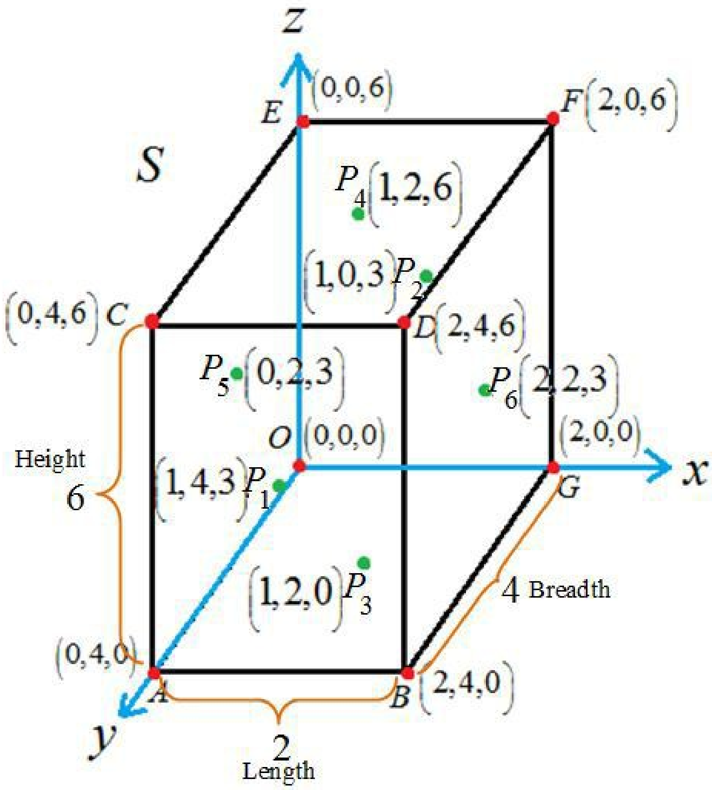
Exercise 16.7

Chapter 16 Vector Calculus Exercise 16.7 1E

Use Riemann sum to approximate the integral $\iint_S e^{-0.1(x+y+z)}dS$.

Here, $f(x,y,z)=e^{-0.1(x+y+z)}$ and S be the surface of the box bounded by the planes $x=0, x=2, y=0, y=4, z=0,$ and $z=6.$

Consider the following figure, which shows the surface of the box.



The area of the face $ABCD$ in the plane $y=4$ is,

$$\begin{aligned}\Delta S_1 &= (\text{length})(\text{height})\\ &= (2)(6)\\ &= 12\end{aligned}$$

The midpoint of the face $ABCD$ is $P_1=(1,4,3)$.

Find the function value $f(x,y,z)$ at $P_1=(1,4,3)$.

$$\begin{aligned}f(x,y,z) &= e^{-0.1(x+y+z)}\\ f(1,4,3) &= e^{-0.1(1+4+3)}\\ &= e^{-0.1(8)}\\ &= e^{-0.8}\end{aligned}$$

$$f(P_1)\approx 0.4493289641$$

The area of the face $OEFG$ in the plane $y=0$ is,

$$\begin{aligned}\Delta S_2 &= (\text{length})(\text{height}) \\ &= (2)(6) \\ &= 12\end{aligned}$$

The midpoint of the face $OEFG$ is $P_2=(1,0,3)$.

Find the function value $f(x,y,z)$ at $P_2=(1,0,3)$.

$$\begin{aligned}f(x,y,z) &= e^{-0.1(x+y+z)} \\ f(1,0,3) &= e^{-0.1(1+0+3)} \\ &= e^{-0.1(4)} \\ &= e^{-0.4} \\ f(P_2) &\approx 0.6703200460\end{aligned}$$

The area of the face $OABG$ in the plane $z=0$ is,

$$\begin{aligned}\Delta S_3 &= (\text{length})(\text{Breadth}) \\ &= (2)(4) \\ &= 8\end{aligned}$$

The midpoint of the face $OEFG$ is $P_3=(1,2,0)$.

Find the function value $f(x,y,z)$ at $P_3=(1,2,0)$.

$$\begin{aligned}f(x,y,z) &= e^{-0.1(x+y+z)} \\ f(1,2,0) &= e^{-0.1(1+2+0)} \\ &= e^{-0.1(3)} \\ &= e^{-0.3} \\ f(P_3) &\approx 0.7408182207\end{aligned}$$

The area of the face $CDFE$ in the plane $z=6$ is,

$$\begin{aligned}\Delta S_4 &= (\text{length})(\text{Breadth}) \\ &= (2)(4) \\ &= 8\end{aligned}$$

The midpoint of the face $OEFG$ is $P_4=(1,2,6)$.

Find the function value $f(x,y,z)$ at $P_4=(1,2,6)$.

$$\begin{aligned}f(x,y,z) &= e^{-0.1(x+y+z)} \\ f(1,2,6) &= e^{-0.1(1+2+6)} \\ &= e^{-0.1(9)} \\ &= e^{-0.9} \\ f(P_4) &\approx 0.4065696597\end{aligned}$$

The area of the face $OACE$ in the plane $x=0$ is,

$$\begin{aligned}\Delta S_5 &= (\text{Breadth})(\text{Height})\\ &= (4)(6)\\ &= 24\end{aligned}$$

The midpoint of the face $OEFG$ is $P_5 = (0,2,3)$.

Find the function value $f(x,y,z)$ at $P_5 = (0,2,3)$.

$$\begin{aligned}f(x,y,z) &= e^{-0.1(x+y+z)}\\ f(0,2,3) &= e^{-0.1(0+2+3)}\\ &= e^{-0.1(5)}\\ &= e^{-0.5}\end{aligned}$$

$$f(P_5) \approx 0.6065306597$$

The area of the face $BGFD$ in the plane $x=2$ is,

$$\begin{aligned}\Delta S_6 &= (\text{Breadth})(\text{Height})\\ &= (4)(6)\\ &= 24\end{aligned}$$

The midpoint of the face $OEFG$ is $P_6 = (2,2,3)$.

Find the function value $f(x,y,z)$ at $P_6 = (2,2,3)$.

$$\begin{aligned}f(x,y,z) &= e^{-0.1(x+y+z)}\\ f(2,2,3) &= e^{-0.1(2+2+3)}\\ &= e^{-0.1(7)}\\ &= e^{-0.7}\end{aligned}$$

$$f(P_6) \approx 0.4965853038$$

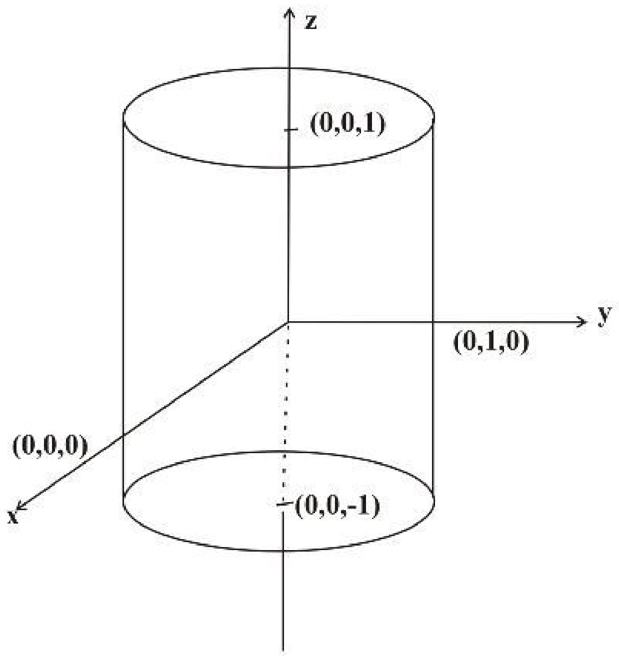
Evaluate the integral:

$$\begin{aligned}\iint_S e^{-0.1(x+y+z)}dS &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}\\ &= \Delta S_1(f(P_1)) + \Delta S_2(f(P_2)) + \Delta S_3(f(P_3))\\ &\quad + \Delta S_4(f(P_4)) + \Delta S_5(f(P_5)) + \Delta S_6(f(P_6))\end{aligned}$$

$$\begin{aligned}&\approx (12)(0.4493289641) + (12)(0.6703200460) + (8)(0.7408182207)\\ &\quad + (8)(0.4065696597) + (24)(0.6065306597) + (24)(0.4965853038)\\ &\approx 5.391947569 + 8.043840552 + 5.926545766\\ &\quad + 3.252557278 + 14.55673583 + 11.91804729\\ &\approx 49.08967429\\ &\approx 49.09\end{aligned}$$

Therefore, the approximate value of the integral $\iint_S e^{-0.1(x+y+z)}dS$ is $\boxed{49.09}$.

Chapter 16 Vector Calculus Exercise 16.7 2E



The surface S consists of cylinder $x^2+y^2=1$, $-1\le z\le 1$
We know if f is a function of three variables whose domain includes S, and if we divide S into patches S_{ij} with area ΔS_{ij} and P_{ij}^* a point in each path then using Riemann sum

$$\iint_S f(x,y,z)\,ds=\lim_{m,n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} f\left(P_{ij}^*\right)\Delta S_{ij}$$

Now if we divide the given cylinder S into four quarter cylinders and the top and bottom disks then a point in each quarter cylinder is (0, 1, 0), (0, -1, 0), (1, 0, 0) and (-1, 0, 0) and a point in top disk is (0, 0, 1) and a point in bottom disk is $(0,0,-1)$

Now the area of each quarter cylinder is:

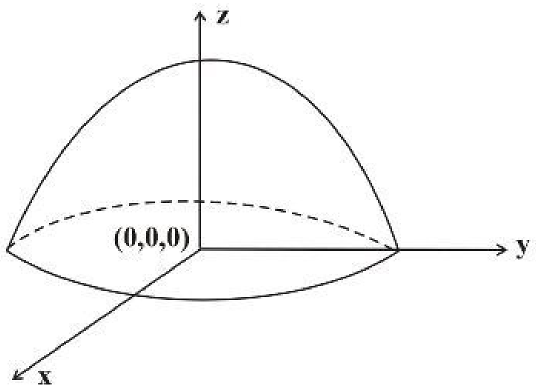
$$=\frac{1}{4}\times 2\pi(1)(1+2)$$
$$=\frac{3\pi}{2}$$

And area of each circular disk is $\pi(1)^2=\pi$

Therefore $\iint_S f(x,y,z)\,ds$

$$=f(1,0,0)\times\frac{3\pi}{2}+f(-1,0,0)\times\frac{3\pi}{2}+f(0,1,0)\times\frac{3\pi}{2}+$$
$$f(0,7,0)\times\frac{3\pi}{2}+f(0,0,1)\times\pi+f(0,0,-1)\pi$$
$$=2\times\frac{3\pi}{2}+2\times\frac{3\pi}{2}+3\times\frac{3\pi}{2}+3\times\frac{3\pi}{2}+4\times\pi+4\pi$$
$$=15\pi+8\pi$$
$$=\boxed{23\pi}$$

Chapter 16 Vector Calculus Exercise 16.7 3E



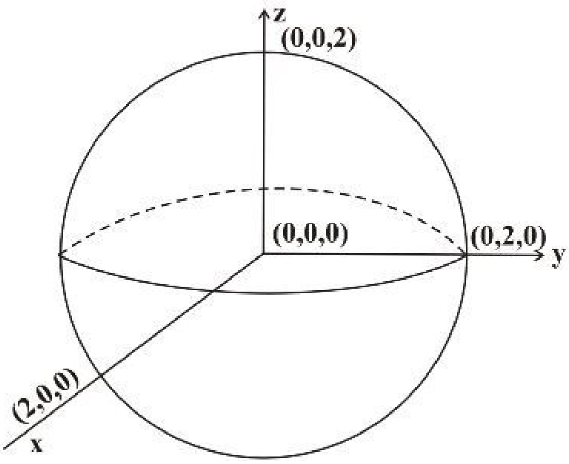
The given surface H is a hemisphere $x^2+y^2+z^2=50$
If we divide this hemisphere into four quarter hemispheres then a point in each quarter hemisphere is $(3,4,5),(3,-4,5),(-3,4,5),(-3,-4,5)$

And area of each patch is $\frac{1}{4}\times 2\pi\left(\sqrt{50}\right)^2$
$= 25\pi$

Then using Riemann sum

$$\begin{aligned}\iint_H f(x,y,z)ds &= f(3,4,5)\times 25\pi + f(3,-4,5)\times 25\pi \\ &+ f(-3,4,5)\times 25\pi + f(-3,-4,5)\times 25\pi \\ &= (7+8+9+12)\times 25\pi \\ &= 36\times 25\pi \\ &= \boxed{900\pi}\end{aligned}$$

Chapter 16 Vector Calculus Exercise 16.7 4E



If we divide the sphere $x^2+y^2+z^2=4$ into four quarter spheres and takes a point in each sphere as:

$(0,2,0)$, $(0,-2,0)$, $(2,0,0)$, $(-2,0,0)$

The area of each quarter is: $\frac{1}{4}\times 4\pi(2)^2$
$= 4\pi$

Then using Riemann sum we have

$$\begin{aligned}\iint_S f(x,y,z)ds &= f(0,2,0)4\pi + f(0,-2,0)4\pi + f(2,0,0)4\pi \\ &+ f(-2,0,0)4\pi \\ &= \left[g\left(\sqrt{0+4+0}\right)+g\left(\sqrt{0+4+0}\right)+g\left(\sqrt{4+0+0}\right)+g\left(\sqrt{4+0+0}\right)\right]4\pi \\ &= \left[g(2)+g(2)+g(2)+g(2)\right]4\pi \\ &= 4g(2)4\pi \\ &= 4(-5)(4\pi) \\ &= (-20)(4\pi) \\ &= \boxed{-80\pi}\end{aligned}$$

Chapter 16 Vector Calculus Exercise 16.7 5E

The cross product of $\mathbf{r}_u$ and $\mathbf{r}_v$ is calculated as,

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix}$$
$$= \mathbf{i}(1-(-2)) - \mathbf{j}(1-2) + \mathbf{k}(-1-1)$$
$$= 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$
$$= \langle 3,1,-2 \rangle$$

The modulus of $\mathbf{r}_u \times \mathbf{r}_v$ is,

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{3^2 + 1^2 + (-2)^2}$$
$$= \sqrt{14}$$

Determine $f(\mathbf{r}(u,v))$ is calculated as follows:

$$x + y + z = (u+v) + (u-v) + (1+2u+v)$$
$$= 4u + v + 1$$

The parameter domain is given by,

$$D = \{(u,v) \mid 0 \le u \le 2, 0 \le v \le 1\}$$

Write the surface integral as follows:

$$\iint_S f(x,y,z)\ dS = \iint_D f(\mathbf{r}(u,v)) |\mathbf{r}_u \times \mathbf{r}_v|\ dA$$

$$\iint_S (x+y+z)\ dS = \int_0^2 \int_0^1 (4u+v+1)\sqrt{14}\ dv\ du$$
$$= \sqrt{14} \int_0^2 \int_0^1 (4u+v+1)\ dv\ du$$

$$= \sqrt{14} \int_0^2 \left[(4u+1)v + \frac{1}{2}v^2 \right]_{v=0}^{v=1} du$$
$$= \sqrt{14} \int_0^2 4u + 1 + \frac{1}{2}\ du$$
$$= \sqrt{14} \int_0^2 4u + \frac{3}{2}\ du$$
$$= \sqrt{14} \left[2u^2 + \frac{3}{2}u \right]_{u=0}^{u=2}$$
$$= \sqrt{14} \left[2(2)^2 + \frac{3}{2}(2) - [0] \right]$$
$$= \sqrt{14}[8+3]$$
$$= 11\sqrt{14}$$

Therefore, $\iint_S (x+y+z)\ dS = \boxed{11\sqrt{14}}$.

Chapter 16 Vector Calculus Exercise 16.76E

Consider the surface

$$S = \left\{ (x,y,z) : x = u\cos v, y = u \sin v, z = u, 0 \le u \le 1, 0 \le v \le \frac{\pi}{2} \right\}$$

The parametric equation is

$$\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u\mathbf{k}$$

Recall that, if the surface S has a vector function

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

Then,

$$\iint_S f(x,y,z)\,dzdydx = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA \quad \ldots\ldots (1)$$

Evaluate $\mathbf{r}_u$,

Differentiate $\mathbf{r}$ with respect to u

$$\begin{aligned}\frac{\partial}{\partial u}\mathbf{r}(u,v) &= \frac{\partial}{\partial u}(u\cos v)\,\mathbf{i}+\frac{\partial}{\partial u}(u\sin v)\,\mathbf{j}+\frac{\partial}{\partial u}(u)\,\mathbf{k}\\ &= \cos v\mathbf{i}+\sin v\mathbf{j}+1\mathbf{k}\\ &= \cos v\mathbf{i}+\sin v\mathbf{j}+\mathbf{k}\end{aligned}$$

Therefore

$$\mathbf{r}_u = \cos v\mathbf{i}+\sin v\mathbf{j}+\mathbf{k}$$

Evaluate $\mathbf{r}_u$,

Differentiate $\mathbf{r}$ with respect to v

$$\begin{aligned}\frac{\partial}{\partial v}\mathbf{r}(u,v) &= \frac{\partial}{\partial v}(u\cos v)\,\mathbf{i}+\frac{\partial}{\partial v}(u\sin v)\,\mathbf{j}+\frac{\partial}{\partial v}(u)\,\mathbf{k}\\ &= -u\sin v\mathbf{i}+u\cos v\mathbf{j}+0\mathbf{k}\\ &= -u\sin v\mathbf{i}+u\cos v\mathbf{j}\end{aligned}$$

Therefore

$$\mathbf{r}_v = -u\sin v\mathbf{i}+u\cos v\mathbf{j}$$

Now, evaluate the vector $\mathbf{r}_u\times\mathbf{r}_v$

$$\begin{aligned}\mathbf{r}_u\times\mathbf{r}_v &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos v & \sin v & 1\\ -u\sin v & u\cos v & 0\end{vmatrix}\\ &= \mathbf{i}(0-u\cos v)-\mathbf{j}(0+u\sin v)+\mathbf{k}\left(u\cos^2 v+u\sin^2 v\right)\\ &= -u\cos v\mathbf{i}-u\sin v\mathbf{j}+u\mathbf{k} \qquad \text{Since } \cos^2\theta+\sin^2\theta=1\end{aligned}$$

Therefore,

$$\mathbf{r}_u\times\mathbf{r}_v = -u\cos v\mathbf{i}-u\sin v\mathbf{j}+u\mathbf{k}$$

Find the magnitude of the vector $\mathbf{r}_u\times\mathbf{r}_v$

$$\begin{aligned}\left|\mathbf{r}_u\times\mathbf{r}_v\right| &= \sqrt{(-u\cos v)^2+(-u\sin v)^2+u^2}\\ &= \sqrt{u^2\cos^2 v+u^2\sin^2 v+u^2}\\ &= \sqrt{u^2\left(\cos^2 v+\sin^2 v\right)+u^2}\end{aligned}$$

$$\begin{aligned}&= \sqrt{u^2+u^2} \qquad \text{Since } \cos^2\theta+\sin^2\theta=1\\ &= \sqrt{2u^2}\\ &= u\sqrt{2}\end{aligned}$$

Use equation (1) to evaluate the integral $\iint_S xyz\,dS$

Here $f(x,y,z)=xyz$, then

$$\begin{aligned}f(r(u,v)) &= (u\cos v)(u\sin v)(u)\\ &= u^3\cos v\sin v\end{aligned}$$

So the integral becomes

$$\begin{aligned}\iint_S xyz\,dS &= \iint_D f(\mathbf{r}(u,v))\left|\mathbf{r}_u\times\mathbf{r}_v\right|dA\\ &= \iint_D \left(u^3\cos v\sin v\right)\left(\sqrt{2}u\right)dA\\ &= \sqrt{2}\iint_D u^4\cos v\sin v\,dA\end{aligned}$$

Since the domain for (u,v) is

$D=\left\{(u,v):0\le u\le 1, 0\le v\le \frac{\pi}{2}\right\}$ The integral is

$$\iint_S xyz\,dS=\sqrt{2}\iint_D u^4\cos v\sin v\,dA$$
$$=\sqrt{2}\int_0^{\frac{\pi}{2}}\int_0^1\left(u^4\cos v\sin v\right)du\,dv$$
$$=\sqrt{2}\int_0^{\frac{\pi}{2}}\cos v\sin v\left(\frac{u^5}{5}\right)_0^1 dv$$
$$=\frac{\sqrt{2}}{10}\int_0^{\frac{\pi}{2}}2\cos v\sin v\,dv$$
$$=\frac{\sqrt{2}}{10}\int_0^{\frac{\pi}{2}}\sin 2v\,dv \qquad \text{Since } 2\sin\theta\cos\theta=\sin 2\theta$$
$$=\frac{\sqrt{2}}{10}\left(-\frac{\cos 2v}{2}\right)_0^{\frac{\pi}{2}}$$
$$=\frac{\sqrt{2}}{10}\left(-\frac{(-1)}{2}+\frac{1}{2}\right)$$
$$=\frac{\sqrt{2}}{10}$$

Therefore, the integral is

$$\boxed{\iint_S xyz\,dS=\frac{\sqrt{2}}{10}}$$

Chapter 16 Vector Calculus Exercise 16.7 7E

Consider the surface of helicoid

$\mathbf{r}(u,v)=\langle u\cos v+u\sin v+v\rangle, 0\le u\le 1, 0\le v\le \pi$

Recall that, if the surface S has a vector function

$\mathbf{r}(u,v)=x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k}$

Then,

$$\iint_S f(x,y,z)\,dzdydx=\iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u\times\mathbf{r}_v|\,dA \quad \ldots\ldots (1)$$

Evaluate $\mathbf{r}_u$,

Differentiate $\mathbf{r}$ with respect to u

$$\frac{\partial}{\partial u}\mathbf{r}(u,v)=\left\langle\frac{\partial}{\partial u}(u\cos v),\frac{\partial}{\partial u}(u\sin v),\frac{\partial}{\partial u}(v)\right\rangle$$
$$\frac{\partial}{\partial u}\mathbf{r}(u,v)=\langle\cos v,\sin v,0\rangle$$

Therefore

$\mathbf{r}_u=\langle\cos v,\sin v,0\rangle$

Evaluate $\mathbf{r}_u$,

Differentiate $\mathbf{r}$ with respect to v

$$\frac{\partial}{\partial v}\mathbf{r}(u,v)=\left\langle\frac{\partial}{\partial v}(u\cos v),\frac{\partial}{\partial v}(u\sin v),\frac{\partial}{\partial v}(v)\right\rangle$$
$$=\langle -u\sin v,u\cos v,1\rangle$$

Therefore

$\mathbf{r}_v=\langle -u\sin v,u\cos v,1\rangle$

Now, evaluate the vector $\mathbf{r}_u \times \mathbf{r}_v$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}$$

$$= \mathbf{i}(\sin v) - \mathbf{j}(\cos v) + \mathbf{k}\left(u\cos^2 v + u\sin^2 v\right)$$

$$= \sin v\mathbf{i} - \cos v\mathbf{j} + u\mathbf{k} \qquad \text{Since } \cos^2\theta + \sin^2\theta = 1$$

Therefore,

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u \rangle$$

Find the magnitude of the vector $\mathbf{r}_u \times \mathbf{r}_v$

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{(\sin v)^2 + (-\cos v)^2 + u^2}$$

$$= \sqrt{\sin^2 v + \cos^2 v + u^2}$$

$$= \sqrt{1 + u^2} \qquad \text{Since } \cos^2\theta + \sin^2\theta = 1$$

Use equation (1) to evaluate the integral $\iint_S y\,dS$

Here $f(x, y, z) = y$, then

$$f(r(u, v)) = u\sin v$$

So the integral becomes

$$\iint_S y\,dS = \iint_D f(\mathbf{r}(u, v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

$$= \iint_D (u\sin v)\left(\sqrt{1 + u^2}\right)dA$$

Since the domain for (u, v) is

$D = \{(u, v) : 0 \le u \le 1, 0 \le v \le \pi\}$ The integral is

$$\iint_S y\,dS = \iint_D \sin v\left(u\sqrt{1 + u^2}\right)dA$$

$$= \int_0^\pi \int_0^1 \sin v\left(u\sqrt{1 + u^2}\right)du\,dv$$

First solve the integral $\int_0^1 u\sqrt{1+u^2}\,du$, substitute $1+u^2=t$, then $du=\frac{1}{2}dt$

The limits of the integration changes

If $u=0 \text{ then } t=1$

If $u=1 \text{ then } t=2$

Integral becomes

$$\int_0^1 u\sqrt{1+u^2}\,du=\int_1^2 \sqrt{t}\frac{dt}{2}$$

$$=\frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right)_1^2$$

$$=\frac{1}{3}\left(2\sqrt{2}-1\right)$$

Substitute $\int_0^1 u\sqrt{1+u^2}\,du=\frac{1}{3}\left(2\sqrt{2}-1\right)$ in $\int_0^\pi\int_0^1 \sin v\left(u\sqrt{1+u^2}\right)dudv$,

$$\int_0^\pi\int_0^1 \sin v\left(u\sqrt{1+u^2}\right)dudv=\int_0^\pi \sin v dv\int_0^1\left(u\sqrt{1+u^2}\right)du$$

$$=\frac{1}{3}\left(2\sqrt{2}-1\right)\int_0^\pi \sin v dv$$

$$=\frac{1}{3}\left(2\sqrt{2}-1\right)\left(-\cos v\right)_0^\pi$$

$$=\frac{1}{3}\left(2\sqrt{2}-1\right)\left(-(-1)+1\right)$$

$$=\frac{2}{3}\left(2\sqrt{2}-1\right)$$

Therefore, the integral is

$$\boxed{\iint_S y\,dS=\frac{2}{3}\left(2\sqrt{2}-1\right)}$$

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Consider,

$$\iint_S \left(x^2+y^2\right)dS.$$

Here, S is the surface with vector equation $\mathbf{r}(u,v)=\left\langle 2uv,u^2-v^2,u^2+v^2\right\rangle, u^2+v^2\le 1$.

Compare this parametric surface with its standard form, then

$x=2uv, y=u^2-v^2, z=u^2+v^2$.

The surface integral of $f(x,y,z)$ determined by parameters u and v is given by the following formula:

$$\iint_S f(x,y,z)\ dS=\iint_D f(\mathbf{r}(u,v))\left|\mathbf{r}_u\times\mathbf{r}_v\right| dA$$

Here, D is the domain of integration.

Need to calculate $\left|\mathbf{r}_u\times\mathbf{r}_v\right|$.

The partial derivative of the parametric equation with respect to u and v are calculate as,

$$\begin{aligned}\mathbf{r}(u,v)&=\left\langle 2uv,u^2-v^2,u^2+v^2\right\rangle\\ \mathbf{r}_u(u,v)&=\left\langle 2v,2u,2u\right\rangle\\ \mathbf{r}_v(u,v)&=\left\langle 2u,-2v,2v\right\rangle\end{aligned}$$

Now the cross product of these two vectors is,

$$\begin{aligned}\mathbf{r}_u\times\mathbf{r}_v&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2v&2u&2u\\2u&-2v&2v\end{vmatrix}\\ &=\left\langle 8uv,4v^2-4u^2,-4v^2-4u^2\right\rangle\\ &=4\left\langle 2uv,v^2-u^2,-\left(v^2+u^2\right)\right\rangle\end{aligned}$$

The modulus of this vector is,

$$\begin{aligned}\left|\mathbf{r}_u\times\mathbf{r}_v\right|&=4\sqrt{(2uv)^2+\left(v^2-u^2\right)^2+\left(-\left(v^2+u^2\right)\right)^2}\\ &=4\sqrt{4u^2v^2+v^4-2u^2v^2+u^4+v^4+2u^2v^2+u^4}\\ &=4\sqrt{2v^4+4u^2v^2+2u^4}\\ &=4\sqrt{2\left[\left(v^2\right)^2+2u^2v^2+\left(u^2\right)^2\right]}\end{aligned}$$

$$\begin{aligned}&=4\sqrt{2}\sqrt{\left(v^2+u^2\right)^2}\\ &=4\sqrt{2}\left(v^2+u^2\right)\end{aligned}$$

Now calculate $f(\mathbf{r}(u,v))$ as,

$$\begin{aligned} x^2+y^2 &= (2uv)^2+(u^2-v^2)^2 \\ &= 4u^2v^2+u^4-2u^2v^2+v^4 \\ &= u^4+2u^2v^2+v^4 \\ &= (u^2+v^2)^2 \end{aligned}$$

The parameter domain to this equation is

$$D=\{(u,v)\,|\,u^2+v^2\le 1\}$$

Write the surface integral

$$\begin{aligned} \iint_S f(x,y,z)\ dS &= \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u\times\mathbf{r}_v|\ dA \\ &= \iint_{u^2+v^2\le 1}(u^2+v^2)^2\left(4\sqrt{2}(u^2+v^2)\right)\ dA \\ &= 4\sqrt{2}\iint_{u^2+v^2\le 1}(u^2+v^2)^3\ dA \end{aligned}$$

Use polar coordinate system $u=r\cos\theta, v=r\sin\theta$ then the integral reduced as,

$$\iint_S(x^2+y^2)\ dS = 4\sqrt{2}\int_0^{2\pi}\int_0^1 r^7\ dr\ d\theta.$$

$$=4\sqrt{2}(2\pi)\left[\frac{1}{8}r^8\right]_{r=0}^{r=1}$$

$$=4\sqrt{2}(2\pi)\left(\frac{1}{8}\right)$$

$$=\pi\sqrt{2}$$

Hence, $\iint_S(x^2+y^2)dS=\boxed{\pi\sqrt{2}}$.

Chapter 16 Vector Calculus Exercise 16.7 9E

Consider the following surface integral:

$$\iint_S x^2yz dS$$

Here S is the part of the plane $z=1+2x+3y$ that lies above the rectangle.

The plane is $z=1+2x+3y$.

Differentiate partially with respect to 'x'.

$$\frac{\partial z}{\partial x}=2$$

Differentiate partially with respect to 'y'.

$$\frac{\partial z}{\partial y}=3$$

If S is a surface with equation $z=g(x,y)$ and D is its projection is as follows:

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA$$

Here $D=\{(x,y):\ 0\le x\le 3,\ \ 0\le y\le 2\}$.

Then $\iint_S x^2yz\,dS = \iint_D x^2y(1+2x+3y)\sqrt{4+9+1}\,dA$

$$=\sqrt{14}\int_0^2\int_0^3(x^2y+2x^3y+3x^2y^2)\,dx\,dy$$

$$=\sqrt{14}\int_0^2\left[\frac{x^3y}{3}+\frac{1}{2}x^4y+x^3y^2\right]_{x=0}^{x=3}dy \quad \text{Since } \int x^n dx=\frac{x^{n+1}}{n+1}+C$$

$$=\sqrt{14}\int_0^2\left(\frac{27y}{3}+\frac{81y}{2}+27y^2\right)dy$$

Now evaluate the following integral:

$$\sqrt{14}\int_0^2\left(\frac{27y}{3}+\frac{81y}{2}+27y^2\right)dy \;=\sqrt{14}\left[9\frac{y^2}{2}+\frac{81}{2}\frac{y^2}{2}+27\frac{y^3}{3}\right]_0^2$$

$$=\sqrt{14}\left[\frac{9y^2}{2}+\frac{81y^2}{4}+9y^3\right]_0^2$$

$$=\sqrt{14}\left[\frac{36}{2}+\frac{324}{4}+72\right]$$

$$=\sqrt{14}\left(\frac{72+324}{4}+72\right)$$

$$=\sqrt{14}\left(\frac{396}{4}+72\right)$$

$$=\sqrt{14}(99+72)$$

$$=171\sqrt{14}$$

Therefore, the surface integral is $\boxed{\iint_S x^2yz\,dS=171\sqrt{14}}$.

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Now the surface integral can be evaluated as,

$$S=\iint_D xz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\;dA$$

$$=\iint_D x(4-2x-2y)\sqrt{(-2)^2+(-2)^2+1}\;dA$$

$$=\iint_D\left(4x-2x^2-2xy\right)\sqrt{9}\;dA$$

$$=3\int_0^2\int_0^{2-x}\left(4x-2x^2-2xy\right)\,dy\,dx$$

$$=3\int_0^2\int_0^{2-x}\left(4x-2x^2-2xy\right)\,dy\,dx$$

$$=3\int_0^2\left[\left(4x-2x^2\right)y-xy^2\right]_{y=0}^{y=2-x}dx$$

$$=3\int_0^2\left[\left(4x-2x^2\right)(2-x)-x(2-x)^2-[0]\right]dx$$

$$=3\int_0^2\left[8x-4x^2-4x^2+2x^3-x^3+4x^2-4x\right]dx$$

$$=3\int_0^2\left(4x-4x^2+x^3\right)dx$$

$$=3\left[2x^2-\frac{4}{3}x^3+\frac{1}{4}x^4\right]_{x=0}^{x=2}$$

$$=3\left[2(2)^2-\frac{4}{3}(2)^3+\frac{1}{4}(2)^4\right]$$

$$=3\left[8-\frac{32}{3}+4\right]$$

$$=4$$

Hence, the value of the surface integral is

$S=\boxed{4}$.

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Since $x \le 0, y \ge 0, z \ge 0$, if we project z onto the xy-plane, $y = 2x - 2$ in the fourth quadrant, so the domain of integration is the triangle formed by the line $y = 2x - 2$ bounded by the coordinate axis is

$D = \{(x, y) \mid 0 \le x \le 1, 2x - 2 \le y \le 0\}$.

The partial derivatives of $z = -4x + 2y + 4$ are

$\frac{\partial z}{\partial x} = -4, \frac{\partial z}{\partial y} = 2$.

The surface integral is calculated as,

$$S = \iint_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

$$= \iint_D x\sqrt{(-4)^2 + (2)^2 + 1}\, dA$$

$$= \iint_D x\sqrt{21}\, dA$$

$$= \sqrt{21}\int_0^1 \int_{2x-2}^0 x\, dy\, dx$$

$$= \sqrt{21}\int_0^1 [xy]_{y=2x-2}^{y=0}\, dx$$

$$= \sqrt{21}\int_0^1 [x(0) - x(2x-2)]\, dx$$

$$= -\sqrt{21}\int_0^1 (2x^2 - 2x)\, dx$$

$$= -\sqrt{21}\left[\frac{2}{3}x^3 - x^2\right]_{x=0}^{x=1}$$

$$= -\sqrt{21}\left[\frac{2}{3} - 1 - (0)\right]$$

$$= -\sqrt{21}\left[-\frac{1}{3}\right]$$

$$= \frac{\sqrt{21}}{3}$$

Therefore, the surface integral value is

$$S = \boxed{\frac{\sqrt{21}}{3}}.$$

Chapter 16 Vector Calculus Exercise 16.7 12E

Consider the following surface

$$z = \frac{2}{3}\left(x^{3/2} + y^{3/2}\right), 0 \le x \le 1, 0 \le y \le 1$$

Find $\iint_S y\, ds$.

Recall the surface integral

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations $x = x, y = y, z = g(x, y)$.

$$\iint_S f(x, y, z)\, ds = \iint_D f(x, y, g(x, y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

The derivative of z partially with respect to x is

$$\frac{\partial z}{\partial x} = x^{1/2}$$

The derivative of z partially with respect to y is

$$\frac{\partial z}{\partial y} = y^{1/2}$$

The surface integral is

$$\int_S y\,ds = \int_0^1\int_0^1 y\sqrt{1+x+y}\,dx\,dy$$

$$= \frac{2}{3}\int_0^1 y\left[(1+x+y)^{3/2}\right]_{x=0}^{x=1} dy$$

$$= \frac{2}{3}\int_0^1\left[y(2+y)^{3/2} - y(1+y)^{3/2}\right]dy$$

$$= \frac{2}{3}\left[\frac{2}{5}y(2+y)^{5/2} - \frac{2}{5}\times\frac{2}{7}(2)^{5/2}(2+y)^{7/2} - \frac{2}{5}y(1+y)^{5/2} + \frac{2}{5}\times\frac{2}{7}(1+y)^{7/2}\right]_0^1$$

$$= \frac{2}{3}\left[\frac{2}{5}(3)^{5/2} - \frac{4}{35}(3)^{7/2} - \frac{2}{5}(2)^{5/2} + \frac{4}{35}(2)^{7/2} + \frac{4}{35}(2)^{7/2} - \frac{4}{35}\right]$$

$$= \frac{2}{3}\left[\frac{18}{5}\sqrt{3} - \frac{108}{35}\sqrt{3} - \frac{8}{5}\sqrt{2} + \frac{32}{35}\sqrt{2} + \frac{32}{35}\sqrt{2} - \frac{4}{35}\right]$$

$$= \frac{2}{3}\left[\frac{126\sqrt{3} - 108\sqrt{3} - 56\sqrt{2} + 64\sqrt{2} - 4}{35}\right]$$

$$= \frac{2}{3}\left[\frac{18\sqrt{3} + 8\sqrt{2} - 4}{35}\right]$$

$$= \frac{4\left(9\sqrt{3} + 4\sqrt{2} - 2\right)}{105}$$

Therefore, the surface integral is

$$\iint_s y\,ds = \boxed{\frac{4}{105}\left(9\sqrt{3} + 4\sqrt{2} - 2\right)}$$

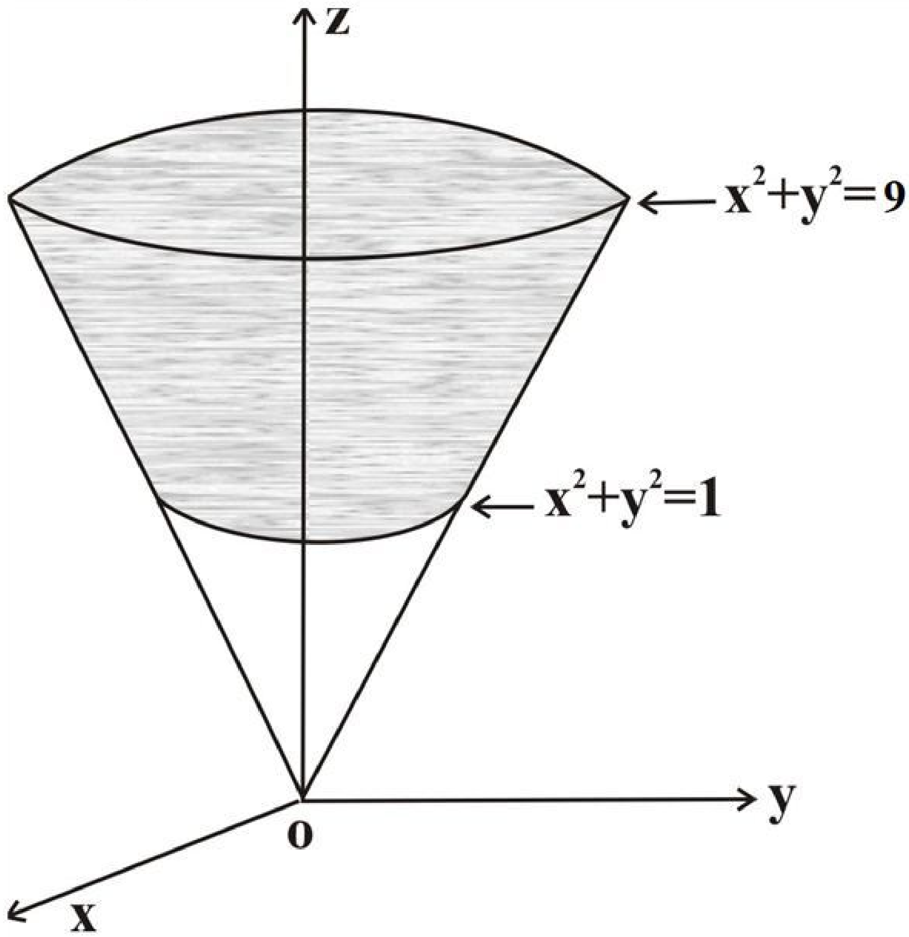
Chapter 16 Vector Calculus Exercise 16.7 13E

Consider S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$.

The objective is to evaluate the surface integral $\iint_S x^2 z^2\, dS$.

The cone $z^2 = x^2 + y^2$ meets plane z = 1 in circle $x^2 + y^2 = 1$ and it meets plane z=3 in circle $x^2 + y^2 = 9$.

Sketch the region is shown below:



The surface integral is,

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{1+z_x^2+z_y^2}\,dA$$

where *D* is the projection of *S* on *xy*-plane.

The domain of integration is the is the part of the cone $z^2 = x^2+y^2$ that lies between the planes $z=1$ and $z=3$. This means $1 \le x^2+y^2 \le 9$ and the domain of integration can be written in polar coordinates as,

$$D = \{(r,\theta) \mid 1 \le r \le 3, 0 \le \theta \le 2\pi\}$$

Consider,

$$z^2 = x^2 + y^2$$

Then $\frac{\partial z}{\partial x} = \frac{x}{z}$ and $\frac{\partial z}{\partial y} = \frac{y}{z}$

And,

$$\sqrt{1+z_x^2+z_y^2} = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}$$

$$= \sqrt{1+\frac{(x^2+y^2)}{z^2}}$$

$$= \sqrt{1+1}$$

$$= \sqrt{2}$$

Therefore,

$$\iint_S x^2z^2\,ds = \iint_D x^2\left(x^2+y^2\right)\sqrt{2}\,dA$$
$$= \int_0^{2\pi}\int_1^3\left(r^2\cos^2\theta\right)\left(r^2\right)\sqrt{2}\,r\,dr\,d\theta$$
$$= \sqrt{2}\int_0^{2\pi}\int_1^3 r^5\cos^2\theta\,dr\,d\theta$$
$$= \sqrt{2}\int_0^{2\pi}\left(\frac{r^6}{6}\right)_1^3\cos^2\theta\,d\theta$$
$$= \sqrt{2}\int_0^{2\pi}\left(\frac{3^6}{6}-\frac{1^6}{6}\right)_1^3\cos^2\theta\,d\theta$$
$$= \sqrt{2}\left(\frac{3^6-1}{6}\right)\int_0^{2\pi}\cos^2\theta\,d\theta$$
$$= \sqrt{2}\left(\frac{728}{6}\right)\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= \sqrt{2}\left(\frac{364}{3}\right)\cdot\frac{1}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)_0^{2\pi}$$
$$= \frac{364\sqrt{2}}{3}\cdot\frac{1}{2}\left[(2\pi+0)-(0-0)\right]$$
$$= \frac{364\sqrt{2}}{3}\cdot\pi$$
$$= \frac{364\pi\sqrt{2}}{3}$$

Thus, $\iint_S x^2z^2\,ds = \boxed{\frac{364\pi\sqrt{2}}{3}}$.

Chapter 16 Vector Calculus Exercise 16.7 14E

Consider the following surface integral,

$\iint_S z\,dS$, where, S is the part of the surface $x = y+2z^2$, $0 \le y \le 1,\ 0 \le z \le 1$.

The objective is to compute the value of the surface integral over the given surface.

The surface integral of an equation $x = g(y,z)$ is given by,

$$\iint_D f\left(g(y,z),y,z\right)\sqrt{\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2+1}\,dA$$

where, the D is the domain of integration.

Here, the domain of integration is the part of the surface $x = y+2z^2$.

$D = \left\{(y,z)\,|\,0 \le y \le 1, 0 \le z \le 1\right\}$.

The partial derivatives of $x = y + 2z^2$ are,

$\frac{\partial x}{\partial y} = 1, \quad \text{and} \quad \frac{\partial x}{\partial z} = 4z$.

Thus, the surface integral is,

$$\begin{aligned}\iint_S z\, dS &= \iint_D z\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \\ &= \iint_D z\sqrt{2 + 16z^2}\, dA \\ &= \int_0^1 \int_0^1 z\sqrt{2+16z^2}\, dy\, dz \\ &= \int_0^1 z\sqrt{2+16z^2}\, dz\end{aligned}$$

$$= \left[\frac{1}{48}\left(2+16z^2\right)^{\frac{3}{2}}\right]_{z=0}^{z=1}$$

$$= \left[\frac{1}{48}(18)^{\frac{3}{2}} - \frac{1}{48}(2)^{\frac{3}{2}}\right]$$

$$= \frac{13\sqrt{2}}{12}$$

Therefore, the value of the surface integral is,

$$\boxed{\iint_S z\, dS = \frac{13\sqrt{2}}{12}}.$$

Chapter 16 Vector Calculus Exercise 16.7 15E

Consider the function:

$f(x, y) = \ln(xy)$

Here $x = 3r + 2s, y = 5r + 3s$

The objective is to find $\frac{\partial f}{\partial s}$ at the point $(r, s) = (1, 0)$

Rewrite the function as,

$$\begin{aligned}f(x, y) &= \ln(xy) \\ &= \ln x + \ln y\end{aligned}$$

Using chain rule, $\frac{\partial f}{\partial s}$ can be written as,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Calculate the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}$:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(\ln x + \ln y) \\ &= \frac{1}{x}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(\ln x + \ln y) \\ &= \frac{1}{y}\end{aligned}$$

$$\begin{aligned}\frac{\partial x}{\partial s} &= \frac{\partial}{\partial s}(3r + 2s) \\ &= 2\end{aligned}$$

$$\begin{aligned}\frac{\partial y}{\partial s} &= \frac{\partial}{\partial s}(5r + 3s) \\ &= 3\end{aligned}$$

Next calculate the partial derivative $\frac{\partial f}{\partial s}$ as follows:

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} \\ &= \frac{1}{x}(2)+\frac{1}{y}(3) \\ &= \frac{2}{x}+\frac{3}{y} \\ &= \frac{2}{3r+2s}+\frac{3}{5r+3s}\end{aligned}$$

Now at $(r,s)=(1,0)$

$$\begin{aligned}\left(\frac{\partial f}{\partial s}\right)_{(1,0)} &= \frac{2}{3(1)+2(0)}+\frac{3}{5(1)+3(0)} \\ &= \frac{2}{3}+\frac{3}{5} \\ &= \frac{10+9}{15} \\ &= \boxed{\frac{19}{15}}\end{aligned}$$

Chapter 16 Vector Calculus Exercise 16.7 16E

Recall that, the surface integral of an equation $z=g(x,y)$ is given by the following formula:

$$\iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA.$$

Here, the D is the domain of integration.

The domain of integration is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$, or in polar coordinates.

$D=\{(r,\theta)\,|\,0\le r\le 1, 0\le\theta\le 2\pi\}$.

Since the part of the sphere that lies above the xy-plane is

$z=\sqrt{4-(x^2+y^2)}$.

The partial derivatives of $z=\sqrt{4-(x^2+y^2)}$ are calculated as,

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}\left(\sqrt{4-(x^2+y^2)}\right) \\ &= \frac{1}{2\sqrt{4-(x^2+y^2)}}(-2x) \\ &= \frac{-x}{\sqrt{4-x^2-y^2}}\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}\left(\sqrt{4-(x^2+y^2)}\right) \\ &= \frac{1}{2\sqrt{4-(x^2+y^2)}}(-2y) \\ &= \frac{-y}{\sqrt{4-x^2-y^2}}\end{aligned}$$

Substitute all the values, and then the surface integral can be calculated as,

$$S = \iint_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\ dA$$

$$= \iint_D y^2\sqrt{\left(\frac{-x}{\sqrt{4-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4-x^2-y^2}}\right)^2 + 1}\ dA$$

$$= \iint_D y^2\sqrt{\frac{x^2}{4-x^2-y^2} + \frac{y^2}{4-x^2-y^2} + 1}\ dA$$

$$= \iint_D y^2\sqrt{\frac{x^2+y^2+4-x^2-y^2}{4-x^2-y^2}}\ dA$$

$$= \iint_D y^2\sqrt{\frac{4}{4-x^2-y^2}}\ dA$$

$$= \iint_D \frac{2y^2}{\sqrt{4-x^2-y^2}}\ dA$$

Use polar coordinates, $x = r\cos\theta, y = r\sin\theta, r^2 = x^2 + y^2$, then the surface integral can be written as,

$$S = \int_0^{2\pi}\int_0^1 \left(\frac{2r^2\sin^2\theta}{\sqrt{4-r^2}}\right) r\ dr\ d\theta$$

$$= 2\int_0^{2\pi}\sin^2\theta\ d\theta \int_0^1 \left(\frac{r^2}{\sqrt{4-r^2}}\right) r\ drd\theta \ \ldots\ldots (1)$$

First evaluate the integral $\int_0^1 \left(\frac{r^2}{\sqrt{4-r^2}}\right) r\ dr$ as follows:

Making the substitution $u = 4 - r^2,\ du = -2r\ dr$, then the integral changed as,

$$\int_0^1 \left(\frac{r^2}{\sqrt{4-r^2}}\right) r\ dr = \int_4^3 \frac{4-u}{\sqrt{u}}\left(-\frac{1}{2}\right) du$$

$$= \frac{1}{2}\int_3^4 \left(4u^{\frac{-1}{2}} - u^{\frac{1}{2}}\right) du$$

$$= \frac{1}{2}\left[\frac{4u^{\frac{1}{2}}}{\frac{1}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{u=3}^{u=4}$$

$$= \frac{1}{2}\left[8u^{\frac{1}{2}} - \frac{2u^{\frac{3}{2}}}{3}\right]_{u=3}^{u=4}$$

$$= \left(4(4)^{\frac{1}{2}} - \frac{(4)^{\frac{3}{2}}}{3}\right) - \left(4(3)^{\frac{1}{2}} - \frac{(3)^{\frac{3}{2}}}{3}\right)$$

$$= \left(8 - \frac{8}{3}\right) - \left(4\sqrt{3} - \frac{3\sqrt{3}}{3}\right)$$

$$= \frac{16}{3} - 3\sqrt{3}$$

Evaluating the second integral $\int_0^{2\pi} \sin^2\theta\, d\theta$ as follows:

$$\begin{aligned}\int_0^{2\pi} \sin^2\theta\, d\theta &= \int_0^{2\pi} \frac{1}{2}-\frac{1}{2}\cos 2\theta\, d\theta \\ &= \left[\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right]_{\theta=0}^{\theta=2\pi} \\ &= [\pi-0] \\ &= \pi\end{aligned}$$

From (1) and use the results the surface integral evaluated as,

$$S = 2\int_0^{2\pi}\sin^2\theta\, d\theta\int_0^1\left(\frac{r^2}{\sqrt{4-r^2}}\right) r\, dr$$

$$= 2(\pi)\left(\frac{16}{3}-3\sqrt{3}\right)$$

$$= \frac{2\pi}{3}\left(16-9\sqrt{3}\right)$$

Hence, the surface integral value is

$$S = \boxed{\frac{2\pi}{3}\left(16-9\sqrt{3}\right)}.$$

Chapter 16 Vector Calculus Exercise 16.7 17E

S is the hemisphere $x^2+y^2+z^2=4$, $z \ge 0$

Then the region of integration D is the region below surface $z^2=4-x^2-y^2$ and above the circle $x^2+y^2=4$.

Hemi sphere is,

$$z^2 = 4-x^2-y^2$$

Then,

$$2z\frac{\partial z}{\partial x} = -2x \qquad 2z\frac{\partial z}{\partial y} = -2y$$

$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \text{ and } \quad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

And then $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}$

$$= \sqrt{\frac{z^2+x^2+y^2}{z^2}}$$

$$= \sqrt{\frac{4}{z^2}} \qquad \left(z^2+x^2+y^2=4\right)$$

$$= \frac{2}{z}$$

Also in polar co – ordinates

$$D = \{(r,\theta):\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$

So that, $\iint_S \left(x^2z+y^2z\right)ds=\iint_D \left(x^2+y^2\right)z\times\frac{2}{z}\,dA$

$$=2\int_0^{2\pi}\int_0^2 r^2.r\,dr\,d\theta$$

$$=2\int_0^{2\pi}d\theta\int_0^2 r^3\,dr$$

$$=2[\theta]_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2$$

$$=2(2\pi-0)\left(\frac{16}{4}-0\right)$$

$$=4\pi(4)$$

$$=16\pi$$

Therefore,

$$\iint_s \left(x^2z+y^2z\right)ds=\boxed{16\pi}$$

Chapter 16 Vector Calculus Exercise 16.7 18E

Consider the following surface integral.

$$\iint_S xz\ dS$$

The objective is to evaluate the surface integral $\iint_S xz\ dS$, where S is the boundary of the region enclosed by the cylinder $y^2+z^2=9$ and the planes $x=0$ and $x+y=5$

The surface integral of $f(x,y,z)$ determined by parameters u and v is given by the following formula:

$$\iint_S f(x,y,z)\ dS=\iint_D f(\mathbf{r}(u,v))\left|\mathbf{r}_u\times\mathbf{r}_v\right|dA$$

Where the D is the domain of integration.

Recall from the fundamental theorem of calculus that

$$\int_a^b f(x)\ dx=F(b)-F(a)$$

where $F(x)$ is the antiderivative of $f(x)$.

First parameterize the surface $y^2+z^2=9$ and determine the magnitude of its normal vector

$$\mathbf{r}(x,\theta)=\langle x,3\cos\theta,3\sin\theta\rangle$$
$$\mathbf{r}_x(x,\theta)=\langle 1,0,0\rangle$$
$$\mathbf{r}_\theta(x,\theta)=\langle 0,-3\sin\theta,3\cos\theta\rangle$$

$$\mathbf{r}_x\times\mathbf{r}_\theta=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\0&-3\sin\theta&3\cos\theta\end{vmatrix}=\langle 0,-3\cos\theta,-3\sin\theta\rangle$$

$$\begin{aligned}\left|\mathbf{r}_x\times\mathbf{r}_\theta\right|&=\sqrt{0^2+(-3\cos\theta)^2+(-3\sin\theta)^2}\\&=\sqrt{9\cos^2\theta+9\sin^2\theta}\\&=3\end{aligned}$$

>

Next, determine $f(\mathbf{r}(x,\theta))$

$$xz=x(3\sin\theta)$$

The parameter domain is given is from the surface the cylinder $y^2+z^2=9$ between the planes $x=0$ and $x+y=5$

$$\mathbf{r}(x,\theta)=\langle x,3\cos\theta,3\sin\theta\rangle,\ 0\le x\le 5-y=5-3\cos\theta,\ 0\le\theta\le 2\pi$$

$$D=\{(x,\theta)\,|\,0\le x\le 5-3\cos\theta,0\le\theta\le 2\pi\}$$

Now, write the surface integral

$$\iint_S f(x,y,z)\ dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\ dA$$

$$\iint_S xz\ dS = \int_0^{2\pi}\int_0^{5-3\cos\theta} x(3\sin\theta)3\ dx\ d\theta$$
$$= 3\int_0^{2\pi}\int_0^{5-3\cos\theta} x(3\sin\theta)\ dx\ d\theta$$

Evaluating the integral, we have

$$3\int_0^{2\pi}\int_0^{5-3\cos\theta} x(3\sin\theta)\ dx\ d\theta = 3\int_0^{2\pi}(3\sin\theta)\left[\frac{1}{2}x^2\right]_{x=0}^{x=5-3\cos\theta} d\theta$$
$$= 3\int_0^{2\pi}(3\sin\theta)\left[\frac{1}{2}(5-3\cos\theta)^2\right]d\theta$$

$$3\int_0^{2\pi}(3\sin\theta)\left[\frac{1}{2}(5-3\cos\theta)^2\right]d\theta = 3\left[\frac{1}{6}(5-3\cos\theta)^3\right]_{\theta=0}^{\theta=2\pi}$$
$$= 3\left[\frac{1}{6}(2)^3 - \frac{1}{6}(2)^3\right]$$
$$= \boxed{0}$$

Thus, the surface integral is $\boxed{\iint_S xz\ dS = 0}$.

Chapter 16 Vector Calculus Exercise 16.7 19E

The objective is to evaluate the surface integral.

Consider the following surface integral.

$$\iint_S (z + x^2 y)\ dS$$

The objective is to evaluate the surface integral $\iint_S (z + x^2 y)\ dS$.

Here, *S* is the part of the cylinder $y^2 + z^2 = 1$ and it lies between the planes $x = 0$ and $x = 3$ in the first octant.

For the θ and x as parameters and write its parametric representation as,

$x = x, y = \cos\theta, z = \sin\theta$, where $0 \le x \le 3,\quad 0 \le \theta \le \frac{\pi}{2}$.

The surface integral of $f(x,y,z)$ determined by parameters *u* and *v* is given by the following formula,

$$\iint_S f(x,y,z)\ dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\ dA$$

where the *D* is the domain of integration.

Determine $\mathbf{r}_\theta$ and $\mathbf{r}_x$ from $\mathbf{r}(x,\theta)$.

$$\mathbf{r}(x,\theta) = \langle x, \cos\theta, \sin\theta\rangle$$
$$\mathbf{r}_x(x,\theta) = \langle 1,0,0\rangle$$
$$\mathbf{r}_\theta(x,\theta) = \langle 0, -\sin\theta, \cos\theta\rangle$$

Calculate $\mathbf{r}_x \times \mathbf{r}_\theta$

$$\mathbf{r}_x \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta \end{vmatrix}$$
$$= \langle 0, -\cos\theta, -\sin\theta\rangle$$

$$|\mathbf{r}_x \times \mathbf{r}_\theta| = \sqrt{0^2 + (-\cos\theta)^2 + (-\sin\theta)^2}$$
$$= \sqrt{\cos^2\theta + \sin^2\theta}$$
$$= 1$$

Next determine $f\left(\mathbf{r}(x,\theta)\right)$.

$$z+x^2y=\sin\theta+x^2\cos\theta$$

The parameter domain is given is from the surface the cylinder $y^2+z^2=1$ between the planes $x=0$ and $x=3$ in the first octant.

$$\mathbf{r}(x,\theta)=\langle x,3\cos\theta,3\sin\theta\rangle,\ 0\le x\le 3,\ 0\le\theta\le\frac{\pi}{2}$$

$$D=\left\{(x,\theta)\,|\,0\le x\le 3, 0\le\theta\le\frac{\pi}{2}\right\}$$

Now, write the surface integral

$$\iint_S f(x,y,z)\ dS=\iint_D f\left(\mathbf{r}(u,v)\right)\left|\mathbf{r}_u\times\mathbf{r}_v\right|\ dA$$

$$\iint_S\left(z+x^2y\right)\ dS=\int_0^{\frac{\pi}{2}}\int_0^3\left(\sin\theta+x^2\cos\theta\right)\cdot 1\cdot\ dx\ d\theta$$

Evaluating the integral,

$$\int_0^{\frac{\pi}{2}}\int_0^3\left(\sin\theta+x^2\cos\theta\right)\ dx\ d\theta=\int_0^{\frac{\pi}{2}}\left[x\sin\theta+\frac{1}{3}x^3\cos\theta\right]_{x=0}^{x=3}\ d\theta$$

$$=\int_0^{\frac{\pi}{2}}(3\sin\theta+9\cos\theta)\ d\theta$$

$$=\left[-3\cos\theta+9\sin\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}}$$

$$=\left[0+9-(-3+0)\right]$$

$$=12$$

Therefore, the value of the integral is, $\boxed{12}$.

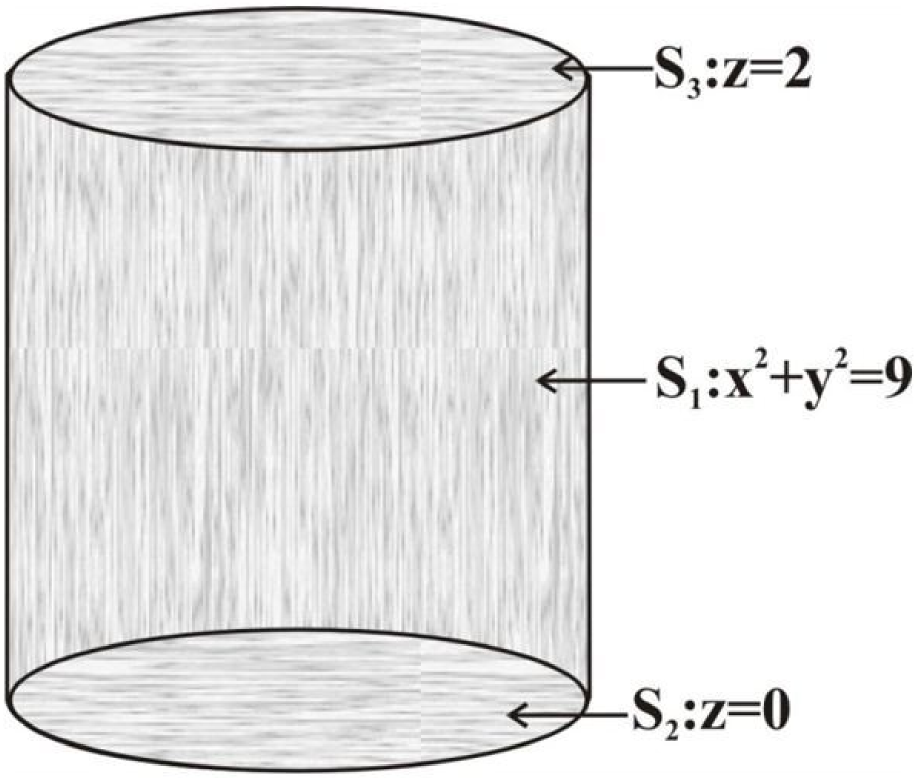
Chapter 16 Vector Calculus Exercise 16.7 20E

Consider the following integral:

$$\iint_S\left(x^2+y^2+z^2\right)ds$$

Here, S is the part of the cylinder $x^2+y^2=9$ with its top and bottom discs $z=0$ and $z=2$.

The region is as shown below:



The surface S is shown above it consists of three surfaces S1, S2 and S3

For S1, taking θ and z as parameters, the parametric equations are as follows:

$$x = 3\cos\theta,\ \ y = 3\sin\theta,\ z = z,\ 0 \le \theta \le 2\pi,\ \ 0 \le z \le 2$$

Then $\vec{r}(\theta, z) = <3\cos\theta, 3\sin\theta, z>$

$$\vec{r}_\theta = <-3\sin\theta, 3\cos\theta, 0>$$

$$\vec{r}_z = <0,0,1>$$

Compute $\vec{r}_\theta \times \vec{r}_z$ as follows:

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} i & j & k \\ -3\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 3\cos\theta i + 3\sin\theta j$$

$$\vec{r}_\theta \times \vec{r}_z = 3\cos\theta \hat{i} + 3\sin\theta \hat{j}$$

Then $|\vec{r}_\theta \times \vec{r}_z| = 3$

Compute the value of the integral is as follows:

$$\iint_{S_1} (x^2 + y^2 + z^2)\, dS = \iint_D (x^2 + y^2 + z^2) |\vec{r}_\theta \times \vec{r}_z|\, dA$$

$$= \int_0^{2\pi}\int_0^2 (9 + z^2) 3\ dz\, d\theta$$

$$= 3\int_0^{2\pi} \left(9z + \frac{z^3}{3}\right)_0^2 d\theta$$

$$= 3\int_0^{2\pi} \left(18 + \frac{8}{3}\right) d\theta$$

$$= 62(\theta)_0^{2\pi}$$

$$= 124\pi$$

Therefore, the value of the integral along the surface S_1 is $\boxed{124\pi}$.

Evaluate the integral along the surface S_2 is as follows:

For S2: z = 0

Then $\iint_{S_2} (x^2 + y^2 + z^2)\, dS = \iint_D (x^2 + y^2) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dA$

$$= \iint_D (x^2 + y^2)\, dA$$

Use polar coordinates to evaluate the integral.

In polar co – ordinates $D = \{(r, \theta):\ 0 \le \theta \le 2\pi,\ \ 0 \le r \le 3\}$

$$\iint_{S_2} (x^2 + y^2 + z^2)\, dS = \int_0^{2\pi}\int_0^3 (r^2\cos^2\theta + r^2\sin^2\theta) r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^3 r^3\, dr$$

$$= (\theta)_0^{2\pi} \left(\frac{r^4}{4}\right)_0^3$$

$$= 2\pi\left(\frac{81}{4}\right)$$

$$= \frac{81}{2}\pi$$

Evaluate the integral along the surface S_3 is as follows:

For S3: it is the region under plane z = 2 and above the circle $x^2+y^2=9$.

Then changing to polar co – ordinates

$$\iint_{S_3}(x^2+y^2+z^2)dS=\iint_D(x^2+y^2+z^2)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$=\int_D(x^2+y^2+4)\sqrt{1+0+0}\,dA$$

$$=\int_0^{2\pi}\int_0^3(r^2\cos^2\theta+r^2\sin^2\theta+4)r\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^3(r^2+4)r\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^3(r^3+4r)dr$$

$$=(\theta)_0^{2\pi}\left(\frac{r^4}{4}+2r^2\right)_0^3$$

$$=2\pi\left(\frac{81}{4}+18\right)$$

$$=\frac{153}{2}\pi$$

The total value of the integral along the surface S is as follows:

$$\iint_S(x^2+y^2+z^2)dS=\iint_{S_1}(x^2+y^2+z^2)dS+\iint_{S_2}(x^2+y^2+z^2)dS$$

$$+\iint_{S_3}(x^2+y^2+z^2)dS$$

$$=124\pi+\frac{81}{2}\pi+\frac{153}{2}\pi$$

$$=\boxed{241\pi}$$

Therefore, the value of the integral is $\boxed{241\pi}$.

Chapter 16 Vector Calculus Exercise 16.7 21E

Consider the vector field

$$\mathbf{F}(x,y,z)=ze^{xy}\mathbf{i}-3ze^{xy}\mathbf{j}+xy\mathbf{k}$$

The surface S is the parallelogram

$$S=\{(x,y,z):x=u+v,y=u-v,z=1+2u+v,0\le u\le 2,0\le v\le 1\}$$

The parametric equation is

$$\mathbf{r}(u,v)=(u+v)\mathbf{i}+(u-v)\mathbf{j}+(1+2u+v)\mathbf{k}$$

Recall that, if $\mathbf{F}$ is a continuation vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the surface integral $\mathbf{F}$ over $\mathbf{S}$ is

$$\iint_S\mathbf{F}.d\mathbf{S}=\iint_D\mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)dA \quad \ldots\ldots (1)$$

Evaluate $\mathbf{r}_u$.

Differentiate $\mathbf{r}$ with respect to u

$$\frac{\partial}{\partial u}\mathbf{r}(u,v)=\frac{\partial}{\partial u}(u+v)\mathbf{i}+\frac{\partial}{\partial u}(u-v)\mathbf{j}+\frac{\partial}{\partial u}(1+2u+v)\mathbf{k}$$
$$=\mathbf{i}+\mathbf{j}+2\mathbf{k}$$

Therefore

$$\mathbf{r}_u=\mathbf{i}+\mathbf{j}+2\mathbf{k}$$

Evaluate $\mathbf{r}_u$,

Differentiate $\mathbf{r}$ with respect to v

$$\frac{\partial}{\partial v}\mathbf{r}(u,v) = \frac{\partial}{\partial v}(u+v)\mathbf{i} + \frac{\partial}{\partial v}(u-v)\mathbf{j} + \frac{\partial}{\partial v}(1+2u+v)\mathbf{k}$$
$$= 1\mathbf{i} + (-1)\mathbf{j} + 1\mathbf{k}$$
$$= \mathbf{i} - \mathbf{j} + \mathbf{k}$$

Therefore

$$\mathbf{r}_v = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

Now, evaluate the vector $\mathbf{r}_u \times \mathbf{r}_v$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix}$$
$$= \mathbf{i}(1+2) - \mathbf{j}(1-2) + \mathbf{k}(-1-1)$$
$$= 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

Therefore,

$$\mathbf{r}_u \times \mathbf{r}_v = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

Use equation (1) to evaluate the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$

Consider the integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$
$$= \iint_D \left(ze^{xy}\mathbf{i} - 3ze^{xy}\mathbf{j} + xy\mathbf{k}\right) \cdot (3\mathbf{i} + \mathbf{j} - 2\mathbf{k})\, dA$$
$$= \iint_D \left(3ze^{xy} - 3ze^{xy} - 2xy\right) dA$$
$$= \iint_D -2xy dA$$

Substitute the parameters $x = u+v, y = u-v$ in $\iint_D -2xydA$ and the domain for (u,v) is

$D = \{(u,v) : 0 \le u \le 2, 0 \le v \le 1\}$ The integral is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D -2(u+v)(u-v)\, dA$$
$$= 2\int_0^1 \int_0^2 \left(v^2 - u^2\right) du dv$$
$$= 2\int_0^1 \left(v^2 u - \frac{u^3}{3}\right)_0^2 dv$$
$$= 2\int_0^1 \left(2v^2 - \frac{8}{3}\right) dv$$
$$= 2\left(\frac{2v^3}{3} - \frac{8}{3}v\right)_0^1$$
$$= 2\left(\frac{2}{3} - \frac{8}{3}\right)$$
$$= -4$$

Therefore

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = -4}$$

Here negative sign represents the direction of the flow of the flux in to the surface S.

Hence the flux of vector function $\mathbf{F}(x,y,z) = ze^{xy}\mathbf{i} - 3\mathbf{z}e^{xy}\mathbf{j} + xy\mathbf{k}$ across S is 4.

Chapter 16 Vector Calculus Exercise 16.7 22E

We have $F = zi + yj + xk$. Let $x = u\cos v$, $y = u\sin v$, and $z = u$, for $0 \le u \le 1$

And $0 \le v \le \pi$. Then, $\mathbf{r}_u = \langle \cos v, \sin v, 0 \rangle$ and $\mathbf{r}_v = \langle -u\sin v, u\cos v, 1 \rangle$.

Also, $\mathbf{r}_u \times \mathbf{r}_v = \sin v\mathbf{i} - \cos v\mathbf{j} + u\mathbf{k}$ and $\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = v\sin v - \dfrac{u\sin 2v}{2} + u^2\cos v$.

Now, find $\iint_S \vec{\mathbf{F}}\,ds$ given by $\iint_S \vec{\mathbf{F}} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\,dA$.

$$\iint_S \vec{\mathbf{F}}\,ds = \int_0^1\int_0^\pi v\sin v - \frac{u\sin 2v}{2} + u^2\cos v\,dv\,du$$

$$= \int_0^1\int_0^\pi v\sin v\,dv\,du - \int_0^1 \frac{u}{2}\,du\int_0^\pi \sin 2v\,dv + \int_0^1 u^2\,du\int_0^\pi \cos v\,dv$$

We know that $\int_0^\pi \cos v\,dv = 0$ and $\int_0^\pi \sin 2v\,dv = 0$.

$$\iint_S \vec{\mathbf{F}}\,ds = \int_0^\pi v\sin v\,dv\,du$$

$$= \pi + [\sin v]_0^\pi$$

$$= \pi$$

Thus, we get $\boxed{\iint_S \vec{\mathbf{F}}\,ds = \pi}$.

Chapter 16 Vector Calculus Exercise 16.7 23E

Consider the vector field,

$$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$$

The objective is to evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field **F** and the oriented surface *S*.

Here, *S* is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$, and has upward orientation.

Use the following formula to calculate its surface integral, where *P*, *Q*, and *R* are the component functions of F in the **i**, **j**, **k** directions, and using the partial derivatives of *g* with respect to both *x* and *y*:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

Calculate the partial derivatives of $z = 4 - x^2 - y^2$ we have

$$\frac{\partial z}{\partial x} = -2x, \frac{\partial z}{\partial y} = -2y$$

Next determine $-P\dfrac{\partial g}{\partial x} - Q\dfrac{\partial g}{\partial y} + R$

Here, $P = xy, Q = yz, R = zx$

$$-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R = -xy(-2x) - yz(-2y) + zx$$

$$= 2x^2y + 2y^2z + zx$$

$$= 2x^2y + 2y^2\left(4 - x^2 - y^2\right) + \left(4 - x^2 - y^2\right)x$$

$$= \left(4x - x^3\right) + 2x^2y + \left(8 - x - 2x^2\right)y^2 - 2y^4$$

The parameter domain is given from the problem, which is the square

$D=\{(x,y)\mid 0\le x\le 1, 0\le y\le 1\}$

Evaluate the surface integral as follows:

$$\iint_S \mathbf{F}\bullet d\mathbf{S}=\iint_D\left(-P\frac{\partial g}{\partial x}-Q\frac{\partial g}{\partial y}+R\right)dA$$

$$=\int_0^1\int_0^1\left[\left(4x-x^3\right)+2x^2y+\left(8-x-2x^2\right)y^2-2y^4\right]dydx$$

$$=\int_0^1\left[\left(4x-x^3\right)y+x^2y^2+\frac{1}{3}\left(8-x-2x^2\right)y^3-\frac{2}{5}y^5\right]_{y=0}^{y=1}dx$$

$$=\int_0^1\left(-x^3+\frac{1}{3}x^2+\frac{11}{3}x+\frac{34}{15}\right)dx$$

$$=\left[-\frac{1}{4}x^4+\frac{1}{9}x^3+\frac{11}{6}x^2+\frac{34}{15}x\right]_{x=0}^{x=1}$$

$$=\left[-\frac{1}{4}+\frac{1}{9}+\frac{11}{6}+\frac{34}{15}-(0)\right]$$

$$=\boxed{\frac{713}{180}}$$

Chapter 16 Vector Calculus Exercise 16.7 24E

Consider the following vector function:

$\mathbf{F}(x,y,z)=-x\,\mathbf{i}-y\,\mathbf{j}+z^3\,\mathbf{k}$

The objective is to evaluate the surface integral $\iint_S \mathbf{F}\bullet d\mathbf{S}$ for the given vector field **F** and the oriented surface *S*.

Here, *S* is the part of the cone $z=\sqrt{x^2+y^2}$ between the planes $z=1$ and $z=3$ with downward orientation.

To find the surface integral $\iint_S \mathbf{F}\bullet d\mathbf{S}$, use the following:

$$\iint_S \mathbf{F}\bullet d\mathbf{S}=\iint_D\left(-P\frac{\partial g}{\partial x}-Q\frac{\partial g}{\partial y}+R\right)dA$$

Here, $F=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$

Calculating the partial derivatives of $z=\sqrt{x^2+y^2}$ we have

$$\frac{\partial z}{\partial x}=x\left(x^2+y^2\right)^{-\frac{1}{2}},\frac{\partial z}{\partial y}=y\left(x^2+y^2\right)^{-\frac{1}{2}}$$

Consider the vector function $\mathbf{F}(x,y,z)=-x\,\mathbf{i}-y\,\mathbf{j}+z^3\,\mathbf{k}$

Here, $P=-x, Q=-y$ and $R=z^3$

Next determine $-P\frac{\partial g}{\partial x}-Q\frac{\partial g}{\partial y}+R$ as follows:

$$-P\frac{\partial g}{\partial x}-Q\frac{\partial g}{\partial y}+R=-\left(-x\left(x\left(x^2+y^2\right)^{-\frac{1}{2}}\right)\right)-\left(-y\left(y\left(x^2+y^2\right)^{-\frac{1}{2}}\right)\right)+z^3$$

$$=x^2\left(x^2+y^2\right)^{-\frac{1}{2}}+y^2\left(x^2+y^2\right)^{-\frac{1}{2}}+z^3$$

$$=\left(x^2+y^2\right)\left(x^2+y^2\right)^{-\frac{1}{2}}+z^3$$

$$=\left(x^2+y^2\right)^{\frac{1}{2}}+\left(x^2+y^2\right)^{\frac{3}{2}}$$

The parameter domain of integration is the is the part of the cone $z^2=x^2+y^2$ that lies between the planes $z=1$ and $z=3$. This means $1\le x^2+y^2\le 9$ and the parameter domain of integration can be written in polar coordinates as

$$D=\{(r,\theta)\,|\,1\le r\le 3, 0\le\theta\le 2\pi\}$$

Write the surface integral as follows:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D\left(-P\frac{\partial g}{\partial x}-Q\frac{\partial g}{\partial y}+R\right)dA\\ &= \iint_D (x^2+y^2)^{\frac{1}{2}}+(x^2+y^2)^{\frac{3}{2}}\,dA\\ &= \int_0^{2\pi}\int_1^3\left((r^2)^{\frac{1}{2}}+(r^2)^{\frac{3}{2}}\right)r\,dr\,d\theta\\ &= \int_0^{2\pi}\int_1^3 r^2+r^4\,dr\,d\theta\end{aligned}$$

$$=2\pi\left[\frac{1}{3}r^3+\frac{1}{5}r^5\right]_{r=1}^{r=3}$$

$$=2\pi\left[\frac{1}{3}(3)^3+\frac{1}{5}(3)^5-\left(\frac{1}{3}+\frac{1}{5}\right)\right]$$

$$=2\pi\left[\frac{856}{15}\right]$$

$$=\boxed{\frac{1712\pi}{15}}$$

Chapter 16 Vector Calculus Exercise 16.7 25E

Consider the following vector filed.

$$\mathbf{F}(x,y,z)=x\,\mathbf{i}-z\,\mathbf{j}+y\,\mathbf{k}$$

The objective is to evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ for the given vector field **F** and the oriented surface *S*. Where, *S* is the hemisphere $x^2+y^2+z^2=4$, in the first octant, with orientation toward the origin.

The surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ of a vector field **F** over a parametric surface by a vector function $\mathbf{r}(u,v)$, we use the following formula where *D* is the parameter domain of the vector function:

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=\iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA$$

Recall from the fundamental theorem of calculus that

$$\int_a^b f(x)\,dx=F(b)-F(a)$$

where *F(x)* is the antiderivative of *f(x)*.

First, we determine the normal vector and parameterization of the quarter-hemisphere. From p. 818, the parameterization and normal vectors of the sphere are given as follows, with respect to θ and ϕ, where $0\le\theta\le 2\pi, 0\le\phi\le\pi$.

$$\mathbf{r}(\phi,\theta)=\langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$$
$$\mathbf{r}_\phi\times\mathbf{r}_\theta=\langle a^2\sin^2\phi\cos\theta, a^2\sin^2\phi\sin\theta, a^2\sin\phi\cos\phi\rangle$$

For this particular problem, we have $0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2}, a=2$ and

$$\mathbf{r}(\phi,\theta)=\langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi\rangle$$
$$\mathbf{r}_\phi\times\mathbf{r}_\theta=\langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi\rangle$$

The given normal vector is for an outward orientation. For the inward orientation we multiply this normal vector by -1

$$-(\mathbf{r}_\phi\times\mathbf{r}_\theta)=\langle -4\sin^2\phi\cos\theta, -4\sin^2\phi\sin\theta, -4\sin\phi\cos\phi\rangle$$

Next we determine $\mathbf{F} \cdot -(\mathbf{r}_\phi \times \mathbf{r}_\theta)$

$$\mathbf{F}(x,y,z) = \langle x, -z, y \rangle$$
$$\mathbf{F}(\mathbf{r}(\phi,\theta)) = \langle 2\sin\phi\cos\theta, -2\cos\phi, 2\sin\phi\sin\theta \rangle$$
$$\begin{aligned}\mathbf{F} \cdot -(\mathbf{r}_\phi \times \mathbf{r}_\theta) &= \langle 2\sin\phi\cos\theta, -2\cos\phi, 2\sin\phi\sin\theta \rangle \cdot \langle -4\sin^2\phi\cos\theta, -4\sin^2\phi\sin\theta, -4\sin\phi\cos\phi \rangle \\ &= -8\sin^3\phi\cos^2\theta + 8\sin^2\phi\sin\theta\cos\phi - 8\sin^2\phi\sin\theta\cos\phi \\ &= -8\sin^3\phi\cos^2\theta \\ &= -4(\sin\phi - \sin\phi\cos^2\phi)(1+\cos 2\theta)\end{aligned}$$

The parameter domain is given from the parameterization of the sphere in the first octant

$$D = \left\{(\phi,\theta) \mid 0 \le \phi \le \frac{\pi}{2}, 0 \le \theta \le \frac{\pi}{2}\right\}$$

Now, write the surface integral

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot -(\mathbf{r}_\phi \times \mathbf{r}_\theta)\, dA \\ &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} -4(\sin\phi - \sin\phi\cos^2\phi)(1+\cos 2\theta)\, d\phi\, d\theta \\ &= -4\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\, d\theta \int_0^{\frac{\pi}{2}}(\sin\phi - \sin\phi\cos^2\phi)\, d\phi\end{aligned}$$

Evaluating the integral we have

$$\begin{aligned}-4\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\, d\theta \int_0^{\frac{\pi}{2}}(\sin\phi - \sin\phi\cos^2\phi)\, d\phi &= -4\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}}\left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_{\phi=0}^{\phi=\frac{\pi}{2}} \\ &= -4\left[\frac{\pi}{2} - 0\right]\left[0 - \left(-\frac{1}{3}+1\right)\right] \\ &= -4\left[\frac{\pi}{2}\right]\left[\frac{2}{3}\right] \\ &= \boxed{-\frac{4\pi}{3}}\end{aligned}$$

Thus, the integral is $\iint_S \mathbf{F} \cdot d\mathbf{S} = \boxed{-\frac{4\pi}{3}}$.

Chapter 16 Vector Calculus Exercise 16.7 26E

Consider the following vector filed.

$$\mathbf{F}(x,y,z) = xz\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$$

The objective is to evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field **F** and the oriented surface *S*. Where, *S* is the hemisphere $x^2+y^2+z^2=25,\ y \ge 0$, oriented in the direction of the positive *y*-axis.

The surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ of a vector field **F** over a parametric surface by a vector function $\mathbf{r}(u,v)$, we use the following formula where *D* is the parameter domain of the vector function:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Recall from the fundamental theorem of calculus that

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Where *F(x)* is the antiderivative of *f(x)*.

First, we determine the normal vector and parameterization of the hemisphere. From p. 818, the parameterization and normal vectors of the sphere are given as follows, with respect to θ and ϕ, where $0 \le \theta \le 2\pi, 0 \le \phi \le \pi$.

$$\mathbf{r}(\phi,\theta) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$$
$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle a^2\sin^2\phi\cos\theta, a^2\sin^2\phi\sin\theta, a^2\sin\phi\cos\phi\rangle$$

For this particular problem, we have $0 \le \theta \le \pi, 0 \le \phi \le \dfrac{\pi}{2}, a = 5$ and

$$\mathbf{r}(\phi,\theta) = \langle 5\sin\phi\cos\theta, 5\sin\phi\sin\theta, 5\cos\phi\rangle$$
$$\begin{aligned}\mathbf{r}_\phi \times \mathbf{r}_\theta &= \langle 25\sin^2\phi\cos\theta, 25\sin^2\phi\sin\theta, 25\sin\phi\cos\phi\rangle \\ &= \left\langle 25\sin^2\phi\cos\theta, 25\sin^2\phi\sin\theta, \frac{25}{2}\sin 2\phi\right\rangle\end{aligned}$$

Next we determine $\mathbf{F} \cdot \mathbf{r}_\phi \times \mathbf{r}_\theta$

$$\mathbf{F}(x,y,z) = \langle xz, x, y\rangle$$
$$\begin{aligned}\mathbf{F}(\mathbf{r}(\phi,\theta)) &= \langle (5\sin\phi\cos\theta)(5\cos\phi), 5\sin\phi\cos\theta, 5\sin\phi\sin\theta\rangle \\ &= \left\langle \frac{25}{2}\sin 2\phi\cos\theta, 5\sin\phi\cos\theta, 5\sin\phi\sin\theta\right\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{F} \cdot -(\mathbf{r}_\phi \times \mathbf{r}_\theta) &= \left\langle \frac{25}{2}\sin 2\phi\cos\theta, 5\sin\phi\cos\theta, 5\sin\phi\sin\theta\right\rangle \cdot \left\langle 25\sin^2\phi\cos\theta, 25\sin^2\phi\sin\theta, \frac{25}{2}\sin 2\phi\right\rangle \\ &= \frac{625}{2}\sin 2\phi\cos^2\theta + 625\sin^3\phi\cos\theta\sin\theta + \frac{625}{2}\sin 2\phi\sin\phi\sin\theta \\ &= \frac{625}{2}\sin 2\phi\left(\frac{1+\cos 2\theta}{2}\right) + \frac{625}{2}\sin^3\phi\sin 2\theta + \frac{625}{2}\sin 2\phi\sin\phi\sin\theta \\ &= \frac{625}{2}\left[\sin 2\phi\left(\frac{1+\cos 2\theta}{2}\right) + \sin^3\phi(\sin 2\theta) + \sin 2\phi\sin\phi(\sin\theta)\right]\end{aligned}$$

The parameter domain is given from the parameterization of the sphere in the first octant

$$D = \left\{(\phi,\theta) \mid 0 \le \phi \le \frac{\pi}{2}, 0 \le \theta \le \frac{\pi}{2}\right\}$$

We now write the surface integral

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot \mathbf{r}_\phi \times \mathbf{r}_\theta \, dA \\ &= \int_0^\pi \int_0^\pi \frac{625}{2}\left[\sin 2\phi\left(\frac{1+\cos 2\theta}{2}\right) + \sin^3\phi(\sin 2\theta) + \sin 2\phi\sin\phi(\sin\theta)\right] d\theta\, d\phi \\ &= \frac{625}{2}\int_0^\pi \int_0^\pi \left[\sin 2\phi\left(\frac{1+\cos 2\theta}{2}\right) + \sin^3\phi(\sin 2\theta) + \sin 2\phi\sin\phi(\sin\theta)\right] d\theta\, d\phi\end{aligned}$$

Evaluating the integral, we have

$$\begin{aligned}&\frac{625}{2}\int_0^\pi \int_0^\pi \left[\sin 2\phi\left(\frac{1+\cos 2\theta}{2}\right) + \sin^3\phi(\sin 2\theta) + \sin 2\phi\sin\phi(\sin\theta)\right] d\theta\, d\phi \\ &= \frac{625}{2}\int_0^\pi \left[\sin 2\phi\left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right) + \sin^3\phi\left(-\frac{1}{4}\cos 2\theta\right) + \sin 2\phi\sin\phi(-\cos\theta)\right]_{\theta=0}^{\theta=\pi} d\phi \\ &= \frac{625}{2}\int_0^\pi \left[\sin 2\phi\left(\frac{\pi}{2} - 0\right) + \sin^3\phi\left(-\frac{1}{4}(1-1)\right) + \sin 2\phi\sin\phi(-(-1-1))\right] d\phi \\ &= \frac{625}{2}\int_0^\pi \left[\frac{\pi}{2}\sin 2\phi + 2\sin 2\phi\sin\phi\right] d\phi\end{aligned}$$

Continue to the above step,

$$\frac{625}{2}\int_0^{\pi}\left[\frac{\pi}{2}\sin 2\phi+2\sin 2\phi\sin\phi\right]d\phi=\frac{625}{2}\int_0^{\pi}\left[\frac{\pi}{2}\sin 2\phi+4\sin^2\phi\cos\phi\right]d\phi$$

$$=\frac{625}{2}\left[-\frac{\pi}{4}\cos 2\phi+\frac{4}{3}\sin^3\phi\right]_{\phi=0}^{\phi=\pi}$$

$$=\frac{625}{2}\left[-\frac{\pi}{4}(1-1)+\frac{4}{3}(0)\right]$$

$$=\boxed{0}$$

Thus, the surface integral $\boxed{\iint_S \mathbf{F}\cdot d\mathbf{S}=0}$.

Chapter 16 Vector Calculus Exercise 16.727E

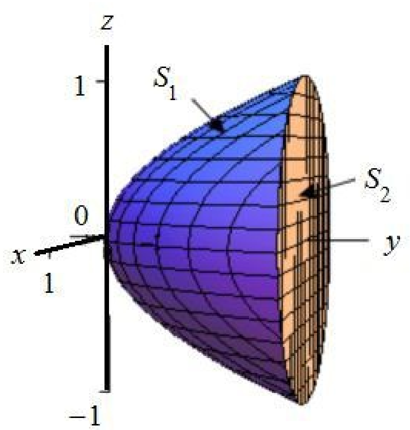
Consider the following vector field:

$$\mathbf{F}(x,y,z)=y\mathbf{j}-z\mathbf{k}$$

The objective is to evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ for the given vector field over the surface S.

Where, S is the surface given by the paraboloid $y=x^2+z^2, 0\le y\le 1$ and the disk $x^2+z^2\le 1, y=1.$

The graph of the paraboloid and disk as shown below:



Here, the surface S consists of two surfaces explained as follows:

$$S_1: y=x^2+z^2,\ 0\le y\le 1$$
$$S_2: x^2+z^2\le 1, y=1$$

The surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ of a vector field **F** over a parametric surface by a vector function $\mathbf{r}(u,v)$, use the following formula where D is the parameter domain of the vector function:

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=\iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA$$

First, parameterize the surface $S_1: y=x^2+z^2,\ 0\le y\le 1$ and determine its normal vector.

$$\mathbf{r}(x,z)=\langle x,x^2+z^2,z\rangle$$
$$\mathbf{r}_x(x,\theta)=\langle 1,2x,0\rangle$$
$$\mathbf{r}_z(z,\theta)=\langle 0,2z,1\rangle$$

Then, the normal vector is given by,

$$\mathbf{r}_x\times\mathbf{r}_z=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2x & 0\\ 0 & 2z & 1\end{vmatrix}$$
$$=\langle 2x,-1,2z\rangle$$

Determine the value of $\mathbf{F}\cdot\mathbf{r}_x\times\mathbf{r}_z$ as follows:

$$\mathbf{F}(x,y,z)=\langle 0,y,-z\rangle$$
$$\mathbf{F}(\mathbf{r}(x,z))=\langle 0,x^2+z^2,-z\rangle$$
$$\begin{aligned}\mathbf{F}\cdot\mathbf{r}_x\times\mathbf{r}_y&=\langle 0,x^2+z^2,-z\rangle\cdot\langle 2x,-1,2z\rangle\\&=0-(x^2+z^2)-2z^2\\&=-\left((x^2+z^2)+2z^2\right)\end{aligned}$$

The parameter domain is given from $y=x^2+z^2,\ 0\le y\le 1$, which means $0\le x^2+z^2\le 1$, or in polar coordinates $D=\{(r,\theta)\,|\,0\le r\le 1, 0\le\theta\le 2\pi\}$.

Now, evaluate the surface integral over S_1.

$$\begin{aligned}\iint_{S_1}\mathbf{F}\bullet d\mathbf{S}&=\iint_D\mathbf{F}\bullet\mathbf{r}_x\times\mathbf{r}_z\ dA\\&=\iint_D-\left((x^2+z^2)+2z^2\right)dA\\&=-\int_0^{2\pi}\int_0^1\left(r^2+2r^2\cos^2\theta\right)r\ dr\ d\theta\\&=-\int_0^{2\pi}(2+\cos 2\theta)\ d\theta\int_0^1 r^3\ dr\end{aligned}$$
$$=-\left[2\theta+\frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\theta=2\pi}\left[\frac{1}{4}r^4\right]_{r=0}^{r=1}$$
$$=-[4\pi]\left[\frac{1}{4}\right]$$
$$=\boxed{-\pi}$$

Now, set up the integral $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S}$, where $S_2: x^2+z^2\le 1, y=1$.

Parameterize the surface to obtain,

$$\mathbf{r}(x,z)=\langle x,1,z\rangle$$
$$\mathbf{r}_x\times\mathbf{r}_z=\langle 0,1,0\rangle$$
$$\mathbf{F}(\mathbf{r}(x,z))=\langle 0,1,-z\rangle$$
$$\mathbf{F}\cdot\mathbf{r}_x\times\mathbf{r}_y=1$$

The parametric domain for $S_2: x^2+z^2\le 1, y=1$ is $D=\{(x,z)\,|\,x^2+z^2\le 1\}$.

Evaluate $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S}$ as follows:

$$\begin{aligned}\iint_{S_2}\mathbf{F}\cdot d\mathbf{S}&=\iint_D\mathbf{F}\cdot\mathbf{r}_x\times\mathbf{r}_z\ dA\\&=\iint_D 1\ dA\\&=\iint_{x^2+z^2\le 1}1\ dA\\&=\pi\end{aligned}$$

Compute the surface integral $\iint_S\mathbf{F}\cdot d\mathbf{S}$ as follows:

$$\begin{aligned}\iint_S\mathbf{F}\cdot d\mathbf{S}&=\iint_{S_1}\mathbf{F}\cdot d\mathbf{S}+\iint_{S_2}\mathbf{F}\cdot d\mathbf{S}\\&=-\pi+\pi\\&=\boxed{0}\end{aligned}$$

Thus, the surface integral is $\iint_S\mathbf{F}\cdot d\mathbf{S}=\boxed{0}$.

Chapter 16 Vector Calculus Exercise 16.7 28E

Consider the vector $\mathbf{F}=\left(xy\mathbf{i}+4x^2\mathbf{j}+yz\mathbf{k}\right)$ and the surface is $z=xe^y;0\le x\le 1,0\le y\le 1$.

To find the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ of a vector field **F** over a surface $z=g\left(x,y\right)$ use the formula $\iint_S \mathbf{F}\cdot d\mathbf{S}=\iint_D\left(-P\frac{\partial z}{\partial x}-Q\frac{\partial z}{\partial y}+R\right)dA$

Recall from the fundamental theorem of calculus that

$$\int_a^b f\left(x\right)\,dx=F\left(b\right)-F\left(a\right)$$

Where $F\left(x\right)$ is the anti-derivative of $f\left(x\right)$.

Calculate the partial derivatives of $z=xe^y$.

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left(xe^y\right)=e^y \quad \text{And} \quad \frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left(xe^y\right)=xe^y$$

Determine $-P\frac{\partial z}{\partial x}-Q\frac{\partial z}{\partial y}+R$.

Here, $P=xy,Q=4x^2$, and $R=yz$.

$$\begin{aligned}-P\frac{\partial z}{\partial x}-Q\frac{\partial z}{\partial y}+R&=-xy\left(e^y\right)-4x^2\left(xe^y\right)+yz\\&=-xye^y-4x^3e^y+xye^y\\&=-4x^3e^y\end{aligned}$$

From the statement, the parameter domain

$$D=\left\{\left(x,y\right)|\,0\le x\le 1,0\le y\le 1\right\}$$

Write the surface integral.

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S}&=\iint_D\left(-P\frac{\partial z}{\partial x}-Q\frac{\partial z}{\partial y}+R\right)dA\\&=\int_0^1\int_0^1\left(-4x^3e^y\right)dy\,dx\\&=-\int_0^1\left(4x^3\int_0^1 e^y\,dy\right)dx\\&=-\int_0^1\left(4x^3\left[e^y\right]_0^1\right)dx\\&=-\int_0^1\left(4x^3\left[e^1-e^0\right]\right)dx\\&=-\int_0^1\left(4x^3\left[e-1\right]\right)dx\end{aligned}$$

On continuation,

$$\begin{aligned}&=-\left(e-1\right)\int_0^1 4x^3dx\\&=-\left(e-1\right)\left[\frac{4x^4}{4}\right]_0^1\\&=-\left(e-1\right)\left[x^4\right]_0^1\\&=-\left(e-1\right)\left[1-0\right]\\&=-\left(e-1\right)\\&=1-e\end{aligned}$$

Hence, the value of the surface integral of the vector ***F*** with the surface $z=xe^y;0\le x\le 1,0\le y\le 1$ is $\boxed{\left(1-e\right)}$.

Chapter 16 Vector Calculus Exercise 16.7 29E

Consider the following vector filed.

$\mathbf{F}(x,y,z) = y\ \mathbf{j} - z\ \mathbf{k}$

The objective is to evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field **F** and the oriented surface *S*. Where, *S* is the paraboloid $y = x^2 + z^2 = 25,\ 0 \le y \le 1$, and the disk $x^2 + z^2 \le 1, y = 1$

The surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ of a vector field **F** over a parametric surface by a vector function $\mathbf{r}(u,v)$, we use the following formula where *D* is the parameter domain of the vector function:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Recall from the fundamental theorem of calculus that

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where *F(x)* is the antiderivative of *f(x)*.

In this particular problem, *S* is composed of two surfaces

$$S_1 : y = x^2 + z^2,\ 0 \le y \le 1$$
$$S_2 : x^2 + z^2 \le 1, y = 1$$

First, parameterize the surface $S_1 : y = x^2 + z^2,\ 0 \le y \le 1$ and determine its normal vector

$$\mathbf{r}(x,z) = \langle x, x^2 + z^2, z \rangle$$
$$\mathbf{r}_x(x,\theta) = \langle 1, 2x, 0 \rangle$$
$$\mathbf{r}_z(z,\theta) = \langle 0, 2z, 1 \rangle$$
$$\mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 2z & 1 \end{vmatrix} = \langle 2x, -1, 2z \rangle$$

Next, determine $\mathbf{F} \cdot \mathbf{r}_x \times \mathbf{r}_z$

$$\mathbf{F}(x,y,z) = \langle 0, y, -z \rangle$$
$$\mathbf{F}(\mathbf{r}(x,z)) = \langle 0, x^2 + z^2, -z \rangle$$
$$\begin{aligned} \mathbf{F} \cdot \mathbf{r}_x \times \mathbf{r}_y &= \langle 0, x^2 + z^2, -z \rangle \cdot \langle 2x, -1, 2z \rangle \\ &= 0 - (x^2 + z^2) - 2z^2 \\ &= -((x^2 + z^2) + 2z^2) \end{aligned}$$

The parameter domain is given from $y = x^2 + z^2,\ 0 \le y \le 1$, which means $0 \le x^2 + z^2 \le 1$, or in polar coordinates

$$D = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$$

Now, write the surface integral

$$\begin{aligned} \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot \mathbf{r}_x \times \mathbf{r}_z\ dA \\ &= \iint_D -((x^2 + z^2) + 2z^2)\, dA \\ &= -\int_0^{2\pi} \int_0^1 (r^2 + 2r^2 \cos^2 \theta) r\ dr\ d\theta \\ &= -\int_0^{2\pi} (2 + \cos 2\theta)\ d\theta \int_0^1 r^3\ dr \end{aligned}$$

Evaluating the integral we have

$$\begin{aligned} -\int_0^{2\pi} (2 + \cos 2\theta)\ d\theta \int_0^1 r^3\ dr &= -\left[2\theta + \frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\theta=2\pi} \left[\frac{1}{4} r^4\right]_{r=0}^{r=1} \\ &= -[4\pi]\left[\frac{1}{4}\right] \\ &= \boxed{-\pi} \end{aligned}$$

Next set up the integral $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$, where we have $S_2 : x^2 + z^2 \le 1, y = 1$. First, we parameterize the surface to obtain

$$\mathbf{r}(x,z) = \langle x,1,z \rangle$$
$$\mathbf{r}_x \times \mathbf{r}_z = \langle 0,1,0 \rangle$$
$$\mathbf{F}(\mathbf{r}(x,z)) = \langle 0,1,-z \rangle$$
$$\mathbf{F} \cdot \mathbf{r}_x \times \mathbf{r}_y = 1$$

Next, obtain the parameter domain for $S_2 : x^2 + z^2 \le 1, y = 1$, which is the disk

$$D = \{(x,z) \mid x^2 + z^2 \le 1\}$$

Next, write and evaluate $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$

$$\begin{aligned}\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot \mathbf{r}_x \times \mathbf{r}_z \, dA \\ &= \iint_D 1 \, dA \\ &= \iint_{x^2+z^2 \le 1} 1 \, dA \\ &= \pi\end{aligned}$$

Now, compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} \\ &= -\pi + \pi \\ &= \boxed{0}\end{aligned}$$

Thus, the integral is $\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = 0}$.

Chapter 16 Vector Calculus Exercise 16.7 30E

Consider,

$$F(x,y,z) = x\mathbf{i} + y\mathbf{j} + 5\mathbf{k}.$$

There are three surfaces of the region described: the cylinder, the plane $y = 0$, and the plane $x + y = 2$.

Parametrize the cylinder in cylindrical coordinates:

$$r(\theta, y) = \langle \cos\theta, y, \sin\theta \rangle, 0 \le \theta \le 2\pi, 0 \le y \le 2 - \cos\theta.$$

The partial derivatives of $r(\theta, y) = \langle \cos\theta, y, \sin\theta \rangle$ is shown below:

$$r_\theta = \langle -\sin\theta, 0, \cos\theta \rangle, r_y = \langle 0,1,0 \rangle.$$

The cross product of these two vectors is

$$\begin{aligned}r_\theta \times r_y &= \langle -\sin\theta, 0, \cos\theta \rangle \times \langle 0,1,0 \rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{vmatrix} \\ &= \mathbf{i}(0 - \cos\theta) - \mathbf{j}(0 - 0) + \mathbf{k}(-\sin\theta - 0) \\ &= \langle -\cos\theta, 0, -\sin\theta \rangle\end{aligned}$$

Since the surface S is closed, we will orient with outward-pointing normal vector.

On the cylinder, this means the normal vector should point away from the $y-$axis.

To orient the surface correctly, should use the surface $r_y \times r_\theta = \langle \cos\theta, 0, \sin\theta \rangle$.

The surface integral can be evaluated as,

$$\begin{aligned}\iint_S \mathbf{F} \cdot dS &= \int_0^{2\pi} \int_0^{2-\cos\theta} \langle \cos\theta, y, 5 \rangle \cdot \langle \cos\theta, 0, \sin\theta \rangle \, dy d\theta \\ &= \int_0^{2\pi} \int_0^{2-\cos\theta} \left(\cos^2\theta + 5\sin\theta \right) dy d\theta \\ &= \int_0^{2\pi} \left(\cos^2\theta + 5\sin\theta \right) \left(2 - \cos\theta \right) d\theta \\ &= \int_0^{2\pi} \left(2\cos^2\theta - \cos^3\theta + 10\sin\theta - 5\sin\theta\cos\theta \right) d\theta\end{aligned}$$

$$= \int_0^{2\pi} \left(2\cos^2\theta - \cos^3\theta + 10\sin\theta - 5\sin\theta\cos\theta \right) d\theta$$

$$= \int_0^{2\pi} \left[\left(1 + \cos 2\theta\right) - \frac{3\cos\theta + \cos 3\theta}{4} + 10\sin\theta - \frac{5}{2}\sin 2\theta \right] d\theta$$

$$= \int_0^{2\pi} \left[\left(1 + \cos 2\theta\right) - \frac{3\cos\theta + \cos 3\theta}{4} + 10\sin\theta - \frac{5}{2}\sin 2\theta \right] d\theta$$

Use $\int_0^{2\pi} \left(1 + \cos 2\theta\right) d\theta = 2\pi, \int_0^{2\pi} \left(\frac{3\cos\theta + \cos 4\theta}{4} \right) d\theta = 0, \int_0^{2\pi} \left(10\sin\theta - \frac{5}{2}\sin 2\theta \right) d\theta = 0$

So the integral value is,

$$\iint_S \mathbf{F} \cdot dS = 2\pi$$

Parametrize the plane $y = 0$ by

$$r(x,z) = \langle x, 0, z \rangle, x^2 + z^2 \le 1.$$

The partial derivatives of the vector $r(x,z) = \langle x, 0, z \rangle$ are calculated as,

$$r_x = \langle 1, 0, 0 \rangle, r_z = \langle 0, 0, 1 \rangle.$$

The cross product of r_x and r_z is,

$$\begin{aligned}r_x \times r_z &= \langle 1, 0, 0 \rangle \times \langle 0, 0, 1 \rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \langle 0, -1, 0 \rangle\end{aligned}$$

To have S oriented outward, the plane $y = 0$ is oriented in the negative $y-$direction.

So, the correct orientation is

$$\begin{aligned}\iint_S \mathbf{F} \cdot dS &= \iint_D \langle x, 0, 5 \rangle \cdot \langle 0, -1, 0 \rangle \, dx dy \\ &= \iint_D 0 \cdot dx dy \\ &= 0\end{aligned}$$

Parametrize the plane $x + y = 1$ by

$$r(x,z) = \langle x, 2 - x, z \rangle, x^2 + z^2 \le 1.$$

The partial derivative of the above vector is

$$r_x = \langle 1, -1, 0 \rangle, r_z = \langle 0, 0, 1 \rangle.$$

The cross product of the these two vectors is

$$\begin{aligned}r_x \times r_z &= \langle 1, -1, 0 \rangle \times \langle 0, 0, 1 \rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \langle -1, -1, 0 \rangle\end{aligned}$$

To have S oriented outward, the plane $x+y=1$ is oriented in the positive $y-$direction.

Need to use $r_x \times r_z = \langle 1,1,0 \rangle$.

So the correct orientation is

$$\begin{aligned}\iint_S \mathbf{F}\cdot dS &= \iint_D \langle x, 2-x, 5\rangle \cdot \langle 1,1,0\rangle\, dxdy \\ &= \iint_D 2dxdy \\ &= 2\pi\end{aligned}$$

Add the three results obtained in the above, the flux of $\mathbf{F}$ through the surface S is

$$\begin{aligned}\iint_S \mathbf{F}\cdot dS &= 2\pi + 0 + 2\pi \\ &= \boxed{4\pi}\end{aligned}$$

Chapter 16 Vector Calculus Exercise 16.7 31E

Consider the following vector filed.

$$\mathbf{F}(x,y,z) = x^2\ \mathbf{i} + y^2\ \mathbf{j} + z^2\ \mathbf{k}$$

The objective is to evaluate the surface integral $\iint_S \mathbf{F} \bullet d\mathbf{S}$ for the given vector field **F** and the oriented surface *S*. Where, *S* is the boundary of the solid half-cylinder $0 \le z \le \sqrt{1-y^2}$, $0 \le x \le 2$.

The surface integral $\iint_S \mathbf{F} \bullet d\mathbf{S}$ of a vector field **F** over a parametric surface by a vector function $\mathbf{r}(u,v)$, we use the following formula where *D* is the parameter domain of the vector function:

$$\iint_S \mathbf{F} \bullet d\mathbf{S} = \iint_D \mathbf{F} \bullet (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

The surface integral $\iint_S \mathbf{F} \bullet d\mathbf{S}$ of a vector field **F** over a surface given by the equation $z = g(x,y)$, we use the following formula to calculate its surface integral, where *P*, *Q*, and *R* are the component functions of F in the **i**, **j**, **k** directions, and using the partial derivatives of *g* with respect to both *x* and *y*:

$$\iint_S \mathbf{F} \bullet d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

Recall from the fundamental theorem of calculus that

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where *F(x)* is the antiderivative of *f(x)*.

In this particular problem we have the following four surfaces

$$S_1 : z = \sqrt{1-y^2}, 0 \le x \le 2, -1 \le y \le 1,\ S_2 : z = 0, 0 \le x \le 2, -1 \le y \le 1,$$
$$S_3 : x = 2, -1 \le y \le 1, 0 \le z \le \sqrt{1-y^2},\ S_4 : x = 0, -1 \le y \le 1, 0 \le z \le \sqrt{1-y^2}$$

First we setup the surface integral for S_1, which is the graph

$$z = \sqrt{1-y^2}, 0 \le x \le 2, -1 \le y \le 1$$

Calculating the partial derivatives of $z = \sqrt{1-y^2}$ we have

$$\frac{\partial z}{\partial x} = 0, \frac{\partial z}{\partial y} = -y\left(1-y^2\right)^{-\frac{1}{2}}$$

Next ,determine $-P\frac{\partial g}{\partial x}-Q\frac{\partial g}{\partial y}+R$

$$\mathbf{F}(x,y,z)=\left\langle x^2,y^2,z^2\right\rangle$$

$$\begin{aligned}-P\frac{\partial g}{\partial x}-Q\frac{\partial g}{\partial y}+R&=-\left(-x^2(0)\right)-\left(y^2\left(-y\left(1-y^2\right)^{-\frac{1}{2}}\right)\right)+z^2\\&=y^3\left(1-y^2\right)^{-\frac{1}{2}}+z^2\\&=y^3\left(1-y^2\right)^{-\frac{1}{2}}+\left(1-y^2\right)\end{aligned}$$

Now, write the surface integral

$$\begin{aligned}\iint_{S_1}\mathbf{F}\cdot d\mathbf{S}&=\iint_D\left(-P\frac{\partial g}{\partial x}-Q\frac{\partial g}{\partial y}+R\right)dA\\&=\iint_D y^3\left(1-y^2\right)^{-\frac{1}{2}}+\left(1-y^2\right)\ dA\\&=\int_0^2\int_{-1}^1 y^3\left(1-y^2\right)^{-\frac{1}{2}}+\left(1-y^2\right)dy\ dx\end{aligned}$$

Before evaluating the integral, we need to find the anti-derivative of $y^3\left(1-y^2\right)^{-\frac{1}{2}}$. Using the substitution $u=1-y^2, du=-2y\ dy$, we have

$$\begin{aligned}\int y^3\left(1-y^2\right)^{-\frac{1}{2}}\ dy&=\int(1-u)u^{-\frac{1}{2}}\left(-\frac{1}{2}du\right)\\&=\frac{1}{2}\int u^{\frac{1}{2}}-u^{-\frac{1}{2}}\ du\\&=\frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}-2u^{\frac{1}{2}}+C\right]\\&=\frac{1}{3}u^{\frac{3}{2}}-u^{\frac{1}{2}}+C\end{aligned}$$

Reverting back to the original integral we have

$$\int y^3\left(1-y^2\right)^{-\frac{1}{2}}\ dy=\frac{1}{3}\left(1-y^2\right)^{\frac{3}{2}}-\left(1-y^2\right)^{\frac{1}{2}}+C$$

Evaluating the integral we have

$$\begin{aligned}\int_0^2\int_{-1}^1 y^3\left(1-y^2\right)^{-\frac{1}{2}}+\left(1-y^2\right)dy\ dx&=2\left[-\sqrt{1-y^2}+\frac{1}{3}\left(1-y^2\right)^{\frac{3}{2}}+y-\frac{1}{3}y^3\right]_{y=-1}^{y=1}\\&=2\left[1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right]\\&=2\left[\frac{4}{3}\right]\\&=\frac{8}{3}\end{aligned}$$

On S_2 we have $\mathbf{F}=\langle x^2, y^2, 0\rangle$, $\mathbf{n}=\langle 0,0,-1\rangle$ and

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S}=\iint_{S_2} 0\, dS=0$$

On S_3 we have $\mathbf{F}=\langle 4, y^2, z^2\rangle$, $\mathbf{n}=\langle 1,0,0\rangle$, and

$$\iint_{S_3} \mathbf{F}\cdot d\mathbf{S}=\iint_D \mathbf{F}\cdot\mathbf{n}\, dA=\int_{-1}^{1}\int_0^{\sqrt{1-y^2}} 4\, dz\, dy=4\left(\frac{1}{2}\right)\pi(1)^2=2\pi$$

On S_4 we have $\mathbf{F}=\langle 0, y^2, z^2\rangle$, $\mathbf{n}=\langle -1,0,0\rangle$ and

$$\iint_{S_4} \mathbf{F}\cdot d\mathbf{S}=\iint_{S_4} 0\, dS=0$$

Now, evaluate the surface integral

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=\sum_{i=1}^{4}\iint_{S_i} \mathbf{F}\cdot d\mathbf{S}=\frac{8}{3}+0+2\pi+0=\boxed{\frac{8}{3}+2\pi}$$

Thus, the integral is $\iint_S \mathbf{F}\cdot d\mathbf{S}=\boxed{\frac{8}{3}+2\pi}$.

Chapter 16 Vector Calculus Exercise 16.732E

We split the tetrahedron into its four sides, calculate the surface integral of each, and add them together.

We will need the following formula for calculating the surface integrals of a positively oriented surface:

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=\iint_D\left(-P\frac{\partial g}{\partial x}-Q\frac{\partial g}{\partial y}+R\right)dA \quad \ldots\ldots (1)$$

Where $\mathbf{F}=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ and the surface S is given by the graph $z=g(x,y)$ over the region D in the xy-plane (with normal vector of the surface pointing in the positive z-direction).
Note that this can also be written symmetrically for the other coordinate axes as

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=\iint_D\left(-P\frac{\partial h}{\partial x}+Q-R\frac{\partial h}{\partial z}\right)dA \quad \ldots\ldots (2)$$

Where $\mathbf{F}=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ and the surface S is given by the graph $y=h(x,z)$ over the region D in the xz-plane (with normal vector of the surface pointing in the positive y-direction), and

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=\iint_D\left(P-Q\frac{\partial k}{\partial y}-R\frac{\partial k}{\partial z}\right)dA \quad \ldots\ldots (3)$$

Where $\mathbf{F}=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ and the surface S is given by the graph $x=k(x,z)$ over the region D in the yz-plane (with normal vector of the surface pointing in the positive x-direction).

We first divide the region into its four faces. Three of the faces are the coordinate planes, $x=0$, $y=0$, and $z=0$. The fourth (slanted) face has a normal vector that points out along the three-dimensional diagonal, a normal vector of $\langle 1,1,1\rangle$. The Cartesian equation for the plane is therefore
$x+y+z=d$

Since the components of the normal vector are the coefficients of the variables in the equation. We plug in a point in the plane that is given, say, (1, 0, 0), to find that $d=1$, so we have the equation
$x+y+z=1$
As the equation for the slanted face of the tetrahedron.

We look at the slanted face first.
Since the surface is oriented positively, the normal vectors are all pointed outward, and the slanted face therefore has an upward (positive) orientation when taken separately.
We rewrite the equation for this plane as $z=1-x-y$ and use this as the equation for the surface $g(x,y)$ when applying (1). Find the necessary partial derivatives:

$$\frac{\partial g}{\partial x}=-1$$
$$\frac{\partial g}{\partial y}=-1$$

The region of integration D beneath $g(x,y)$ will be the projection in the xy-plane, which is the same shape as the triangular base of the tetrahedron. The y limit will range from $y=0$ to the diagonal $y=1-x$ and the x limit will range from 0 to 1.

Plug into (1) with $\mathbf{F}(x,y,z)=y\mathbf{i}+(z-y)\mathbf{k}+x\mathbf{k}$ (given):

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \left(-P\frac{\partial g}{\partial x}-Q\frac{\partial g}{\partial y}+R\right)dA\\ &= \int_0^1\int_0^{1-x}(-y(-1)-(z-y)(-1)+x)dydx\\ &= \int_0^1\int_0^{1-x}(y+z-y+x)dydx\\ &= \int_0^1\int_0^{1-x}(z+x)dydx\end{aligned}$$

Plug in the equation for z, $z=1-x-y$:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^1\int_0^{1-x}(1-x-y+x)dydx\\ &= \int_0^1\int_0^{1-x}(1-y)dydx\end{aligned}$$

Integrate in terms of y:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^1\left(y-\frac{y^2}{2}\right)\Bigg|_0^{1-x}dx\\ &= \int_0^1\left((1-x)-\frac{(1-x)^2}{2}-0\right)dx\\ &= \int_0^1\left(\frac{2-2x}{2}-\frac{1-2x+x^2}{2}\right)dx\\ &= \frac{1}{2}\int_0^1\left(2-2x-1+2x-x^2\right)dx\\ &= \frac{1}{2}\int_0^1\left(1-x^2\right)dx\end{aligned}$$

Integrate in terms of x:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \frac{1}{2}\left(x-\frac{x^3}{3}\right)\Bigg|_0^1\\ &= \frac{1}{2}\left(1-\frac{1^3}{3}-0\right)\\ &= \frac{1}{3}\end{aligned}$$

The flux through the slanted face is 1/3.

Next we look at the base of the tetrahedron, where $z=0$.
Since the surface is oriented positively, the normal vectors are all pointed outward, and the base of the tetrahedron therefore has a downward (negative) orientation when taken separately. We will therefore have to add a negative when applying (1) to account for this negative orientation.

The base of the tetrahedron is in the plane $z=0$; we use this as the equation for the surface $g(x,y)$ when applying (1). Find the necessary partial derivatives:

$$\frac{\partial g}{\partial x}=0$$
$$\frac{\partial g}{\partial y}=0$$

The region of integration D beneath $g(x,y)$ is just itself, the triangular base of the tetrahedron. The y limit will range from $y=0$ to the diagonal $y=1-x$ and the x limit will range from 0 to 1.

Plug into (1) with $\mathbf{F}(x,y,z)=y\mathbf{i}+(z-y)\mathbf{k}+x\mathbf{k}$ (given), remembering to add a negative for the negative orientation:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$
$$= -\int_0^1\int_0^{1-x} (-y(0)-(z-y)(0)+x)\,dydx$$
$$= -\int_0^1\int_0^{1-x} (x)\,dydx$$

Integrate in terms of y:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\int_0^1 (xy)\Big|_0^{1-x} dx$$
$$= -\int_0^1 (x(1-x)-0)\,dx$$
$$= -\int_0^1 (x-x^2)\,dx$$

Integrate in terms of x:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Bigg|_0^1$$
$$= -\left(\frac{1^2}{2}-\frac{1^3}{3}-0\right)$$
$$= -\frac{1}{6}$$

The flux through the base face is -1/6.

Next we look at the side of the tetrahedron where $y=0$. Since there is no way to write this equation in terms of z, we want to apply formula (2) instead.

Since the surface is oriented positively, the normal vectors are all pointed outward and therefore the normal vectors for the $y=0$ side point out through it from the positive y and therefore point in the negative y-direction. The surface is therefore oriented in a downward (negative) orientation when taken separately and looked at in the y-direction. We will therefore have to add a negative when applying (2) to account for this negative orientation.

The equation for this side of the tetrahedron is in the plane $y=0$; we use this as the equation for the surface $h(x,y)$ when applying (2). Find the necessary partial derivatives:

$$\frac{\partial h}{\partial x}=0$$
$$\frac{\partial h}{\partial z}=0$$

The region of integration D is just the surface itself, the triangular side of the tetrahedron in the xz-plane. The z limit will range from $z=0$ to the diagonal $z=1-x$ and the x limit will range from 0 to 1.

Plug into (2) with $\mathbf{F}(x,y,z)=y\mathbf{i}+(z-y)\mathbf{k}+x\mathbf{k}$ (given), remembering to add a negative for the negative orientation:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D \left(-P\frac{\partial h}{\partial x} + Q - R\frac{\partial h}{\partial z}\right) dA$$
$$= -\int_0^1\int_0^{1-x} (-y(0)+(z-y)-x(0))\,dzdx$$
$$= -\int_0^1\int_0^{1-x} (z-y)\,dzdx$$

Since $y=0$ in this case, this becomes:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\int_0^1\int_0^{1-x} (z-0)\,dzdx$$
$$= -\int_0^1\int_0^{1-x} (z)\,dzdx$$

Integrate in terms of z:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\int_0^1 \left(\frac{z^2}{2}\right)\Bigg|_0^{1-x} dx$$
$$= -\int_0^1 \left(\frac{(1-x)^2}{2} - 0\right)dx$$
$$= -\frac{1}{2}\int_0^1 \left(1-2x+x^2\right)dx$$

Integrate in terms of x:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\frac{1}{2}\left(x - x^2 + \frac{x^3}{3}\right)\Bigg|_0^1$$
$$= -\frac{1}{2}\left(1 - 1^2 + \frac{1^3}{3} - 0\right)$$
$$= -\frac{1}{6}$$

The flux through this face is -1/6.

Next we look at the side of the tetrahedron where $x = 0$. Since there is no way to write this equation in terms of z, we want to apply formula (3) instead.

Since the surface is oriented positively, the normal vectors are all pointed outward and therefore the normal vectors for the $x = 0$ side point out through it from the positive x and therefore point in the negative x-direction. The surface is therefore oriented in a downward (negative) orientation when taken separately and looked at in the x-direction. We will therefore have to add a negative when applying (3) to account for this negative orientation.

The equation for this side of the tetrahedron is in the plane $x = 0$; we use this as the equation for the surface $k(x,y)$ when applying (3). Find the necessary partial derivatives:

$$\frac{\partial k}{\partial y} = 0$$
$$\frac{\partial k}{\partial z} = 0$$

The region of integration D is just the surface itself, the triangular side of the tetrahedron in the yz-plane. The z limit will range from $z = 0$ to the diagonal $z = 1 - y$ and the y limit will range from 0 to 1.

Plug into (3) with $\mathbf{F}(x,y,z) = y\mathbf{i} + (z-y)\mathbf{k} + x\mathbf{k}$ (given), remembering to add a negative for the negative orientation:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D \left(P - Q\frac{\partial k}{\partial y} - R\frac{\partial k}{\partial z}\right)dA$$
$$= -\int_0^1\int_0^{1-y} (y - (z-y)(0) - x(0))dzdy$$
$$= -\int_0^1\int_0^{1-y} (y)dzdy$$

Integrate in terms of z:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\int_0^1 (yz)\Big|_0^{1-y} dy$$
$$= -\int_0^1 (y(1-y) - 0)dy$$
$$= -\int_0^1 \left(y - y^2\right)dy$$

Integrate in terms of y:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^1$$
$$= -\left(\frac{1}{2} - \frac{1^3}{3} - 0\right)$$
$$= -\frac{1}{6}$$

The flux through this face is -1/6.

The total flux through the surface is the sum of the surface integrals of all four faces, or

$$\frac{1}{3}-3\left(\frac{1}{6}\right)=\boxed{-\frac{1}{6}}$$

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The formula for the surface integral when parameterized in x and y is:

$$\iint_S f(x,y,z)dS=\iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA \quad \ldots\ldots (1)$$

Where S is a surface over domain D.

The surface is $z=xy$. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$\frac{\partial z}{\partial x}=y$$
$$\frac{\partial z}{\partial y}=x$$

The limits of integration are $0\le x\le 1$ and $0\le y\le 1$. We now have everything necessary to plug into (1). We plug in $f(x,y,z)=x^2yz$, the equation $z=xy$, the partial derivatives of z, and the limits of integration and simplify:

$$\iint_S\left(x^2yz\right)dS=\int_0^1\int_0^1\left(x^2y(xy)\right)\sqrt{1+(y)^2+(x)^2}\,dydx$$
$$=\int_0^1\int_0^1\left(x^3y^2\right)\sqrt{1+y^2+x^2}\,dydx$$

The problem indicates we will need to use computer software or a calculator to solve this but specifies an exact solution. Plug in the integral to find that the exact solution to the surface integral is

$$\boxed{\frac{-\left(15\ln\left(256\left(10864\sqrt{3}+18817\right)\left(\sqrt{2}+1\right)\right)-120\ln 2-48\sqrt{3}-317\sqrt{2}\right)}{2880}}$$

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Given that $z=3-2x^2-y^2$

we have to find the value of

$$\frac{\partial z}{\partial x}=-4x,\quad \frac{\partial z}{\partial y}=-2y$$

The boundaries of the region are $3-2x^2-y^2\ge 0$, $-\sqrt{\frac{3}{2}}\le x\le\sqrt{\frac{3}{2}}$, and $-\sqrt{3-2x^2}\le y\le\sqrt{3-2x^2}$.

Using a CAS, we calculate

$$=\int_{-\sqrt{3/2}}^{\sqrt{3/2}}\int_{-\sqrt{3-2x^2}}^{\sqrt{3-2x^2}}x^2y^2\left(3-2x^2-y^2\right)^2\sqrt{16x^2+4y^2+1}\,\mathrm{d}y\,\mathrm{d}x$$

$$=3.4895$$

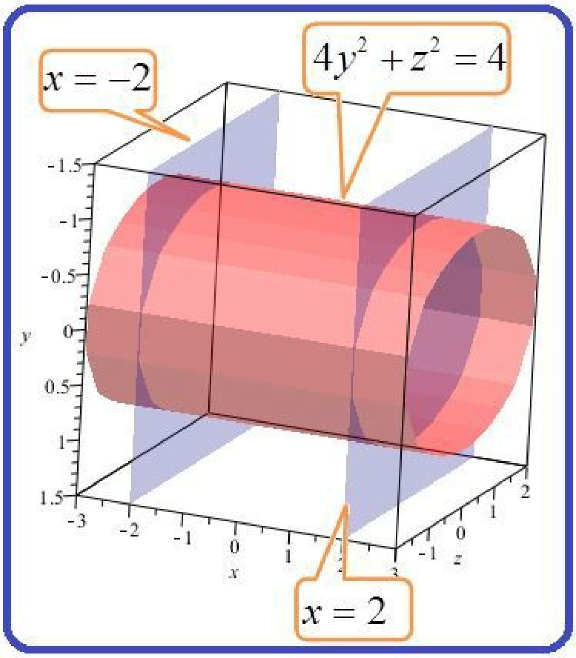
Consider the vector filed

$$\mathbf{F}(x,y,z)=\sin(xyz)\,\mathbf{i}+x^2y\,\mathbf{j}+z^2e^{x/5}\,\mathbf{k}$$

And the upward oriented surface *S* consist a cylinder $4y^2+z^2=4$ that lies above the *xy*-plane and between the planes $x=-2$ and $x=2.$

Determine the flux of **F** across the surface *S*.

The oriented surface *S* is shown below:



Parameterize the surface *S*:

Here the cylinder $4y^2+z^2=4$ is an elliptical cylinder.

$$y^2+\frac{z^2}{4}=1.$$

Also, use *x* and θ as parameters.

Then the parametric equations are:

$$x=x \quad y=\cos\theta \quad z=2\sin\theta$$

$$\mathbf{r}(x,\theta)=\langle x,\cos\theta,2\sin\theta\rangle$$

But, the cylinder $4y^2+z^2=4$ that lies above the *xy*-plane and between the planes $x=-2$ and $x=2.$

So,

$$-2\le x\le 2, 0\le\theta\le 2\pi.$$

Next find $\mathbf{r}_x$ and $\mathbf{r}_\theta$:

$$\mathbf{r}_x(x,\theta) = \frac{\partial}{\partial x}\langle x, \cos\theta, 2\sin\theta\rangle$$
$$= \langle 1,0,0\rangle$$

And

$$\mathbf{r}_\theta(x,\theta) = \frac{\partial}{\partial \theta}\langle x, \cos\theta, 2\sin\theta\rangle$$
$$= \langle 0, -\sin\theta, 2\cos\theta\rangle$$

Then $\mathbf{r}_x \times \mathbf{r}_\theta$:

$$\mathbf{r}_x \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & -\sin\theta & 2\cos\theta \end{vmatrix}$$
$$= \mathbf{i}(2\cos\theta\cdot 0 + 0\cdot\sin\theta) - \mathbf{j}(2\cos\theta\cdot 1 - 0) + \mathbf{k}(-\sin\theta\cdot 1 - 0\cdot 0)$$
$$= \mathbf{i}(0) - \mathbf{j}(2\cos\theta) - \mathbf{k}\sin\theta$$
$$= \langle 0, -2\cos\theta, -\sin\theta\rangle$$

The vector filed is defined as

$$\mathbf{F}(x,y,z) = \langle \sin(xyz), x^2y + z^2e^{x/5}\rangle$$
$$\mathbf{F}(\mathbf{r}(x,\theta)) = \langle \sin(x\cdot\cos\theta\cdot 2\sin\theta), x^2\cos\theta, (2\sin\theta)^2 e^{x/5}\rangle \text{ Here } y=1$$
$$= \langle \sin(x\sin 2\theta), x^2\cos\theta, 4e^{x/5}\sin^2\theta\rangle$$

Find $\mathbf{F}\cdot(\mathbf{r}_x \times \mathbf{r}_\theta)$:

$$\langle \sin(x\sin 2\theta), x^2\cos\theta, 4e^{x/5}\sin^2\theta\rangle \cdot \langle 0, -2\cos\theta, -\sin\theta\rangle$$
$$= 0\cdot\sin(x\sin 2\theta) + (x^2\cos\theta)\cdot(-2\cos\theta) + (4e^{x/5}\sin^2\theta)\cdot(-\sin\theta)$$
$$= 0 - 2x^2\cos^2\theta - 4e^{x/5}\sin^3\theta$$
$$= -2x^2\cos^2\theta - 4e^{x/5}\sin^3\theta$$

Recollect, suppose **F** is a continuous vector filed defined on an oriented surface *S* given by a vector function $\mathbf{r}(u,v)$.

Then, the **surface integral of F over *S*** is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Here the parameter domain *D* is given by

$D = \{(x,\theta) \mid -2 \le x \le 2, 0 \le \theta \le 2\pi\}$.

Find the surface integral of **F** over ***S***:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_x \times \mathbf{r}_\theta)\,dA$$

$$= \int_0^{2\pi}\int_{-2}^{2}\left(-2x^2\cos^2\theta - 4e^{x/5}\sin^3\theta\right)dx\,d\theta$$

$$= -2\int_0^{2\pi}\cos^2\theta\int_{-2}^{2}x^2dx\,d\theta - 4\int_0^{2\pi}\sin^3\theta\int_{-2}^{2}e^{x/5}\,dx\,d\theta$$

$$= -2\int_0^{2\pi}\cos^2\theta\left[\frac{x^3}{3}\right]_{-2}^{2}d\theta - 4\int_0^{2\pi}\sin^3\theta\left[\frac{e^{x/5}}{1/5}\right]_{-2}^{2}d\theta$$

$$= -2\int_0^{2\pi}\cos^2\theta\left[\frac{8}{3}-\frac{(-8)}{3}\right]d\theta - 4\int_0^{2\pi}\sin^3\theta\left[5e^{2/5}-5e^{-2/5}\right]d\theta$$

$$= -2\cdot\frac{16}{3}\int_0^{2\pi}\cos^2\theta d\theta - 20\left(e^{2/5}-e^{-2/5}\right)\int_0^{2\pi}\sin^3\theta\,d\theta$$

Use the facts that $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ and $\sin^2\theta = 1-\cos^2\theta$.

$$= -\frac{32}{3}\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta - 20\left(e^{2/5}-e^{-2/5}\right)\int_0^{2\pi}\sin\theta\sin^2\theta\,d\theta$$

$$= -\frac{16}{3}\int_0^{2\pi}(1+\cos 2\theta)d\theta - 20\left(e^{2/5}-e^{-2/5}\right)\int_0^{2\pi}\sin\theta\left(1-\cos^2\theta\right)d\theta$$

$$= -\frac{16}{3}\int_0^{2\pi}(1+\cos 2\theta)d\theta - 20\left(e^{2/5}-e^{-2/5}\right)\int_0^{2\pi}\left(\sin\theta-\sin\theta\cos^2\theta\right)d\theta$$

$$= -\frac{16}{3}\int_0^{2\pi}(1+\cos 2\theta)d\theta - 20\left(e^{2/5}-e^{-2/5}\right)\left[\int_0^{2\pi}(\sin\theta)\,d\theta - \int_0^{2\pi}\sin\theta\cos^2\theta\,d\theta\right]$$

Continuation of above simplification:

$$= -\frac{16}{3}\left(\theta+\frac{\sin 2\theta}{2}\right)_0^{2\pi} - 20\left(e^{2/5}-e^{-2/5}\right)\left[\left(-\cos\theta\right)_0^{2\pi} - \left(\frac{\cos^3\theta}{3}\right)_0^{2\pi}\right]$$

$$= -\frac{16}{3}\left(\theta+\frac{\sin 2\theta}{2}\right)_0^{2\pi} + 20\left(e^{2/5}-e^{-2/5}\right)\left[\left(\cos\theta\right)_0^{2\pi} + \left(\frac{\cos^3\theta}{3}\right)_0^{2\pi}\right]$$

$$= -\frac{16}{3}\left(2\pi+\frac{\sin 4\pi}{2}-0-\frac{\sin 2(0)}{2}\right)$$

$$+20\left(e^{2/5}-e^{-2/5}\right)\left[\cos 2\pi - \cos(0) + \left(\frac{\cos^3 2\pi}{3}\right) - \left(\frac{\cos^3(0)}{3}\right)\right]$$

$$= -\frac{16}{3}(2\pi+0-0-0) + 20\left(e^{2/5}-e^{-2/5}\right)\left[1-1+\left(\frac{1}{3}\right)-\left(\frac{1}{3}\right)\right]$$

$$= -\frac{16}{3}(2\pi) + 20\left(e^{2/5}-e^{-2/5}\right)[0]$$

$$= -\frac{32}{3}\pi$$

Therefore, the flux of **F** across the surface *S* is $\boxed{-\frac{32}{3}\pi}$.

Next, use Maple software to sketch the cylinder and the vector filed on same screen:

Step 1: Open the worksheet mode in the Maple.

Next, upload with(plots) and with(VectorCalculus) packages.

Input command:

with(plots);

with(VectorCalculus);

The input and output will be displayed as shown:

Output:

> *with*(*plots*);

[*animate*, *animate3d*, *animatecurve*, *arrow*, *changecoords*, *complexplot*, *complexplot3d*, *conformal*, *conformal3d*, *contourplot*, *contourplot3d*, *coordplot*, *coordplot3d*, *densityplot*, *display*, *dualaxisplot*, *fieldplot*, *fieldplot3d*, *gradplot*, *gradplot3d*, *implicitplot*, *implicitplot3d*, *inequal*, *interactive*, *interactiveparams*, *intersectplot*, *listcontplot*, *listcontplot3d*, *listdensityplot*, *listplot*, *listplot3d*, *loglogplot*, *logplot*, *matrixplot*, *multiple*, *odeplot*, *pareto*, *plotcompare*, *pointplot*, *pointplot3d*, *polarplot*, *polygonplot*, *polygonplot3d*, *polyhedra_supported*, *polyhedraplot*, *rootlocus*, *semilogplot*, *setcolors*, *setoptions*, *setoptions3d*, *spacecurve*, *sparsematrixplot*, *surfdata*, *textplot*, *textplot3d*, *tubeplot*]

Output:

> *with*(*VectorCalculus*);

[&x, `*`, `+`, `-`, `.`, <,>, <|>, *About*, *AddCoordinates*, *ArcLength*, *BasisFormat*, *Binormal*, *Compatibility*, *ConvertVector*, *CrossProduct*, *Curl*, *Curvature*, D, *Del*, *DirectionalDiff*, *Divergence*, *DotProduct*, *Flux*, *GetCoordinateParameters*, *GetCoordinates*, *GetNames*, *GetPVDescription*, *GetRootPoint*, *GetSpace*, *Gradient*, *Hessian*, *IsPositionVector*, *IsRootedVector*, *IsVectorField*, *Jacobian*, *Laplacian*, *LineInt*, *MapToBasis*, *Nabla*, *Norm*, *Normalize*, *PathInt*, *PlotPositionVector*, *PlotVector*, *PositionVector*, *PrincipalNormal*, *RadiusOfCurvature*, *RootedVector*, *ScalarPotential*, *SetCoordinateParameters*, *SetCoordinates*, *SpaceCurve*, *SurfaceInt*, *TNBFrame*, *Tangent*, *TangentLine*, *TangentPlane*, *TangentVector*, *Torsion*, *Vector*, *VectorField*, *VectorPotential*, *VectorSpace*, *Wronskian*, *diff*, *eval*, *evalVF*, *int*, *limit*, *series*]

Next use the filedplot3d to draw the vector filed for **F** and use implicitplot3d to draw the surface of the cylinder $4y^2 + z^2 = 4$.

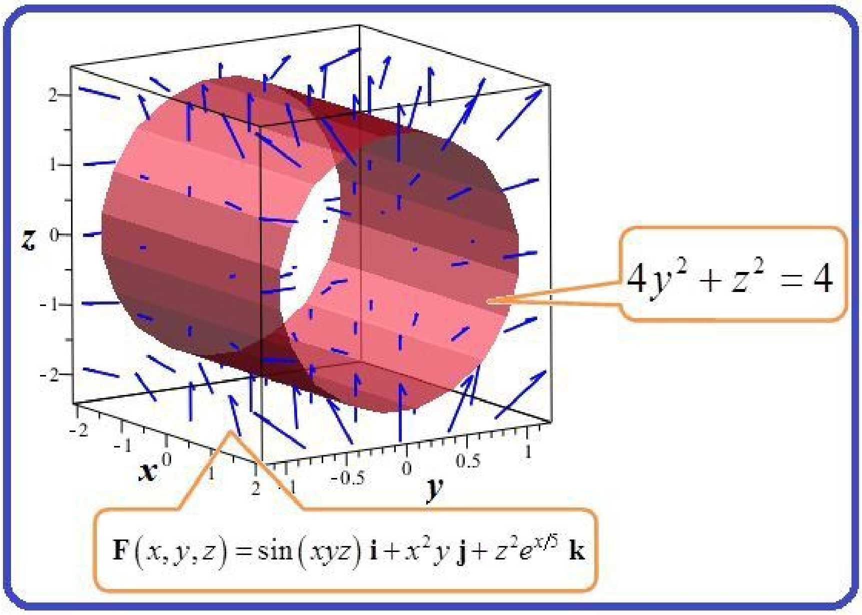
To get both vector filed and cylinder on the same screen use display command.

Input command:

display([fieldplot3d(,x=-2..2,y=-1..1,z=-2..2,grid=[5,5,5],color=blue)], [implicitplot3d(4y^2+z^2=4,x=-2..2,y=-1..1,z=-2..2,grid=[8,8,8])]);

The input and output will be displayed as shown:

$$display\left(\left[fieldplot3d\left(\left\langle \sin(x\cdot y\cdot z), x^2\cdot y, z^2\cdot\exp\left(\frac{x}{5}\right)\right\rangle, x=-2..2, y=-1..1, z=-2..2, grid=[5, 5, 5], color=blue\right)\right], \left[implicitplot3d\left(4y^2+z^2=4, x=-2..2, y=-1..1, z=-2..2, grid=[8, 8, 8]\right)\right]\right);$$



Chapter 16 Vector Calculus Exercise 16.7 37E

The surface S is given by y = h(x, z). We can find $\hat{n}$, the unit normal vector to S, by considering $f(x,y,z) = y - h(x,z)$, where S is a level surface to $f(x,y,z) = 0$

Let (x, y, z) be any point on the surface. Then $\vec{\nabla} f(x,y,z)$ is normal to the surface S_1 and then the unit normal vector is

$$\hat{n} = \frac{\vec{\nabla} f(x,y,z)}{\left\|\vec{\nabla} f(x,y,z)\right\|}$$

$$= \frac{-h_x(x,z)\hat{i} + \hat{j} - h_z(x,z)\hat{k}}{\sqrt{\left[h_x(x,z)\right]^2 + 1 + \left[h_z(x,z)\right]^2}}$$

Since $\hat{j}$ component is positive, this is an outward unit normal vector.

So the unit normal vector to the left is $-\hat{n} = \dfrac{h_x(x,z)\hat{i} - \hat{j} + h_z(x,z)\hat{k}}{\sqrt{\left[h_x(x,z)\right]^2 + 1 + \left[h_z(x,z)\right]^2}}$

Take $\vec{F}(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$

Then $\iint_S \vec{F}.d\vec{s} = \iint_S \vec{F}.\hat{n}\,ds$

$$= \iint_D \left(P\hat{i} + Q\hat{j} + R\hat{k}\right).\frac{h_x(x,z)\hat{i} - \hat{j} + h_z(x,z)\hat{k}}{\sqrt{h_x^2 + h_z^2 + 1}}$$

$$\times\sqrt{\left(\frac{\partial h}{\partial x}\right) + \left(\frac{\partial h}{\partial z}\right)^2 + 1}\; dA$$

$$= \iint_D \left(P\frac{\partial h}{\partial x} - Q + R\frac{\partial h}{\partial z}\right) dA$$

Where D is projection on xz – plane

Chapter 16 Vector Calculus Exercise 16.738E

The surface S is given by $x = k(y,z)$. We can find $\hat{n}$, the unit normal vector to S, by considering $f(x,y,z) = x - k(y,z)$. Where S is a level surface to $f(x,y,z) = 0$

Let (x,y,z) be any point on the surface.

Then $\vec{\nabla} f(x,y,z)$ is normal to the surface S_1 and then the unit normal vector is:

$$\hat{n} = \frac{\vec{\nabla} f(x,y,z)}{\left\|\vec{\nabla} f(x,y,z)\right\|}$$
$$= \frac{\hat{i} - k_y(y,z)\hat{j} - k_z(y,z)\hat{k}}{\sqrt{1+k_y^2+k_z^2}}$$

Since $\hat{i}$ component is positive, this is an outward unit normal vector.

Take $\vec{F}(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$

Then $\iint_S \vec{F}.d\vec{s} = \iint_S \vec{F}.\hat{n}\,ds$

$$= \iint_D \left(P\hat{i} + Q\hat{j} + R\hat{k}\right). \frac{\hat{i} + k_y\hat{j} - k_z(y,z)\hat{k}}{\sqrt{1+k_y^2+k_z^2}} \times$$
$$\times\sqrt{\left(\frac{\partial k}{\partial y}\right)^2 + \left(\frac{\partial k}{\partial z}\right)^2 + 1}\; dA$$
$$= \iint_D \left(P - Q\frac{\partial k}{\partial y} - R\frac{\partial k}{\partial z}\right) dA$$

Where D is projection on yz – plane

Chapter 16 Vector Calculus Exercise 16.7 39E

$\rho(x,y,z) =$ constant k (say)

Then $m = \iint_S \rho(x,y,z)\,dS$

Where S is the hemisphere $x^2 + y^2 + z^2 = a^2, \quad z \ge 0$

Then $m = k\iint_S ds$

Now the surface S is the region under $z = \sqrt{a^2 - (x^2+y^2)}$

And above the circle $x^2 + y^2 = a^2$

Now $\frac{\partial z}{\partial x} = -\frac{x}{z}$ and $\frac{\partial z}{\partial y} = -\frac{y}{z}$

Then $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}$

$$= \sqrt{\frac{x^2+y^2+z^2}{z^2}}$$
$$= \frac{a}{z}$$

Then $m = k\iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dA$

i.e. $m = k\iint_D \frac{a}{z} dA$

$$= k\iint_D \frac{a}{\sqrt{a^2 - (x^2+y^2)}} dA$$

Changing to polar co – ordinates

$$m = ak\int_0^{2\pi}\int_0^{a}\left(a^2 - r^2\right)^{-1/2} r\,dr\,d\theta$$

$$= -ak\left(\theta\right)_0^{2\pi}\left(\sqrt{a^2 - r^2}\right)_{r=0}^{r=a}$$

$$= -ak\left(2\pi\right)\left(-a\right)$$

i.e. $m = 2\pi a^2 k$

Changing to polar co – ordinates

$$m = ak\int_0^{2\pi}\int_0^{a}\left(a^2 - r^2\right)^{-1/2} r\,dr\,d\theta$$

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$$= -ak\left(2\pi\right)\left(-a\right)$$

i.e. $m = 2\pi a^2 k$

By the symmetry of the surface $\bar{x} = \bar{y} = 0$

Now $\bar{z} = \frac{1}{m}\iint_S z\,\rho\left(x, y, z\right)dS$

$$= \frac{k}{2\pi a^2 k}\iint_D z\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\;dA$$

$$= \frac{k}{2\pi a^2 k}\iint_D z\frac{a}{z}dA$$

$$= \frac{1}{2\pi a^2}.a\iint_D dA$$

$$= \frac{1}{2\pi a}\int_0^{2\pi}\int_0^{a} r\,dr\,d\theta$$

$$= \frac{1}{2\pi a}\left(\theta\right)_0^{2\pi}\left(\frac{r^2}{2}\right)_0^{a}$$

$$= \frac{1}{2\pi a}\times 2\pi\times\frac{a^2}{2}$$

$$= \frac{a}{2}$$

Hence the centre of mass is $\boxed{\left(0, 0, \frac{a}{2}\right)}$

Chapter 16 Vector Calculus Exercise 16.7 40E

$\rho\left(x, y, z\right) = 10 - z$

We know the mass is given by

$$m = \iint_S \rho\left(x, y, z\right)ds$$

Now the surface S has parametric representation

$$x = z\cos\theta, \quad y = z\sin\theta, \quad z = z$$

$$1 \le z \le 4, \quad 0 \le \theta \le 2\pi$$

Then $\vec{r}\left(z, \theta\right) = < z\cos\theta,\ z\sin\theta,\ z >$

$$\vec{r}_z = < \cos\theta, \sin\theta, 1 >$$

$$\vec{r}_\theta = < -z\sin\theta,\ z\cos\theta,\ 0 >$$

\>

Then $\vec{r}_z \times \vec{r}_\theta = < -z\cos\theta,\ -z\sin\theta,\ z >$

And $\left|\vec{r}_z \times \vec{r}_\theta\right| = \sqrt{z^2\cos^2\theta + z^2\sin^2\theta + z^2}$

$$= \sqrt{2z^2}$$

$$= z\sqrt{2}$$

Therefore $m = \iint_D \rho(x,y,z)\left|\vec{r}_z \times \vec{r}_\theta\right| dA$

$$= \iint_D (10-z)\, z\sqrt{2}\, dA$$

$$= \iint_D \sqrt{2}\left(10z - z^2\right) dA$$

$$= \sqrt{2}\int_0^{2\pi}\int_1^4 \left(10z - z^2\right) dz\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} d\theta \int_1^4 \left(10z - z^2\right) dz$$

$$= \sqrt{2}\,(\theta)_0^{2\pi}\left(5z^2 - \frac{1}{3}z^3\right)_1^4$$

i.e. m $= \sqrt{2}\,(2\pi)\left(80 - \frac{64}{3} - 5 + \frac{1}{3}\right)$

$$= 2\sqrt{2}\,\pi\left(75 - \frac{63}{3}\right)$$

$$= 2\sqrt{2}\pi(75-21)$$

$$= 2\sqrt{2}\pi(54)$$

$$= 108\sqrt{2}\pi$$

Hence the mass of the funnel is $\boxed{108\sqrt{2}\pi}$

Chapter 16 Vector Calculus Exercise 16.7 41E

(A)

If ρ is the density function, then the moment of inertia about z – axis is given by

$$I_z = \iint_S \left(x^2 + y^2\right) \rho(x,y,z)\, dS$$

(B)

The surface has parametric representation

$$x = z\cos\theta, y = z\sin\theta,\ z = z$$

$$1 \le z \le 4,\ \ 0 \le \theta \le 2\pi$$

(B)

The surface has parametric representation

$$x = z\cos\theta, y = z\sin\theta,\ z = z$$

$$1 \le z \le 4,\ \ 0 \le \theta \le 2\pi$$

Then $\vec{r}(z,\theta) = \langle z\cos\theta,\ z\sin\theta,\ z \rangle$

$$\vec{r}_z = \langle \cos\theta,\ \sin\theta,\ 1 \rangle$$

$$\vec{r}_\theta = \langle -z\sin\theta,\ z\cos\theta, 0 \rangle$$

Then $I_z = \iint_S \left(x^2 + y^2\right)(10 - z)\, ds$

$$= \iint_D \left(z^2\cos^2\theta + \sin^2\theta\right)(10-z)\left(z\sqrt{2}\right) dA$$

$$= \iint_D \sqrt{2}z^3\,(10-z)\, dA$$

$$= \sqrt{2}\int_0^{2\pi}\int_1^4 z^3\,(10-z)\, dz\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_1^4 \left(10z^3 - z^4\right) dz\, d\theta$$

i.e. $I_z = \sqrt{2}\int_0^{2\pi} d\theta \int_1^4 \left(10z^3 - z^4\right) dz$

$$= \sqrt{2}\,(\theta)_0^{2\pi}\left(\frac{5}{2}z^4 - \frac{1}{5}z^5\right)_1^4$$

$$= \sqrt{2}\,(2\pi)\left(640 - \frac{1024}{5} - \frac{5}{2} + \frac{1}{5}\right)$$

$$= 2\sqrt{2}\Pi\left(\frac{4329}{10}\right)$$

$$= \frac{4329\sqrt{2}\pi}{5}$$

Hence $\boxed{I_z = \frac{4329\sqrt{2}\pi}{5}}$

Chapter 16 Vector Calculus Exercise 16.7 42E

Using spherical coordinates to parametrize the sphere we have **r**(φ,θ) = 5 sin φ cos θ **i** + 5 sin φ sin θ **j** + 5 cos φ **k**, and $\left|r_\phi \times r_\theta\right| = 25\sin\phi$. S is the portion of the sphere where z ≥ 4, so $0 \le \phi \le \tan^{-1}\left(\frac{3}{4}\right)$ and $0 \le \theta \le 2\pi$.

(a)

$$m = \iint_S \rho\left(x, y, z\right)dS = \int_0^{2\pi}\int_0^{\tan^{-1}(3/4)} k(25\sin\phi)\,d\phi\,d\theta = 25k\int_0^{2\pi} d\theta\int_0^{\tan^{-1}(3/4)}$$

$$= 25k\left(2\pi\right)\left[-\cos\left(\tan^{-1}\frac{3}{4}\right) + 1\right] = 50\pi k\left(-\frac{4}{5} + 1\right) = 10\pi k$$

Because S has a constant density, $\bar{x} = \bar{y} = 0$ by symmetry and so we find that

$$\bar{z} = \frac{1}{m}\iint_S z\rho\left(x, y, z\right)dS = \frac{1}{10\pi k}\int_0^{2\pi}\int_0^{\tan^{-1}(3/4)} k(5\cos\phi)\left(25\sin\phi\right)d\phi\,d\theta$$

$$= \frac{1}{10\pi k}\left(125k\right)\int_0^{2\pi} d\theta\int_0^{\tan^{-1}(3/4)}\sin\phi\cos\phi\,d\phi = \frac{1}{10\pi k}\left(125k\right)\left(2\pi\right)\left[\frac{1}{2}\sin^2\phi\right]_0^1$$

and so the center of mass is $\left(\bar{x}, \bar{y}, \bar{z}\right) = \left(0, 0, \frac{9}{2}\right)$.

(b)

The moment of inertia about the z-axis is

$$I_z = \iint_S\left(x^2 + y^2\right)\rho\left(x, y, z\right)dS = \int_0^{2\pi}\int_0^{\tan^{-1}(3/4)} k\left(25\sin^2\phi\right)\left(25\sin\phi\right)d\phi\,d\theta$$

$$= 625k\int_0^{2\pi} d\theta\int_0^{\tan^{-1}(3/4)}\sin^3\phi\,d\phi = 625k(2\pi)\left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\tan^{-1}(3/4)}$$

$$= 1250\pi k\left[\frac{1}{3}\left(\frac{4}{5}\right)^3 - \frac{4}{5} - \frac{1}{3} + 1\right] = 1250\pi k\left(\frac{14}{375}\right) = \frac{140}{3}\pi k$$

Chapter 16 Vector Calculus Exercise 16.7 43E

Consider that the density of a fluid is $870\ \text{kg/m}^3$ and velocity is $\mathbf{v} = z\mathbf{i} + y^2\mathbf{j} + x^2\mathbf{k}$.

Determine the rate of flow outward through the cylinder $x^2 + y^2 = 4$.

Take the surface $S : x^2 + y^2 = 4$ which is oriented outward.

Parameterize the surface S:

Here the cylinder $x^2 + y^2 = 4$ has $r = 2$ in cylindrical coordinates so, use θ and z as parameters.

Then the parametric equations are:

$$x = 2\cos\theta \quad y = 2\sin\theta \quad z = z$$

$$\mathbf{r}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z\rangle$$

Where, $0 \le z \le 1, 0 \le \theta \le 2\pi.$

Next find $\mathbf{r}_\theta$ and $\mathbf{r}_z$:

$$\begin{aligned}\mathbf{r}_\theta(\theta, z) &= \frac{\partial}{\partial\theta}\langle 2\cos\theta, 2\sin\theta, z\rangle \\ &= \langle -2\sin\theta, 2\cos\theta, 0\rangle\end{aligned}$$

And

$$\begin{aligned}\mathbf{r}_z(\theta, z) &= \frac{\partial}{\partial z}\langle 2\cos\theta, 2\sin\theta, z\rangle \\ &= \langle 0, 0, 1\rangle\end{aligned}$$

Then $\mathbf{r}_\theta \times \mathbf{r}_z$:

$$\begin{aligned}\mathbf{r}_\theta \times \mathbf{r}_z &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \mathbf{i}(2\cos\theta\cdot 1 - 0) - \mathbf{j}(-2\sin\theta\cdot 1 - 0) + \mathbf{k}(-2\sin\theta\cdot 0 - 2\cos\theta\cdot 0) \\ &= \mathbf{i}(2\cos\theta) + \mathbf{j}(2\sin\theta) + \mathbf{k}(0) \\ &= \langle 2\cos\theta, 2\sin\theta, 0\rangle\end{aligned}$$

The vector filed is defined as

$$\mathbf{v}(x, y, z) = \langle z, y^2, x^2\rangle$$

$$\mathbf{v}(\mathbf{r}(z,\theta)) = \left\langle z, (2\sin\theta)^2, (2\cos\theta)^2\right\rangle \text{ Here } y = 1$$

$$= \langle z, 4\sin^2\theta, 4\cos^2\theta\rangle$$

Find $\mathbf{v}\cdot(\mathbf{r}_\theta \times \mathbf{r}_z)$:

$$\begin{aligned}&\langle z, 4\sin^2\theta, 4\cos^2\theta\rangle\cdot\langle 2\cos\theta, 2\sin\theta, 0\rangle \\ &= z\cdot 2\cos\theta + (4\sin^2\theta)\cdot(2\sin\theta) + (4\cos^2\theta)\cdot 0 \\ &= 2z\cos\theta + 8\sin^3\theta\end{aligned}$$

Recollect, suppose **F** is a continuous vector filed defined on an oriented surface S given by a vector function $\mathbf{r}(u, v)$.

Then, the **surface integral of F over** ***S*** is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Here the parameter domain D is given by

$$D = \{(\theta, z) \mid 0 \le \theta \le 2\pi, 0 \le z \le 1\}$$

Thus the net flow rate out of the cylinder is defined as

$$\rho\iint_S \mathbf{v}\cdot d\mathbf{S} = \rho\iint_D \mathbf{v}\cdot(\mathbf{r}_\theta \times \mathbf{r}_z)\,dA$$

Where, ρ is the density of the fluid and is 870 kg/m^3.

Find the surface integral of **F** over ***S***:

$$\begin{aligned}\iint_S \mathbf{v}\cdot d\mathbf{S} &= \iint_D \mathbf{v}\cdot(\mathbf{r}_\theta \times \mathbf{r}_z)\,dA\\ &= \int_0^1\int_0^{2\pi}\left(2z\cos\theta + 8\sin^3\theta\right)d\theta dz\\ &= \int_0^1\int_0^{2\pi} 2z\cos\theta d\theta dz + \int_0^1\int_0^{2\pi} 8\sin^3\theta d\theta dz\end{aligned}$$

$$= 2\int_0^1 zdz\int_0^{2\pi}\cos\theta d\theta + 8\int_0^1 zdz\int_0^{2\pi}\sin^3\theta d\theta$$

$$= 2\int_0^1 zdz\cdot\left[-\sin\theta\right]_0^{2\pi} + 8\int_0^1 zdz\int_0^{2\pi}\left(1-\cos^2\theta\right)\sin\theta d\theta$$

$$= 2\int_0^1 zdz\cdot\left[-0+0\right] + 8\int_0^1 zdz\int_0^{2\pi}\left(\cos^2\theta - 1\right)\left(-\sin\theta d\theta\right)$$

$$\iint_S \mathbf{v}\cdot d\mathbf{S} = 8\int_0^1 zdz\int_0^{2\pi}\left(\cos^2\theta-1\right)\left(-\sin\theta d\theta\right) \quad \text{...... (1)}$$

Use the substitution to solve the integral $\int_0^{2\pi}\left(\cos^2\theta-1\right)\left(-\sin\theta d\theta\right)$:

Take $u = \cos\theta$.

Differentiate:

$$du = -\sin\theta d\theta$$

When $\theta = 0$, the value of u becomes

$$\begin{aligned}u &= \cos(0)\\ &= 1\end{aligned}$$

When $\theta = 2\pi$, the value of u becomes

$$\begin{aligned}u &= \cos(2\pi)\\ &= 1\end{aligned}$$

Substitute $u = \cos\theta$ and $du = -\sin\theta d\theta$ in $\int_0^{2\pi}\left(\cos^2\theta-1\right)\left(-\sin\theta d\theta\right)$:

$$\begin{aligned}\int_0^{2\pi}\left(\cos^2\theta-1\right)\left(-\sin\theta d\theta\right) &= \int_1^1\left(u^2-1\right)(du)\\ &= \left[\frac{u^3}{3}-u\right]_1^1\\ &= 0\end{aligned}$$

Substitute $\int_0^{2\pi}\left(\cos^2\theta-1\right)\left(-\sin\theta d\theta\right) = 0$ in (1):

$$\begin{aligned}\iint_S \mathbf{v}\cdot d\mathbf{S} &= 8\int_0^1 zdz\int_0^{2\pi}\left(\cos^2\theta-1\right)\left(-\sin\theta d\theta\right)\\ &= 8\int_0^1 zdz(0)\\ &= 0\end{aligned}$$

Therefore, the net flow rate out of the cylinder is $\boxed{0\text{ kg/s}}$.

Chapter 16 Vector Calculus Exercise 16.7 44E

Consider that the density of seawater is $1025\ \text{kg/m}^3$ and velocity filed $\mathbf{v} = y\,\mathbf{i} + x\,\mathbf{j}$.

Determine the rate of flow outward through the hemisphere $x^2 + y^2 + z^2 = 9, z \geq 0$.

Take the surface $S: x^2 + y^2 + z^2 = 9, z \geq 0$, which is oriented outward.

Parameterize the surface S:

Here the hemisphere $x^2 + y^2 + z^2 = 9$ has $r = 3$ in spherical coordinates so, use ϕ and θ as parameters.

Then the parametric equations are:

$$x = 3\sin\phi\cos\theta \quad y = 3\sin\phi\sin\theta \quad z = 3\cos\phi$$
$$\mathbf{r}(\theta, z) = \langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi \rangle$$

Where, $0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$.

Next find $\mathbf{r}_\phi$ and $\mathbf{r}_\theta$:

$$\mathbf{r}_\phi(\theta, z) = \frac{\partial}{\partial \phi}\langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi \rangle$$
$$= \langle 3\cos\phi\cos\theta, 3\cos\phi\sin\theta, -3\sin\phi \rangle$$

And

$$\mathbf{r}_\theta(\theta, z) = \frac{\partial}{\partial \theta}\langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi \rangle$$
$$= \langle -3\sin\phi\sin\theta, 3\sin\phi\cos\theta, 0 \rangle$$

“>

Then $\mathbf{r}_\phi \times \mathbf{r}_\theta$:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos\phi\cos\theta & 3\cos\phi\sin\theta & -3\sin\phi \\ -3\sin\phi\sin\theta & 3\sin\phi\cos\theta & 0 \end{vmatrix}$$
$$= \mathbf{i}(3\cos\phi\sin\theta \cdot 0 + 3\sin\phi\cos\theta \cdot 3\sin\phi)$$
$$-\mathbf{j}(3\cos\phi\cos\theta \cdot 0 - 3\sin\phi\sin\theta \cdot 3\sin\phi)$$
$$+\mathbf{k}(3\cos\phi\cos\theta \cdot 3\sin\phi\cos\theta + 3\sin\phi\sin\theta \cdot 3\cos\phi\sin\theta)$$
$$= \mathbf{i}(9\sin^2\phi\cos\theta) + \mathbf{j}(9\sin^2\phi\sin\theta)$$
$$+\mathbf{k}(9\sin\phi\cos\phi\cos^2\theta + 9\sin\phi\cos\phi\sin^2\theta)$$
$$= \langle 9\sin^2\phi\cos\theta, 9\sin^2\phi\sin\theta, 9\sin\phi\cos\phi \rangle$$

“>

The vector filed is defined as

$$\mathbf{v}(x, y, z) = \langle y, x, 0 \rangle$$
$$\mathbf{v}(\mathbf{r}(z, \theta)) = \langle 3\sin\phi\sin\theta, 3\sin\phi\cos\theta, 0 \rangle \quad \text{Here } y = 1$$

Find $\mathbf{v} \cdot (\mathbf{r}_\theta \times \mathbf{r}_z)$:

$$\langle 3\sin\phi\sin\theta, 3\sin\phi\cos\theta, 0 \rangle \cdot \langle 9\sin^2\phi\cos\theta, 9\sin^2\phi\sin\theta, 9\sin\phi\cos\phi \rangle$$
$$= 27\sin^3\phi\sin\theta\cos\theta + 27\sin^3\phi\sin\theta\cos\theta + 0$$
$$= 54\sin^3\phi\sin\theta\cos\theta$$

“>

Recollect, suppose **F** is a continuous vector filed defined on an oriented surface *S* given by a vector function $\mathbf{r}(u, v)$.

Then, the **surface integral of F over *S*** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Here the parameter domain *D* is given by

$$D = \{(\theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$$

“>

Thus the net flow rate out of the cylinder is defined as

$$\rho\iint_S \mathbf{v}\cdot d\mathbf{S} = \rho\iint_D \mathbf{v}\cdot\left(\mathbf{r}_\theta\times\mathbf{r}_z\right)dA$$

Where, ρ is the density of the fluid and is 1025 kg/m^3.

Find the surface integral of **F** over ***S***:

$$\iint_S \mathbf{v}\cdot d\mathbf{S} = \iint_D \mathbf{v}\cdot\left(\mathbf{r}_\phi\times\mathbf{r}_\theta\right)dA$$

$$= \int_0^1\int_0^{2\pi}\left(54\sin^3\phi\sin\theta\cos\theta\right)d\phi d\theta$$

$$= 54\int_0^{2\pi}\sin\theta\cos\theta d\theta\int_0^{\pi/2}\sin^3\phi d\phi$$

$$\iint_S \mathbf{v}\cdot d\mathbf{S} = 54\int_0^{2\pi}\sin\theta\cos\theta\, d\theta\int_0^{\pi/2}\sin^3\phi\, d\phi \quad \ldots\ldots (1)$$

Use the substitution to solve the integral $\int_0^{2\pi}\sin\theta\cos\theta\, d\theta$:

Take $u = \sin\theta$.

Differentiate:

$$du = \cos\theta d\theta$$

When $\theta = 0$, the value of u becomes

$$u = \sin(0)$$
$$= 0$$

When $\theta = 2\pi$, the value of u becomes

$$u = \sin(2\pi)$$

“>

$$= 0$$

Substitute $u = \sin\theta$ and $du = \cos\theta d\theta$ in $\int_0^{2\pi}\sin\theta\cos\theta\, d\theta$:

$$\int_0^{2\pi}\sin\theta\cos\theta\, d\theta = \int_0^0 u(du)$$

$$= \left[\frac{u^2}{2}\right]_0^0$$

$$= 0$$

Substitute $\int_0^{2\pi}\sin\theta\cos\theta\, d\theta = 0$ in (1):

$$\iint_S \mathbf{v}\cdot d\mathbf{S} = 54\int_0^{2\pi}\sin\theta\cos\theta\, d\theta\int_0^{\pi/2}\sin^3\phi\, d\phi$$

$$= 0\cdot 54\int_0^{\pi/2}\sin^3\phi\, d\phi$$

$$= 0$$

Therefore, the flow rate out of the sea water is $\boxed{0 \text{ kg / s}}$.

“>

Chapter 16 Vector Calculus Exercise 16.7 45E

Consider the electric filed $\mathbf{E}(x, y, z) = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}.$

Find the charge contained in the solid hemisphere $x^2 + y^2 + z^2 \le a^2, z \ge 0$ by using the Gauss's Law.

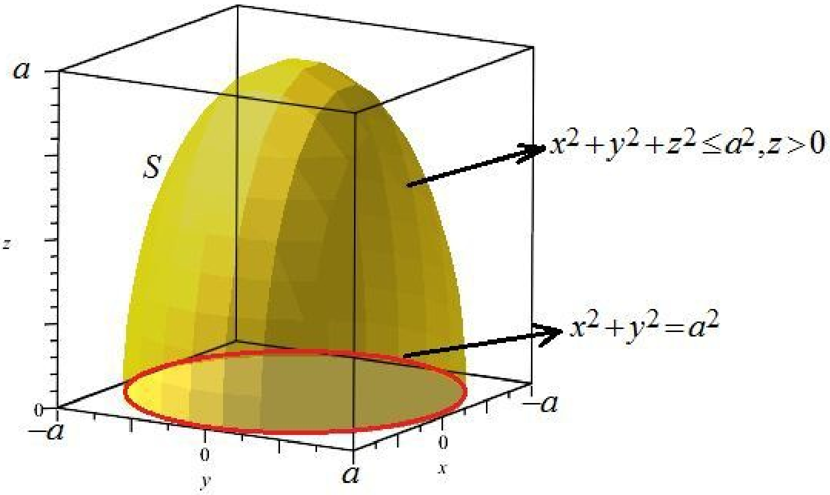
Gauss's Law:

The net charge enclosed by a closed surface S is $Q = \varepsilon_0\iint_S \mathbf{E}\cdot d\mathbf{S}.$

Here, ε_0 is a constant (The permittivity of free space).

For the given problem, the surface S is a hemisphere and its boundary is a disk $x^2+y^2=a^2$.

The closed surface S is shown in the below figure:



The parametric representation of the upper hemisphere $x^2+y^2+z^2 \le a^2, z \ge 0$ is given below:

$$x = a\sin\phi\cos\theta$$
$$y = a\sin\phi\sin\theta$$
$$z = a\cos\phi$$
$$0 \le \phi \le \frac{\pi}{2}, 0 \le \theta \le 2\pi.$$

The corresponding vector equation is,

$$\mathbf{r}(\phi,\theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$$

So, the parameter domain is the rectangle $D = \left[0,\frac{\pi}{2}\right]\times[0,2\pi]$.

The partial derivatives of $\mathbf{r}$ are

$$\mathbf{r}_\phi(\phi,\theta) = (a\cos\phi\cos\theta)\mathbf{i} + (a\cos\phi\sin\theta)\mathbf{j} - a\sin\phi\mathbf{k}$$
$$\mathbf{r}_\theta(\phi,\theta) = (-a\sin\phi\sin\theta)\mathbf{i} + (a\sin\phi\cos\theta)\mathbf{j}$$

The normal vector $\mathbf{r}_\phi \times \mathbf{r}_\theta$ is,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$
$$= a^2\left(\sin^2\phi\cos\theta\mathbf{i} + \sin^2\phi\sin\theta\mathbf{j} + \sin\phi\cos\phi\mathbf{k}\right)$$

Rewrite the field $\mathbf{E}(x,y,z) = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$ using the parametric equations.

Then, $\mathbf{E}(\mathbf{r}(\phi,\theta)) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (2a\cos\phi)\mathbf{k}.$

$$\mathbf{E}\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta) = \begin{pmatrix} (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} \\ +(2a\cos\phi)\mathbf{k} \end{pmatrix} \cdot a^2 \begin{pmatrix} \sin^2\phi\cos\theta\mathbf{i} + \sin^2\phi\sin\theta\mathbf{j} \\ +\sin\phi\cos\phi\mathbf{k} \end{pmatrix}$$
$$= a^3\left(\sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta + 2\sin\phi\cos^2\phi\right)$$
$$= a^3\left(\sin^3\phi\left(\cos^2\theta + \sin^2\theta\right) + 2\sin\phi\cos^2\phi\right)$$
$$= a^3\left(\sin^3\phi(1) + 2\sin\phi\cos^2\phi\right)$$
$$= a^3\sin\phi\left(\sin^2\phi + 2\cos^2\phi\right)$$
$$= a^3\sin\phi\left(\sin^2\phi + \cos^2\phi + \cos^2\phi\right)$$
$$= a^3\sin\phi\left(1 + \cos^2\phi\right)$$

The net charge enclosed by a closed surface S is,

$$Q = \varepsilon_0 \iint_S \mathbf{E}\cdot d\mathbf{S}$$

$$= \varepsilon_0 \iint_D \left[\mathbf{E}\cdot\left(\mathbf{r}_\phi \times \mathbf{r}_\theta\right)\right] dA$$

$$= \varepsilon_0 \int_0^{2\pi}\int_0^{\frac{\pi}{2}} a^3\left(\sin\phi\left(1+\cos^2\phi\right)\right) d\phi\, d\theta$$

$$= a^3\varepsilon_0\left[\int_0^{2\pi} d\theta\right]\left[\int_0^{\frac{\pi}{2}}\left(\sin\phi\left(1+\cos^2\phi\right)\right)d\phi\right]$$

$$= a^3\varepsilon_0\left[\theta\right]_0^{2\pi}\left[-\int_1^0\left(1+t^2\right)dt\right] \qquad \left[\begin{array}{l}\text{Put } \cos\phi = t \Rightarrow -\sin\phi\, d\phi = dt \\ \text{when } \phi\to 0, t\to 1 \\ \text{when } \phi\to\frac{\pi}{2}, t\to 0\end{array}\right]$$

$$= -a^3\varepsilon_0\left[2\pi\right]\left[t+\frac{t^3}{3}\right]_1^0$$

$$= -a^3\varepsilon_0\left[2\pi\right]\left[0+\frac{0^3}{3}-1-\frac{1^3}{3}\right]$$

$$= -a^3\varepsilon_0\left[2\pi\right]\left[-\frac{4}{3}\right]$$

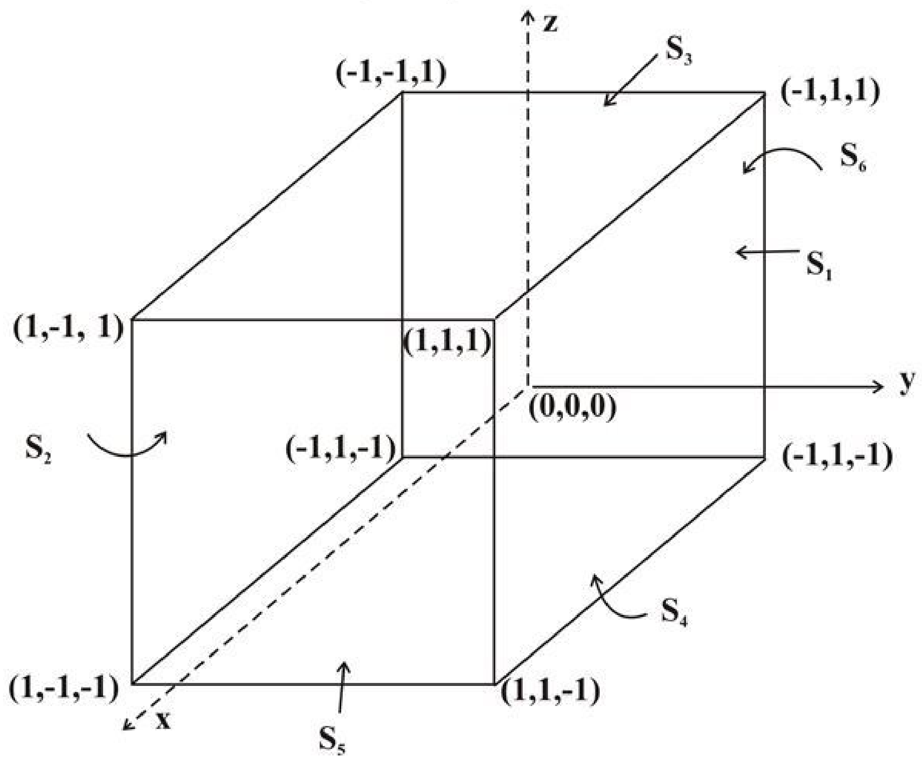
$$= \frac{8a^3\varepsilon_0}{3}\pi$$

Therefore, the net charge contained in the solid hemisphere S is $Q = \boxed{\frac{8}{3}\pi a^3\varepsilon_0}$.

Chapter 16 Vector Calculus Exercise 16.7 46E

Use gauss law to find the charge enclosed by the cue with vertices $(\pm1,\pm1,\pm1)$ if the electric charge is given by the vector function $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

At first, draw the cube with vertices $(\pm1,\pm1,\pm1)$.



From the above figure it is observed that, the region containing the electric charge is enclosed by 6 surfaces which are planes.

According to the gauss law, the net charge enclosed by the surface *S* is

$$Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S} \ \left(\text{Electric Flux of } \mathbf{E} \text{ through the surface } \mathbf{S.}\right)$$

Where ε_0 is permittivity of free space (In SI system $\varepsilon_0 \approx 8.8542 \times 10^{-12} \mathrm{C}^2 / \mathrm{N} \cdot \mathrm{m}^2$)

Now the cube has six boundary surfaces S1, S2, S3, S4, S5 and S6 as shown in the figure.

So the total electric flux of $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ through the surface $\mathbf{S}$

$\left(\text{i.e. Cube with vertices } (\pm1, \pm1, \pm1)\right)$ is the sum of the electric flux of $\mathbf{E}$ through the surfaces S1, S2, S3, S4, S5 and S6.

That is

$$\begin{aligned} Q &= \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S} \\ &= \varepsilon_0 \iint_{S_1} \mathbf{E} \cdot d\mathbf{S}_1 + \varepsilon_0 \iint_{S_2} \mathbf{E} \cdot d\mathbf{S}_2 + \varepsilon_0 \iint_{S_3} \mathbf{E} \cdot d\mathbf{S}_3 + \varepsilon_0 \iint_{S_4} \mathbf{E} \cdot d\mathbf{S}_4 + \varepsilon_0 \iint_{S_5} \mathbf{E} \cdot d\mathbf{S}_5 + \varepsilon_0 \iint_{S_6} \mathbf{E} \cdot d\mathbf{S}_6 \\ &= \varepsilon_0 \left(\iint_{S_1} \mathbf{E} \cdot d\mathbf{S}_1 + \iint_{S_2} \mathbf{E} \cdot d\mathbf{S}_2 + \iint_{S_3} \mathbf{E} \cdot d\mathbf{S}_3 + \iint_{S_4} \mathbf{E} \cdot d\mathbf{S}_4 + \iint_{S_5} \mathbf{E} \cdot d\mathbf{S}_5 + \iint_{S_6} \mathbf{E} \cdot d\mathbf{S}_6 \right) \end{aligned}$$

Find the electric flux of $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ through the surface $\mathbf{S}_1$.

Observe the vertices of the surface $\mathbf{S}_1$, in which the *y*-coordinate is same and it is equal to1. So the surface $\mathbf{S}_1$ is the plane parallel to $yz-$plane with equation $y = 1$.

Since the plane is parallel to $yz-$plane, the outward normal is parallel to *y*-axis.

So the unit outward normal to surface $\mathbf{S}_1$ is $\mathbf{j}$

That is $\mathbf{n} = \mathbf{j} = \langle 0, 1, 0 \rangle$, over the surface $\mathbf{S}_1$ the variable *x* varies from -1 to 1 and the variable *z* varies from -1 to 1. That is, $-1 \le x \le 1, -1 \le z \le 1$

Then

$$\begin{aligned} \iint_{S_1} \mathbf{E} \cdot d\mathbf{S}_1 &= \iint_D \mathbf{E} \cdot \mathbf{n}\ d\mathbf{A} \\ &= \iint_D \left(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\right) \cdot \mathbf{j}\ d\mathbf{A} \\ &= \int_{-1}^{1} \int_{-1}^{1} y\, dx\, dz \\ &= \int_{-1}^{1} \int_{-1}^{1} (1)\, dx\, dz \\ &= (2)(2) \\ &= 4 \end{aligned}$$

Hence the electric flux of the electric field given by the vector function $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

Through the surface $\mathbf{S}_1 \left(\text{Plane: } y = 1\right)$ is $\iint_{S_1} \mathbf{E} \cdot d\mathbf{S}_1 = 4.$

We use the same criteria as in the step-3, to find the electric flux of the given electric field through the remaining surfaces.

Similarly on $\mathbf{S}_2$, $y = -1, \quad -1 \le x \le 1, \qquad -1 \le z \le 1$

And $\mathbf{n} = -\mathbf{j} = \langle 0, -1, 0 \rangle$

Then

$$\iint_{S_2} \mathbf{E}\cdot d\mathbf{S}_2 = \iint_D \mathbf{E}\cdot\mathbf{n}\, d\mathbf{A}_2$$
$$= \iint_D (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(-\mathbf{j})\, d\mathbf{A}_2$$
$$= \int_{-1}^{1}\int_{-1}^{1} -y\,dx\,dz$$
$$= \int_{-1}^{1}\int_{-1}^{1} -(-1)\,dx\,dz$$
$$= (2)(2)$$
$$= 4$$

$$\Rightarrow \iint_{S_2} \mathbf{E}\cdot d\mathbf{S}_2 = 4$$

Similarly on $\mathbf{S}_3$, $z=1,\ -1\le x\le 1,\ -1\le y\le 1$

And $\mathbf{n}=\mathbf{k}=\langle 0,0,1\rangle$

Then $\iint_{\mathbf{S}_3} \mathbf{E}.\, d\mathbf{S}_3 = 4$

Similarly on $\mathbf{S}_4$, $z=-1,\ -1\le x\le 1,\ -1\le y\le 1$

And $\mathbf{n}=-\mathbf{k}=\langle 0,0,-1\rangle$

Then $\iint_{\mathbf{S}_4} \mathbf{E}.\, d\mathbf{S}_4 = 4$

Similarly on $\mathbf{S}_5$, $x=1,\ -1\le y\le 1,\ -1\le z\le 1$

And $\mathbf{n}=\mathbf{i}=\langle 1,0,0\rangle$

Then $\iint_{\mathbf{S}_5} \mathbf{E}.\, d\mathbf{S}_5 = 4$

Similarly on $\mathbf{S}_6$, $x=-1,\ -1\le y\le 1,\ -1\le z\le 1$

And $\mathbf{n}=-\mathbf{i}=\langle -1,0,0\rangle$

Then $\iint_{\mathbf{S}_6} \mathbf{E}.\, d\mathbf{S}_6 = 4$

Therefore, the net charge enclosed in the cube is given by

$$Q=\varepsilon_0\left(\iint_{S_1}\mathbf{E}\cdot d\mathbf{S}_1+\iint_{S_2}\mathbf{E}\cdot d\mathbf{S}_2+\iint_{S_3}\mathbf{E}\cdot d\mathbf{S}_3+\iint_{S_4}\mathbf{E}\cdot d\mathbf{S}_4+\iint_{S_5}\mathbf{E}\cdot d\mathbf{S}_5+\iint_{S_6}\mathbf{E}\cdot d\mathbf{S}_6\right)$$
$$=\varepsilon_0(4+4+4+4+4+4)$$
$$=\boxed{24\varepsilon_0}.$$

Chapter 16 Vector Calculus Exercise 16.7 47E

$$u(x,y,z)=2y^2+2z^2$$

Then $\vec{\nabla}u(x,y,z)=4y\hat{j}+4z\hat{k}$

Then heat flow is

$$\vec{F}(x,y,z)=-k\vec{\nabla}u$$
$$=-6.5(4y\hat{j}+4z\hat{k})$$
$$=-26y\hat{j}-26z\hat{k}$$

(Where k is the conductivity of metal)

Therefore the rate of heat flow across S is

$$\iint_S \vec{F}.d\vec{s}$$

Now S has parametric representation

$$x = x,\quad y = \sqrt{6}\cos\theta,\quad z = \sqrt{6}\sin\theta$$

$$0 \le x \le 4,\quad 0 \le \theta \le 2\pi$$

i.e. $\vec{r}(x,\theta) = \langle x, \sqrt{6}\cos\theta, \sqrt{6}\sin\theta \rangle$

Then $\vec{r}_x = \langle 1,0,0 \rangle$

And $\vec{r}_\theta = \langle 0, -\sqrt{6}\sin\theta, \sqrt{6}\cos\theta \rangle$

Therefore $\vec{r}_x \times \vec{r}_\theta = \langle 0, -\sqrt{6}\cos\theta, -\sqrt{6}\sin\theta \rangle$

(Since heat flow is inward we take $\vec{r}_x \times \vec{r}_\theta$)

Then $|\vec{r}_x \times \vec{r}_\theta| = \sqrt{6\cos^2\theta + 6\sin^2\theta} = \sqrt{6}$

Now $\vec{F}(\vec{r}) = -26\sqrt{6}\cos\theta\,\hat{j} - 26\sqrt{6}\sin\theta\,\hat{k}$

Therefore $\iint_S \vec{F}.d\vec{s}$

$$= \int_D \vec{F}.(\vec{r}_x \times \vec{r}_\theta)\,dA$$

$$= \iint_D \left(-26\sqrt{6}\cos\theta\,\hat{j} - 26\sqrt{6}\sin\theta\,\hat{k}\right).\left(-\sqrt{6}\cos\theta\,\hat{j} - \sqrt{6}\sin\theta\,\hat{k}\right)dA$$

$$= \iint_D (26\times 6)\cos^2\theta + (26\times 6)\sin^2\theta\,dA$$

$$= 156\iint_D dA$$

$$= 156\int_0^{2\pi}\int_0^4 dx\,d\theta$$

$$= 156(2\pi)(4)$$

$$= \boxed{1248\pi}$$

Chapter 16 Vector Calculus Exercise 16.7 48E

Taking the centre of the ball to be origin we have

$$u(x,y,z) = \frac{c}{\sqrt{x^2+y^2+z^2}}$$

Where c is constant of proportionality

Then the heat flow is

$$\vec{F}(x,y,z) = -k\vec{\nabla}u$$

$$= -kc\left[\frac{-x\hat{i}}{(x^2+y^2+z^2)^{3/2}} - \frac{y\hat{j}}{(x^2+y^2+z^2)^{3/2}} - \frac{z\hat{k}}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= kc\left[\frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}}\right]$$

Where k is the conductivity of metal

We know the outward unit normal to the sphere $x^2+y^2+z^2 = a^2$ at the point (x,y,z) is

$$\hat{n} = \frac{1}{a}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

Then $\vec{F}.\hat{n} = \frac{kc}{a}\left[\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right]$

$$= \frac{kc}{a} \times \frac{1}{x^2+y^2+z^2}$$

$$= \frac{kc}{a} \times \frac{1}{a}$$

$$= \frac{kc}{a^2}$$

Therefore the rate of flow across S is

$$\iint_S \vec{F}.d\vec{s} = \iint_S \vec{F}.\hat{n}\,ds$$
$$= \frac{kc}{a^2}\iint_S ds$$
$$= \frac{kc}{a^2}\left(4\pi a^2\right)$$
$$= 4kc\pi$$

Hence $\boxed{\iint_S \vec{F}.d\vec{s} = 4kc\pi}$

Chapter 16 Vector Calculus Exercise 16.749E

Consider the inverse square field **F**

$$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3}$$

Where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and c is the constant.

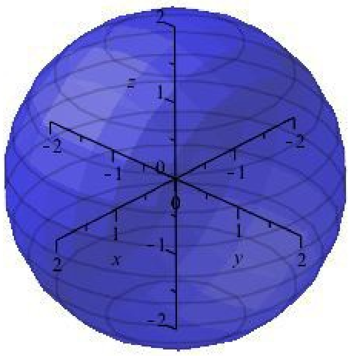
Determine the flux of **F** across a surface S is a sphere with center at origin and independent of radius.

Use the fact that the equation of sphere with radius a and center as origin is defined as

$S:\ x^2 + y^2 + z^2 = a^2$.

Parameterize the surface S:

The surface of the sphere is shown as



Here the sphere $x^2 + y^2 + z^2 = a^2$ has $r = a$ in spherical coordinates so, use ϕ and θ as parameters.

Then the parametric equations are:

$$x = a\sin\phi\cos\theta \quad y = a\sin\phi\sin\theta \quad z = a\cos\phi$$
$$\mathbf{r}(\phi,\theta) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$$

Where, $0 \le \phi \le \pi, 0 \le \theta \le 2\pi.$

Next find $\mathbf{r}_\phi$ and $\mathbf{r}_\theta$:

$$\mathbf{r}_\phi(\theta,z) = \frac{\partial}{\partial\phi}\langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$$
$$= \langle a\cos\phi\cos\theta, a\cos\phi\sin\theta, -a\sin\phi\rangle$$

And

$$\mathbf{r}_\theta(\theta,z) = \frac{\partial}{\partial\theta}\langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$$
$$= \langle -a\sin\phi\sin\theta, a\sin\phi\cos\theta, 0\rangle$$

Then $\mathbf{r}_\phi \times \mathbf{r}_\theta$:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \mathbf{i}\left(a\cos\phi\sin\theta\cdot 0 + a\sin\phi\cos\theta\cdot a\sin\phi\right)$$
$$-\mathbf{j}\left(a\cos\phi\cos\theta\cdot 0 - a\sin\phi\sin\theta\cdot a\sin\phi\right)$$
$$+\mathbf{k}\left(a\cos\phi\cos\theta\cdot a\sin\phi\cos\theta + a\sin\phi\sin\theta\cdot a\cos\phi\sin\theta\right)$$
$$= \mathbf{i}\left(a^2\sin^2\phi\cos\theta\right) + \mathbf{j}\left(a^2\sin^2\phi\sin\theta\right)$$
$$+\mathbf{k}\left(a^2\sin\phi\cos\phi\cos^2\theta + a^2\sin\phi\cos\phi\sin^2\theta\right)$$
$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \left\langle a^2\sin^2\phi\cos\theta, a^2\sin^2\phi\sin\theta, a^2\sin\phi\cos\phi\right\rangle$$

First find $|\mathbf{r}|^3$:

$$|\mathbf{r}|^3 = \left(\sqrt{(a\sin\phi\cos\theta)^2 + (a\sin\phi\sin\theta)^2 + (a\cos\phi)^2}\right)^3$$
$$= \left(\sqrt{a^2\sin^2\phi\cos^2\theta + a^2\sin^2\phi\sin^2\theta + a^2\cos^2\phi}\right)^3$$
$$= \left(\sqrt{a^2\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right) + a^2\cos^2\phi}\right)^3$$
$$= \left(\sqrt{a^2\sin^2\phi + a^2\cos^2\phi}\right)^3$$
$$= \left(\sqrt{a^2\left(\sin^2\phi + \cos^2\phi\right)}\right)^3$$
$$= \left(\sqrt{a^2}\right)^3$$
$$= a^3$$

Then the vector filed **F** is defined as

$$\mathbf{F}(x,y,z) = \langle x,y,z\rangle$$
$$\mathbf{v}\left(\mathbf{r}(\phi,\theta)\right) = \frac{c}{|\mathbf{r}|^3}\mathbf{r}(\phi,\theta)$$
$$= \frac{c}{a^3}\langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$$

Find $\mathbf{F}\cdot\left(\mathbf{r}_\phi \times \mathbf{r}_\theta\right)$:

$$\mathbf{F}\cdot\left(\mathbf{r}_\phi \times \mathbf{r}_\theta\right) = \frac{c}{a^3}\langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle\cdot$$
$$\left\langle a^2\sin^2\phi\cos\theta, a^2\sin^2\phi\sin\theta, a^2\sin\phi\cos\phi\right\rangle$$
$$= \frac{c}{a^3}\left(a^3\sin^3\phi\cos^2\theta + a^3\sin^3\phi\sin^2\theta + a^3\sin\phi\cos^2\phi\right)$$
$$= \frac{c}{a^3}\left(a^3\sin^3\phi\left(\cos^2\theta + \sin^2\theta\right) + a^3\sin\phi\cos^2\phi\right)$$
$$= \frac{c}{a^3}\left(a^3\sin^3\phi + a^3\sin\phi\cos^2\phi\right)$$
$$= \frac{c}{a^3}\cdot a^3\sin\phi\left(\sin^2\phi + \cos^2\phi\right)$$
$$= c\sin\phi$$

Recollect, suppose **F** is a continuous vector filed defined on an oriented surface *S* given by a vector function $\mathbf{r}(u,v)$.

Then, the **surface integral of F over *S*** is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot\left(\mathbf{r}_u \times \mathbf{r}_v\right)dA$$

Here the parameter domain *D* is given by

$$D = \left\{(\phi,\theta)\mid 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\right\}$$

Find the surface integral of **F** over ***S***:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot\left(\mathbf{r}_\phi \times \mathbf{r}_\theta\right) dA$$

$$= \int_0^{2\pi}\int_0^{\pi} c\sin\phi \, d\phi \, d\theta$$

$$= c\int_0^{2\pi} \left[-\cos\phi\right]_0^{\pi} d\theta$$

$$= -c\int_0^{2\pi} \left[\cos(\pi) - \cos(0)\right] d\theta$$

$$= -c\int_0^{2\pi} \left[-1-1\right] d\theta$$

$$= 2c\int_0^{2\pi} d\theta$$

$$= 2c\left[\theta\right]_0^{2\pi}$$

$$= 2c\left[2\pi - 0\right]$$

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = 4\pi c$$

Therefore, the flux of **F** across a sphere *S* with center at origin and independent of radius is

$\boxed{4\pi c}$.