mathbf{G}| = 60$ is

(A) $4x^2y^2 + 4x^4 + 9z^4 + 2x^2 = 96$

(B) $2x^2y^2 + x^4 + 9z^4 + 2x^2 = 96$

(C) $2x^2y^2 + 4x^4 + 9z^4 + 2x^2 = 96$

(D) $4x^2y^2 + x^4 + 9z^4 + 2x^2 = 96$

13. A vector field is given by

$$\mathbf{E} = 4zy^2\mathbf{u}_z + 2y\sin 2x\,\mathbf{u}_y + y^2\sin 2x\,\mathbf{u}_z$$

The surface on which $\mathbf{E}_y = 0$ is

(A) Plane $y = 0$ (B) Plane $x = 0$

(C) Plane $x = \frac{3\pi}{2}$ (D) all

14. The vector field $\mathbf{E}$ is given by

$$\mathbf{E} = 6zy^2\cos 2x\,\mathbf{u}_x + 4xy\sin 2x\,\mathbf{u}_y + y^2\sin 2x\,\mathbf{u}_z$$

The region in which $\mathbf{E} = 0$ is

(A) $y = 0$ (B) $x = 0$

(C) $z = 0$ (D) $x = \frac{n\pi}{2}$

15. Two vector fields are $\mathbf{F} = -10\mathbf{u}_x + 20x(y-1)\mathbf{u}_y$ and $\mathbf{G} = 2x^2y\mathbf{u}_x - 4\mathbf{u}_y + 2\mathbf{u}_z$. At point A(2, 3, –4) a unit vector in the direction of $\mathbf{F} - \mathbf{G}$ is

(A) $0.18\mathbf{u}_x + 0.98\mathbf{u}_y - 0.05\mathbf{u}_z$

(B) $-0.18\mathbf{u}_x - 0.98\mathbf{u}_y + 0.05\mathbf{u}_z$

(C) $-0.37\mathbf{u}_x + 0.92\mathbf{u}_y + 0.02\mathbf{u}_z$

(D) $0.37\mathbf{u}_x - 0.92\mathbf{u}_y - 0.02\mathbf{u}_z$

16. A field is given as

$$\mathbf{G} = \frac{25}{x^2 + y^2}(x\mathbf{u}_x + y\mathbf{u}_y)$$

The unit vector in the direction of $\mathbf{G}$ at P(3, 4, –2) is

(A) $0.6\mathbf{u}_x + 0.8\mathbf{u}_y$ (B) $0.8\mathbf{u}_x + 0.6\mathbf{u}_y$

(C) $0.6\mathbf{u}_y + 0.8\mathbf{u}_z$ (D) $0.6\mathbf{u}_z + 0.6\mathbf{u}_x$

17. A field is given as $\mathbf{F} = xy\mathbf{u}_x + yz\mathbf{u}_y + zx\mathbf{u}_x$ The value of the double integral $I = \int_0^4\int_0^2 \mathbf{F}\cdot\mathbf{u}_y dzdx$ in the plane $y = 7$ is

(A) 128 (B) 56

(C) 190 (D) 0

18. Two vector extending from the origin are given as $\mathbf{R}_1 = 4\mathbf{u}_x + 3\mathbf{u}_y - 2\mathbf{u}_z$ and $\mathbf{R}_2 = 3\mathbf{u}_x - 4\mathbf{u}_y - 6\mathbf{u}_z$. The area of the triangle defined by $\mathbf{R}_1$ and $\mathbf{R}_2$ is

(A) 12.47 (B) 20.15

(C) 10.87 (D) 15.46

19. The four vertices of a regular tetrahedron are located at O (0, 0, 0), A(0, 1, 0), B(0.5 $\sqrt{3}$, 0.5, 0) and C ($\frac{0.5}{\sqrt{3}}$, 0.5, $\frac{\sqrt{2}}{3}$). The unit vector perpendicular (outward) to the face ABC is

(A) $0.41\mathbf{u}_x + 0.71\mathbf{u}_y + 0.29\mathbf{u}_z$

(B) $0.47\mathbf{u}_x + 0.82\mathbf{u}_y + 0.33\mathbf{u}_z$

(C) $-0.47\mathbf{u}_x - 0.82\mathbf{u}_y - 0.33\mathbf{u}_z$

(D) $-0.41\mathbf{u}_x - 0.71\mathbf{u}_y - 0.29\mathbf{u}_z$

20. The two vector are $\mathbf{R}_{AM} = 20\mathbf{u}_x + 18\mathbf{u}_y - 18\mathbf{u}_z$ and $\mathbf{R}_{AN} = -10\mathbf{u}_x + 8\mathbf{u}_y + 15\mathbf{u}_z$. The unit vector in the plane of the triangle that bisects the interior angle at A is

(A) $0.168\mathbf{u}_x + 0.915\mathbf{u}_y + 0.367\mathbf{u}_z$

(B) $0.729\mathbf{u}_x + 0.134\mathbf{u}_y - 0.672\mathbf{u}_z$

(C) $0.729\mathbf{u}_x + 0.134\mathbf{u}_y + 0.672\mathbf{u}_z$

(D) $0.168\mathbf{u}_x + 0.915\mathbf{u}_y - 0.367\mathbf{u}_z$

21. Two points in cylindrical coordinates are A($\rho = 5$, $\phi = 70°$, $z = -3$) and B($\rho = 2$, $\phi = 30°$, $z = 1$). A unit vector at A towards B is

(A) $0.03\mathbf{u}_x - 0.82\mathbf{u}_y + 0.57\mathbf{u}_z$

(B) $0.03\mathbf{u}_x + 0.82\mathbf{u}_y + 0.57\mathbf{u}_z$

(C) $-0.82\mathbf{u}_x + 0.003\mathbf{u}_y + 0.57\mathbf{u}_z$

(D) $0.003\mathbf{u}_x - 0.82\mathbf{u}_y + 0.57\mathbf{u}_z$

22. A field in cartesian form is given as

$$\mathbf{D} = x\mathbf{u}_x + \frac{y\mathbf{u}_y}{x^2 + y^2}$$

In cylindrical form it will be

(A) $\mathbf{D} = \frac{\mathbf{u}_\rho}{\rho}$ (B) $\mathbf{D} = \frac{\mathbf{u}_\rho}{\rho} + \frac{\mathbf{u}_\phi}{\cos\phi}$

(C) $\mathbf{D} = \rho\mathbf{u}_\rho$ (D) $\mathbf{D} = \rho\mathbf{u}_\rho + \cos\phi\,\mathbf{u}_\phi$

23. A vector extends from A($\rho = 4$, $\phi = 40°$, $z = -2$) to B($\rho = 5$, $\phi = -110°$, $z = 1$). The vector $\mathbf{R}_{AB}$ is

(A) $4.77\mathbf{u}_x + 7.30\mathbf{u}_y + 4\mathbf{u}_z$

(B) $-4.77\mathbf{u}_x - 7.30\mathbf{u}_y + 4\mathbf{u}_z$

(C) $-7.30\mathbf{u}_x - 4.77\mathbf{u}_y + 4\mathbf{u}_z$

(D) $7.30\mathbf{u}_x + 4.77\mathbf{u}_y + 4\mathbf{u}_z$

24. The surface $\rho = 3$, $\rho = 5$, $\phi = 100°$, $\phi = 130°$, $z = 3$ and $z = 4.5$ define a closed surface. The enclosed volume is

(A) 480 (B) 5.46

(C) 360 (D) 6.28

25. The surface $\rho = 2$, $\rho = 4$, $\phi = 45°$, $\phi = 135°$, $z = 3$ and $z = 4$ define a closed surface. The total area of the enclosing surface is

(A) 34.29 (B) 20.7

(C) 32.27 (D) 16.4

26. The surface $\rho = 3$, $\rho = 5$, $\phi = 100°$, $\phi = 130°$, $z = 3$ and $z = 4.5$ define a closed volume. The length of the longest straight line that lies entirely within the volume is

(A) 3.21 (B) 3.13

(C) 4.26 (D) 4.21

27. A vector field **H** is

$$\mathbf{H} = \rho z^2 \sin\phi\,\mathbf{u}_\rho + e^{-z}\sin\left(\frac{\phi}{2}\right)\mathbf{u}_\phi + \rho^3\mathbf{u}_z$$

At point $\left(2, \frac{\pi}{3}, 0\right)$ the value of $\mathbf{H}\cdot\mathbf{u}_x$ is

(A) 0.25 (B) 0.433

(C) –0.433 (D) –0.25

28. A vector is $\mathbf{A} = y\mathbf{u}_x + (x+z)\mathbf{u}_y$. At point P(–2, 6, 3) **A** in cylindrical coordinate is

(A) $-0.949\mathbf{u}_\rho - 6.008\mathbf{u}_\phi$ (B) $0.949\mathbf{u}_\rho - 6.008\mathbf{u}_\phi$

(C) $-6.008\mathbf{u}_\rho - 0.949\mathbf{u}_\phi$ (D) $6.008\mathbf{u}_\rho + 0.949\mathbf{u}_\phi$

29. The vector

$$\mathbf{B} = \frac{10}{r}\mathbf{u}_r + r\cos\theta\,\mathbf{u}_\theta + \mathbf{u}_\phi$$

in cartesian coordinates at (-3, 4, 0) is

(A) $\mathbf{u}_x - 2\mathbf{u}_y$ (B) $-2\mathbf{u}_x + \mathbf{u}_y$

(C) $1.36\mathbf{u}_x + 2.72\mathbf{u}_y$ (D) $-2.72\mathbf{u}_x + 1.36\mathbf{u}_x$

30. The two point have been given A(20, 30°, 45°) and B(30, 115°, 160°). The $|\mathbf{R}_{AB}|$ is

(A) 22.2 (B) 44.4

(C) 11.1 (D) 33.3

31. The surface $r = 2$ and 4, $\theta = 30°$ and 60°, $\phi = 20°$ and 80° identify a closed surface. The enclosed volume is

(A) 11.45 (B) 7.15

(C) 6.14 (D) 8.26

32. The surface $r = 2$ and 4, $\theta = 30°$ and 50° and $\phi = 20°$ and 60° identify a closed surface. The total area of the enclosing surface is

(A) 6.31 (B) 18.91

(C) 25.22 (D) 12.61

33. At point P($r = 4$, $\theta = 0.2\pi$, $\phi = 0.8\pi$), $\mathbf{u}_r$ in cartesian component is

(A) $0.48\mathbf{u}_x + 0.35\mathbf{u}_y + 0.81\mathbf{u}_z$

(B) $0.48\mathbf{u}_x - 0.35\mathbf{u}_y - 0.81\mathbf{u}_z$

(C) $-0.48\mathbf{u}_x + 0.35\mathbf{u}_y + 0.81\mathbf{u}_z$

(D) $0.48\mathbf{u}_x - 0.35\mathbf{u}_y - 0.81\mathbf{u}_z$

34. The expression for $\mathbf{u}_y$ in spherical coordinates at P($r = 4$, $\theta = 0.2\pi$, $\phi = 0.8\pi$) is

(A) $0.48\mathbf{u}_r + 0.35\mathbf{u}_\theta - 0.81\mathbf{u}_\phi$

(B) $0.35\mathbf{u}_r + 0.48\mathbf{u}_\theta - 0.81\mathbf{u}_\phi$

(C) $-0.48\mathbf{u}_r + 0.35\mathbf{u}_\theta - 0.81\mathbf{u}_\phi$

(D) $-0.35\mathbf{u}_r + 0.48\mathbf{u}_\theta - 0.81\mathbf{u}_\phi$

35. Given a vector field

$$\mathbf{D} = r\sin\phi\,\mathbf{u}_r - \frac{1}{r}\sin\theta\cos\phi\,\mathbf{u}_\theta + r^2\mathbf{u}_\phi$$

The component of **D** tangential to the spherical surface $r = 10$ at P(10, 150°, 330°) is

(A) $0.043\mathbf{u}_\theta + 100\mathbf{u}_\phi$

(B) $-0.043\mathbf{u}_\theta - 100\mathbf{u}_\phi$

(C) $110\mathbf{u}_\theta + 0.043\mathbf{u}_\phi$

(D) $0.043\mathbf{u}_\theta - 100\mathbf{u}_\phi$

36. The circulation of $\mathbf{F} = x^2\mathbf{u}_x - xz\mathbf{u}_y - y^2\mathbf{u}_z$ around the path shown in fig. P8.1.36 is

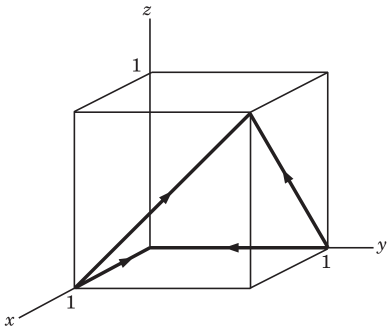

Fig. P8.1.36

(A) $-\frac{1}{3}$ (B) $\frac{1}{6}$

(C) $-\frac{1}{6}$ (D) $\frac{1}{3}$

37. The circulation of $\mathbf{A} = \rho \cos\phi\, \mathbf{u}_\rho + z \sin\phi\, \mathbf{u}_z$ around the edge L of the wedge shown in Fig. P8.1.37 is

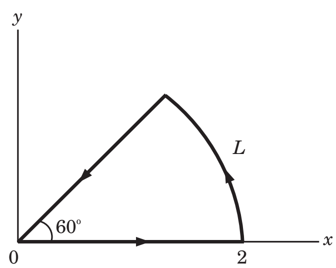

Fig. P8.1.37

(A) 1 (B) –1

(C) 0 (D) 3

38. The gradient of field $f = y^2x + xyz$ is

(A) $y(y+z)\mathbf{u}_x + x(2y+z)\mathbf{u}_y + xy\mathbf{u}_z$

(B) $y(2x+z)\mathbf{u}_x + x(x+z)\mathbf{u}_y + xy\mathbf{u}_z$

(C) $y^2\mathbf{u}_x + 2yx\mathbf{u}_y + xy\mathbf{u}_z$

(D) $y(2y+z)\mathbf{u}_x + x(2y+z)\mathbf{u}_y + xy\mathbf{u}_z$

39. The gradient of the field $f = \rho^2 z \cos 2\phi$ at point (1, 45°, 2) is

(A) $4\mathbf{u}_\phi$ (B) $4\sqrt{2}\mathbf{u}_\phi$

(C) $-4\mathbf{u}_\phi$ (D) $-4\sqrt{2}\mathbf{u}_\phi$

40. The gradient of the function $G = r^3 \sin 2\theta \sin 2\phi \sin\theta$ at point $P(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is

(A) $1.41\mathbf{u}_\rho + 3\mathbf{u}_z$ (B) $\mathbf{u}_x + \mathbf{u}_y + \mathbf{u}_z$

(C) $3.46\mathbf{u}_r + 9.3\mathbf{u}_\theta$ (D) All

41. The directional derivative of function $\Phi = xy + yz + zx$ at point P(3, –3, – 3) in the direction toward point Q(4, –1, –1) is

(A) –3 (B) 1

(C) –2 (D) 0

42. The temperature in a auditorium is given by $T = 2x^2 + y^2 - 2z^2$. A mosquito located at (2, 2, 1) in the auditorium desires to fly in such a direction that it will get warm as soon as possible. The direction, in that it must fly is

(A) $8\mathbf{u}_x + 8\mathbf{u}_y - 4\mathbf{u}_z$

(B) $2\mathbf{u}_x + 2\mathbf{u}_y - \mathbf{u}_z$

(C) $4\mathbf{u}_x + 4\mathbf{u}_y - 4\mathbf{u}_z$

(D) $-(2\mathbf{u}_x + 2\mathbf{u}_y - \mathbf{u}_z)$

43. The angle between the normal to the surface $x^2y + z = 3$ and $x \ln z - y^2 = -4$ at the point of intersection (–1, 2, 1) is

(A) 73.4° (B) 36.3°

(C) 16.6° (D) 53.7°

44. The divergence of vector $\mathbf{A} = yz\mathbf{u}_x + 4xy\mathbf{u}_y + y\mathbf{u}_z$ at point P(1, –2, 3) is

(A) 2 (B) -2

(C) 0 (D) 4

45. The divergence of the vector $\mathbf{A} = 2r\cos\theta\cos\phi\,\mathbf{u}_r + r^{1/2}\mathbf{u}_\phi$ at point P(1, 30°, 60°) is

(A) 2.6 (B) 1.5

(C) 4.5 (D) -4.5

46. The divergence of the vector

$$\mathbf{A} = \rho z^2 \cos\phi\,\mathbf{u}_\rho + z\sin^2\phi\,\mathbf{u}_z \text{ is}$$

(A) $2\rho z^2 \cos\phi\,\mathbf{u}_\rho + \sin^2\phi\,\mathbf{u}_z$

(B) $2\rho z^2 \cos\phi\,\mathbf{u}_\rho + \sin^2\phi\,\mathbf{u}_z$

(C) $2z^2 \cos\phi\,\mathbf{u}_\rho + \sin^2\phi\,\mathbf{u}_z$

(D) $z\sin 2\frac{\phi}{\rho}\mathbf{u}_\rho + 2\rho z\cos\phi\,\mathbf{u}_\phi + z^2\sin\phi\,\mathbf{u}_z$

47. The flux of $\mathbf{D} = \rho^2 \cos^2 \phi \mathbf{u}_\rho + 3 \sin \phi \mathbf{u}_\phi$ over the closed surface of the cylinder $0 \le z < 3$, $\rho = 3$ is

(A) 324 (B) 81π

(C) 81 (D) 64π

48. The curl of vector $\mathbf{A} = e^{xy}\mathbf{u}_x + \sin xy\, \mathbf{u}_y + \cos^2 xz\, \mathbf{u}_z$ is

(A) $y\, e^{xy}\mathbf{u}_x + x \cos xy\, \mathbf{u}_y - 2x \sin 2xz\, \mathbf{u}_z$

(B) $z \sin 2xy\, \mathbf{u}_y + (y \cos xy - xe^{xy})\mathbf{u}_z$

(C) $z \sin 2xy\, \mathbf{u}_y + (x \cos xy - xe^{xy})\mathbf{u}_z$

(D) $xy\, e^{xy}\mathbf{u}_x + xy \cos xy\, \mathbf{u}_y - 2xz \sin 2xz\, \mathbf{u}_z$

49. The curl of vector field

$\mathbf{A} = \rho z \sin \phi\, \mathbf{u}_\rho + 3\rho z^2 \cos \phi\, \mathbf{u}_\rho$ at point (5, 90°, 1) is

(A) 0 (B) $12\mathbf{u}_\theta$

(C) $6\mathbf{u}_r$ (D) $5\mathbf{u}_\phi$

50. The curl of vector field

$\mathbf{A} = r \cos \theta\, \mathbf{u}_r - \frac{1}{r} \sin \theta\, \mathbf{u}_\theta + 2r^2 \sin \theta\, \mathbf{u}_\phi$ is

(A) $\cos \theta\, \mathbf{u}_r - \frac{1}{r} \cos \theta\, \mathbf{u}_\theta$

(B) $2r^2 \cos \theta\, \mathbf{u}_r - 4r \sin \theta\, \mathbf{u}_\theta + \left(\frac{1}{r^2} \sin \theta - r \sin \theta\right)\mathbf{u}_\phi$

(C) $4r \cos \theta\, \mathbf{u}_r - 6r \sin \theta\, \mathbf{u}_\theta + \sin \theta\, \mathbf{u}_\phi$

(D) 0

51. If $\mathbf{A} = (3y^2 - 2z)\mathbf{u}_x - 2x^2 z\, \mathbf{u}_y + (x + 2y)\mathbf{u}_z$, the value of $\nabla \times \nabla \times \mathbf{A}$ at P(–2, 3, –1) is

(A) $-(6\mathbf{u}_x + 4\mathbf{u}_y)$ (B) $8(\mathbf{u}_x + \mathbf{u}_y)$

(C) $-8(\mathbf{u}_x + \mathbf{u}_y)$ (D) 0

52. The **grad** $\cdot \nabla \times \mathbf{A}$ of a vector field

$\mathbf{A} = x^2 y \mathbf{u}_x + y^2 z \mathbf{u}_y - 2xz\mathbf{u}_z$ is

(A) $2xy + 2yz - 2x$

(B) $x^2 y + y^2 z - 2xz$

(C) $2x^2 y + 2y^2 z - 2xz$

(D) 0

53. If $V = xy - x^2 y + y^2 z^2$, the value of the **div grad** V is

(A) 0

(B) $z + x^2 + 2y^2 z$

(C) $2y(z^2 - yz - x)$

(D) $2(z^2 - y^2 - y)$

54. If $V = x^2 y^2 z^2$, the laplacian of the field V is

(A) $2(xy^2 + yz^2 + zx^2)$

(B) $2(x^2 y^2 + y^2 z^2 + z^2 x^2)$

(C) $(x^2 y^2 + y^2 z^2 + z^2 x^2)$

(D) 0

55. The value of $\nabla^2 V$ at point P(3, 60°, –2) is if $V = \rho^2 z(\cos \phi + \sin \phi)$

(A) –8.2 (B) 12.3

(C) –12.3 (D) 0

56. If the scalar field $V = r^2 (1 + \cos \theta \sin \phi)$ then $\nabla^2 V$ is

(A) $1 + 2(1 - r^2) \cos \theta \sin \phi$

(B) $6 + 4 \cos \theta \sin \phi - \cot \theta \operatorname{cosec} \theta \sin \phi$

(C) $2 + 2(1 - r^2) \cos \theta \sin \phi$

(D) 0

57. $\nabla \ln \rho$ is equal to

(A) $\nabla \times (\phi \mathbf{u}_z)$ (B) $\nabla \times (z\mathbf{u}_\phi)$

(C) $\nabla \times (\rho \mathbf{u}_\phi)$ (D) $\nabla \times (\rho \mathbf{u}_z)$

58. If $\mathbf{r} = x\mathbf{u}_x + y\mathbf{u}_y + z\mathbf{u}_x$ then $(\mathbf{r} \cdot \nabla) r^2$ is equal to

(A) $2r^2$ (B) $3r^2$

(C) $4r^2$ (D) 0

59. If $\mathbf{r} = x\mathbf{u}_x + y\mathbf{u}_y + z\mathbf{u}_x$ is the position vector of point P(x, y, z) and $r = |\mathbf{r}|$ then $\nabla \cdot r^n \mathbf{r}$ is equal to

(A) nr^n (B) $(n + 3)r^n$

(C) $(n + 2)r^n$ (D) 0

60. If $\mathbf{F} = x^2 y \mathbf{u}_x - y\mathbf{u}_y$, the circulation of vector field $\mathbf{F}$ around closed path shown in fig. P8.1.60 is,

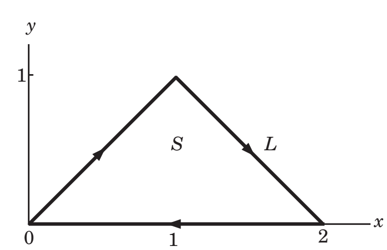

Fig. P8.1.60

(A) $\frac{7}{3}$ (B) $-\frac{7}{6}$

(C) $\frac{7}{6}$ (D) $-\frac{7}{3}$

61. If $\mathbf{A} = \rho \sin\phi\, \mathbf{u}_\rho + \rho^2 \mathbf{u}_\rho$, and L is the contour of fig. P8.1.61, then circulation $\oint_L \mathbf{A}.d\mathbf{L}$ is

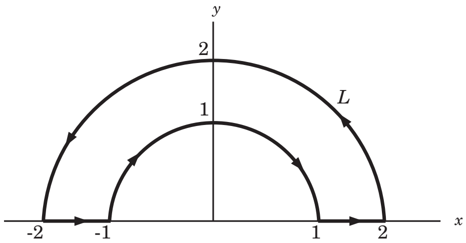

Fig. P8.1.61

(A) $7\pi + 2$ (B) $7\pi - 2$

(C) 7π (D) 0

62. The surface integral of vector

$$\mathbf{F} = 2\rho^2 z^2\, \mathbf{u}_\rho + \rho \cos^2\phi\, \mathbf{u}_z$$

over the region defined by $2 \le \rho \le 5$, $-1 < z < 1$, $0 < \phi < 2\pi$ is

(A) 44 (B) 176

(C) 88 (D) 352

63. If $\mathbf{D} = xy\mathbf{u}_x + yz\mathbf{u}_y + zx\mathbf{u}_z$, then the value of $\oiint \mathbf{A} \cdot d\mathbf{S}$ is, where S is the surface of the cube defined by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$

(A) 0.5 (B) 3

(C) 0 (D) 1.5

64. If $\mathbf{D} = 2\rho z \mathbf{u}_\rho + 3z \sin\phi\, \mathbf{u}_\rho - 4\rho \cos\phi\, \mathbf{u}_z$ and S is the surface of the wedge $0 < \rho < 2$, $\theta < \phi < 45°$, $0 < z < 5$, then the surface integral of $\mathbf{D}$ is

(A) 24.89 (B) 131.57

(C) 63.26 (D) 0

65. If the vector field

$$\mathbf{F} = (\alpha xy + \beta z^3)\mathbf{u}_x + (3x^2 - \gamma z)\mathbf{u}_y + (3\,xz^2 - y)\mathbf{u}_z$$

is irrotational, the value of α , β and γ is

(A) $\alpha = \beta = \gamma = 1$ (B) $\alpha = \beta = 1, \quad \gamma = 0$

(C) $\alpha = 0, \quad \beta = \gamma = 1$ (D) $\alpha = \beta = \gamma = 0$

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SOLUTIONS

1. (D) $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

$= \sqrt{(4-2)^2 + (-6-3)^2 + (3-(-1))^2} = \sqrt{4 + 81 + 16} = \sqrt{101}$

2. (A) $\mathbf{R}_{AB} = \mathbf{R}_B - \mathbf{R}_A$

$= (3\mathbf{u}_x + 0\mathbf{u}_y + 2\mathbf{u}_z) - (5\mathbf{u}_x - \mathbf{u}_y + 0\mathbf{u}_z)$

$= -2\mathbf{u}_x + \mathbf{u}_y + 2\mathbf{u}_z$

$|\mathbf{R}_{AB}| = \sqrt{2^2 + 1 + 2^2} = 3$

$\mathbf{u}_R = -\frac{2}{3}\mathbf{u}_x + \frac{1}{3}\mathbf{u}_y + \frac{2}{3}\mathbf{u}_z$

3. (C) The component of $\mathbf{F}$ parallel to $\mathbf{G}$ is

$= \frac{\mathbf{F}\cdot\mathbf{G}}{G^2}\mathbf{G} = \frac{(10, -6, 5)\cdot(0.1, 0.2, 0.3)}{0.1^2 + 0.2^2 + 0.3^2}(0.1, 0.2, 0.3)$

$= 9.3(0.1, 0.2, 0.3) = (0.93, 1.86, 2.79)$

4. (C) The vector component of $\mathbf{F}$ perpendicular to $\mathbf{G}$ is

$= \mathbf{F} - \frac{\mathbf{F}\cdot\mathbf{G}}{G^2}\mathbf{G} = (3, 2, 1) - \frac{(3, 2, 1)\cdot(4, 4, -2)}{4^2 + 4^2 + 2^2}(4, 4, -2)$

$= (3, 2, 1) - (2, 2, -1) = (1, 0, 2) = \mathbf{u}_x + 2\mathbf{u}_z$

5. (C) $\mathbf{R} = 3\mathbf{u}_x + 4\mathbf{M} - \mathbf{N}$

$= 3\mathbf{u}_x + 4(2\mathbf{u}_x + 3\mathbf{u}_y - 4\mathbf{u}_z) - (-4\mathbf{u}_x + 4\mathbf{u}_y + 3\mathbf{u}_z)$

$= 15\mathbf{u}_x + 8\mathbf{u}_y - 19\mathbf{u}_z$

$|\mathbf{R}| = \sqrt{15^2 + 8^2 + 19^2} = 25.5 = 25.5$

6. (B) $\mathbf{R} = -\mathbf{M} + 2\mathbf{N}$

$= -(8\mathbf{u}_x + 4\mathbf{u}_y - 8\mathbf{u}_z) + 2(8\mathbf{u}_x + 6\mathbf{u}_y - 2\mathbf{u}_z)$

$= 8\mathbf{u}_x + 8\mathbf{u}_y + 4\mathbf{u}_z$

$\mathbf{u}_R = \frac{8\mathbf{u}_x + 8\mathbf{u}_y + 4\mathbf{u}_z}{\sqrt{8^2 + 8^2 + 4^2}}$

$= \frac{2}{3}\mathbf{u}_x + \frac{2}{3}\mathbf{u}_y + \frac{1}{3}\mathbf{u}_z = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

$-\mathbf{M} + 2\mathbf{N} = -(8, 4, -8) + 2(8, 6, -2) = (8, 8, 4)$

$\mathbf{u}_R = \frac{(8, 8, 4)}{\sqrt{8^2 + 8^2 + 4^2}} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

$\mathbf{u}_R = \frac{2}{3}\mathbf{u}_x + \frac{2}{3}\mathbf{u}_y + \frac{1}{3}\mathbf{u}_z$

7. (C) Mid point is $\left(\frac{1+7}{2`}, \frac{-6-2}{2}, \frac{4+0}{2}\right) = (4, -4, 2)$

$\mathbf{u}_R = \frac{(4, -4, 2)}{\sqrt{4^2 + 4^2 + 2^2}} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$

$= \frac{2}{3}\mathbf{u}_x - \frac{2}{3}\mathbf{u}_y + \frac{1}{3}\mathbf{u}_z$

8. (A) $\mathbf{G} = 24(1)(2)\mathbf{u}_x + 12(1+2)\mathbf{u}_y + 18(-1)^2\mathbf{u}_z$

$= 48\mathbf{u}_x + 36\mathbf{u}_y + 18\mathbf{u}_z$

9. (A) $\mathbf{A} = (6, -2, -4)$, $\mathbf{B} = k\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$

$|\mathbf{B} - \mathbf{A}| = 10$

$\left(6 - \frac{2}{3}k\right)^2 + \left(-2 + \frac{2}{3}k\right)^2 + \left(-4 - \frac{1}{3}k\right)^2 = 100$

$k^2 - 8k - 44 = 0 \quad \Rightarrow \quad k = 11.75,$

$B = 11.75\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$

$= (7.83, -7.83, -3.92)$

10. (D) $\mathbf{G} = \frac{13}{(-2)^2 + (3)^2}(3\mathbf{u}_x + 4\mathbf{u}_y - 2\mathbf{u}_z)$

$= 3\mathbf{u}_x + 4\mathbf{u}_y - 2\mathbf{u}_z$

11. (A) Let θ be the angle between $\mathbf{F}$ and $\mathbf{u}_x$,

Magnitude of $\mathbf{F}$ is $|\mathbf{F}| = \sqrt{y^2 + z^2 + x^2}$

$\mathbf{F}\cdot\mathbf{u}_x = (\mathbf{F})(1)\cos\theta = y$

$\cos\theta = \frac{y}{\sqrt{y^2 + z^2 + x^2}} = \frac{2}{\sqrt{2^2 + 2^2}} = \frac{1}{\sqrt{2}}$

$\theta = 45°$

12. (D) $|\mathbf{G}| = 60$

$\sqrt{(12xy)^2 + (6(x^2+2))^2 + (18z^2)^2} = 60$

$\Rightarrow \quad 144x^2y^2 + 36(x^4 + 4x^2 + 4) + 324z^4 = 3600$

$\Rightarrow \quad 4x^2y^2 + (x^4 + 4x^2 + 4) + 9z^4 = 100$

$\Rightarrow \quad 4x^2y^2 + x^4 + 9z^4 + 4x^2 = 96$

13. (D) For $E_y = 0$, $2y\sin 2x = 0 \quad \Rightarrow \quad y = 0$

$\sin 2x = 0, \Rightarrow \quad 2x = 0, \ \pi, 3\pi, \Rightarrow \quad x = 0, \ \frac{3\pi}{2}$

Hence (D) is correct.

14. (A)

$\mathbf{E} = y(6zy\cos 2x\,\mathbf{u}_x + 4x\sin 2x\,\mathbf{u}_y + y\sin 2x\,\mathbf{u}_z)$

Hence in plane $y = 0$, $\quad E = 0$.

15. (C) $\mathbf{R} = \mathbf{F} - \mathbf{G}$

$= (-10\mathbf{u}_x + 20x(y-1)\mathbf{u}_y) - (2x^2y\mathbf{u}_x - 4\mathbf{u}_y + 2\mathbf{u}_z)$

At P(2, 3, – 4),

$\mathbf{R} = \mathbf{F} - \mathbf{G} = (-10\mathbf{u}_x + 80\mathbf{u}_y) - (24\mathbf{u}_x - 4\mathbf{u}_y + 2\mathbf{u}_z)$

$= -34\mathbf{u}_x + 84\mathbf{u}_y - 2\mathbf{u}_z$

$\mathbf{u}_R = \frac{-34\mathbf{u}_x + 84\mathbf{u}_y - \mathbf{u}_z}{\sqrt{34^2 + 84^2 + 2^2}}$

$= -0.37\mathbf{u}_x + 0.92\mathbf{u}_y - 0.02\mathbf{u}_z$

16. (A) At P(3, 4, –2)

$\mathbf{G} = \frac{25}{3^2 + 4^2}(3\mathbf{u}_x + 4\mathbf{u}_y) = 3\mathbf{u}_x + 4\mathbf{u}_y$

$\mathbf{u}_G = \frac{3\mathbf{u}_x + 4\mathbf{u}_y}{\sqrt{3^2 + 4^2}} = 0.6\mathbf{u}_x + 0.8\mathbf{u}_y$

17. (B) $\mathbf{F}\cdot\mathbf{u}_y = F_y = yz$

$I = \int_0^4\int_0^2 yz\,dz\,dx = \int_0^4\left(\int_0^2 yz\,dz\right)dx = \int_0^4 2y\,dx = 2(4)y = 8y$

At $y = 7$, $I = 8(7) = 56$

18. (B) Area $= \frac{1}{2}|\mathbf{R}_1 \times \mathbf{R}_2| = \frac{1}{2}\begin{bmatrix}\mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z\\ 4 & 3 & -2\\ 3 & -4 & -6\end{bmatrix}$

$= \mathbf{u}_x(-18-8) - \mathbf{u}_y(-24+6) + \mathbf{u}_z(-16-9)$

$= -26\mathbf{u}_x + 18\mathbf{u}_y - 25\mathbf{u}_z$

$|\mathbf{R}_1 \times \mathbf{R}_2| = \sqrt{26^2 + 18^2 + 25^2} = 40.31$

area $= \frac{40.31}{2} = 20.15$

19. (B) $\mathbf{R}_{BA} = (0, 1, 0) - (0.5\sqrt{3}, 0.5, 0) = (-0.5\sqrt{3}, 0.5, 0)$

$\mathbf{R}_{BC} = \left(\frac{0.5}{\sqrt{3}}, 0.5, \frac{\sqrt{2}}{3}\right) - (0.5\sqrt{3}, 0.5, 0) = \left(-\frac{1}{\sqrt{3}}, 0, \frac{\sqrt{2}}{3}\right)$

$\mathbf{R}_{BA} \times \mathbf{R}_{BC} = \begin{bmatrix}\mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z\\ 0.5\sqrt{3} & 0.5 & 0\\ \frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{3}\end{bmatrix}$

$= \mathbf{u}_x 0.5\sqrt{\frac{2}{3}} - \mathbf{u}_y(0.5\sqrt{2}) + \mathbf{u}_z\frac{0.5}{\sqrt{3}}$

$= 0.41\mathbf{u}_x - 0.71\mathbf{u}_y + 0.29\mathbf{u}_z$

The required unit vector is

$= \frac{0.41\mathbf{u}_x + 0.71\mathbf{u}_y + 0.29\mathbf{u}_z}{\sqrt{0.41^2 + 0.71^2 + 0.29^2}}$

$= 0.47\mathbf{u}_x + 0.81\mathbf{u}_y + 0.33\mathbf{u}_z$

20. (A) The non-unit vector in the required direction is

$= \frac{1}{2}(\mathbf{u}_{AN} + \mathbf{u}_{AM})$

$\mathbf{u}_{AN} = \frac{(-10, 8, 15)}{\sqrt{100 + 64 + 225}} = (-0.507, 0.406, 0.761)$

$$\mathbf{u}_{AM} = \frac{(20,\ 18,\ -10)}{\sqrt{400+324+100}} = (0.697,\ 0.627,\ -0.348)$$

$$\frac{1}{2}(\mathbf{u}_{AM} + \mathbf{u}_{AN})$$

$$= \frac{1}{2}\left[(0.697, 0.627, -0.348) + (-0.507, 0.406, 0.761)\right]$$

$$= (0.095,\ 0.516,\ 0.207)$$

$$\mathbf{u}_{bis} = \frac{(0.095,\ 0.516,\ 0.207)}{\sqrt{0.095^2 + 0.516^2 + 0.261^2}} = (0.168,\ 0.915,\ 0.367)$$

Hence (A) is correct.

21. (D) In cartesian coordinates

$$A\,(5\cos 70°,\ 5\sin 70°, -3) = A(1.71, 4.70, -3)$$

$$B\,(2\cos(-30°), 2\sin(-30°), -3) = B(1.73, -1, 1)$$

$$\mathbf{R}_{AB} = \mathbf{R}_B - \mathbf{R}_A = (1.73, -1, 1) - (1.71, 4.70, -3)$$

$$= (0.02, -5.70, 4)$$

$$\mathbf{u}_{AB} = \frac{(0.02, -5.70, 4)}{\sqrt{0.02^2 + 5.70^2 + 4^2}} = (0.003, -0.82, 0.57)$$

22. (A) $x = \rho\cos\phi, \quad y = \rho\sin\phi$

$$\mathbf{D} = \frac{\rho\cos\phi\,\mathbf{u}_x + \rho\sin\phi\,\mathbf{u}_y}{\rho^2\cos^2\phi + \rho^2\sin^2\phi} = \frac{1}{\rho}(\cos\phi\,\mathbf{u}_x + \sin\phi\,\mathbf{u}_y)$$

$$D_\rho = \mathbf{D}\cdot\mathbf{u}_\rho = \frac{1}{\rho}[\cos\phi(\mathbf{u}_x\cdot\mathbf{u}_\rho) + \sin\phi(\mathbf{u}_y\cdot\mathbf{u}_\rho)]$$

$$= \frac{1}{\rho}[\cos^2\phi + \sin^2\phi] = \frac{1}{\rho}$$

$$D_\phi = \mathbf{D}\cdot\mathbf{u}_\phi = \frac{1}{\rho}[\cos\phi(\mathbf{u}_x\cdot\mathbf{u}_\phi) + \sin\phi(\mathbf{u}_y\cdot u_\phi)]$$

$$= \frac{1}{\rho}[\cos\phi(-\sin\phi) + \sin\phi(\cos\phi)] = 0$$

Therefore $\mathbf{D} = \frac{1}{\rho}\mathbf{u}_\rho$

23. (B) $A(4\cos 40°,\ 4\sin 40°, -2) = A(3.06, 2.57, -2)$

$$B\,(5\cos(-110°), 5\sin(-110°), 2) = B\,(-1.71, -4.7, 2)$$

$$\mathbf{R}_{AB} = \mathbf{R}_B - \mathbf{R}_A = (-1.71, -4.7,\ 2) - (3.06, 2.57,\ -2)$$

$$= (-4.77, -7.3,\ 4)$$

24. (D) $Vol = \int_3^{4.5}\int_{100°}^{130°}\int_3^5 \rho\, d\rho\, d\phi\, dz = 6.28$

25. (C) Area is

$$= \mathbf{2}\int_{45°}^{135°}\int_2^4 \rho\, \boldsymbol{d}\rho \boldsymbol{d}\phi + \int_3^4\int_{45°}^{135°} \mathbf{2}\boldsymbol{d}\phi \boldsymbol{dz} + \int_3^4\int_{45°}^{135°} \mathbf{4}\boldsymbol{d}\phi \boldsymbol{dz} + \mathbf{2}\int_3^4\int_2^4 \boldsymbol{d}\rho \boldsymbol{dz}$$

$$= 2\left[\frac{\rho^2}{2}\right]_2^4\left(\frac{\pi}{2}\right) + (2)(1)\left(\frac{\pi}{2}\right) = 32.27$$

26. (A) $A(\rho = 3,\ \phi = 100°,\ z = 3) = A(-0.52, 2.95,\ 3)$

$$B(\rho = 5,\quad \phi = 130°,\quad z = 4.5) = B(-3.21, 3.83,\ 4.5)$$

$$\text{length} = |\mathbf{B} - \mathbf{A}|$$

$$\mathbf{B} - \mathbf{A} = (-3.21, 3.83, 4.5) - (-0.52, 2.95,\ 3)$$

$$= (-2.69,\ 0.88,\ 1.5)$$

$$|\mathbf{B} - \mathbf{A}| = |(-2.69,\ 0.88, 1.5)| = \sqrt{2.69^2 + 0.88^2 + 1.5^2} = 3.21$$

27. (C) At $P\left(2, \frac{\pi}{3}, 0\right)$, $\mathbf{H} = 0.5\mathbf{u}_\phi + 8\mathbf{u}_z$

$$\mathbf{u}_x = \cos\phi\,\mathbf{u}_\rho - \sin\phi\,\mathbf{u}_\phi = \frac{1}{2}(\mathbf{u}_\rho - \sqrt{3}\mathbf{u}_\phi)$$

$$\mathbf{H}\cdot\mathbf{u}_x = (0.5)\left(-\frac{\sqrt{3}}{2}\right) = -0.433$$

28. (A) $\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$

At P (–2, 6, 3)

$$\mathbf{A} = 6\mathbf{u}_x + \mathbf{u}_y,\ \phi = \tan^{-1}\left(\frac{6}{-2}\right) = 108.43°$$

$$\cos\phi = -0.316,\ \sin\phi = 0.948$$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} -0.316 & 0.948 & 0 \\ -0.948 & -0.316 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

$$A_\rho = 6\,(-0.316) + 0.948 = -0.949,$$

$$A_\phi = 6(-0.948) - 0.316 = -6.008,\ A_z = 0$$

Hence (A) is correct option.

29.(B) At P (–3, 4, 0)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{3^2 + 4^2 + 0^2} = 5$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} = \frac{\pi}{2}$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{4}{-3} = 126.87°$$

$$\mathbf{B} = 2\mathbf{u}_r + \mathbf{u}_\phi$$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \cos\theta\cos\phi & \cos\phi \\ \cos\theta & -\sin\theta & 1 \end{bmatrix}\begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$= \begin{bmatrix} -0.6 & 0 & -0.8 \\ 0.8 & 0 & -0.6 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$B_x = 2(-0.6) + 0.8 = -2$$

$$B_y = 2(0.2) - 0.6 = 1$$

$$B_z = 0$$

Along 3, $C_3 = \int_0^2 \rho \cos\phi \, d\rho \Big|_{\phi=60°} = \frac{\rho^2}{2} = -1$

$\int_L \mathbf{A} \cdot d\mathbf{L} = C_1 + C_2 + C_3 = 1$

38. (A) $\nabla f = \mathbf{u}_x \frac{\partial f}{\partial x} + \mathbf{u}_y \frac{\partial f}{\partial y} + \mathbf{u}_z \frac{\partial f}{\partial z}$

$= y(y+z)\mathbf{u}_x + x(2y+z)\mathbf{u}_y + xy\mathbf{u}_z$

39. (C) $\nabla f = \mathbf{u}_\rho \frac{\partial f}{\partial x} + \mathbf{u}_\phi \frac{1}{\rho}\frac{\partial f}{\partial y} + \mathbf{u}_z \frac{\partial f}{\partial z}$

$= 2\rho^2 z \cos 2\phi \, \mathbf{u}_\rho - 2\rho z \sin 2\phi \, \mathbf{u}_\phi + \rho^2 \cos 2\phi \, \mathbf{u}_z$

At P (1, 45°, 2), $\nabla f = -4\mathbf{u}_\phi$

40. (B) $r\sin\theta\cos\phi = x$, $r\sin\theta\sin\phi = y$, $r\cos\theta = z$

$G = r^3 \sin 2\theta \sin 2\phi \sin\theta$

$= r^3(2\sin\theta\cos\theta)(2\sin\phi\cos\phi)\sin\theta$

$= 4(r\sin\theta\cos\phi)(r\sin\theta\sin\phi)(r\cos\theta) = 4xyz$

$\nabla G = \mathbf{u}_x \frac{\partial(4xyz)}{\partial x} + \mathbf{u}_y \frac{\partial(4xyz)}{\partial x} + \mathbf{u}_z \frac{\partial(4xyz)}{\partial z}$

$= 4yz\mathbf{u}_x + 4xz\mathbf{u}_y + 4xy\mathbf{u}_z$

At P $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\nabla G = \mathbf{u}_x + \mathbf{u}_y + \mathbf{u}_z$

41. (C) $\nabla\Phi = (y+z)\mathbf{u}_x + (x+z)\mathbf{u}_y + (x+y)\mathbf{u}_z$

At P(3, –3, – 3),

$\nabla\Phi = -6\mathbf{u}_x$, $\mathbf{R}_{PQ} = (4, -1, -1) - (3, -3, -3) = (1, 2, 2)$

$\nabla\Phi \cdot \mathbf{u}_R = \frac{(-6\mathbf{u}_x)\cdot(\mathbf{u}_x + 2\mathbf{u}_y + 2\mathbf{u}_z)}{3} = -2$

42. (C) $\nabla T = \mathbf{u}_x \frac{\partial T}{\partial x} + \mathbf{u}_y \frac{\partial T}{\partial y} + \mathbf{u}_z \frac{\partial T}{\partial z}$

$= 2x\mathbf{u}_x + 2y\mathbf{u}_z - 4z\mathbf{u}_z$

At P(2, 2, 1), $\nabla T = 4\mathbf{u}_x + 4\mathbf{u}_y - 4\mathbf{u}_z$

43. (A) Let $f = x^2 y + z - 3$, $g = x\ln z - y^2 + 4$

$\nabla f = 2xy\mathbf{u}_x + x^2\mathbf{u}_y + \mathbf{u}_z$

$\nabla g = \ln z \, \mathbf{u}_x - 2y\mathbf{u}_y + \frac{x}{z}\mathbf{u}_z$

At point P (–1, 2, 1)

$\mathbf{u}_f = \frac{-4\mathbf{u}_x + \mathbf{u}_y + \mathbf{u}_z}{\sqrt{18}}$, $\mathbf{u}_g = \frac{-4\mathbf{u}_y - \mathbf{u}_z}{\sqrt{17}}$

$\cos\theta = \pm\, \mathbf{u}_f \cdot \mathbf{u}_g = \pm \frac{-5}{\sqrt{18\times 17}} = 0.286$

$\theta = \cos^{-1} 0.28 = 73.4°$

44. (D) $\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 + 4x + 0$

At P (1, –2, 3), $\nabla\cdot\mathbf{A} = 4$

45. (A)

$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\phi)}{\partial\phi}$

$= \frac{1}{r^2}(6r^2\cos\theta\cos\phi) + 0 + 0$

At P (1, 30°, 60°), $\nabla\cdot\mathbf{A} = 6(1)(\cos 30°)(\cos 60°) = 2.6$

46. (C) $\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial(A_\phi)}{\partial\theta} + \frac{\partial(A_z)}{\partial z}$

$= \frac{1}{\rho}\frac{\partial(\rho\rho z^2\cos\phi)}{\partial\rho} + \frac{\partial(z\sin^2\phi)}{\partial z} = 2z^2\cos\phi + \sin^2\phi$

47. (B) The flux is $\int_S \mathbf{D}\cdot d\mathbf{S}$, By divergence theorem

$\int_S \mathbf{D}\cdot d\mathbf{S} = \int_v \nabla\cdot\mathbf{D}\, dv$

$\nabla\cdot\mathbf{D} = \frac{1}{\rho}\frac{\partial(\rho\rho^2\cos^2\phi)}{\partial\rho} + \frac{1}{\rho}\frac{\partial(z\sin\phi)}{\partial\rho} = 3\rho\cos^2\phi + \frac{3}{\rho}\sin\phi$

$\int_v \nabla\cdot\mathbf{D}\, dv = \int_v \left(3\rho\cos^2\phi + \frac{z}{\rho}\sin\phi\right)\rho\, d\phi\, dz\, d\rho$

$= \int_0^3 dz \int_0^3 z\rho^2 d\rho \int_0^{2\pi}\cos^2\phi\, d\phi + \int_0^3 z dz \int_0^3 d\rho \int_0^{2\pi}\sin\phi\, d\phi = 81\pi$

48. (B) $\nabla\times\mathbf{A} = \begin{bmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix} = \begin{bmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & \sin xy & \cos^2 xz \end{bmatrix}$

$\mathbf{u}_x(0-0) - \mathbf{u}_y(2\cos xz\,(-\sin xz)z) + \mathbf{u}_z(y\cos xy - xe^{xy})$

$= z\sin 2xy\, \mathbf{u}_y + (y\cos xy - xe^{xy})\mathbf{u}_z$

49. (D) $\nabla\times\mathbf{A} = \frac{1}{\rho}\begin{bmatrix} \mathbf{u}_\rho & \mathbf{u}_\phi & \mathbf{u}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & A_\phi & A_z \end{bmatrix}$

$= \frac{1}{\rho}\begin{bmatrix} \mathbf{u}_\rho & \mathbf{u}_\phi & \mathbf{u}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ \rho z\sin\phi & 2\rho^2 z^2\cos\phi & 0 \end{bmatrix}$

$= \frac{1}{\rho}\mathbf{u}_\rho(-6\rho^2 z\cos\phi) - \mathbf{u}_\phi(-\rho\sin\phi)\frac{1}{\rho}\mathbf{u}_z(6\rho z^2\cos\phi - \rho z\cos\phi)$

At point P(5, 90°, 1), $\nabla\times\mathbf{A} = 5\mathbf{u}_\phi$

50. (C) $\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \mathbf{u}_r & r\mathbf{u}_\theta & r\sin\theta\mathbf{u}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$

$$= \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{u}_r & r\mathbf{u}_\theta & r\sin\theta\,\mathbf{u}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ r\cos\theta & -\sin\theta & 2r^3\sin^2\theta \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\mathbf{u}_r(2r^3\sin\theta - 0) - \frac{1}{r\sin\theta}\mathbf{u}_\theta(6r^2\sin^2\theta - 0) + \frac{1}{r}\mathbf{u}_\phi(0 + r\sin\theta)$$

$$= 4r\cos\theta\,\mathbf{u}_r - 6r\sin\theta\,\mathbf{u}_\theta + \sin\theta\,\mathbf{u}_\phi$$

51. (A) $\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3y^2 - 2z) & (-2x^2z) & (x + 2y) \end{vmatrix}$

$$= \mathbf{u}_x(2 + 2x^2) - \mathbf{u}_y(1 - (-2)) + \mathbf{u}_z(-4xz - 6y)$$

$$\nabla \times \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2 + 2x^2) & 3 & -(4xz + 6y) \end{vmatrix}$$

$$= \mathbf{u}_x(-6) - \mathbf{u}_y(-4z) + \mathbf{u}_z(0) = -6\mathbf{u}_x + 4z\mathbf{u}_y$$

At P(−2, 3, −1),

$$\nabla \times \nabla \times \mathbf{A} = -6\mathbf{u}_x - 4\mathbf{u}_y = -(6\mathbf{u}_x + 4\mathbf{u}_y)$$

52. (D) $\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & -2xz \end{vmatrix}$

$$= -y^2\mathbf{u}_x + 2z\mathbf{u}_y - x^2\mathbf{u}_z$$

$$\nabla(\nabla \times \mathbf{A}) = 0$$

53. (D) $\nabla V = \mathbf{u}_x\frac{\partial V}{\partial x} + \mathbf{u}_y\frac{\partial V}{\partial y} + \mathbf{u}_z\frac{\partial V}{\partial z}$

$$= (z - 2xy)\mathbf{u}_x + (2yz^2 - x^2)\mathbf{u}_y + (x - 2y^2z)\mathbf{u}_z$$

$$\nabla\cdot(\nabla V) = \frac{\partial(z - 2xy)}{\partial x} + \frac{\partial(2yz^2 - x^2)}{\partial y} + \frac{\partial(x - 2y^2z)}{\partial z}$$

$$= -2y + 2z^2 - 2y^2 = 2(z^2 - y^2 - y)$$

54. (B) $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

$$= 2(y^2z^2 + x^2z^2 + x^2y^2) = 2(x^2y^2 + y^2z^2 + z^2x^2)$$

55. (A) $\nabla^2 V = \frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$

$$= 4z(\cos\phi + \sin\phi) - z(\cos\phi + \sin\phi) + 0 = 3z(\cos\phi + \sin\phi)$$

At P(3, 60°, −2), $\nabla^2 V = 3(-2)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = -8.2$

56.(B)

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

$$= 6(1 + \cos\theta\sin\phi) - \frac{2\sin\theta}{\sin\theta}\cos\theta\sin\phi - \frac{\cos\theta\sin\phi}{\sin^2\theta}$$

$$= 6 + 4\cos\theta\sin\phi - \cot\theta\,\text{cosec}\,\theta\sin\phi$$

57. (A) $\rho = \sqrt{x^2 + y^2}$

$$\nabla\ln\rho = \mathbf{u}_x\frac{\partial\ln\rho}{\partial x} + \mathbf{u}_y\frac{\partial\ln\rho}{\partial y} + \mathbf{u}_z\frac{\partial\ln\rho}{\partial z} = \frac{x}{\rho^2}\mathbf{u}_x + \frac{y}{\rho^2}\mathbf{u}_y$$

$$\nabla \times \phi\mathbf{u}_z = \nabla \times \tan^{-1}\frac{y}{x}\mathbf{u}_z = \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \tan^{-1}\frac{y}{x} \end{vmatrix}$$

$$= \frac{x}{x^2 + y^2}\mathbf{u}_x + \frac{y}{x^2 + y^2}\mathbf{u}_y = \frac{x}{\rho}\mathbf{u}_x + \frac{y}{\rho}\mathbf{u}_y$$

58. (A) $(\mathbf{r}\cdot\nabla)r^2 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)$

$$= x(2x) + y(2y) + z(2z) = 2(x^2 + y^2 + z^2) = 2r^2$$

59. (B) $\nabla\cdot r^n\,\mathbf{r} = \frac{\partial(xr^n)}{\partial x} + \frac{\partial(yr^n)}{\partial y} + \frac{\partial(zr^n)}{\partial z}$

where $r^n = (x^2 + y^2 + z^2)^{\frac{n}{2}}$

$$\nabla\cdot r^n\,\mathbf{r} = 2x^2\left(\frac{n}{2}\right)(x^2 + y^2 + z^2)^{\frac{n}{2}-1}$$

$$+ 2y^2\left(\frac{n}{2}\right)(x^2 + y^2 + z^2)^{\frac{n}{2}-1} + 2z^2\left(\frac{n}{2}\right)(x^2 + y^2 + z^2)^{\frac{n}{2}-1}$$

$$+ r^n + r^n + r^n$$

$$= n(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{\frac{n}{2}-1} + 3r^n$$

$$= nr^n + 3r^n = (n + 3)r^n$$

60. (C) By Stokes theorem $\oint_L \mathbf{F}\cdot d\mathbf{L} = \oint_S (\nabla \times \mathbf{F})\cdot d\mathbf{S}$

$$\nabla \times \mathbf{F} = -x^2\mathbf{u}_z$$

$$d\mathbf{S} = dxdy\,(-\mathbf{u}_z)$$

$$\oint_S (\nabla \times \mathbf{F})\cdot d\mathbf{S} = -\iint(-x^2)dxdy$$

$$= \int_0^1\int_0^x x^2dydx + \int_1^2\int_0^{-x+2} x^2dydx = \int_0^1 x^3dx + \int_1^2 x^2(2 - x)dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_1^2$$

$$= \frac{1}{4} + \left[\frac{16}{3} - 4\right] - \left[\frac{2}{3} - \frac{1}{4}\right] = \frac{1}{4} + \frac{14}{3} - 4 + \frac{1}{4} = \frac{7}{6}$$

61. (C) $\oint_C \mathbf{A}\cdot d\mathbf{L} = \left(\int_{ab} + \int_{bc} + \int_{cd} + \int_{da}\right)\mathbf{A}\cdot d\mathbf{L}$

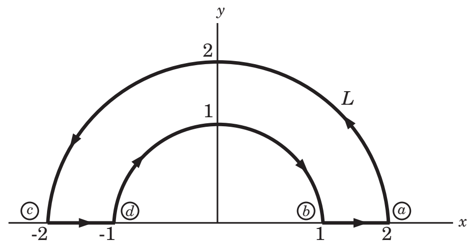

Fig. S8.1.61

Along ab, $d\phi = 0$, $\phi = 0$,

$\mathbf{A}\cdot d\mathbf{L} = 0$, $\int_a^b \mathbf{A}\cdot d\mathbf{L} = 0$

Along bc, $d\rho = 0$, $\mathbf{A}\cdot d\mathbf{L} = \rho^3 d\phi$

$$\int_b^c \mathbf{A}\cdot d\mathbf{L} = \int_0^\pi \rho^3 d\phi = (2)^3\pi = 8\pi$$

Along cd, $d\phi = 0$, $\phi = \pi$, $\mathbf{A}\cdot d\mathbf{L} = 0$,

$$\int_c^d \mathbf{A}\cdot d\mathbf{L} = 0$$

Along da, $d\rho = 0$, $\mathbf{A}\cdot d\mathbf{L} = \rho^3 d\phi$

$$\int_d^a \mathbf{A}\cdot d\mathbf{L} = \rho^3\int_\pi^0 d\phi = (1)^3(-\pi) = -\pi$$

$$\int \mathbf{A}\cdot d\mathbf{L} = 0 + 8\pi + 0 - \pi = 7\pi$$

62. (B) Using divergence theorem

$$\oint \mathbf{F}\cdot d\mathbf{S} = \int_v \nabla\cdot\mathbf{F}dv$$

$$\nabla\cdot\mathbf{F} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(2\rho^2 z^2) = 4z^2 = 4z^2$$

$$\int_v \nabla\cdot\mathbf{F}\,dv = \iiint 4z^2\rho d\rho d\phi dz = 4\int_{-1}^{1} z^2 dz\int_2^5 \rho d\rho\int_0^{2\pi} d\phi = 176$$

63. (D) $\oint_\mathbf{S} \mathbf{A}\cdot d\mathbf{S} = \oint_v (\nabla\cdot\mathbf{A})dv$

$$\nabla\cdot\mathbf{A} = \frac{\partial(xy)}{\partial x} + \frac{\partial(yz)}{\partial y} + \frac{\partial(zx)}{\partial z}x = y + z + x$$

$$\oint_\mathbf{S} \mathbf{A}\cdot d\mathbf{S} = \int_0^1\int_0^1\int_0^1 (y + z + x)dxdydz$$

$$= 3\left(\int_0^1 xdx\int_0^1 dy\int_0^1 dz\right) = 3\left(\frac{1}{2}\right) = 1.5$$

64. (B) $\nabla\cdot\mathbf{D} = \frac{1}{\rho}\frac{\partial(\rho D_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial(D_\phi)}{\partial\theta} + \frac{\partial(D_z)}{\partial z}$

$$= 4z + \frac{3}{\rho}z\cos\phi$$

$$\oint_\mathbf{S} \mathbf{D}\cdot d\mathbf{S} = \oint_v (\nabla\cdot\mathbf{D})dv$$

$$\oint \mathbf{D}\cdot d\mathbf{S} = \iiint\left(4z + \frac{3z}{\rho}\cos\phi\right)\rho d\rho d\phi dz$$

$$= 4\int_0^2 \rho d\rho\int_0^5 zdz\int_0^{45^\circ} d\phi + 3\int_0^2 d\rho\int_0^5 zdz\int_0^{45^\circ}\cos\phi\,d\phi$$

$$= 4\left(\frac{4}{2}\right)\left(\frac{25}{2}\right)\left(\frac{\pi}{4}\right) + 3(2)\left(\frac{25}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = 131.57$$

65. (A) $\nabla\times\mathbf{F} = \begin{bmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha x + \beta z^2 & 3x^2 - \gamma z & 3xz^2 - y \end{bmatrix}$

$$= (-1 + \gamma)\mathbf{u}_x + (3\beta z^2 - 3z^2)\mathbf{u}_y + (6x - \alpha x)\mathbf{u}_z$$

If $\mathbf{F}$ is irrotational, $\nabla\times\mathbf{F} = 0$

i.e. $\alpha = 1 = \beta = \gamma$.

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