# Matrices

- A **matrix** is an ordered rectangular array of numbers or functions. The numbers or functions are called the **elements** or the **entries** of the matrix.

  For example, $\begin{bmatrix} -10 & \sin x & \log x \\ e^x & 2 & -9 \end{bmatrix}$ is a matrix having 6 elements. In this matrix, number of rows = 2 and number of columns = 3

- A matrix having $m$ rows and $n$ columns is called a matrix of order $m \times n$. In such a matrix, there are $mn$ numbers of elements.

  For example, the order of the matrix $\begin{bmatrix} \sin x & \cos x \\ -1 & 1+\sin x \\ 0 & \cos x \end{bmatrix}$ is $3 \times 2$ as the numbers of rows and columns of this matrix are 3 and 2 respectively.

- A matrix $A$ is said to be a **row matrix**, if it has only one row. In general, $A = [a_{ij}]_{1\times n}$ is a row matrix of order $1 \times n$.

  For example, $\begin{bmatrix} -9 & 6 & 5 & e & \sin x \end{bmatrix}$ is a row matrix of order $1 \times 5$.

- A matrix $B$ is said to be a **column matrix**, if it has only one column. In general, $B = [b_{ij}]_{m\times 1}$ is a column matrix of order $m \times 1$.

  For example, $B = \begin{bmatrix} -6 \\ 19 \\ 13 \end{bmatrix}$ is a column matrix of order $3 \times 1$.

- A matrix $C$ is said to be a **square matrix**, if the number of rows and columns of the matrix are equal. In general, $C = [b_{ij}]_{m\times n}$ is a square matrix, if $m = n$

  For example, $C = \begin{bmatrix} -1 & 9 \\ 5 & 1 \end{bmatrix}$ is a square matrix.

- A square matrix $A$ is said to be a **diagonal matrix**, if all its non-diagonal elements are zero. In general, $A = [a_{ij}]_{m\times n}$ is a diagonal matrix, if $a_{ij} = 0$ for $i \neq j$

- A matrix is said to be a **rectangular matrix,** if the number of rows is not equal to the number of columns.

  For example: $\begin{bmatrix} 8 & 3 & 9 \\ 1 & 6 & 7 \end{bmatrix}$ is a rectangular matrix.

- If $A$ is a matrix of order $m \times n$, then the matrix obtained by interchanging the rows and columns of $A$ is called the transpose of matrix $A$. The transpose of $A$ is denoted by $A'$ or $A^T$. In other words, if $A = [a_{ij}]_{m\times n}$, then $A' = [a_{ij}]_{n\times m}$

  For example, the transpose of the matrix $\begin{bmatrix} 2 & 8 & -3 \\ 1 & 11 & 9 \end{bmatrix}$ is $\begin{bmatrix} 2 & 1 \\ 8 & 11 \\ -3 & 9 \end{bmatrix}$.

- For any matrices $A$ and $B$ of suitable orders, the properties of transpose of matrices are given as:
  - $(A')' = A$
  - $(kA)' = kA'$, where $k$ is a constant
  - $(A+B)' = A' + B'$
  - $(AB)' = B'A'$

- Two matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are said to be equal (denoted as $A = B$) if they are of the same order and each element of $A$ is equal to the corresponding element of $B$ i.e., $a_{ij} = b_{ij}$ for all $i$ and $j$.

For example: $\begin{bmatrix} 15 & 11 \\ 7 & 2 \end{bmatrix}$ and $\begin{bmatrix} 15 & 11 \\ 7 & 2 \end{bmatrix}$ are equal but $\begin{bmatrix} 15 & 11 \\ 7 & 2 \end{bmatrix}$ and $\begin{bmatrix} 7 & 2 \\ 15 & 11 \end{bmatrix}$ are not equal.

**Example:** If $\begin{bmatrix} 7 & x-y \\ 13 & 3y+z \end{bmatrix} = \begin{bmatrix} 2x+y & 5 \\ 2x+y+z & 3 \end{bmatrix}$, then find the values of $x$, $y$ and $z$.

**Solution:**

Since the corresponding elements of equal matrices are equal,

$2x + y = 7$… (1)

$x - y = 5$… (2)

$2x + y + z = 13$… (3)

$3y + z = 3$… (4)

On solving equations (1) and (2), we obtain $x = 4$ and $y = -1$.

On substituting the value of $y$ in equation (4), we obtain $z = 6$.

Thus, the values of $x$, $y$ and $z$ are 4, −1 and 6 respectively.

- Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ can be added, if they are of the same order.

  The sum of two matrices $A$ and $B$ of same order $m \times n$ is defined as matrix $C = [c_{ij}]_{m\times n}$, where $c_{ij} = a_{ij} + b_{ij}$ for all possible values of $i$ and $j$.

- The difference of two matrices $A$ and $B$ is defined, if and only if they are of same order. The difference of the matrices $A$ and $B$ is defined as $A - B = A + (-1)B$
- If $A$, $B$, and $C$ are three matrices of same order, then they follow the following properties related to addition:
  - Commutative law: $A + B = B + A$
  - Associative law: $A + (B + C) = (A + B) + C$
  - Existence of additive identity: For every matrix A, there exists a matrix $O$ such that $A + O = O + A = A$. In this case, $O$ is called the additive identity for matrix addition.
  - Existence of additive inverse: For every matrix $A$, there exists a matrix $(-A)$ such that $A + (-A) = (-A) + A = O$. In this case, $(-A)$ is called the additive inverse or the negative of $A$.

**Example:** Find the value of $x$ and $y$, if:

$$\begin{bmatrix} 2x+3y & 9 \\ -2 & 4x-7y \end{bmatrix} + 2\begin{bmatrix} 3x+\frac{5}{2}y & -11 \\ -13 & 3x-\frac{3}{2}y \end{bmatrix} = \begin{bmatrix} 56 & -13 \\ -28 & 30 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 2x+3y & 9 \\ -2 & 4x-7y \end{bmatrix} + 2\begin{bmatrix} 3x+\frac{5}{2}y & -11 \\ -13 & 3x-\frac{3}{2}y \end{bmatrix} = \begin{bmatrix} 56 & -13 \\ -28 & 30 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+3y & 9 \\ -2 & 4x-7y \end{bmatrix} + \begin{bmatrix} 6x+5y & -22 \\ -26 & 6x-3y \end{bmatrix} = \begin{bmatrix} 56 & -13 \\ -28 & 30 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8(x+y) & -13 \\ -28 & 10(x-y) \end{bmatrix} = \begin{bmatrix} 56 & -13 \\ -28 & 30 \end{bmatrix}$$

Therefore, we have

$8(x+y) = 56$ and $10(x-y) = 30$

$\Rightarrow x + y = 7$ … (1)

And

$x - y = 3$ … (2)

Solving equation (1) and (2), we obtain $x = 5$ and $y = 2$

- The multiplication of a matrix $A$ of order $m \times n$ by a scalar $k$ is defined as

  $kA = kA = k[a_{ij}]_{m\times n} = [k(a_{ij})]_{m\times n}$

- If $A$ and $B$ are matrices of same order and $k$ and $l$ are scalars, then
  - $k(A + B) = kA + kB$
  - $(k + l) A = kA + lA$
- The negative of a matrix $B$ is denoted by –B and is defined as $(-1)B$.

- The product of two matrices $A$ and $B$ is defined, if the number of columns of $A$ is equal to the number of rows of $B$.
- If $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$ are two matrices, then their product is defined as $AB = C = [c_{ik}]_{m\times p}$, where $c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$

For example, if $A=\begin{bmatrix}2 & -3 & 7\\ 0 & 1 & -9\end{bmatrix}$ and $B=\begin{bmatrix}-5 & 9\\ 7 & 2\\ 0 & 1\end{bmatrix}$, then

$$AB=\begin{bmatrix}2 & -3 & 7\\ 0 & 1 & -9\end{bmatrix}\times\begin{bmatrix}-5 & 9\\ 7 & 2\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}2\times(-5)+(-3)\times7+7\times0 & 2\times9+(-3)\times2+7\times1\\ 0\times(-5)+1\times7+(-9)\times0 & 0\times9+1\times2+(-9)\times1\end{bmatrix}$$

$$=\begin{bmatrix}-31 & 19\\ 7 & -7\end{bmatrix}$$

- If $A$, $B$, and $C$ are any three matrices, then they follow the following properties related to multiplication:
  - Associative law: $(AB)\,C = A\,(BC)$
  - Distribution law: $A\,(B + C) = AB + AC$ and $(A + B)\,C = AC + BC$, if both sides of equality are defined.
  - Existence of multiplicative identity: For every square matrix $A$, there exists an identity matrix $I$ of same order such that $IA = AI = A$. In this case, $I$ is called the multiplicative identity.
- Multiplication of two matrices is not commutative. There are many cases where the product $AB$ of two matrices $A$ and $B$ is defined, but the product $BA$ need not be defined.

  For example, if $A=\begin{bmatrix}-1 & 5\end{bmatrix}_{1\times2}$ and $B=\begin{bmatrix}0 & 1 & -4\\ 3 & 2 & -1\end{bmatrix}_{2\times3}$, then $AB$ is defined where as $BA$ is not defined.