17. Theorems on concurrence

Let us do ourselves 17.1

1. Question

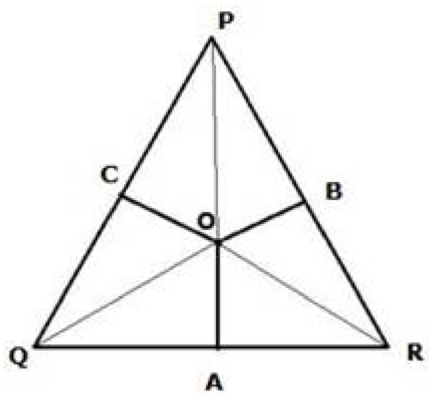
I draw an acute angled triangle PQR and let us prove that the perpendicular bisectors PQ, QR and RP are concurrent. Hence let us write where the circumcentre lies (inside/outside/on the side) of the acute angle triangle.

Answer

Let us consider an acute angle triangle PQR.

OA, OB and OC perpendicularly bisects the sides.

Joined OP, OQ and OR!



In Δ AOQ and Δ AOR

AO = AO (Common side)

AQ = AR(A is the midpoint of QR)

∠OAQ = ∠OAR (OA is the perpendicular bisector)

Hence Δ QOA and Δ ROA are congruent to each other by S.A.S. axiom of congruency

Hence OQ = OR (Corresponding parts of Congruent Triangles)

In Δ QOC and Δ POC

CO = CO (Common side)

CQ = CP (C is the midpoint of QP)

∠OCQ = ∠OCP (OC is the perpendicular bisector)

Hence Δ QOC and Δ POC are congruent to each other by S.A.S. axiom of congruency

Hence OQ = OP (Corresponding parts of Congruent Triangles)

Hence OP = OQ = OR

So it is proved that the perpendicular bisectors are concurrent

The circumcentre lies inside in an acute angled triangle

2. Question

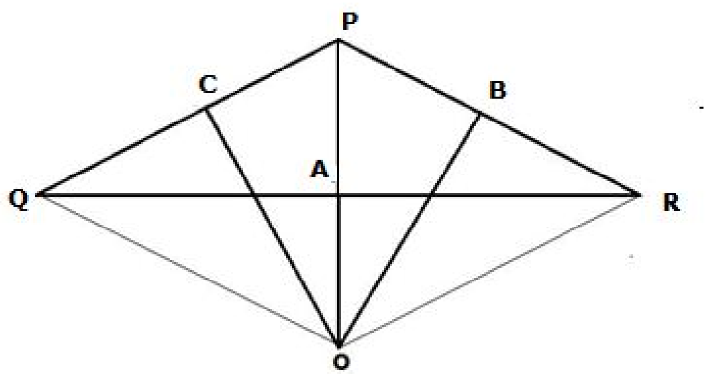
I draw an obtuse angled triangle and let us prove that the perpendicular bisectors of sides are concurrent. Let us write where the circumcentre lies (inside/outside/on the side) of the triangular region.

Answer

Let us consider an obtuse angled triangle PQR.

OA, OB and OC perpendicularly bisect the sides.

Joined OP, OQ and OR!



In Δ AOQ and Δ AOR

AO = AO (Common side)

AQ = AR (A is the midpoint of QR)

∠OAQ = ∠OAR (OA is the perpendicular bisector)

Hence Δ QOA and Δ ROA are congruent to each other by S.A.S. axiom of congruency

Hence OQ = OR (Corresponding parts of Congruent Triangles)

In Δ QOC and Δ POC

CO = CO (Common side)

CQ = CP(C is the midpoint of QP)

∠OCQ = ∠OCP (OC is the perpendicular bisector)

Hence Δ QOC and Δ POC are congruent to each other by S.A.S. axiom of congruency

Hence OQ = OP (Corresponding parts of Congruent Triangles)

Hence OP = OQ = OR

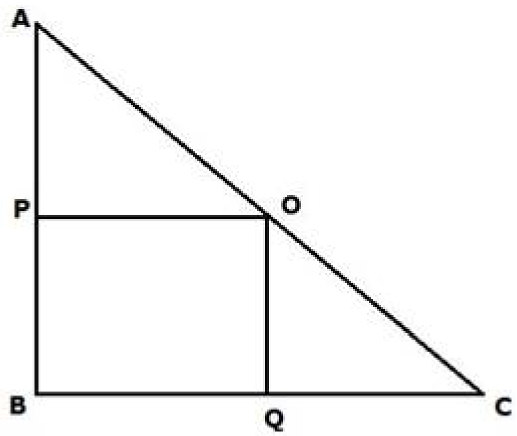
So it is proved that the perpendicular bisectors are concurrent

The circum centre lies outside in an obtuse angled triangle

3. Question

Rita draws a right angled triangle. Let us prove logically that the perpendicular bisectors of sides are concurrent and where the position of the circum centre (inside/outside/on the side) is

Answer



PO is perpendicular to AB and OQ is perpendicular to BC

Since Δ ABC is right angled at B so POQB forms a rectangle

Hence we can say that

PO = BQ (Opposite sides of rectangle)

BP = QO(Opposite sides of a rectangle)

In Δ AOP and Δ QOC

∠APO = ∠OQC (OP and OQ are perpendiculars)

∠OAP = ∠COQ (Corresponding angles)

PO = BQ = CQ (Q is midpoint of BC)

So Δ AOP and Δ COQ are congruent by A.A.S. axiom of congruency

Hence AO = OC

So O is the midpoint of AC

Hence the perpendiculars are concurrent

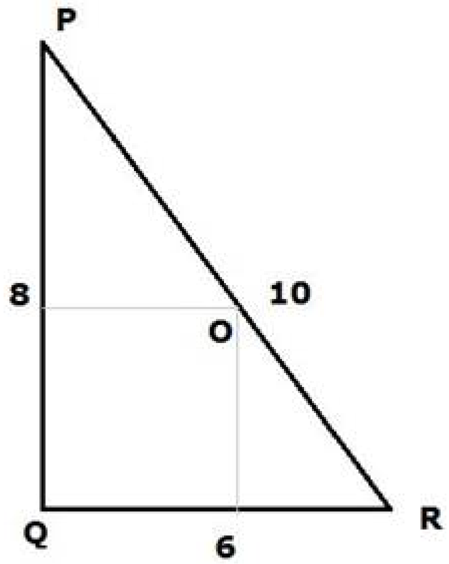
The circum centre lies on the midpoint of the hypotenuse of the triangle.

Let us do ourselves 17.2

1. Question

Let us write what is the length of circum radius of a triangle lengths of sides of which are 6 cm, 8 cm and 10 cm.

Answer



The Δ PQR is right angled at Q

Let PQ = 8 cm

QR = 6 cm

PR = 10 cm

Squaring and adding the smaller two sides of the triangle we get

$PQ^2 + QR^2 = 6^2 + 8^2$

$PQ^2 + QR^2 = 36 + 64$

$\Rightarrow PQ^2 + QR^2 = 100$

$\Rightarrow PQ^2 + QR^2 = 10^2$

$\Rightarrow PQ^2 + QR^2 = PR^2$

So we see the sum of squares of the two smaller sides is equal to the square of the larger side.

Hence the triangle is a right angled triangle.

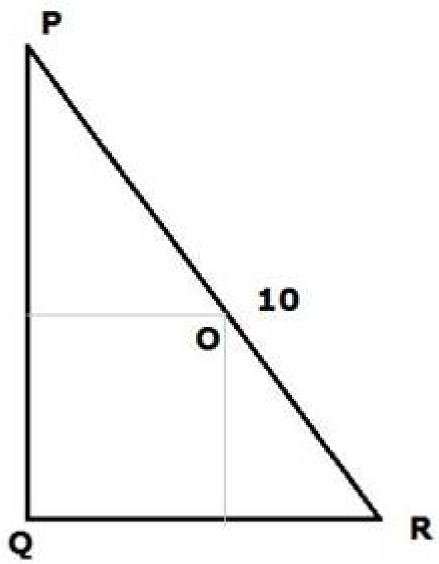
The circum radius of a right angled triangle is always half of the length of the hypotenuse.

Circum radius = $\frac{10}{2} = 5$ cm

2. Question

If the length of circum radius of a right-angled triangle is 10 cm, then let us write how much length of hypotenuse of triangle is.

Answer



In a right angled triangle the length of hypotenuse is always twice the length of its circum radius

The Δ PQR is right angled at Q

PR is the hypotenuse

OP = OR is the circum radius

Length of circum radius = 10 cm

⇒ Length of hypotenuse = 2×10 = 20 cm

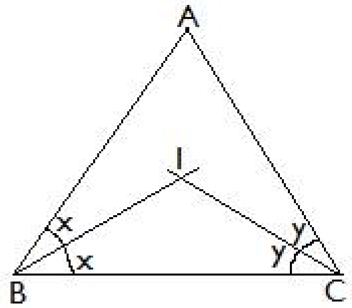
Let us do 17

1. Question

The bisectors of ∠B and ∠C of ΔABC intersect at the point I. Let us prove that

$\angle BIC = 90° + \frac{\angle BAC}{2}$

Answer



Let ∠ABI = ∠CBI = x

And ∠ACI = ∠BCI = y

In Δ ABC, using angle sum property,

∠A + 2x + 2y = 180° ....(1)

In Δ BIC, using angle sum property,

∠I + x + y = 180° ....(2)

Divide (1) by 2

We get,

$$\frac{\angle A}{2} + x + y = 90° \ ...(3)$$

Subtract (3) from (2)

$$\angle I - \frac{\angle A}{2} = 90°$$

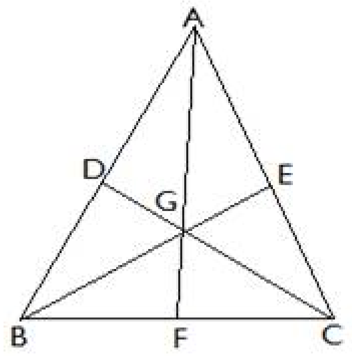
Or

$$\angle BIC = \frac{\angle BAC}{2} + 90°$$

2. Question

If the length of three medians of a triangle are equal. Let us prove that the triangle is an isosceles triangle.

Answer



In Δ BGD and Δ CGE,

∠BGD = ∠CGE (vertically opposite angles) ...(1)

BE = DC (medians are equal)

Since, centroid divides the median in ratio 2:1

So, $BG = \frac{2}{3}BE$ and $CG = \frac{2}{3}CD$

⇒ BG = CG (as BE = DC) ...(2)

And $GE = \frac{1}{3}BE$ and $DG = \frac{1}{3}CD$

⇒ GE = DG (BE = DC) .....(3)

Hence, by SAS congruency, Δ BGD and Δ CGE are congruent.

By CPCT, BD = EC

2× BD = 2 × EC

⇒ AB = AC

Hence, the triangle is isosceles.

3. Question

Let us prove that in an equilateral triangle, circumcentre, incentre, centriod, orthocenter will coincide.

Answer

We know,

Circumcenter of a triangle: The intersection of perpendicular bisectors of the triangles.

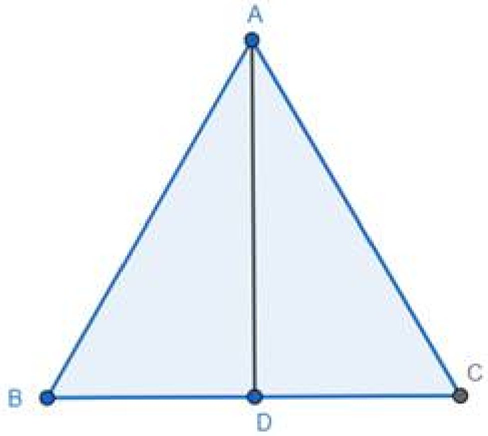
Incenter: The intersection of angle bisectors of the triangle.

Centroid: The intersection of the medians of the triangle.

Orthocentr: The intersection of the altitudes of the triangle

In order to prove that, In an equilateral triangle, circumcenter, incentre, centroid and orthocenter coincide, it is sufficient to prove that for any side median, altitude, perpendicular bisector and angle bisector of angle opposite to that side is common.

Let us consider an equilateral ΔABC, such that AD is a median to side BC.



To Prove:

(i) $AD \perp BC$ [AD is Altitude and perpendicular bisector of BC]

(ii) $\angle BAD = \angle CAD$ [AD is angle bisector of $\angle A$]

Proof:

In ΔABD and ΔACD

AB = AC [Sides of equilateral triangle]

AD = AD [Common side]

BD = CD [As, AD is median to BC]

$\Rightarrow \Delta ABD \cong \Delta ACD$ [By SSS property of congruent triangles]

Now, As corresponding parts of congruent triangles are equal [CPCT], we have

$\angle ADB = \angle ADC$

Also,

$\angle ADB + \angle ADC = 180°$ [Linear pair]

$\Rightarrow \angle ADB + \angle ADB = 180°$

$\Rightarrow \angle ADB = \angle ADC = 90°$

$\Rightarrow AD \perp BC$

Since, $AD \perp BC$, and BD = BC

∴ AD is perpendicular bisector of BC and as well as altitude from A to BC.

Now,

∠BAD = ∠CAD [By CPCT]

AD is angle bisector of ∠A.

∴ AD is median, perpendicular bisector, altitude and angle- bisector.

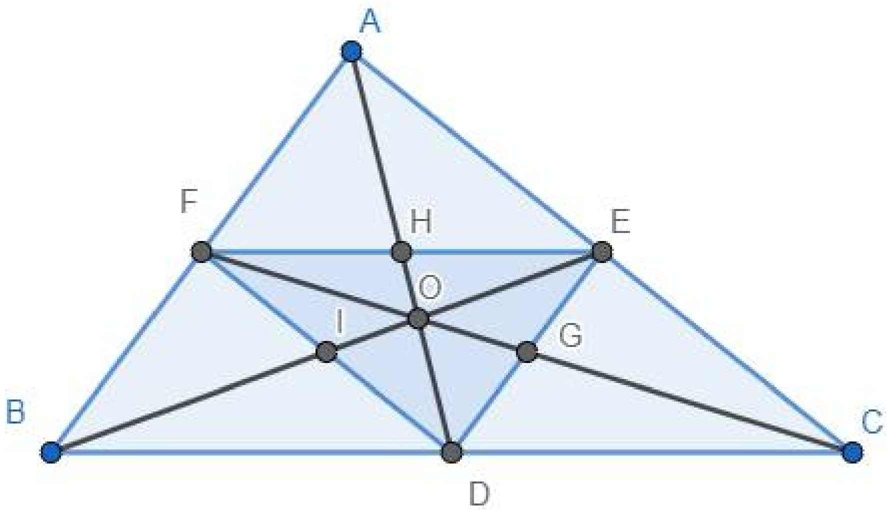
We can prove this result for every median, and since lines are same, their intersection will also be same.

Hence Proved!

4. Question

AD, BE and CF are three medians of a triangle ABC. Let us prove that the centriod of ABC and DEF are the same point.

Answer



Given: A triangle ABC, AD BE and CF are medians, let us join DEF to make a triangle and label the points where medians of ΔABC intersects the sides of ΔDEF. Let O be the intersection of medians of ΔABC i.e. O is centroid of ΔABC

To Prove: O is centroid of ΔDEF

Proof:

As, CF and BE are medians to sides AB and AC respectively,

F is mid-point of AB and E is mid-point of AC

By mid-point theorem [The line segment connecting the **midpoints** of two sides of a triangle is parallel to the third side]

EF || BC

⇒ FH || BD and HE || DC

In ΔABD and ΔAFH

∠FAH = ∠BAD [Common]

∠AFH = ∠ABD [Alternate angles]

ΔABD ~ ΔAFH [By AA similarity criterion]

$\Rightarrow \frac{FH}{BD} = \frac{AH}{AD}$ ......[1] [Similar triangles have equal ratio of Corresponding sides]

In ΔADC and ΔAHE

∠HAE = ∠DAC [Common]

∠AHE = ∠ADC [Alternate angles]

ΔADC ~ ΔAHE [By AA similarity criterion]

$\Rightarrow \frac{HE}{DC} = \frac{AH}{AD}$ ......[2] [Similar triangles have equal ratio of Corresponding sides]

From [1] and [2], we have

$$\frac{FH}{BD} = \frac{HE}{DC}$$

Here, BD = DC ∵ AD is median

⇒ FH = HE

⇒ DH is a median from D to side FH

Similarly, we can prove that EI and FG are medians, and as intersection of FH, EI and FG is O.

⇒ O is centroid for ΔDEF

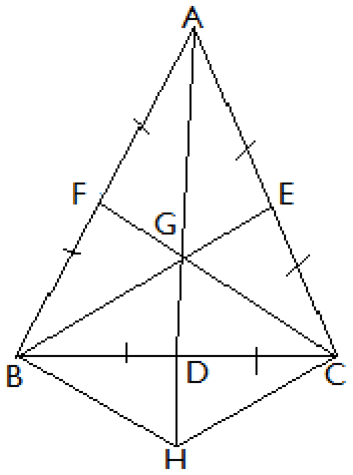
⇒ ΔDEF and ΔABC have same centroid.

Hence, Proved!

5. Question

Let us prove that two medians of triangle are together greater than the third median.

Answer



Given: AD, BE and FC are medians.

Extend AD to H such that (AG = HG) then join BH and CH.

In Δ ABH,

F is mid-point and G is centroid, so use mid-point theorem,

So, FG || BH

Similarly, GC || BH ….(1)

And BG || HC …..(2)

From (1) and (2),

We get,

BGCH is a parallelogram,

So, BH = GC …..(3) (opp. Sides of a parallelogram are equal)

In Δ BGH,

BG + GH > BH (sum of two sides greater than third side)

⇒ BG + AG > GC

Similarly, BE + CF > AD and AD + CF > BE

Hence proved.

6. Question

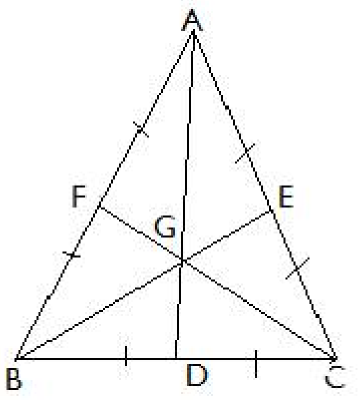
AD, BE and CF are three medians of ΔABC. Let us prove that

(i) 4 (AD + BE + CF) > 3 (AB + BC + CA)

(ii) 3(AB + BC + CA) > 2(AD + BE + CF)

Answer

(i)



As we know, if G is centroid, and centroid divides the median in 2:1

Then $\frac{2}{3}$ BE = BG and $\frac{2}{3}$ FC = CG

In

Δ BGC,

BG + GC > BC (sum of two sides greater than 3rd side)

$\Rightarrow \frac{2}{3}BE + \frac{2}{3}FC > BC$

⇒ 2BE + 2FC > 3BC

Or

3BC < 2BE + 2FC ….(1)

Similarly,

3CA < 2CF + 2AD ….(2)

And 3AB < 2AD + 2BE ….(3)

Add (1), (2) and (3)

We get,

3 BC + 3CA + 3AB < 2BE + 2FC + 2CF + 2AD + 2AD + 2BE

⇒ 3 (AB + BC + CA) = 4 BE + 4 FC + 4 AD

Or 4 (AD + BE + CF) > 3 (AB + BC + CA)

Hence proved.

(ii) In ΔACD

AC + CD > AD (sum of two sides greater than 3rd side)

Since, AD is median

$$\Rightarrow BD = CD = \frac{1}{2}BC$$

$$\Rightarrow AC + \frac{BC}{2} > AD \text{ ......[1]}$$

Similarly,

In ΔABE

$$AB + \frac{1}{2}AC > BE \text{ ......[2]}$$

In ΔBFC

$$BC + \frac{1}{2}AB > CF \text{ ......[3]}$$

Adding [1], [2] and [3], we get

$$AC + \frac{BC}{2} + AB + \frac{AC}{2} + BC + \frac{AB}{2} > AD + BE + CF$$

$$\Rightarrow \frac{3}{2}(AB + BC + CA) > AD + BE + CF$$

$$\Rightarrow 3(AB + BC + CA) > 2(AD + BE + CF)$$

Hence, Proved!

7. Question

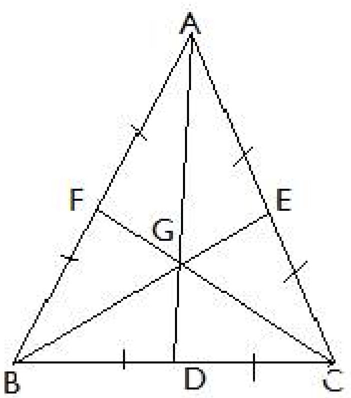
Three medians AD, BE and CF of ΔABC intersect each other at the point G. If area of ΔABC is 36 sq. cm, let us calculate.

(i) Area of ΔAGB

(ii) Area of ΔCGE

(iii) Area of quadrilateral BDGF.

Answer



Given: area of ABC is 36 sq cm

(i) As we know if three medians AD, BE and CF of a ΔABC intersect one another at the point G (centroid) then,

Area of ΔABC = 3 × Δ AGB

⇒ Area of ΔAGB = $\frac{1}{3} \times 36$

⇒ Area of ΔAGB = 12 sq. cm

(ii) As we know if three medians AD, BE and CF of a ΔABC intersect one another at the point G (centroid) then,

Area of ΔABC = 6 × Δ GEC

⇒ Area of ΔGEC = $\frac{1}{6} \times 36$

⇒ Area of ΔGEC = 6 sq. cm

(ii) As we know if three medians AD, BE and CF of a ΔABC intersect one another at the point G (centroid) then,

Area of ΔABC = 3 × Δ BDGF

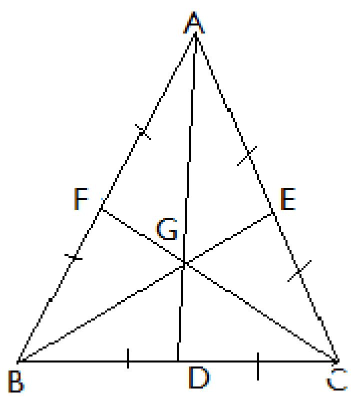
⇒ Area of ΔBDGF = $\frac{1}{3} \times 36$

⇒ Area of ΔBDGF = 12 sq. cm

8. Question

AD, BE and CF are the medians of ΔABC. If $\frac{2}{3}$ AD = BC, then let us prove that the angle between two medians is 90°.

Answer



Given: $\frac{2}{3}$ AD = BC

As we know, if G is centroid,

Then $\frac{2}{3}$ AD = AG

⇒ AG = BC

⇒ if AG = 2 then GD = 1 ….(1)

Also, BC = 2 (BG = AG)

BD = DC = 1 (median) ….(2)

From (1) and (2)

BD = DC = GD = 1

⇒ ∠GDB = 90° and ∠GDC = 90°

Because, BD = GD

⇒ ∠GBD = ∠DGB = x (let) (isosceles triangle)

Hence,

x + x + 90° = 180°

2x = 90°

⇒ x = 45°

Similarly,

Because, DC = DG

⇒ ∠GCD = ∠DGC = y (let) (isosceles triangle)

Hence,

y + y + 90° = 180°

2y = 90°

⇒ y = 45°

And x + y = 45° + 45°

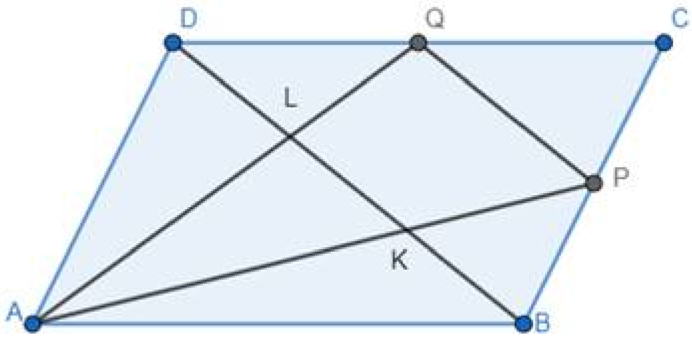
⇒ ∠BGC = 90°

i.e. angle between median is 90° .

9. Question

P and Q are the mid points of sides BC and CD of a parallelogram ABCD respectively; the diagonals AP and AQ cut BD at the points K and L. Let us prove that, BK = KL = LD.

Answer



Given: P and Q are the mid points of sides BC and CD of a parallelogram ABCD respectively; the diagonals AP and AQ cut BD at the points K and L

To Prove: BK = KL = LD.

1 A. Question

O is the circumcentre of ABC; if ∠BOC = 80° the ∠BAC is

A. 40°

B. 160°

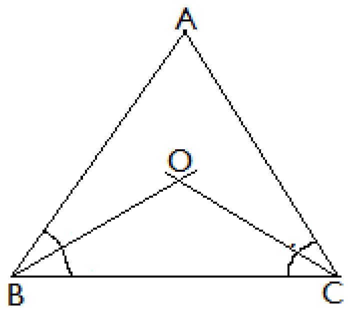
C. 130°

D. 110°

Answer: A

Answer

Given:



As we know,

If O is circumcenter of Δ ABC then:

∠BOC = 2 ∠BAC (Application 1)

$$\Rightarrow \angle BAC = \frac{\angle BOC}{2}$$

$\Rightarrow \angle BAC = \frac{80°}{2}$

⇒ ∠BAC = 40°

Option (A) is correct.

1 B. Question

O is the orthocenter of ABC; if ∠BAC = 40° the ∠BOC is

A. 80°

B. 140°

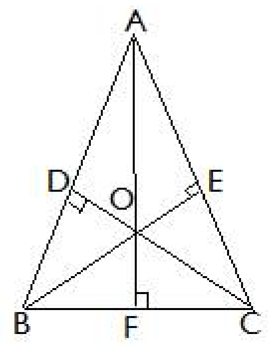
C. 110°

D. 40°

Answer: B

Answer

Given:



As we know,

If O is orthocenter Δ ABC then sum of ∠BOC and ∠BAC is180°:

i.e. ∠BOC + ∠BAC =180°:

∠BOC =180° - ∠BAC

⇒ ∠BOC = 180° - 40°

⇒ ∠BAC = 140°

Option (B) is correct.

1 C. Question

O is the incentre of ΔABC; if ∠BAC = 40° then ∠BOC is

A. 80°

B. 110°

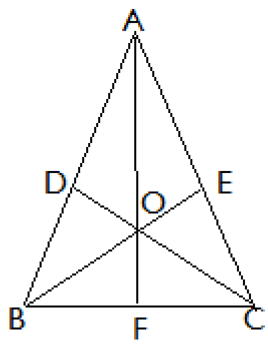
C. 140°

D. 40°

Answer: B

Answer

Given:



As we know,

If O is incenter of ΔABC then:

$$\angle BOC = 90° + \frac{\angle BAC}{2}$$

$$\Rightarrow \angle BOC = 90° + \frac{40°}{2}$$

$\Rightarrow \angle BOC = 90° + 20°$

$\Rightarrow \angle BOC = 110°$

Option (B) is correct.

1 D. Question

G is the centriod of triangle ABC; if area of GBC is 12 sq cm, then the area of ABC is

A. 24 sq cm.

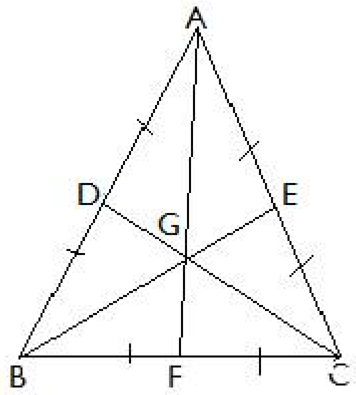
B. 6 sq cm.

C. 36 sq cm.

D. none of them

Answer: C

Answer



Given: area of GBC is 12 sq cm

As we know if three medians AD, BE and CF of a ΔABC intersect one another at the point G (centroid) then,

Area of ΔABC = 3 × Δ BGC

⇒ Area of ΔABC = 3 × 12

⇒ Area of ΔABC = 36 sq. cm

Option (C) ic correct.

1 E. Question

If the length of circumradius of a right angled triangle is 5 cm, then the length of hypotenuse is

A. 2.5 cm

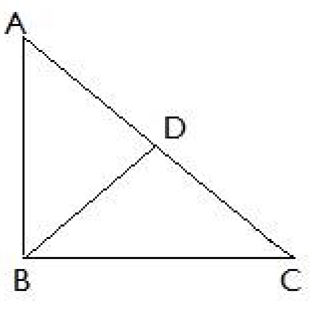
B. 10 cm

C. 5 cm

D. none of this

Answer: B

Answer



Given: length of circumradius of a right angled triangle = 5 cm

As we know,

length of circumradius of a right angled triangle = 1/2 length of hypotenuse

⇒ BD = 1/2 AC

⇒ AC = 2 BD

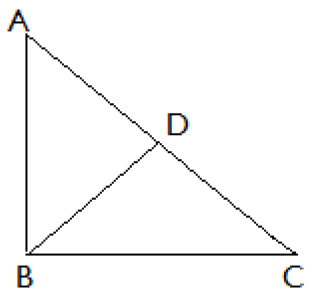
⇒ AC = 2 (5) = 10cm

Option (B) ic correct.

11 A. Question

If the lengths of sides of triangle are 6 cm, 8 cm, and 10 cm, then let us write where the circumcentre of this triangle lies.

Answer



Given: lengths of sides of triangle are 6 cm, 8 cm, and 10 cm.

Let AC = 10cm, BC = 8cm and AB = 6cm

To be a right angle triangle,

$(AC)^2 = (AB)^2 + (BC)^2$

$(10)^2 = (8)^2 + (6)^2$

100 = 64 + 36

100 = 100

Hence, LHS = RHS

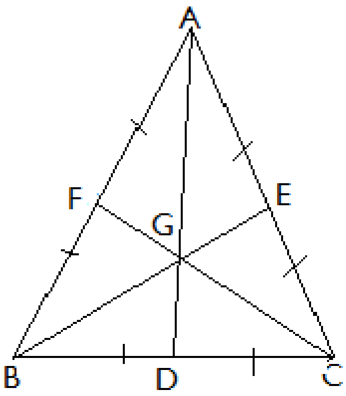
So, it is a right angle triangle.

In a right angle triangle circumcentre lies on hypotenuse.

11 B. Question

AD is the median and G is the centriod of an equilateral triangle. If the length of side $3\sqrt{3}$ m, then let us write the length of AG.

Answer



Given: length of side = a = $3\sqrt{3}$ m

As we know,

Centroid is equidistant from the corners of an equilateral triangle.

Hence, AG is actually the radius of circumcircle to be inscribed around triangle.

As we know, in an equilateral triangle,

Circumradius of an equilateral triangle (r) = $\frac{a}{\sqrt{3}}$

$\Rightarrow r = \frac{3\sqrt{3}}{\sqrt{3}}$

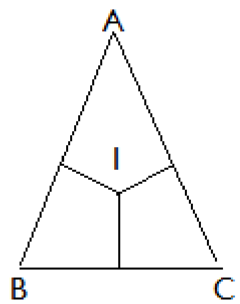
$\Rightarrow$ r = 3

Then, length of AG = 3m

11 C. Question

Let us write how many points are equidistant from sides of a triangle.

Answer



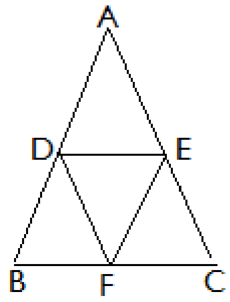
The point which is equidistant from all sides of a triangle is called incenter.

Hence, there is only one point equidistant from sides of a triangle.

11 D. Question

DEF is a pedal triangle of an equilateral triangle ABC. Let us write the measure of ∠FDA

Answer



Given: Pedal triangle is a triangle formed by joining the mid-point of a triangle's sides.

As we know, in an equilateral triangle, triangles formed after joining the mid points is also an equilateral triangle.

Also, the triangle formed joining the vertx and one of the vertex is also an equilateral triangle.

As we know, the measure of an angle in an equilateral triangle is 60° .

Hence, in Δ DEF

∠FDE = 60°

And, in Δ ADE

∠ADE = 60°

So,

∠FDA = ∠FDE + ∠ADE = 60°

⇒ ∠FDA = 60° + 60°

⇒ ∠FDA = 120°

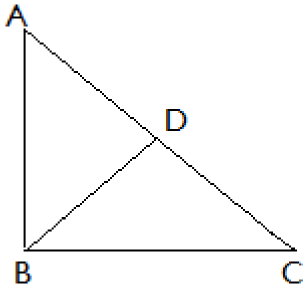
11 E. Question

ABC is an isosceles triangle in which ∠ABC = ∠ACB and Median AD = $\frac{1}{2}$ BC. If AB = √2 cm, let us write the length of circumradius of this triangle.

Answer

Given: Median AD = $\frac{1}{2}$ BC ....(1)

From (1) we can say it is a right angle triangle as well,



AB = √2 cm

AC = √2 cm (isosceles triangle)

In a right angle triangle,

$(BC)^2 = (AB)^2 + (AC)^2$

$\Rightarrow (BC)^2 = (\sqrt{2})^2 + (\sqrt{2})^2$

$\Rightarrow (BC)^2 = 2 + 2$

$\Rightarrow (BC)^2 = 4$

⇒ BC = 2cm

As we know,

length of circumradius of a right angled triangle = 1/2 length of hypotenuse

⇒ AD = 1/2 BC

⇒ AD = 1/2 (2)

⇒ AD = 1cm