CHAPTER 05

Probability

5.1 Some Fundamental Concepts

Sample Space and Event: Consider an experiment whose outcome is not predictable with certainty. Such an experiment is called a **random experiment**. However, although the outcome of the experiment will not be known in advance, let us suppose that the set of all possible outcomes is known. This set of all possible outcomes of an experiment is known as the **sample space** of experiment and is denoted by S.

Some examples follows:

(*i*) If the outcome of an experiment consist in the determination of the sex of a newborn child, then S = {*g*, *b*} where the outcome *g* means that the child is a girl and *b* is the boy.

(*ii*) If the outcome of an experiment consist of what comes up on a single dice, then S = {1, 2, 3, 4, 5, 6}

(*iii*) If the outcome of an experiment is the order of finish in a race among the 7 horses having post positions 1, 2, 3, 4, 5, 6, 7; then S = {all 7! permutations of the (1, 2, 3, 4, 5, 6, 7)}

The outcome (2, 3, 1, 6, 5, 4, 7) means, for instances, that the number 2 horse comes in first, then the number 3 horse, then the number 1 horse, and so on.

Any subset E of the sample space is known as **Event**. That is, an event is a set consisting of some or all of the possible outcomes of the experiment.

If the outcome of the experiment is contained in E, then we say that E has occurred. Always $E \subseteq S$.

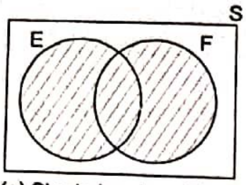
Since E and S are sets, theorems of set theory may be effectively used to represent and solve probability problems which are more complicated.

Examples: In the preceding example – (i) If $E_1 = \{g\}$, then E_1 is the event that the child is a girl. Similarly, if $E_2 = \{b\}$.

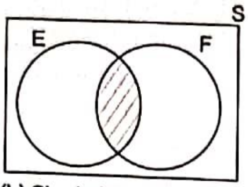
Then E_2 is the event that the child is a boy. These are examples of simple events. Compounded events may consist of more than one outcome. Such as E = {1, 3, 5} for an experiment of throwing a dice. We say event E has happened if the dice comes up 1 or 3 or 5.

For any two events E and F of a sample space S, we define the new event $E \cup F$ to consists of all outcomes that are either in E or in F or in both E and F. That is, the event $E \cup F$ will occur if either E or F or both occurs. For instances, in dice example (i) if event $E_1 = \{1, 2\}$ and $E_2 = \{3, 4\}$, then $E \cup F = \{1, 2, 3, 4\}$.

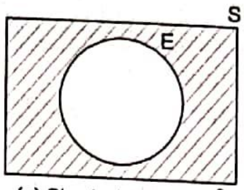
That is $E \cup F$ would be another event consisting of 1 or 2 or 3 or 4. The event $E \cup F$ is called **union** of event E and the event F. Similarly, for any two events E and F we may also define the new event $E \cap F$, called **intersection** of E and F, to consists of all outcomes that are common to both E and F.



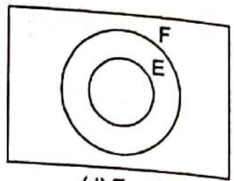
(a) Shaded region : $E \cup F$



(b) Shaded region : $E \cap F$



(c) Shaded region : E^C



(d) $E \subset F$

Mutually Exclusive Events

Two events E and F are mutually exclusive, if $E \cap F = \phi$ i.e. $P(E \cap F) = 0$. In other words, if E occurs, F cannot occur and if F occurs, then E cannot occur (i.e. both cannot occur together).

Collectively Exhaustive Events

Two events E and F are collectively exhaustive, if $E \cup F = S$.i.e. together E and F include all possible outcomes, $P(E \cup F) = P(S) = 1$

DeMorgan's Law

(i) $\left(\bigcup_{i=1}^{n} E_i\right)^C = \bigcap_{i=1}^{n} E_i^C$ (ii) $\left(\bigcap_{i=1}^{n} E_i\right)^C = \bigcup_{i=1}^{n} E_i^C$

Example: $(E_1 \cup E_2)^C = E_1^C \cap E_2^C$ $(E_1 \cup E_2)^C = E_1^C \cup E_2^C$

Note that $E_1^C \cap E_2^C$ is called neither E_1 nor E_2. $E_1 \cup F_2$ is called either E_1 or E_2 (or both).

5.1.1 Approaches to Probability

There are 2 approaches to quantifying probability of an Event E.

1. **Classical Approach:** $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{|E|}{|S|}$

   i.e. the ratio of number of ways an event can happen to the number of ways sample space can happen, is the probability of the event. Classical approach assumes that all outcomes are equally likely.

2. **Frequency Approach:** Since sometimes all outcomes may not be equally likely, a more general approaches is the frequency approach, where probability is defined as the relative frequency of occurrence of E.

   $P(E) = \lim_{N \to \infty} \dfrac{n(E)}{N}$ where N is the number of times exp is performed & n(E) is the no of times the event E occurs.

5.1.2 Axioms of Probability

Consider an experiment whose sample space is S. For each event E of the sample space S we assume that a number P(E) is defined and satisfies the following three axioms.

Axiom-1: $0 \le P(E) \le 1$

Axiom-2: $P(S) = 1$

Axiom-3: For any sequence of mutually exclusive events $E_1, E_2, \ldots$..(that is, events for which $E_i \cap E_j = \phi$ when $i \ne j$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Example: $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ (E_1, E_2 are mutually exclusive)

Some Simple Propositions

It is to be noted that E and E° are always mutually exclusive and since $E \cup E^{\circ} = S$. We have by Axiom-(2) and (3) that : $P(E \cup E^{\circ}) = P(E) + P(E^{\circ}) = P(S) = 1$

Proposition-1: $P(E^{\circ}) = 1 - P(E)$

Proposition-2: If $E \subseteq F$, then $P(E) \le P(F)$

Proposition-3: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Prop - 3 is more general then axiom 3, since here E & F need not be mutually exclusive

Prop - 3 reduces to axiom - 3 when E, F mutually exclusive ($E \cap F = \phi$)

Prop - 3 may be extended for union of more sets as follows:

$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) + P(E \cap F \cap G)$

5.1.3 Conditional Probability

$$E/F = \frac{P(E \cap F)}{P(F)}$$

E/F is called the **conditional probability** of E given F.

Example - 5.1 A coin is flipped twice. What is the conditional probability that both flips result in heads, given that the first flip does?

Solution:

$$E/F = \frac{P(E \cap F)}{P(F)}$$

i.e. P (both are heads I first is heads)

$$= \frac{P(\text{both heads \& first is head})}{P(\text{first is head})}$$

$$= \frac{P(\text{both heads})}{P(\text{first head})} = \frac{1/4}{1/2} = \frac{1}{2}$$

5.1.4 The Multiplication Rule

$$P(E_1 \cap E_2) = P(E_1) * P(E_2/E_1) \quad \ldots(5.1)$$

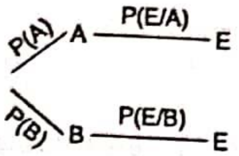
$$= P(E_2) * P(E_1/E_2) \quad \ldots(5.2)$$

Notice that (1) and (2) can be obtained from the following conditional probability formulas after cross multiplication.

$$P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} \quad \text{and} \quad P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

5.1.5 Rule of Total Probability and Bayes Theorem

Consider an event E which occurs via two different events A and B. Further more, Let A and B be mutually exclusive and collectively exhaustive events. This situation may be represented by following tree diagram



Now, the probability of E is given by value of total probability as:

$$P(E) = P(A \cap E) + P(B \cap E)$$
$$= P(A) * P(E/A) + P(B) *(E/B)$$

Sometimes we wish to know that, given that the event E has already occurred, what is the probability that it occurred with A?

i.e. $$P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{P(A \cap E)}{P(A \cap E) + P(B \cap E)}$$

$$= \frac{P(A) * P(E|A)}{P(A) * P(E|A) + P(B) * P(E|B)}$$

Notice that the denominator of Bayes theorem formula is obtained by using rule of total probability.

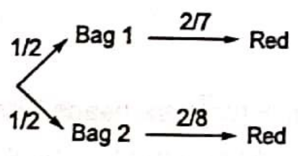
Example-5.2 Suppose we have 2 bags. Bag 1 contains 2 red and 5 green marbles. Bag 2 contains 2 red and 6 green marbles. A person tosses a coin and if it is heads goes to bag 1 and draws a marble. If it is fails he goes to bag 2 and draws a marble. In this situation.

1. What is the probability that the marble drawn this is Red?
2. Given that the marble draw is red, what is probability that it came from bag 1.

Solution:

The tree diagram for above problem,

1. ∴ $$P(Red) = 1/2 \times 2/7 + 1/2 + 2/8$$



2. $$P(bag1|Red) = \frac{P(bag1 \cap Red)}{P(Red)}$$

$$\equiv \frac{\frac{1}{2} \times \frac{2}{7}}{\frac{1}{2} \times \frac{2}{7} \times \frac{1}{2} \times \frac{2}{8}} = \frac{\frac{1}{7}}{\frac{15}{56}} = 8/15$$

5.1.6 Independent Events

Two events are said to be **independent** if equation (A) holds.

$$P(E \cap F) = P(E) * P(F) \quad ...(A)$$

Two events are said to be dependent if they are not independent.

Also it E and F are independent

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E) \times P(F)}{P(F)} = P(E)$$

Similarly, $$P(E|F) = P(F)$$

$P(E|F)$ is called **conditional probability** of E given F and P (E) in called **marginal probability** of E to distinguish it from $P(E|F)$.

P(F) is the marginal probability of F.

Example: A card is selected at random from an ordinary deck of 52 playing cards. If E is the event that the selected card is an ace and F is the event that it is a spade, then

$$P(E \cap F) = P(\text{Ace and Spade}) = \frac{1}{52}$$

$$P(E) = P(\text{Ace}) = \frac{4}{52} \text{ and } P(F) = P(\text{Spade}) = \frac{13}{52}$$

Here,

$$P(E \cap F) = P(F) * P(F)$$

$\therefore$ E and F independent.

Proposition: If E and F are independent, then so are E and F^C, E^C & F, E^C & F^C.

Condition for three Events to be Independent: The events E, F ad G are said to be independent if

P(EFG) = P(E) P(F) P(G)

$$\left.\begin{aligned} &\text{and } P(EF) = P(E)\,P(F) \\ &\text{and } P(EG) = P(E)\,P(G) \\ &\text{and } P(FG) = P(F)\,P(G) \end{aligned}\right. \quad \begin{bmatrix} \text{E, F, G,} \\ \text{pairwise} \\ \text{independent} \end{bmatrix}$$

It should be noted that if E, F and G are independent, then E will be independent of any event formed from F and G. For instance, E is independent of $F \cup G$.

5.2 Mean

Arithmetic Mean

The formula for calculating the arithmetic mean is: $\bar{x} = \frac{\Sigma x}{n}$

$\bar{x}$-arithmetic mean

x-refers to the value of an observation

n-number of observations.

Example - 5.3 **The number of visits made by ten mothers to a clinic were 8 6 5 5 7 4 5 9 7 4. Calculate the average number of visits.**

Solution:

Σx = total of all these numbers of visits, that is the total number of visits made by all mothers.

$$8 + 6 + 5 + 5 + 7 + 4 + 5 + 9 + 7 + 4 = 60$$

Number of mothers n = 10

$$\bar{x} = \frac{\Sigma x}{n} = \frac{60}{10} = 6$$

The Arithmetic Mean of a Frequency Distribution

The formula for the arithmetic mean calculated from a frequency distribution has to be amended to include the frequency. It becomes:

$$\bar{x} = \frac{\Sigma(fx)}{\Sigma f}$$

Arithmetic Mean of a Grouped Data

To show how we can calculate the arithmetic mean of a grouped frequency distribution, there is a example of weights of 75 pigs. The classes and frequencies are given in following table:

Weight (kg)	Midpoint of class x	Number of pigs (frequency)	fx
Under 20	$\simeq 15$	1	15
20 & under 30	25	7	175
30 & under 40	35	8	280
40 & under 40	45	11	495
50 & under 60	55	19	1045
60 & under 70	65	10	650
70 & under 80	75	7	525
80 & under 90	85	5	425
90 & under 100	95	4	380
Over 100	$\simeq 105$	3	215
Total		75	4305

With such a frequency distribution we have a range of values of the variable comprising each group. As our values for x in the formula for the arithmetic mean we use the midpoints of the classes.

In this case $\bar{x} = \dfrac{\Sigma(fx)}{\Sigma f} = \dfrac{4305}{75} = 57.4$ kg

5.3 Median

Arithmetic mean is the central value of the distribution in the sense that positive and negative deviations from the arithmetic mean balance each other. On the other hand, **median** is the central value of the distribution in the sense that the number of values less than the median is equal to the number of values greater than the median.

Median is the central value in a sense different from the arithmetic mean. In case of the arithmetic mean it is the **"numerical magnitude"** of the deviations that balances. But, for the median it is the 'number of values greater than the median which balances against the number of values of less than the median. In general, if we have n values of x, they can be arranged in ascending order as:

$x_1 < x_2 < \ldots\ldots < x_n$

Suppose n is odd, then

$$\text{Median} = \text{the } \frac{(n+1)}{2}\text{-th value}$$

However, if n is even, we have two middle points

$$\text{Median} = \frac{\left(\frac{n}{2}\right)\text{-th value} + \left(\frac{n}{2}+1\right)\text{-th value}}{2}$$

Example-5.4 The heights (in cm) of six students in class are 160, 157, 156, 161, 159, 162. What is median height?

Solution:

Arranging the heights in ascending order

156, 157, 159, 160, 161, 162

Two middle most values are the 3rd and 4th.

$$\text{Median} = \frac{1}{2}(159 + 160) = 159.5$$

Median for Grouped Data

1. Identify the median class which contains the middle observation ($\simeq (n+1/2)^{th}$ observation). This can be done by observing the first class in which the cumulation frequency is equal to or more than $n+1/2$.
2. Calculate Median as follows:

$$\text{Median} = L + \left[\frac{\left(\frac{N+1}{2}\right) - (F+1)}{f_m}\right] \times h$$

Where,

L = Lower limit of median class

N = Total number of data items = ΣF

F = Cumulative frequency of the class immediately preceding the median class

f_m = Frequency of median class

h = width of median class

Median for Grouped Data

Consider the following table giving the marks obtained by students in an exam

Mark Range	f No. of Students	Cumulative Frequency
0 – 20	2	2
20 – 40	3	5
40 – 60	10	15
60 – 80	15	30
80 – 100	20	50

Here $\frac{N+1}{2} = 25.5$

The class 60-80 is the median class since cum-freq is 30 > 25.5

$$\text{Median} = \frac{60 + [25.5 - (15+1)]}{15} \times 20 = 69.66$$

∴ Median marks of the class is approximately 69.7 (at most).

i.e. (at least) half the students got less than 69.7 and (almost) half got more than 69.7 marks.

5.4 Mode and Standard Deviation

Mode: Mode is defined as the value of the variable which occurs most frequently.

Calculation of Mode: Mode is that value of x for which the frequency is maximum. If the values of x are grouped into the classes (such that they are uniformly distributed within any class) and we have a frequency distribution then:

(*i*) Identify the class which has the largest frequency (modal class)

(*ii*) Calculate the mode as:

$$\text{Mode} = L + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h$$

Where,

L = Lower limit of the modal class

f_0 = Largest frequency (frequency of Modal Class)

f_1 = Frequency in the class preceding the modal class

f_2 = Frequency in the class next to the modal class

h = Width of the modal class

Example-5.5 **Data relating to the height of 352 school students are given in the following frequency distribution. Calculate the modal height.**

Heigh (in feet)	Number of students
3.0 – 3.5	12
3.5 – 4.0	37
4.0 – 4.5	79
4.5 – 5.0	152
5.0 – 5.5	65
5.5 – 6.0	7
Total	352

Solution:

Since 152 is the largest frequency, the modal class is (4.5 – 5.0).

Thus $L = 4.5$, $f_0 = 152$, $f_1 = 79$, $f_2 = 65$, $h = 0.5$.

$$\text{Mode} = 4.5 + \frac{152 - 79}{2(152) - 79 - 65} \times 0.5 = 4.73 \text{ (approx.)}$$

While mean, median and mode are measures of central tendency.

5.5 Standard Deviation

Standard Deviation is a measure of dispersion or variation amongst data.

Instead of taking absolute deviation from the arithmetic mean, we may square each deviation and obtain the arithmetic mean of squared deviations. This gives us the 'variance' of the values.

The positive square root of the variance is called the **'Standard Deviation'** of the given values.

Standard Deviation from Raw Data

Suppose $x_1, x_2, \ldots x_n$ are n values of the x, their arithmetic mean is:

$\bar{x} = \frac{1}{N}\Sigma x_i$ and $x_1 - \bar{x}, x_2 - \bar{x} \ldots . x_n - \bar{x}$ are the deviations of the values of x from $\bar{x}$. Then

$\sigma^2 = \frac{1}{n^2}\Sigma(x_i - \bar{x})^2$ is the variance of x. It can be shown that

$$\sigma^2 = \frac{\Sigma(x_i - \bar{x})^2}{n} = \frac{1}{n}\Sigma x_i^2 - \bar{x}^2 = \frac{n\Sigma x_i^2 - (\Sigma x_i)^2}{n^2}$$

It is conventional to represents the variance by the symbol σ^2. Infact, σ is small sigma and Σ is capital sigma.

Square root of the variance is the standard deviation

$$\sigma = +\sqrt{\frac{1}{n}\Sigma(x_i - \bar{x})^2} = \sqrt{\frac{1}{n}\Sigma x_i^2 - \bar{x}^2} = \sqrt{\frac{n\Sigma x_i^2 - (\Sigma x_i)^2}{n^2}}$$

Calculation of Standard Deviation from Grouped Data

Calculation for standard deviation for grouped data can be shown by this example:

The frequency distribution for heights of 150 young ladies in a beauty contest is given below for which we have to calculate standard deviation.

Height (in inches)	Mid values x	Frequency f	$f_i \times x_i$	$f_i \times x_i^2$
62.0 – 63.5	62.75	12	753.00	47250.75
63.5 – 65.0	64.25	20	1285.00	82561.25
65.0 – 66.5	65.75	28	1841.00	121045.75
66.5 – 68.0	67.25	18	1210.50	81406.125
68.0 – 69.5	68.75	19	1306.25	89806.125
69.5 – 71.0	70.25	20	1405.00	89804.6875
71.0 – 72.5	71.75	30	2152.50	98701.25
72.5 – 74.0	73.25	3	219.75	154441.875
				16096.6875
Total		150	10173.00	691308.375

Thus,

$$\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{10173}{150} = 67.82$$

and

$$\frac{\Sigma f_i x_i^2}{\Sigma f_i} = 4608.7225$$

where,

$$N = \Sigma f_i = 150$$

Therefore, the variance of x is

$$\sigma_x^2 = \frac{\Sigma f_i x_i^2}{N} - \bar{x}^2 = \frac{N\Sigma f_i x_i^2 - (\Sigma f_i x_i)^2}{N^2} = 9.1701$$

σ_x = 3.03 (inches) and standard deviation is 3.03 (inches).

5.6 Random Variables

It is frequently the case when an experiment is performed that we are mainly interested in some function of the outcome as opposed to the actual outcome itself.

For instances, in tossing dice we are often interested in the sum of two dice and are not really concerned about the separate value of each die. That is, we may be interested in knowing that the sum is 7 and not be concerned over whether the actual outcome was (1, 6) or (2, 5) or (3, 4) or (4, 3) or (5, 2) or (6, 1).

Also, in coin flipping we may be interested in the total number of heads that occur and not care at all about the actual head tail sequence that results. These quantities of interest, or more formally, these real valued functions defined on the sample space, are known as random variables.

Because the value of a random variable is determined by the outcome of the experiment, we may assign probabilities to the possible values of the random variable.

Types of random variable: Random variable may be discrete or continuous.

Discrete random variable: A variable that can take one value from a discrete set of values.

Example: Let x denotes sum of 2 dice, Now x is a discrete random variable as it can take one value from the set {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}, since the sum of 2 dice can only be one of these values.

Continuous random variable: A variable that can take one value form a continuous range of values.

Example: x denotes the volume of Pepsi in a 500 ml cup. Now x may be a number from 0 to 500, any of which value, x may take.

5.7 Distributions

Based on this we can divide distributions also into **discrete distribution** (based on a disc random variable) or **continuous distribution** (based on a continuous random variable).

Examples of discrete distribution are binomial, poisson and hypergeometric distributions.

Examples of continuous distribution are uniform, normal and exponential distribution.

Properties of Discrete Distribution

- $SP(x) = 1$
- $E(x) = \Sigma x\, P(x)$
- $V(x) = E(x^2) - (E(x))^2$
- $V(x) = \Sigma x^2 P(x) - [\Sigma x\, P(x)]^2$

Properties of Continuous Distribution

- $\int_{-\infty}^{\infty} f(x)dx = 1$
- $F(x) = \int_{-\infty}^{x} f(x)dx$ (cumulative distribution function)
- $E(x) = \int_{-\infty}^{\infty} xf(x)dx$
- $V(x) = \int_{-\infty}^{\infty} x^2 f(x)dx - \left[\int_{-\infty}^{\infty} xf(x)dx\right]^2$
- $P(a < x < b) = P(a \le x \le b) = \int_{a}^{b} f(x)dx$

5.7.1 Types of Discrete Distributions

1. Binomial Distribution

Suppose that a trial or an experiment, whose outcome can be classified as either a success or a failure is performed.

Suppose now that n independent trials, each of which results in a successes with probability p and in a failure with probability $1 - p$, are to be performed.

If X represents the number of successes that occur in the n trials, then X is said to be binomial random variable with parameters (n, p). The Binomial distribution occurs when experiment performed satisfies the three assumptions of bernolli trials:

1. Only 2 outcomes are possible, success and failure
2. Probability of success (p) and failure ($1 - p$) remains same from trial to trial.
3. The trials are statistically independent. i.e The outcome of one trial does not influence subsequent trials.

The probability of x success from n trials is given by $P(X = x) = {}^nC_x\, p^x(1 - p)^{n-x}$.

Where p is the probability of success in any trial and $(1 - p) = q$ is the probability of failure.

Example-5.6 10 dice are thrown. What is the probability of getting exactly 2 sixes?

Solution:

$$P(X = 2) = {}^{10}C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8 = 0.2907$$

Example-5.7 It is known that screws produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

Solution:

If X is the number of defective screws in a packages, then X is a binominal variable with parameters (10, 0.01). Hence, the probability that a package will have to be replaced is:

$$P(X \geq 2) = 1 - [P(X \leq 1)] = 1 - [P\{X = 0\} + P\{X = 1\}]$$

$$= 1 - \left[\binom{10}{0}(0.01)^0(0.99)^{10} + \binom{10}{1}(0.01)^1(0.99)^9\right] \approx 0.004$$

Hence only 0.4% of packages will have to be replaced.

For Binomial distribution:

$$\text{Mean} = E[X] = np$$

$$\text{Variance} = V[X] = np(1 - p)$$

Example-5.8 100 dice are thrown. How many are expected to fall 6. What is the variance in the number of 6's?

Solution:

$$E(x) = np = 100 \times 1/6 = 16.7$$

$$V(x) = np(1 - p) = 100 \times 1/6 \times (1 - 1/6) = 13.9$$

2. Poisson Distribution

A random variable X, taking on one of the values 0, 1, 2 .... is said to be a Poisson random variable with parameter λ if for some $\lambda > 0$,

$$P(x = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

For Poisson distribution:

$$\text{Mean} = E(x) = \lambda$$
$$\text{Variance} = V(x) = \lambda$$

Therefore, expected value and variance of a Poisson random variable are both equal to its parameter λ.

Example-5.9 **A certain airport receives on an average of 4 aircrafts per hour. What is the probability that no aircraft lands in a particular 2 hr period?**

Solution:

α = rate of occurrence of event = 4/hr

λ = average no of occurrences of event in specified observation period $\Delta t = \alpha\Delta t$

In this case α = 4/hr and $\Delta t = 2h$

$$\therefore \quad \lambda = 4 \times 1 = 8$$

Now we wish that no aircraft should land for 2 hrs. i.e. $x = 0$

$$P(x = 0) = \frac{e^{-\lambda}\lambda^0}{0!} = \frac{e^{-8}8^0}{0!} = e^{-8}$$

Frequently, poisson distribution is used to approximate binominal distribution when n is very large and p is very small. Notice that direct computation of $nC_x p^x (1-p)^{n-x}$ may be erroneous or impossible when n is very large and p is very small. Hence, we resort to a poisson approximation with $\lambda = np$.

Example-5.10 **A certain company sells tractors which fail at a rate of 1 out of 1000. If 500 tractors are purchased from this company what is the probability of 2 of them failing within first year.**

Solution:

$$\lambda = np\ 500 \times \frac{1}{1000} = \frac{1}{2}$$

$$P(x = 2) = \frac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^2}{2!} = 0.1011$$

5.7.2 Types of Continuous Distributions

1. Uniform distribution
2. Exponential distribution
3. Normal distribution
4. Standard normal distribution

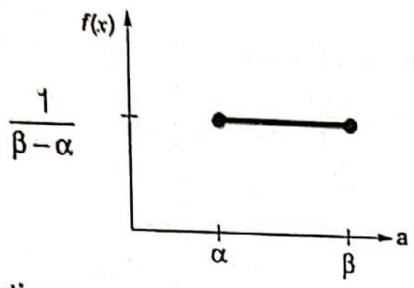
1. Uniform Distribution

In general we say that X is a uniform random variable on the interval (α, β) if its probability density function is given by:

$$f(x) = \begin{cases} \dfrac{1}{\beta-\alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

Since $f(x)$ is a constant, all values of x between α and β are equally likely (uniform).

Graphical representation:



For discrete uniform distribution:

$$\text{Mean} = E[X] = \frac{\beta+\alpha}{2}$$

$$\text{Variance} = V(X) = \frac{(\beta-\alpha)^2}{12}$$

Example-5.11 If X is uniformly distributed over (0, 10), calculate the probability that:

(a) $X < 3$ (b) $X > 6$ (c) $3 < X < 8$

Solution:

$$P\{X < 3\} = \int_0^3 \frac{1}{10} dx = \frac{3}{10}$$

$$P\{X < 6\} = \int_6^{10} \frac{1}{10} dx = \frac{4}{10}$$

$$P\{3 < X < 8\} = \int_3^8 \frac{1}{10} dx = \frac{1}{2}$$

2. Exponential Distribution

A continuous random variable whose probability density function is given for some $\lambda > 0$ by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is said to be exponential random variable with parameter λ. The cumulative distributive function $F(a)$ of an exponential random variable is given by:

$$F(a) = P(X \leq a) = \int_0^a \lambda e^{-\lambda x} dx = \left(-e^{-\lambda x}\right)_0^a = 1 - e^{-\lambda a},\ a \geq 0$$

For Exponential distribution:

$$\text{Mean} = E[X] = \frac{1}{\lambda}$$

$$\text{Variance} = v(x) = \frac{1}{\lambda^2}$$

Example - 5.12 Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = 1/10$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait,

(a) More than 10 minutes (b) Between 10 and 20 minutes

Solution:

Letting X denote the length of the call made by the person in the booth, we have that the desired probabilities are:

(a)
$$\begin{aligned} P\{X > 10\} &= 1 - P(x < 10) \\ &= 1 - F(10) \\ &= 1 - (1 - e^{-\lambda \times 10}) \\ &= e^{-10\lambda} = e^{-1} = 0.368 \end{aligned}$$

(b)
$$\begin{aligned} P\{10 < X < 20\} &= F(20) - F(10) \\ &= (1 - e^{-20\lambda}) - (1 - e^{-10\lambda}) \\ &= e^{-1} - e^{-2} = 0.233 \end{aligned}$$

3. Normal Distribution

We say that X is a normal random variable, or simply that X is normally distributed, with parameters μ and σ^2 if the probability density function is given by:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

The density function is a bell-shaped curve that is symmetric about μ.

For normal distribution:

$$\text{Mean} = E(X) = \mu$$
$$\text{Variance} = V(X) = \sigma^2$$

4. Standard Normal Distribution

Since the for N (μ, σ^2) varies with μ & σ^2 & the integral can only be evaluated numerically, it is more reasonable to reduce this distribution to another distribution called Standard normal distribution N (0, 1) for which, the shape and hence the integral values remain constant.

Since all N (μ, σ^2) problems can be reduced to N(0, 1) problems, we need only to consult a standard table giving calculations of area under N (0, 1) from 0 to any value of z.

The conversion from N (μ, σ^2) to N (0, 1) is effected by the following transformation,

$$Z = \frac{X - \mu}{\sigma}$$

Where Z is called standard normal variate.

For Standard Normal distribution:

$$\text{Mean} = E(X) = 0$$
$$\text{Variance} = V(X) = 1$$

Hence the standard normal distribution is also referred to as the N(0, 1) distribution.

Summary

- Two events E and F are mutually exclusive, if $E \cap F = f$ i.e. $P(E \cap F) = 0$. In other words, if E occurs, F cannot occur and if F occurs, then E cannot occur (i.e. both cannot occur together).
- **Axioms of Probability:**

  Axiom-1: $0 \leq P(E) \leq 1$

  Axiom-2: $P(S) = 1$

  Axiom-3: For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (that is, events for which $E_i \cap E_j = \phi$ when $i \neq j$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

- **Median for Ungrouped Data:**

$$\text{Median} = \text{the } \frac{(n+1)}{2} \text{-th value}$$

  However, if n is even, we have two middle points

$$\text{Median} = \frac{\left(\frac{n}{2}\right)\text{-th value} + \left(\frac{n}{2}+1\right)\text{-th value}}{2}$$

- **Median for Grouped Data:**

$$\text{Median} = L + \left[\frac{\left(\frac{N+1}{2}\right) - (F+1)}{f_m}\right] \times h$$

  Where,

  L = Lower limit of median class

  N = Total number of data items = ΣF

  F = Cumulative frequency of the class immediately preceding the median class

  f_m = Frequency of median class

  h = width of median class

- **Standard Deviation** is a measure of dispersion or variation amongst data. The positive square root of the variance is called the **'Standard Deviation'** of the given values.
- The probability of x success from n trials is given by $P(X = x) = nC_x\, p^x (1 - p)^{n-x}$. Where p is the probability of success in any trial and $(1 - p) = q$ is the probability of failure.
- **Uniform Distribution:**

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Mean} = E[X] = \frac{\beta + \alpha}{2}$$

$$\text{Variance} = V(X) = \frac{(\beta - \alpha)^2}{12}$$

- **Exponential Distribution:**

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{If } x \geq 0 \\ 0 & \text{If } x < 0 \end{cases}$$

$$\text{Mean} = E[X] = \frac{1}{\lambda}$$

$$\text{Variance} = v(x) = 1/\lambda^2$$

- **Normal Distribution:**

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

$$\text{Mean} = E(X) = \mu$$
$$\text{Variance} = V(X) = \sigma^2$$

- **Standard Normal distribution:**

$$\text{Mean} = E(X) = 0$$
$$\text{Variance} = V(X) = 1$$

Hence the standard normal distribution is also referred to as the N(0, 1) distribution.

Student's Assignments

Q.1 Let P (E) denotes the probability of the event E. Given P(A) = 1, P(B) = $\frac{1}{2}$, then if A and B are independent, then the values of $P\left(\frac{A}{B}\right)$ and $P\left(\frac{B}{A}\right)$ respectively are

(a) $\frac{1}{4}, \frac{1}{2}$
(b) $\frac{1}{2}, \frac{1}{4}$
(c) $\frac{1}{2}, 1$
(d) $1, \frac{1}{2}$

Q.2 If P(A) = $\frac{1}{2}$, P(A ∩ B) = $\frac{1}{4}$ then $P\left(\frac{B}{A}\right)$ =

(a) 1
(b) $\frac{1}{2}$
(c) $\frac{3}{4}$
(d) 0

Q.3 A bag contains 5 black, 2 red, and 3 white marbles. Three marbles are drawn simultaneously. The probability that the drawn marbles are of the different color is

(a) $\frac{1}{6}$
(b) $\frac{1}{4}$
(c) $\frac{5}{6}$
(d) None of these

Q.4 A and B are equally likely and independent events. p(A ∪ B) = 0.1. Then what is the value of p(A)?

(a) 0.032
(b) 0.046
(c) 0.513
(d) 0.05

Q.5 The probability of occurrence of an event. A is 0.7, the probability of non-occurrence of an event B is 0.45 and the probability of at least one of A and B not occurring is 0.6. The probability that at least one of A and B occurs is

(a) 0.4
(b) 0.6
(c) 1
(d) 0.85

Q.6 The probability density function of a variable

X	0	1	2	3	4	5	6
P(X)	K	3K	5K	7K	9K	11K	13K

Find the value of K

(a) $\frac{1}{49}$ (b) $\frac{1}{50}$

(c) $\frac{1}{51}$ (d) $\frac{1}{52}$

Q.7 $p(3 < x \le 6)$ = ? (From previous question)

(a) $\frac{33}{49}$ (b) $\frac{17}{25}$

(c) $\frac{34}{51}$ (d) $\frac{1}{4}$

Q.8 Given $P(B) = \frac{3}{4}$, $P(A \cap B \cap \bar{C}) = \frac{1}{3}$ and

$P(\bar{A} \cap B \cap \bar{C}) = \frac{1}{3}$, $P(B \cap C)$ = ?

(a) $\frac{1}{12}$ (b) $\frac{1}{9}$

(c) $\frac{1}{15}$ (d) $\frac{1}{18}$

Q.9 In a lottery, 2 tickets are drawn at a time out of 6 tickets numbered from 1 to 6. The expected value of the sum of the numbers on the tickets drawn is

(a) 7 (b) 6

(c) 5 (d) 4

Q.10 Two dice are thrown simultaneously. The probability that atleast one of them will have 6 facing up is

(a) $\frac{1}{36}$ (b) $\frac{1}{3}$

(c) $\frac{25}{36}$ (d) $\frac{11}{36}$

Q.11 Let X be a contiuous random variable with following distribution

$f(x) = kx \qquad 0 \le x \le 2$

$= 0 \qquad$ elsewhere

The value of k and $P(1 \le x \le 2)$ are respectively,

(a) $\frac{1}{2}, \frac{1}{4}$ (b) $\frac{1}{2}, \frac{3}{4}$

(c) $\frac{1}{4}, \frac{2}{3}$ (d) $\frac{1}{2}, \frac{1}{2}$

Q.12 A gambler has 4 coins in her pocket. Two are double-headed, one is double-tailed, and one is normal. The coins can not be distinguished unless one looks at them. The gambler takes a coin at random, opens her eyes and sees that the upper face of the coin is a head. What is the probability that the lower face is a head?

(a) $\frac{5}{8}$ (b) $\frac{4}{5}$

(c) $\frac{2}{3}$ (d) $\frac{1}{2}$

Q.13 Let X be uniformly distributed on {0, 1, ..., 32} what is $Pr[3x + 12 \equiv 0 (\text{mod } 33)]$?

(a) $\frac{1}{34}$ (b) $\frac{2}{34}$

(c) $\frac{1}{22}$ (d) $\frac{1}{11}$

Q.14 Suppose you are given a bag containing n unbaised coins you are told that $n-1$ of these are normal coins, with heads on one side and tails on the other; however the remaining coin has heads on both its sides.

Suppose you reach in to the bag, pickout a coin uniformly at random, flip it and get a head. What is the (conditional) probability that this coin you choose is the fake (i.e., double-headed) coin?

(a) $\frac{1}{n+1}$ (b) $\frac{2}{n+1}$

(c) $\frac{1}{n-1}$ (d) $\frac{2}{n-1}$

Q.15 Two random variables X and Y are independent if the pair of events X_i and Y_j are independent no matter how you choose the values i and j. Which of the following most accurately expresses the proposition that X and Y are not independent?

(a) for all i, j $Pr[X_i \text{ and } Y_j] \neq Pr[X_i] Pr[Y_j]$
(b) for all i, some j, $Pr[X_i \text{ AND } Y_j] \neq Pr[X_i] Pr[Y_j]$
(c) for some j, all i, $Pr[X_i \text{ AND } Y_j] \neq Pr[X_i] Pr[Y_j]$
(d) for some i, j, $Pr[X_i \text{ AND } Y_j] \neq Pr[X_i] Pr[Y_j]$

Q.16 Suppose you are given a bag containing n unbiased coins. You are total that $n-1$ of these are normal coins, with heads on one side and tails on the other; however, the remaining coin has heads on both its sides.
Suppose you reach into the bag, pick out a coin uniformly at random.
Suppose you flip the coin k times after picking it (instead of just once) and see k heads. What is now the conditional probability that you picked the fake coin?

(a) $\dfrac{2^k}{(n+1+2^k)}$ (b) $\dfrac{2^k}{(n+2^k)}$

(c) $\dfrac{2^k}{(n-1)+2^k}$ (d) None of these

Q.17 The expectation and variance of a random variable $z = X_1 + X_2$ where X_1 and X_2 are independent random variables with expectation μ and variance σ^2.
(a) μ, σ (b) $\mu, 2\sigma$
(c) $2\mu, \sigma^2$ (d) $2\mu, 2\sigma$

Q.18 For each square of an 8×8 checker board, flip a fair coin, and color that square black or red according to whether you get heads or tails. Assume that all coin flips are independent. A same-color row in a row on the board were all squares in the row have the same color (i.e., all red, or all black). Let the random variable X denote the number of same colour rows. What is the Pr(X = 0)

(a) $\dfrac{1}{(2)^7}$ (b) $1-\left(\dfrac{1}{(2^7)}\right)$

(c) $\left(\dfrac{1}{(2)^7}\right)^8$ (d) $\left(1-\left(\dfrac{1}{(2^7)}\right)\right)^8$

Q.19 Suppose you are given a bag containing n unbiased coins and you are told that $n-1$ of these are normal coins, with heads on one side and tails and the other; however, the remaining coin has heads on both its sides.
Suppose you reach in to the bag, pick out a coin uniformly at random.
Suppose you wanted to decide whether the chosen coin was fake by flipping it k times; The decision procedure returns FAKE if all k-flips come up heads, otherwise it returns NORMAL. What is the (unconditional) probability that this procedure makes an error?
(a) $(1/2)^k (n)/(n+1)$
(b) $(1/2)^k (n-1)/n$
(c) $(1/2)^{k+1} (n-1)/n$
(d) None of the above

Q.20 In a multi-user operating system, 20 requests are made to use a particular resource per hour, on an average. The probability that no requests are made in 45 minutes is
(a) e^{-15} (b) e^{-5}
(c) $1-e^{-5}$ (d) $1-e^{-10}$

Common Data Questions (21 and 22):
A random variable x has PDF

$$P(x) = \frac{1}{2}a \text{ for } -a < x < a \text{ and } P(x) = 0, \text{ else}$$

where

Q.21 Find the central moments
(a) All even central moments are zero and odd central moment are $\frac{1}{3}a^2, \frac{1}{5}a^4, \frac{1}{7}a^6$
(b) All odd central moments are zero and even central moments are $\frac{1}{3}a^2, \frac{1}{5}a^4, \frac{1}{7}a^6$
(c) All the odd and even central moments are equal to zero
(d) All the odd and even central moments are not equal to zero.

Q.22 For the above distribution value of is

$P\left(|x| \geq \frac{\sqrt{3}}{2}a\right)$

(a) $\geq \frac{9}{4}$ (b) $\leq \frac{9}{5}$

(c) $\leq \frac{4}{9}$ (d) $\geq \frac{5}{9}$

Common Data Questions (23 and 24):

Analysis of the daily registration at an Examination on a certain day indicated that the source of registration from North India are 15%, South India are 35% and from western part of India are 50%. Further suppose that the probabilities that a registration being a free registration from these parts are 0.01, 0.05, and 0.02, respectively.

Q.23 Find the probability that a registration chosen at random is a free registration

(a) 0.603 (b) 0.029

(c) 0.009 (d) None of these

Q.24 Find the probability that a randomly chosen registration comes from south India, given that it is a free registration.

(a) 60% (b) 3%

(c) 17 % (d) None of these

Q.25 A manufacturer produces IC chips, 1% of which are defective. Find the probability that in a box containing 100 chips, no defective are found. Use Poisson distribution approximation to binomial distribution?

(a) 0.366 (b) 0.368

(c) 0.1 (d) None of these

Answer Key:

1. (d)	**2.** (b)	**3.** (b)	**4.** (c)	**5.** (d)
6. (a)	**7.** (a)	**8.** (a)	**9.** (a)	**10.** (d)
11. (b)	**12.** (b)	**13.** (d)	**14.** (b)	**15.** (d)
16. (c)	**17.** (d)	**18.** (d)	**19.** (b)	**20.** (a)
21. (b)	**22.** (d)	**23.** (b)	**24.** (a)	**25.** (b)

Student's Assignments Explanations

1. (d)

Since A and B independent events

$p(A|B) = p(A) = 1$ and $p(B|A) = p(B) = \frac{1}{2}$.

2. (b)

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

5. (d)

Given,

$$p(A) = 0.7$$
$$p(\bar{B}) = 0.45$$
$$p(\bar{A} \cup \bar{B}) = 0.6$$
$$p(A \cup B) = ?$$
$$p(B) = 1 - p(\bar{B})$$
$$= 1 - 4.5 = 0.55$$
$$p(A \cap B) = 1 - p(\overline{A \cap B})$$
$$= 1 - p(\bar{A} \cup \bar{B})$$
$$= 1 - 0.6 = 0.4$$

Now, $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

$= 0.7 + 0.55 - 0.4 = 0.85$

$\therefore$ $p(A \cup B) = 0.85$

6. (a)

If X is a randon variable, then

$$\Sigma p(X) = 1$$

$\Rightarrow k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

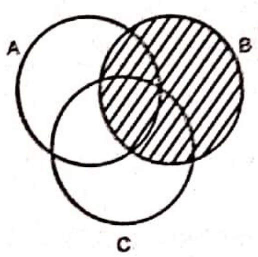
$\Rightarrow$ $k = \frac{1}{49}$

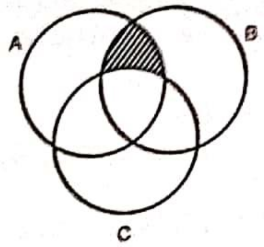
7. (a)

$P(3 < x \leq 6) = 9k + 11k + 13k = 33k$

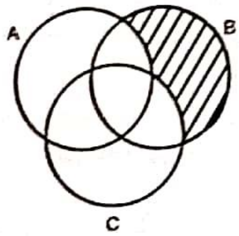
$\therefore$ $P(3 < x \leq 6) = \frac{33}{49}$

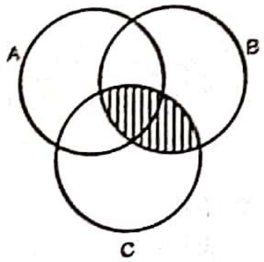
8. (a)





$P(B)\ P(A \cap B \cap \bar{C}) = \frac{1}{3}$





$P(\bar{A} \cap B \cap \bar{C})$ $\quad$ $P(B \cap C)$

From the above Venn diagram

$$P(B \cap C)\ P(B) - P(A \cap B \cap \bar{C}) - P(\bar{A} \cap B \cap \bar{C})$$

$$= \frac{3}{4} - \frac{1}{3} - \frac{1}{3} = \frac{1}{12}$$

9. (a)

Let X be the vandon variable that represents the sum of 2 tickets.

The probability distribution table of X is

X	3	4	5	6	7	8	9	10	11
$p(X)$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{1}{15}$

$$E(X) = \Sigma X p(X)$$

$$= 3 \times \frac{1}{15} + 4 \times \frac{1}{15} + 5 \times \frac{2}{15} + \ldots$$

$$= \frac{105}{15} = 7$$

10. (d)

The possible combinations for at least one dice being 6 is given by 11 ordered pairs below:

(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)

$\therefore$ Probability that at least one dice is 6 $= \frac{11}{36}$

Alternatively we can solve this problem by another method:

p(6 on I dice or 6 on II dice)

$= 1 - p$(not 6 on I dice and not 6 on II dice)

$$= 1 - \frac{5}{6} \times \frac{5}{6} = 1 - \frac{25}{36} = \frac{11}{36}$$

11. (b)

For $f(x)$ to be a probability density function,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_0^2 kx\,dx = 1$$

$$\Rightarrow k\left[\frac{x^2}{2}\right]_0^2 = 1 \Rightarrow 2k = 1$$

$$\Rightarrow k = \frac{1}{2}$$

$$p(1 \le x \le 2) = \int_1^2 f(x)dx$$

$$= \int_1^2 \frac{1}{2}x\,dx = \frac{3}{4}$$

12. (b)

There are 5 faces that are heads out of a total of 8, so the probability is $\frac{5}{8}$. Let A be the event that the upper face is a head, and B be the event that the lower face is heads.

$$Pr[A] = Pr[B] = \frac{5}{8}$$

$$Pr[A \cap B] = \frac{2}{4} = \frac{1}{2}$$

So, $$Pr[B|A] = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{2}}{\frac{5}{8}} = \frac{4}{5}$$

13. (d)

$$3X + 12 \equiv 0 \pmod{33}$$

$$\Rightarrow 3X \equiv -12 \pmod{33}$$

$$3X \equiv 21 \pmod{33}$$

$\Rightarrow \quad X \equiv 7 \bmod 11$

$\Rightarrow \quad X = 7 + 11k$

$\Rightarrow$

Since, $\quad X = 0, 1, 2, \ldots, 32$

Only solutions are 7, 18 and 29

Now,

$pr[3X + 12 \equiv 0 \pmod{33}]$

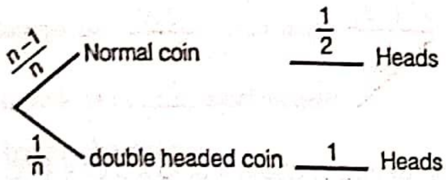
$= pr[(x = 7) \text{ or } (x = 18) \text{ or } (x = 29)]$

$= pr[x = 7] + p[x = 18] + p[x = 29]$

$$= \frac{1}{33} + \frac{1}{33} + \frac{1}{33} = \frac{3}{33} = \frac{1}{11}$$

14. (b)

The tree diagram with probabilities for the given problem is shown below:

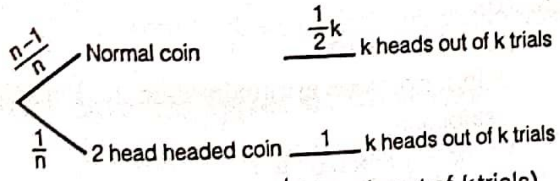


$$p(\text{double head coin} \mid \text{Heads}) = \frac{\frac{1}{n} \times 1}{\frac{n-1}{n} \times \frac{1}{2} + \frac{1}{n} \times 1}$$

$$= \frac{2}{n+1}$$

16. (c)

The tree diagram with probabilities for this problem is shown below:



$\therefore$ $p(2 \text{ headed coin} \mid k \text{ heads out of } k \text{ trials})$

$$= \frac{\frac{1}{n} \times 1}{\frac{n-1}{n} \cdot \frac{1}{2^k} + \frac{1}{n} \times 1} = \frac{2^k}{(n-1) + 2^k}$$

17. (d)

$E(X_1 + X_2) = E(X_1) + E(X_2)$ always

$= \mu + \mu = 2\mu$

$V(aX_1 + bX_2) = a^2 V(X_1) + b^2 V(X_2)$

($\because$ X_1 and X_2 are independent)

Putting, $\quad a = b = 1$

$V(X_1 + X_2) = V(X_1) + V(X_2)$

$= \sigma^2 + \sigma^2 = 2\sigma^2$

18. (d)

p(a row being same color)

$= p$(a row being all black) $+ p$(a row being all red)

$= p$(all heads) $+ p$(all tails)

$$= 8C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^0 + 8C_8 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^8$$

$$= \left(\frac{1}{2}\right)^8 + \left(\frac{1}{2}\right)^8$$

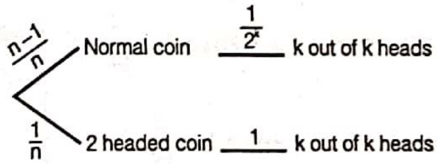
$$= \frac{3}{4}$$

pr(X = 0) = pr(0 out of 8 rows being of same colour)

$$= 8C_0 \left(\frac{1}{2^7}\right)^0 \left(1 - \frac{1}{2^7}\right)^8 = \left(1 - \frac{1}{2^7}\right)^8$$

19. (b)

The tree diagram with probabilities is



p(procedure is in error) $= p$(normal coin and k out of k heads) $+ p$(fake coin and not k out of k heads)

$$= \frac{n-1}{n} \times \left(\frac{1}{2^k}\right) + \frac{1}{n} \times 0$$

$$= \frac{n-1}{n}\left(\frac{1}{2^k}\right) = \left(\frac{1}{2}\right)^k \frac{(n-1)}{n}$$

20. (a)

The arrival pattern follows poisson distribution.

$$p(X = x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

Here $\quad \lambda = \alpha \Delta t$

where, α = number of events/unit time = 20/hr

$$\Delta t = 45 \text{ min} = \frac{3}{4} \text{ hr}$$

$\therefore \quad \lambda = \alpha\Delta t = 20 \times \frac{3}{4} = 15$

$$p(X=0) = \frac{\lambda^0}{0!}e^{-\lambda} = \frac{15^0}{0!}e^{-15} = e^{-15}$$

21. (b)

$$\mu = \int_{-\infty}^{\infty} x p(x)dx = \int_{-a}^{a} \frac{x}{2a}dx$$

$$\mu = \int_{-\infty}^{\infty} (x-\mu)^r p(x)dx = \int_{-a}^{a} \frac{x^r}{2a}dx$$

$$= \frac{1}{2a}\left\{\frac{x^{r+1}}{r+1}\right\}_{-a}^{a} = \begin{cases} 0 \text{ if } r \text{ is odd} \\ \frac{a^r}{r+1} \text{ if } r \text{ is even} \end{cases}$$

$\therefore$ odd central moments are zero, even central moments are

$$\mu_2 = \frac{1}{3}a^2, \mu_2 = \frac{1}{5}a^4, \mu_2 = \frac{1}{7}a^5 \ldots$$

22. (d)

Here, $\mu = 0$ and $\sigma = \frac{a}{\sqrt{3}}$

Using Bienayme-Chebyshev rule,

$$p(\mu - k\sigma \le x \le \mu + k\sigma) \ge 1 - \frac{1}{k^2}$$

$$p(-k\sigma \le x \le +k\sigma) \ge 1 - \frac{1}{k^2}$$

$$k\sigma = \frac{\sqrt{3}}{2}a$$

$$\Rightarrow \quad k\cdot\frac{a}{\sqrt{3}} = \frac{\sqrt{3}}{2}a$$

$$\Rightarrow \quad k = \frac{3}{2}$$

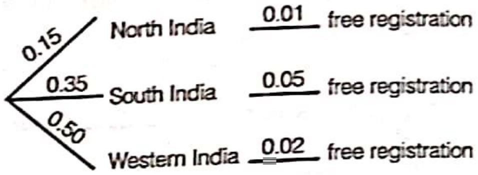
$$\therefore p\left(\frac{-\sqrt{3}}{2}a \le x \le \frac{\sqrt{3}}{2}a\right) \ge 1 - \frac{1}{\left(\frac{3}{2}\right)^2}$$

$$p\left(|x| \ge \frac{\sqrt{3}}{2}a\right) \ge 1 - \left(\frac{2}{3}\right)^3$$

$$p\left(|x| \ge \frac{\sqrt{3}}{2}a\right) \ge \frac{5}{9}$$

23. (b)

The tree diagram is shown below:



$p(\text{free registration}) = 0.15 \times 0.01 + 0.35 \times 0.05 + 0.50 \times 0.02 = 0.029$

24. (a)

$p(\text{South India I free registration})$

$$= \frac{p(\text{South India and free registration})}{p(\text{free registration})}$$

$$= \frac{0.35 \times 0.05}{0.029} = 0.6034 \simeq 60\%$$

25. (b)

$p = 0.01, n = 100$

Using poisson approximation to binomial distribution,

$\lambda = np = 100 \times 0.01 = 1$

$$p(X=0) = \frac{\lambda^0}{0!}e^{-\lambda} = e^{-1} = 0.368$$

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