Chapter 2

Ratio–Proportion–Variation

CHAPTER HIGHLIGHTS

- *Ratio*
- *Proportion*
- *Variation*
- *Direct Variation*
- *Inverse Variation*
- *Joint Variation*

Ratio

If the values of two quantities A and B are 4 and 6, respectively, then we say that they are in the ratio 4:6 (read as 'four is to six'). Ratio is the relation which one quantity bears to another of the same kind, the comparison being made by considering what multiple, part or parts, one quantity is of the other. The ratio of two quantities 'a' and 'b' is represented as a:b and read as 'a is to b'. Here, 'a' is called antecedent, 'b' is the consequent. Since the ratio expresses the number of times one quantity contains the other, it's *an abstract* quantity.

Ratio of any number of quantities is expressed after removing any common factors that ALL the terms of the ratio have. For example, if there are two quantities having values of 4 and 6, their ratio is 4:6, i.e. 2:3 after taking the common factor 2 between them out. Similarly, if there are three quantities 6, 8, and 18, there is a common factor among all three of them. So, dividing each of the three terms by 2, we get the ratio as 3:4:9.

If two quantities whose values are A and B, respectively, are in the ratio a:b, since we know that some common factor $k(> 0)$ would have been removed from A and B to get the ratio a:b, we can write the original values of the two quantities (i.e. A and B) as ak and bk respectively. For example, if the salaries of two persons are in the ratio 7:5, we can write their individual salaries as $7k$ and $5k$, respectively.

A ratio a:b can also be expressed as a/b. So, if two items are in the ratio 2:3, we can say that their ratio is 2/3. If two terms are in the ratio 2, it means that they are in the ratio of 2/1, i.e. 2:1.

'A ratio is said to be a ratio of greater or less inequality or of equality according as antecedent is greater than, less than or equal to consequent'. In other words,

1. The ratio a:b where $a > b$ is called a ratio of greater inequality (example 3:2)
2. The ratio a:b where $a < b$ is called a ratio of less inequality (example 3:5)
3. The ratio a:b where $a = b$ is called a ratio of equality (example 1:1)

From this, we can find that a ratio of greater inequality is diminished and a ratio of less inequality is increased by adding same quantity to both terms, i.e. in the ratio a:b, when we add the same quantity x (positive) to both the terms of the ratio, we have the following results

$$\begin{array}{lll} \text{if } a < b & \text{then} & (a+x):(b+x) > a:b \\ \text{if } a > b & \text{then} & (a+x):(b+x) < a:b \\ \text{if } a = b & \text{then} & (a+x):(b+x) = a:b \end{array}$$

This idea can also be helpful in questions on Data Interpretation when we need to compare fractions to find the larger of two given fractions.

If two quantities are in the ratio a:b, then the first quantity will be $a/(a + b)$ times the total of the two quantities and the second quantity will be equal to $b/(a + b)$ times the total of the two quantities.

Solved Examples

Example 1

If $a:b = 3:4$, find $3a + 4b:4a + 5b$.

Solution

$3a + 4b:4a + 5b$

$$= \frac{3a+4b}{4a+5b} = \frac{\dfrac{3a+4b}{b}}{\dfrac{4a+5b}{b}} = \frac{3\left(\dfrac{a}{b}\right)+4}{4\left(\dfrac{a}{b}\right)+5} = \frac{3\left(\dfrac{3}{4}\right)+4}{4\left(\dfrac{3}{4}\right)+5} = \frac{25}{32}$$

Example 2

The ratio of the number of marbles with Ram and Shyam is 19:13. If Ram gives Shyam 30 marbles, both will have equal number of marbles. Find the number of marbles with Ram.

Solution

Let the number of marbles with Ram and Shyam be $19x$ and $13x$, respectively. Total number of marbles with them = $32x$

If Ram gives Shyam 30 marbles, each will have $\frac{32x}{2} = 16x$ marbles.

$$\therefore \quad 19x - 16x = 30$$
$$x = 10$$
$$19x = 190$$

Example 3

1400 is divided into 4 parts such that half of the first part, one third of the second part, one fourth of the third part, and $\frac{1}{12}$th of the last part are all equal. Find the 4 parts.

Solution

Let the first part, second part, third part, and fourth part be a, b, c, and d respectively.

$$\frac{1}{2}a = \frac{1}{3}b = \frac{1}{4}c = \frac{1}{12}d$$

$$d = 6a,\ b = \frac{3}{2}a,\ c = 2a$$

Given, $\quad a + b + c + d = 1400$

$$\Rightarrow \quad a + \frac{3}{2}a + 2a + 6a = 1400$$

$$\Rightarrow \quad a = \frac{400}{3}$$

$$\therefore \quad b = 200,\ c = \frac{800}{3},\ d = 800$$

Alternative method:

As b is common to both ratios and since it is divisible by 3 (from the first ratio) and it is divisible by 2 (from the second ratio), it is divisible by LCM (3, 2), i.e. 6. Hence if $b = 6$, $a = 4$, and $c = 9$

$$\therefore \quad a{:}b{:}c = 4{:}6{:}9$$

Example 4

There are 2 classes A and B. If 10 students leave class A and join class B, then the ratio of the number of students in class A and class B would reverse. Find the difference in the numbers of students in class A and class B.

Solution

Let the numbers of students in class A and class B be ax and bx, respectively.

Given, $$\frac{ax - 10}{bx + 10} = \frac{b}{a}$$

$$a^2x - 10a = b^2x + 10b$$

$$\Rightarrow \quad a^2x - b^2x - 10a - 10b = 0$$

$$\Rightarrow \quad (ax - bx - 10)(a + b) = 0$$

$$\therefore \quad ax - bx = 10$$

Example 5

A husband's age exceeds that of his wife by 6 years. 10 years ago, the ratio of their ages was 5:4. Find the present age of the husband.

Solution

Let the present age of the husband be x years.

$\Rightarrow$ Present age of the wife = $(x - 6)$ years.

10 years ago, the ages of the husband and the wife will be $(x - 10)$ years and $(x - 16)$ years, respectively.

Given $$x - 10 = \frac{5}{4}(x - 16)$$

$$\therefore \quad x = 40$$

Alternative method:

Let the age of the husband 10 years ago be $5x$ years. Age of his wife at that time = $4x$ years.

The husband would then also be 6 years, older than his wife.

$$\therefore \quad 5x = 4x + 6 \quad \Rightarrow \quad x = 6$$

Hence, the present age of the husband

$$= 5x + 10, \quad \text{i.e. 40 years}$$

Proportion

When two ratios are equal, then the four quantities involved in the two ratios are said to be proportional, i.e. if $a/b = c/d$, then a, b, c, and d are proportional.

This is represented as $a{:}b{::}\ c{:}d$ and is read as 'a is to b (is) as c is to d'.

When a, b, c, and d are in proportion, then a and d are called the EXTREMES and b and c are called the MEANS. We also have the relationship:

Product of the MEANS = Product of the EXTREMES, i.e. $b\,c = adv$

If $\quad a{:}b = c{:}d$ then

$$b{:}a = d{:}c \qquad \text{(A)}$$
$$a{:}c = b{:}d \qquad \text{(B)}$$
$$(a + b){:}b = (c + d){:}d \qquad \text{(C)}$$

(obtained by adding 1 to both sides of the given relationship)

$$(a-b):b=(c-d):d \qquad \text{(D)}$$

(obtained by subtracting 1 from both sides of the given relationship)

$$(a+b):(a-b)=(c+d):(c-d) \qquad \text{(E)}$$

{obtained by dividing relationship (C) above by (D)}

Relationship (A) above is called INVERTENDO;
Relationship (B) is called ALTERNENDO;
Relationship (C) is called COMPONENDO;
Relationship (D) is called DIVIDENDO;
Relationship (E) is called COMPONENDO –DIVIDENDO.

The last relationship, i.e. COMPONENDO–DIVIDENDO is very helpful in simplifying problems. By this rule, whenever we know $a/b = c/d$, then we can write

$$(a+b)/(a-b)=(c+d)/(c-d).$$

The converse of this is also true—whenever we know that $(a + b)/(a - b) = (c + d)/(c - d)$, then we can conclude that $a/b = c/d$.

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}\ldots$, then each of these ratios is equal to $\frac{a+c+e+\ldots}{b+d+f+\ldots}$.

Variation

Two quantities A and B may be such that as one quantities changes in value, the other quantity also changes in value **bearing certain relationship** to the change in the value of the first quantity.

Direct Variation

One quantity A is said to vary directly as another quantity B if the two quantities depend upon each other in such a manner that if B is increased in a certain ratio, A also increases in the same ratio and if B is decreased in a certain ratio, A also decreases in the same ratio.

This is denoted as $A \alpha B$ (A varies directly as B).
If $A \alpha B$ then $A = kB$, where k is a constant. It is called the constant of proportionality.

For example, when the quantity of sugar purchased by a housewife doubles from the normal quantity, the total amount she spends on sugar also doubles, i.e. the quantity and the total amount increase (or decrease) in the same ratio.

From the above definition of direct variation, we can see that when two quantities A and B vary directly with each other, then $A/B = k$ or the ratio of the two quantities is a constant. Conversely, when the ratio of two quantities is a constant, we can conclude that they vary directly with each other.

If X varies directly with Y and we have two sets of values of the variables X and $Y - X_1$ corresponding to Y_1 and X_2 corresponding to Y_2, then, since $X \alpha Y$, we can write down

$$\frac{X_1}{Y_1}=\frac{X_2}{Y_2} \quad \text{or} \quad \frac{X_1}{X_2}=\frac{Y_1}{Y_2}$$

Inverse Variation

A quantity A is said to vary inversely as another quantity B if the two quantities depend upon each other in such a manner that if B is increased in a certain ratio, A gets decreased in the same ratio and if B is decreased in a certain ratio, then A gets increased in the same ratio.

It is the same as saying that A varies directly with $1/B$. It is denoted as $A \alpha 1/B$, i.e. $A = k/B$ where k is k the constant of proportionality.

For example, as the number of men doing a certain work increases, the time taken to do the work decreases and conversely, as the number of men decreases, the time taken to do the work increases.

From the definition of inverse variation, we can see that when two quantities A and B vary inversely with each other, then $AB = a$ constant, i.e. the product of the two quantities is a constant. Conversely, if the product of two quantities is a constant, we can conclude that they vary inversely with each other.

If X varies inversely with Y and we have two sets of values of X and $Y - X_1$ corresponding to Y_1 and X_2 corresponding to Y_2, then since X and Y are inversely related to each other, we can write down

$$X_1Y_1=X_2Y_2 \quad \text{or} \quad \frac{X_1}{X_2}=\frac{Y_2}{Y_1}$$

Joint Variation

If there are three quantities A, B, and C such that A varies with B when C is constant and varies with C when B is constant, then A is said to vary jointly with B and C when both B and C are varying, i.e. $A \alpha B$ when C is constant and $A \alpha C$ when B is a constant; $\Rightarrow A \alpha BC$.

$A \alpha BC \Rightarrow A = kBC$ where k is the constant of proportionality.

Example 6

Find the value(s) of x if $\frac{2x+5}{x+1}=\frac{x+2}{x-1}$.

Solution

$$(2x+5)(x-1)=(x+2)(x+1)$$

$$2x^2+5x-2x-5=x^2+2x+x+2$$

$$\Rightarrow \quad x^2=7$$

$$\therefore \quad x=\pm\sqrt{7}$$

Example 7

X varies directly with $Y^2 + 18$. When $Y = 18$, $X = 18$. Find Y when $X = 1$.

Solution

$$\frac{X_1}{X_2} = \frac{Y_1^2 + 18}{Y_2^2 + 18}\,;\; \frac{18}{1} = \frac{18^2 + 18}{Y_2^2 + 18}$$

$$Y_2^2 + 18 = 19$$

$$Y_2 = \pm 1$$

In these types of problems on variation, there are typically three parts:

1. The relationship between different variables is defined to frame an equation involving the variables and the constant of proportionality.
2. One set of values of all the values of all the variables is given to enable us find the value of the constant of proportionality.
3. The values of all but one variable of a second set are given, and we are asked to find the value of the one variable whose value is not given.

Example 8

The ratio of the monthly incomes of A and B is 4:3. The ratio of their monthly expenditures is 5:4. If A saves one fourth of his monthly income, find the ratio of their monthly savings.

Solution

Let the monthly incomes of A and B be ₹$4x$ and ₹$3x$, respectively.

Let the monthly expenditures of A and B be ₹$5y$ and ₹$4y$ respectively. Monthly savings of A = ₹$(4x - 5y)$. Monthly savings of B = ₹$(3x - 4y)$.

Given that $4x - 5y = \frac{1}{4}(4x)$

$$3x = 5y.$$

$\therefore$ Monthly saving of $B = 3x - 4y = 5y - 4y$, i.e. ₹y.

Required ratio $= \frac{1}{4}(4x) : y = x{:}y = 5{:}3$

Example 9

If x varies directly with y, check whether $x^3 + y^3$ varies directly with $x^3 - y^3$.

Solution

Let $\frac{x}{y} = k$, where k is a constant

$$x = k\,y$$

$$x^3 + y^3 = y^3\,(k^3 + 1)$$

$$x^3 - y^3 = y^3\,(k^3 - 1)$$

$$\frac{x^3 + y^3}{x^3 - y^3} = \frac{k^3 + 1}{k^3 - 1} \text{ a constant}$$

$\therefore$ $(x^3 + y^3)$ varies directly with $(x^3 - y^3)$

Example 10

The monthly expenses of Raja on his car are partly constant and partly vary with the number of kilometres he travels in a month. If he travels 100 km in a month his total car expenses will be ₹3,500. If he travels 200 km in a month, his total car expenses will be ₹4,000. If he travels 250 km in a month, what will be his total car expenses.

Solution

Let his total car expenses be ₹T. Let the fixed expense be ₹F. Let the variable expense be ₹V.

$$T = F + V$$

If he travels D km in a month, $\frac{V}{D} = k$, where k is a constant.

$$\therefore \quad T = F + k\,D$$

Given that

$$3500 = F + 100k \qquad (1)$$

$$4000 = F + 200k \qquad (2)$$

Solving (1) and (2),

$$F = 3000 \text{ and } k = 5.$$

Total car expenses if he travels 250 km

$$= F + 250k = ₹4250.$$

The problems involving ratio and proportion are just different forms of the models of the basic problems we saw above. For example, the problem we just solved above might be reframed bringing in mangoes, bananas, baskets, etc. Here, practice and perseverance pay you a lot. In entrance exams, there will be either direct problems on ratio, proportion, and variation or indirect problems of application of these concepts just discussed to areas like time and work or time and distance.

Exercises

Direction for questions 1 to 18: Select the correct alternative from the given choices.

1. If $p:q = 5:4$ and $p = a + b$ and $q = a - b$, find $a:b$.
(A) 1:9 (B) 9:1
(C) 5:4 (D) 4:5

2. The number of marbles with A and B are in the ratio of 10:11. Which of the following cannot be a possible number of marbles with A and B together?
(A) 189 (B) 210 (C) 231 (D) 153

3. The ratio of the number of students in three classes A, B, and C is 3:7:8. If ten students, are transferred from C to B, B will have 80 students. Find the total number of students in the three classes.
(A) 150 (B) 160 (C) 180 (D) 210

4. Three positive numbers p, q, and r satisfy $\frac{q+r}{p} = \frac{p+r}{q} = \frac{p+q}{r} = K$. $K =$ ______.
(A) 3/2 (B) 5/2 (C) 3 (D) 2

5. What must be subtracted from p and added to q so that the ratio of the resultants becomes 1:3?
(A) $\frac{p+q}{3}$ (B) $\frac{3p-q}{4}$
(C) $\frac{p-q}{p+q}$ (D) $\frac{q-3p}{4}$

6. Vipin's present age is twice the age of Kishore one year ago. What is the sum of their present ages (in years), if the ratio of the sum of their present ages to the difference of their present ages is 19:5?
(A) 21 (B) 19 (C) 24 (D) 34

7. Three different types of balls priced at ₹5, ₹8, and ₹13 per piece are displayed in three different boxes by a trader. Mr. Paul bought from this shop all three types of balls spending a total sum of ₹768. The numbers of the balls he bought, taken in the order in which the prices are mentioned above, are in the ratio 5:4:3. How many balls of the costliest variety did he buy?
(A) 104 (B) 64 (C) 48 (D) 24

8. If $a:b = 2:3$ $b:c = 4:3$ and $c:d = 2:3$, then find $a:b:c:d$.
(A) 8:12:9:27
(B) 16:24:18:27
(C) 18:27:36:8
(D) 12:18:15:20

9. The weights of Bimal and Basu are in the ratio 2:3 and the weights of Basu and Bali are in the ratio 4:3. What is Basu's weight (in kg) if the sum of the weights of Bimal, Basu, and Bali is 203 kg?
(A) 84 (B) 76
(C) 49 (D) 65

10. If $3x - 4y + 2z = 0$ and $4x - 2y - z = 0$, find $x:z:y$.
(A) 8:10:11 (B) 8:11:40
(C) 11:40:8 (D) 8:40:11

11. If $a + b - c:b + c - a:a + c - b = 5:6:7$, then find $a:b:c$.
(A) 12:13:11 (B) 12:11:13
(C) 13:12:11 (D) 13:11:12

12. Which of the following represents a possible value of $p : q$ satisfying $\frac{20p^2 - 40pq}{pq + 4q^2} = 20$?
(A) 3:1 (B) 1:4
(C) 4:1 (D) 5:1

13. Ninety three is divided into two parts such that thrice the first part and twice the second part are in the ratio 25:4. Find the first part.
(A) 60 (B) 75
(C) 50 (D) 70

14. If three is subtracted from the numerator and five is added to the denominator of a fraction, the new fraction formed is 1/2. If two is added to the numerator of the initial fraction, the ratio of the new numerator to the denominator becomes 1:1. Find the original fraction.
(A) 11/13 (B) 18/23
(C) 13/15 (D) 13/11

15. The ratio of the number of students in classes A, B, and C is 3:7:8. If 10 students leave C and join B, the ratio of the number of students in B and C would be reversed. Find the total number of students in the classes A, B, and C.
(A) 144 (B) 162 (C) 180 (D) 198

16. A person has with him a certain number of weighing stones of 100 g, 500 g, and 1 kg in the ratio of 3 : 5 : 1. If a maximum of 5 kg can be measured using weighing stones of 500 g alone, then what is the number of 100 g stones he has?
(A) 6 (B) 3 (C) 9 (D) 5

17. The ratio of the prices of tea last year and this year is 5:6. The ratio of the prices of coffee last year and this year is 7:8. The sum of prices of a kg of tea and a kg of coffee this year is ₹48. Find the price of tea (in ₹) last year if it was $\frac{20}{21}$ of the price of coffee last year.
(A) 15 (B) 20 (C) 25 (D) 10

18. Ajay and Vijay wrote a test. The sum of Ajay's score and twice Vijay's score is 310. The sum of Vijay's score and twice Ajay's score is 290. Find the ratio of the scores of Ajay and Vijay.
(A) 9:11 (B) 13:17
(C) 11:19 (D) 7:13

Direction for questions 19 and 20: These questions are based on the data given below.

A test of 60 minutes contains questions on Mathematics and English only. The time taken to solve a Mathematics question is twice the time taken to answer an English question and the ratio of time taken to solve all Mathematics questions to time taken to answer all English questions is 8/7.

19. What is the ratio of the number of English questions to that of Mathematics?

(A) 11/7 (B) 7/4
(C) 9/4 (D) 7/5

20. If the total number of questions is 22, how many English questions can be answered in 18 minutes?

(A) 8 (B) 10 (C) 11 (D) 9

Direction for questions 21 to 25: Select the correct alternative from the given choices.

21. The pressure of a gas varies directly with the temperature when the volume is constant and varies inversely with the volume when temperature is constant. If the present temperature is 100 K, what will be the increase in temperature if the pressure triples and the volume doubles?

(A) 200 K (B) 600 K
(C) 500 K (D) 100 K

22. For a body starting from rest, the distance travelled (d) is directly proportional to the square of the time elapsed from the start (t). When $t = 4$ s, $d = 48$ m. What is the value of d (in metres) at $t = 7$ s?

(A) 128 (B) 150
(C) 115 (D) 147

23. 'A' varies directly as the sum of two quantities 'B' and 'C'. 'B' in turn varies directly as 'x' and 'C' varies inversely as 'x'. When $x = 1$ or 2, $A = 3$. Find the value of A when $x = 4$.

(A) 5 (B) 4.5
(C) 5.5 (D) 6

24. The extension of a spring from its rest position is directly proportional to the force acting on the spring. An additional force applied on the already stretched spring produces a further extension, which is twice that of the initial extension. What is the ratio of the additional force to the initial force?

(A) 3:1 (B) 2:1
(C) 4:1 (D) 1:3

25. The kinetic energy of a body is directly proportional to the square of its speed when the mass is kept constant and is directly proportional to mass when its speed is kept constant. A body with a mass of 2 kg and a speed of 10 m/s has a kinetic energy of 100 joules. What is the kinetic energy of a body whose mass is 20 kg and speed is 1 m/s?

(A) 100 joules (B) 1000 joules
(C) 10 joules (D) 20 joules

Answer Keys

1. B	**2.** D	**3.** C	**4.** D	**5.** B	**6.** B	**7.** D	**8.** B	**9.** A	**10.** A
11. B	**12.** C	**13.** B	**14.** C	**15.** C	**16.** A	**17.** B	**18.** A	**19.** B	**20.** D
21. C	**22.** D	**23.** B	**24.** B	**25.** C					