Maharashtra State Board
Mathematics and Statistics
Sample Question Paper - 2
Academic Year: 2024-2025

General Instructions: The question paper is divided into four sections.

1. **Section A**: Q.1 contains Eight multiple-choice types of questions, each carrying Two marks. Q.2 contains Four very short answer type questions, each carrying one mark.
2. **Section B**: Q.3 to Q.14 contains Twelve short answer type questions, each carrying Two marks. (Attempt any Eight)
3. **Section C**: Q.15 to Q.26 contain Twelve short answer type questions, each carrying Three marks. (Attempt any Eight)
4. **Section D**: Q. 27 to Q.34 contain Eight long answer type questions, each carrying Four marks. (Attempt any Five)
5. Use of Log table is allowed. Use of calculator is not allowed.
6. Figures to the right indicate full marks.
7. Use of graph paper is not necessary. Only rough sketch of graph is expected.
8. For each multiple-choice type question, it is mandatory to write the correct answer along with its alphabet. e.g., (a) .... /(b) .... /(c) .... /(d) .... ,etc. No mark(s) shall be given if ONLY the correct answer or the alphabet of the correct answer is written. Only the first attempt will be considered for evaluation.
9. Start answer to each section on a new page.

SECTION - A

Q1. Select and write the correct answer for the following multiple-choice type of questions:

1.1. Choose correct alternatives :

The direction ratios of the line which is perpendicular to the two lines

$\frac{x-7}{2}=\frac{y+17}{-3}=\frac{z-6}{1}$ and $\frac{x+5}{1}=\frac{y+3}{2}=\frac{z-4}{-2}$ are ______.

1. 4, 5, 7
2. 4, −5, 7
3. 4, −5, −7
4. −4, 5, 8

Solution

The direction ratios of the line which is perpendicular to the two lines

$\frac{x-7}{2}=\frac{y+17}{-3}=\frac{z-6}{1}$ and $\frac{x+5}{1}=\frac{y+3}{2}=\frac{z-4}{-2}$ are **4, 5, 7**.

1.2. Choose correct alternatives :

The vector equation of line 2x – 1 = 3y + 2 = z – 2 is ______.

1. $\bar{r}=\left(\frac{1}{2}\hat{i}-\frac{2}{3}\hat{j}+2\widehat{k}\right)+\lambda\left(3\hat{i}+2\hat{j}+6\widehat{k}\right)$
2. $\bar{r}=\hat{i}-\hat{j}+\left(2\hat{i}+\hat{j}+\widehat{k}\right)$
3. $\bar{r}=\left(\frac{1}{2}\hat{i}-\hat{j}\right)+\lambda\left(\hat{i}-2\hat{j}+6\widehat{k}\right)$
4. $\bar{r}=\left(\hat{i}+\hat{j}\right)+\lambda\left(\hat{i}-2\hat{j}+6\widehat{k}\right)$

Solution

The vector equation of line 2x – 1 = 3y + 2

= z – 2 is $\underline{\bar{\mathbf{r}}=\left(\frac{\mathbf{1}}{\mathbf{2}}\hat{\mathbf{i}}-\frac{\mathbf{2}}{\mathbf{3}}\hat{\mathbf{j}}+\mathbf{2}\widehat{\mathbf{k}}\right)+\lambda\left(\mathbf{3}\hat{\mathbf{i}}+\mathbf{2}\hat{\mathbf{j}}+\mathbf{6}\widehat{\mathbf{k}}\right)}$

1.3.

If $\int_2^e\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right].dx=a+\frac{b}{\log 2}$, then _____.

1. a = e, b = −2
2. a = e, b = 2
3. a = −e, b = 2
4. a = −e, b = −2

Solution

If $\int_2^e \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right].dx = a + \frac{b}{\log 2}$, then **a = e, b = –2**.

Explanation:

Given that, $\int_2^e \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right].dx = a + \frac{b}{\log 2}$

Put logx = z

⇒ x = e^z

⇒ dx = e^z dz

$\therefore \int_2^e \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right].dx = \int_{\log 2}^1 \left[\frac{1}{z} - \frac{1}{z^2}\right] e^z.dz$

$= \int_{\log 2}^1 e^z \left[\frac{1}{z} + d\left(\frac{1}{z}\right)\right].dz$

$= \left[e^z.\frac{1}{z}\right]_{\log 2}^1$

$= e - \frac{2}{\log 2}$

∴ a = e and b = –2

1.4. Select the correct option from the given alternatives:

The general solution of sec x = $\sqrt{2}$ is ______.

1. $2n\pi \pm \frac{\pi}{4}, n \in Z$
2. $2n\pi \pm \frac{\pi}{2}, n \in Z$
3. $n\pi \pm \frac{\pi}{2}, n \in Z$
4. $2n\pi \pm \frac{\pi}{3}, n \in Z$

Solution

The general solution of sec x = $\sqrt{2}$ is $\underline{\mathbf{2n\pi \pm \frac{\pi}{4}, n \in Z}}$.

1.5. Choose the correct option from the given alternatives :

If f(x) = $\frac{x^2 - 1}{x^2 + 1}$, for every real x, then the minimum value of f is ______.

1. 1
2. 0
3. –1
4. 2

Solution

If f(x) = $\frac{x^2 - 1}{x^2 + 1}$, for every real x, then the minimum value of f is **$\underline{-1}$**.

Explanation:

f(x) = $\frac{x^2 - 1}{x^2 + 1} = \frac{x^2 + 1 - 2}{x^2 + 1} = 1 - \frac{2}{x^2 + 1}$

Therefore, f(x) < 1 ∀x and ≥ –1 ...... $\left(\because \frac{2}{x^2 + 1} \le 2\right)$

Therefore, –1 ≤ f(x) < 1

Hence, f(x) has minimum value −1 and also there is no maximum value.

Alter: We have

f'(x) = $\frac{(x^2 + 1)2x - (x^2 - 1)2x}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}$

f'(x) = 0

⇒ x = 0

Now, f''(x) = $\frac{(x^2 + 1)^2 4 - 4x.2(x^2 + 1)2x}{(x^2 + 1)^4}$

$= \dfrac{(x^2+1)4 - 16x(x)}{(x^2+1)^3}$

$= \dfrac{-12x^2+4}{(x^2+1)^3}$

Therefore, f′′(0) > 0 and there is only one critical point that has minima. Hence, f(x) has the least value at x = 0

f_{min} = f(0) = –1/1 = –1

1.6. Choose correct alternatives :

The shortest distance between the lines

$\vec{r} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) + \lambda\left(\hat{i} - \hat{j} + \hat{k}\right)$ and $\vec{r} = \left(2\hat{i} - \hat{j} - \hat{k}\right) + \mu\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$ is _____.

1. $\dfrac{1}{\sqrt{3}}$
2. $\dfrac{1}{\sqrt{2}}$
3. **$\dfrac{3}{\sqrt{2}}$**
4. $\dfrac{\sqrt{3}}{2}$

Solution

The shortest distance between the lines

$\vec{r} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) + \lambda\left(\hat{i} - \hat{j} + \hat{k}\right)$ and $\vec{r} = \left(2\hat{i} - \hat{j} - \hat{k}\right) + \mu\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$ is $\underline{\boldsymbol{\dfrac{3}{\sqrt{2}}}}$.

1.7. Select the correct option from the given alternatives:

The principal solutions of equation cot θ = $\sqrt{3}$ are ______.

1. **$\dfrac{\pi}{6}, \dfrac{7\pi}{6}$**
2. $\dfrac{\pi}{6}, \dfrac{5\pi}{6}$
3. $\dfrac{\pi}{6}, \dfrac{8\pi}{6}$
4. $\dfrac{7\pi}{6}, \dfrac{\pi}{3}$

Solution

The principal solutions of equation cot θ = $\sqrt{3}$ are $\underline{\frac{\pi}{6}, \frac{7\pi}{6}}$.

1.8. The area of triangle formed by the lines $x^2 + 4xy + y^2 = 0$ and $x - y - 4 = 0$ is ______.

1. $\frac{4}{\sqrt{3}}$ sq units
2. **$\frac{8}{\sqrt{3}}$ sq units**
3. $\frac{16}{\sqrt{3}}$ sq units
4. $\frac{15}{\sqrt{3}}$ sq units

Solution

The area of triangle formed by the lines $x^2 + 4xy + y^2 = 0$ and x - y – 4

= 0 is $\frac{8}{\sqrt{3}}$ **sq units.**

Q2. Answer the following questions:

2.1.

Evaluate: $\int_0^{\frac{\pi}{2}} x \sin x.\, dx$

Solution

$$\int_0^{\frac{\pi}{2}} x \sin x.\, dx$$

$$= \left[x \int \sin x.\, dx\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left[\frac{d}{dx}(x) \int \sin x.\, dx\right].\, dx$$

$$= [x(-\cos x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 1.\,(-\cos x).\, dx$$

$$= -[x \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x.\, dx$$

$$= -\left[\frac{\pi}{2} \cos \frac{\pi}{2} - 0\right] + [\sin x]_0^{\frac{\pi}{2}}$$

$= 0 + \left(\sin\frac{\pi}{2} - \sin 0\right)$

= 1

2.2. Determine the order and degree of the following differential equation:

$$\frac{dy}{dx} = \frac{2\sin x + 3}{\frac{dy}{dx}}$$

Solution

The given Differential equation is

$$\frac{dy}{dx} = \frac{2\sin x + 3}{\frac{dy}{dx}}$$

$\therefore \left(\frac{dy}{dx}\right)^2$ = 2 sin x + 3

This D.E. has highest order derivative $\frac{dy}{dx}$ with power 2.

∴ The given D.E. is of order 1 and degree 2.

2.3. Determine the order and degree of the following differential equation:

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 8\frac{d^2y}{dx^2}$$

Solution

The given D.E. is $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 8\frac{d^2y}{dx^2}$

On squaring both sides, we get

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = 8^2.\left(\frac{d^2y}{dx^2}\right)^2$$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 2.

∴ The given D.E. has order 2 and degree 2.

2.4. Check whether the following matrix is invertible or not:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Solution

Let A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Then, |A| = $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$ = $\cos^2\theta + \sin^2\theta = 1 \neq 0$

∴ A is a non-singular matrix.

Hence, A is invertible.

SECTION – B

Attempt any EIGHT of the following questions:

Q3. Differentiate the following w.r.t.x:

tan[cos(sinx)]

Solution

Let y = tan[cos(sinx)]

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{\tan[\cos(\sin x)]\}$$

$$= \sec^2[\cos(\sin x)].\frac{d}{dx}[\cos(\sin x)]$$

$$= \sec^2[\cos(\sin x)].[-\sin(\sin x)].\frac{d}{dx}(\sin x)$$

$= -\sec^2[\cos(\sin x)]\cdot\sin(\sin x)\cdot\cos x.$

Q4. Find the separate equations of the lines represented by the equation $3x^2 - 10xy - 8y^2 = 0$.

Solution

Given pairs of lines $3x^2 - 10xy - 8y^2 = 0$

$\Rightarrow 3x^2 - 12xy + 2xy - 8y^2 = 0$

⇒ 3x(x – 4y) + 2y(x – 4y) = 0

⇒ (x – 4y)(3x + 2y) = 0

Separated equations are:

3x + 2y = 0 and x – 4y = 0

Q5.

Find the vector equation of the plane passing through the point having position vector $\hat{i}+\hat{j}+\widehat{k}$ and perpendicular to the vector $4\hat{i}+5\hat{j}+6\widehat{k}$.

Solution

Let position vector of point A be $\vec{a}$

$\vec{a}=\hat{i}+\hat{j}+\widehat{k}$,

also $\vec{n}=4\hat{i}+5\hat{j}+6\widehat{k}$

$\therefore \vec{a}.\vec{n}=\left(\hat{i}+\hat{j}+\widehat{k}\right).\left(4\hat{i}+5\hat{j}+6\widehat{k}\right)$

= (1)(4) + (1)(5) + (1)(6)

= 4 + 5 + 6

= 15 ....(1)

∴ The vector equation of plane is

$\vec{r}.\vec{n}=\vec{a}.\vec{n}$

$\vec{r}.\left(4\hat{i}+5\hat{j}+6\widehat{k}\right)$ = 15 ...[From 1]

Q6.

Show that the line $\frac{x-2}{1}=\frac{y-4}{2}=\frac{z+4}{-2}$ passes through the origin.

Solution

The equation of the line is $\frac{x-2}{1}=\frac{y-4}{2}=\frac{z+4}{-2}$

The coordinates of the origin O are (0, 0, 0)

For x = 0, $\frac{x-2}{1} = \frac{0-2}{1}$ = −2

For y = 0, $\frac{y-4}{2} = \frac{0-4}{2}$ = −2

For z = 0, $\frac{z+4}{-2} = \frac{0+4}{-2}$ = −2

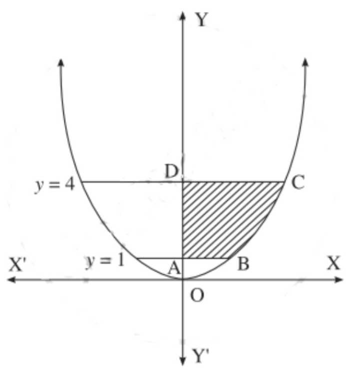
∴ Coordinates of the origin O satisfy the equation of the line.

Hence, the line passes through the origin.

Q7. Solve the following :

Find the area of the region bounded by the curve y = 4x², Y-axis and the lines y = 1, y = 4.

Solution



By symmetry of the parabola, the required area is 2 times the area of the region ABCD.

From the equation of the parabola, $x^2 = \frac{y}{4}$

the first quadrant, x > 0

∴ $x = \frac{1}{2}\sqrt{y}$

∴ required area = $\int_1^4 x \cdot dy$

= $\frac{1}{2}\int_1^4 \sqrt{y} \cdot dy$

$= \frac{1}{2}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$

$= \frac{1}{2} \times \frac{2}{3}\left[4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$

$= \frac{1}{3}\left[\left(2^2\right)^{\frac{3}{2}} - 1\right]$

$= \frac{1}{3}[8 - 1]$

$= \frac{7}{3}$ sq units.

Q8. State if the following is not the probability mass function of a random variable. Give reasons for your answer

Z	3	2	1	0	−1
P(Z)	0.3	0.2	0.4	0	0.05

Solution 1

P.m.f. of random variable should satisfy the following conditions:

a. $0 \le p_i \le 1$

b. $\sum p_i = 1$

Z	3	2	1	0	−1
P(Z)	0.3	0.2	0.4	0	0.05

Here $\sum p_i = 0.3 + 0.2 + 0.4 + 0 + 0.05$

$= 0.95 \neq 1$

Hence, P(Z) cannot be regarded as p.m.f. of the random variable Z.

Solution 2

Here, $p_i \geq 0, \forall i = 1, 2, ..., 5$

Now consider,

$$\sum_{i=1}^{5} P_i = 0.3 + 0.2 + 0.4 + 0 + 0.05$$

$= 0.95 \neq 1$

$\therefore$ Given distribution is not p.m.f.

Q9. Write converse, inverse and contrapositive of the following statement. "If voltage increases then current decreases".

Solution

Let **p:** Voltage increases.

q: Current decreases.

Then the symbolic form of the given statement is $p \to q$.

Converse: $q \to p$ is the converse of $p \to q$.

i.e. If current decreases, then voltage increases.

Inverse: $\sim p \to \sim q$ is the inverse of $p \to q$.

i.e. If voltage does not increase, then-current does not decrease.

Contrapositive: $\sim q \to \sim p$, is the contrapositive of $p \to q$.

i.e. If current does not decrease, then voltage does not increase.

Q10. Find the value of k if lines represented by $kx^2 + 4xy - 4y^2 = 0$ are perpendicular to each other.

Solution

Comparing the equation $kx^2 + 4xy - 4y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = k$, $2h = 4$ and $b = -4$

Since lines represented by $kx^2 + 4xy - 4y^2 = 0$ are perpendicular to each other,

$a + b = 0$

∴ k – 4 = 0

∴ k = 4.

Q11.

If the vectors $2\hat{i} - q\hat{j} + 3\widehat{k}$ and $4\hat{i} - 5\hat{j} + 6\widehat{k}$ are collinear, find q.

Solution

The vectors $2\hat{i} - q\hat{j} + 3\widehat{k}$ and $4\hat{i} - 5\hat{j} + 6\widehat{k}$ are collinear

∴ The coefficients of $\hat{i}, \hat{j}, \widehat{k}$ are proportional

$\therefore \frac{2}{4} = \frac{-q}{-5} = \frac{3}{6}$

$\therefore \frac{q}{5} = \frac{1}{2}$

∴ q = $\frac{5}{2}$

Q12. Given X ~ B(n, p) if p = 0.6 and E(X) = 6, find n and Var(X).

Solution

Given: p = 0.6 and E(X) = 6

E(X) = np

∴ 6 = n × 0.6

n = $\frac{6}{0.6} = 10$

Now, q = 1 – p = 1 – 0.6 = 0.4

∴ Var(X) = npq

= 10 × 0.6 × 0.4 = 2.4

Hence, n = 10, Var(X) = 2.4

Q13. Solve graphically: 3x + 2y ≥ 0

Solution

Consider the line whose equation is 3x + 2y = 0.

The constant term is zero, therefore this line is passing through the origin.

∴ One point on the line is O = (0, 0).

To find another point, we can give any value of x and get the corresponding value of y.

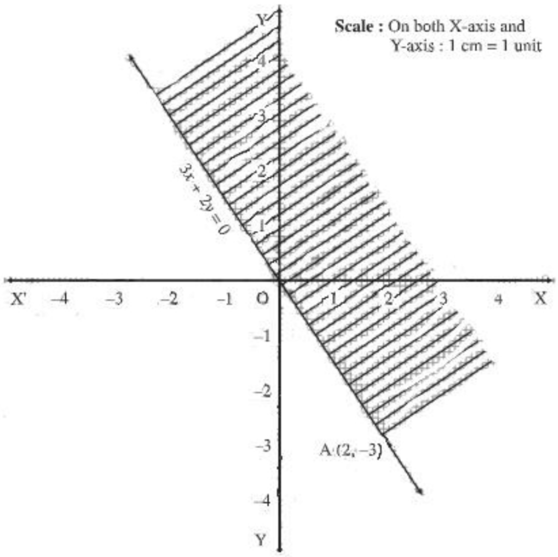
Put x = 2, we get 6 + 2y = 0 i.e. y = –3

∴ A = (2, –3), is another point on the line. Draw the line OA.

To find the solution set, we cannot check (0,0) as it is already on the line.

We can check any other point which is not on the line.

Let us check the point (1, 1).



When x = 1, y = 1, then 3x + 2y = 3 + 2 = 5 which is grreater than zer.

∴ 3x + 2y > 0 in this case.

Hence (1, 1) lies in the required region.

Therefore, the required region is the upper side which is shaded in the graph.

This is the solution set of x + 2y > 0.

Q14. Find the inverse of the following matrix.

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Solution

Let A = $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$\therefore$ |A| = $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ = 4 – 3 = 1 ≠ 0

$\therefore$ A^{-1} exists.

Consider $AA^{-1} = I$

$\therefore$ $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By $R_1 + R_2$, we get,

$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

By $R_2 + R_1$, we get,

$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

By $R_1 + R_2$, we get,

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

$\therefore$ $A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

SECTION – C

Attempt any EIGHT of the following questions:

Q15. Show that the following points are collinear:

P = (4, 5, 2), Q = (3, 2, 4), R = (5, 8, 0).

Solution

Let $\overline{p}, \overline{q}, \overline{r}$ be position vectors of the points.

P = (4, 5, 2), Q = (3, 2, 4), R = (5, 8, 0) respectively.

Then $\overline{p} = 4\hat{i} + 5\hat{j} + 2\hat{k},\ \overline{q} = 3\hat{i} + 2\hat{j} + 4\hat{k},\ \overline{r} = 5\hat{i} + 8\hat{j} + 0\hat{k}$

$\overline{PQ} = \overline{q} - \overline{p}$

$= \left(3\hat{i} + 2\hat{j} + 4\hat{k}\right) - \left(4\hat{i} + 5\hat{j} + 2\hat{k}\right)$

$= -\hat{i} - 3\hat{j} + 2\hat{k}$

$= -\left(\hat{i} + 3\hat{j} - 2\hat{k}\right)$ .....(1)

and $\overline{QR} = \bar{r} - \bar{q}$

$= \left(5\hat{i} + 8\hat{j} + 0\hat{k}\right) - \left(3\hat{i} + 2\hat{j} + 4\hat{k}\right)$

$= 2\hat{i} + 6\hat{j} - 4\hat{k}$

$= 2\left(\hat{i} + 3\hat{j} - 2\hat{k}\right)$

$= 2.\overline{PQ}$ ....[By(1)]

$\therefore$ $\overline{QR}$ is a non-zero scalar multiple of $\overline{PQ}$

$\therefore$ They are parallel to each other.

But they have point Q in common.

$\therefore$ $\overline{PQ}$ and $\overline{QR}$ are collinear vectors.

Hence, the points P, Q, and R are collinear.

Q16. Evaluate the following integrals:

$$\int \frac{2}{\sqrt{x} - \sqrt{x+3}}.dx$$

Solution

$$\int \frac{2}{\sqrt{x} - \sqrt{x+3}}.dx = \int \frac{2}{\sqrt{x} - \sqrt{x+3}} \times \frac{\sqrt{x} + \sqrt{x+3}}{\sqrt{x} + \sqrt{x+3}}.dx$$

$$= \int \frac{2\left(\sqrt{x} + \sqrt{x+3}\right)}{x - (x+3)}.dx$$

$$= -\frac{2}{3}\int \left(\sqrt{x} + \sqrt{x+3}\right).dx$$

$$= -\frac{2}{3}\int x^{\frac{1}{2}}dx - \frac{2}{3}\int (x+3)^{\frac{1}{2}}.dx$$

$$= -\frac{2}{3}.\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} - \frac{2}{3}.\frac{(x+3)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + c$$

$$= -\frac{4}{9}\left[x^{\frac{3}{2}} + (x+3)^{\frac{3}{2}}\right] + c$$

Q17. For the differential equation, find the particular solution $(x - y^2x)\,dx - (y + x^2y)\,dy = 0$ when $x = 2, y = 0$

Solution

$(x - y^2x)\,dx - (y + x^2y)\,dy = 0$, when $x = 2, y = 0$

$\therefore x(1 - y^2)\,dx - y(1 + x^2)\,dy = 0$

$$\therefore \frac{x}{1+x^2}dx - \frac{y}{1-y^2}dy = 0$$

$$\therefore \frac{2x}{1+x^2}dx - \frac{2y}{1-y^2}dy = 0$$

Integrating both sides, we get

$$\int \frac{2x}{1+x^2}dx + \int \frac{-2y}{1-y^2}dy = c_1$$

Each of these integrals is of the type

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$$

$\therefore$ The general solution is $\log|1 + x^2| + \log|1 - y^2| = \log c$, where $c_1 = \log c$

$\therefore \log|(1 + x^2)(1 - y^2)| = \log c$

$\therefore (1 + x^2)(1 - y^2) = c$

When $x = 2, y = 0$, we have

$(1 + 4)(1 - 0) = c$

$\therefore c = 5$

$\therefore$ The particular solution is $(1 + x^2)(1 - y^2) = 5$.

Q18. Differentiate the following w.r.t. x: $x^{\tan^{-1}x}$

Solution

Let $y = x^{\tan^{-1}x}$

Then $\log y = \log\left(x^{\tan^{-1}x}\right) = (\tan^{-1}x)(\log x)$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}\left[\left(\tan^{-1} x\right)(\log x)\right]$$

$$= \left(\tan^{-1} x\right) \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}\left(\tan^{-1} x\right)$$

$$= \left(\tan^{-1} x\right) \times \frac{1}{x} + (\log x) \times \frac{1}{1+x^2}$$

$$\therefore \frac{dy}{dx} = y\left[\frac{\tan^{-1} x}{x} + \frac{\log x}{1+x^2}\right]$$

$$= x^{\tan^{-1} x}\left[\frac{\tan^{-1} x}{x} + \frac{\log x}{1+x^2}\right]$$

Q19. Find the principal solutions of the following equation:

$\sin 2\theta = -\frac{1}{2}$

Solution

$\sin 2\theta = -\frac{1}{2}$

Since, $\theta \in (0, 2\pi)$, $2\theta \in (0, 4\pi)$

$$\sin 2\theta = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right)$$

$= \sin\left(3\pi + \frac{\pi}{6}\right) = \sin\left(4\pi - \frac{\pi}{6}\right)$ ....[$\because \sin(\pi + \theta) = \sin(2\pi - \theta) =$

$\sin(3\pi + \theta) = \sin(4\pi - \theta) = -\sin\theta$]

$$\therefore \sin 2\theta = \sin\frac{7\pi}{6} = \sin\frac{11\pi}{6} = \sin\frac{19\pi}{6} = \sin\frac{23\pi}{6}$$

$$\therefore 2\theta = \frac{7\pi}{6} \text{ or } 2\theta = \frac{11\pi}{6} \text{ or } 2\theta = \frac{19\pi}{6} \text{ or } 2\theta = \frac{23\pi}{6}$$

$$\therefore \theta = \frac{7\pi}{12} \text{ or } \theta = \frac{11\pi}{12} \text{ or } \theta = \frac{19\pi}{12} \text{ or } \theta = \frac{23\pi}{12}$$

Hence, the required principal solutions are $\left\{\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$.

Q20. Evaluate:

$$\int_0^{\pi} \sin^3 x(1+2\cos x)(1+\cos x)^2 . dx$$

Solution

Let I = $\int_0^{\pi} \sin^3 x(1+2\cos x)(1+\cos x)^2 . dx$

= $\int_0^{\pi} \sin^2 x(1+2\cos x)(1+\cos x)^2 . \sin x . dx$

= $\int_0^{\pi} (1-\cos^2 x)(1+\ 2\cos x)(1+\cos x)^2 . \sin x . dx$

Put cos x = t

∴ – sinx.dx = dt.

∴ sinx.dx = –dt

When x = 0, t = cos 0 = 1

When x = π, t = cos π = –1

∴ I = $\int_1^{-1} (1-t^2)(1+2t)(1+t)^2(-dt)$

= $-\int_1^{-1} (1+2t-t^2-2t^3)(1+2t+t^2) . dt$

= $-\int_1^{-1} (1+2t-t^2-2t^3+2t+4t^2-2t^3-4t^4+t^2+2t^3-t^4-2t^5) . dt$

= $\int_1^{-1} (1+4t+4t^2-2t^3-5t^4-2t^5) . dt$

= $-\left[t+4\left(\frac{t^2}{2}\right)+4\left(\frac{t^3}{3}\right)-2\left(\frac{t^4}{4}\right)-5\left(\frac{t^5}{5}\right)-2\left(\frac{t^6}{6}\right)\right]_1^{-1}$

= $-\left[t+2t^2\ \frac{4}{3}t^3-\frac{1}{2}t^4-t^5-\frac{1}{3}t^6\right]_1^{-1}$

= $-\left[\left(-1+2-\frac{4}{3}-\frac{1}{2}+1-\frac{1}{3}\right)-\left(1+2+\frac{4}{3}-\frac{1}{2}-1-\frac{1}{3}\right)\right]$

$= -\left[\left(-1+2-\frac{4}{3}-\frac{1}{2}+1-\frac{1}{3}\right)-\left(1+2+\frac{4}{3}-\frac{1}{2}-1-\frac{1}{3}\right)\right]$

$= -\left[-1+2-\frac{4}{3}-\frac{1}{2}+1-\frac{1}{3}-1-2-\frac{4}{3}+\frac{1}{2}+1+\frac{1}{3}\right]$

$= -\left[-\frac{8}{3}\right]$

$= \frac{8}{3}.$

Q21. In a box of floppy discs, it is known that 95% will work. A sample of three of the discs is selected at random. Find the probability that none of the floppy disc work.

Solution

Let X = number of working discs.

p = probability that a floppy disc works

$\therefore$ p = 95% = $\frac{95}{100} = \frac{19}{20}$

and q = 1 – p = $1 - \frac{19}{20} = \frac{1}{20}$

Given: n = 3

$\therefore X \sim B\left(3, \frac{19}{20}\right)$

The p.m.f. of X is given by P(X = x) = ${}^{n}C_{x}\, p^{x} q^{n-x}$

i.e. p(x) = ${}^{3}C_{x}\left(\frac{19}{20}\right)^{x}\left(\frac{1}{20}\right)^{3-x}$, x = 0, 1, 2, 3

P(none of the floppy discs work) = P(X = 0)

= p(0) = ${}^{3}C_{0}\left(\frac{19}{20}\right)^{0}\left(\frac{1}{20}\right)^{3-0}$

= $1 \times 1 \times \frac{1}{20^3} = \frac{1}{20^3} = \frac{1}{8000}$

Hence, the probability that none of the floppy disc will work = $\frac{1}{8000}$.

Q22. Find the centroid of tetrahedron with vertices K(5, −7, 0), L(1, 5, 3), M(4, −6, 3), N(6, −4, 2)

Solution

Let G be the centroid of the tetrahedron K, L, M, N.

Let $\vec{p}, \vec{l}, \vec{m}, \vec{n}$ be the position vectors of the points K, L, M, N respectively w.r.t. the origin O.

Then, $\vec{p} = 5\hat{i} - 7\hat{j} + 0\widehat{k}$

$\vec{l} = \hat{i} + 5\hat{j} + 3\widehat{k}$

$\vec{m} = 4\hat{i} - 6\hat{j} + 3\widehat{k}$

Let G(g) be the centroid of the tetrahedron.

Then by centroid formula

$$\vec{g} = \frac{\vec{p} + \vec{l} + \vec{m} + \vec{n}}{4}$$

$$= \frac{1}{4}\left[\left(5\hat{i} - 7\hat{j} + 0.\widehat{k}\right) + \left(\hat{i} + 5\hat{j} + 3\widehat{k}\right) + \left(4\hat{i} - 6\hat{j} + 3\widehat{k}\right) + \left(6\hat{i} - 4\hat{j} + 2\widehat{k}\right)\right]$$

$$= \frac{1}{4}\left(16\hat{\mathrm{i}} - 12\hat{\mathrm{j}} + 8\hat{\mathrm{k}}\right)$$

$$= 4\hat{i} - 3\hat{j} + 2\widehat{k}$$

Hence, the centroid of the tetrahedron is G = (4, −3, 2).

Q23.

If $\vec{a}$ and $\vec{b}$ are two vectors perpendicular to each other, prove that $\left(\vec{a} + \vec{b}\right)^2 = \left(\vec{a} - \vec{b}\right)^2$

Solution

$\vec{a}$ and $\vec{b}$ are perpendicular to each other.

$\therefore \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = 0$ ...(i)

LHS = $\left(\vec{a} + \vec{b}\right)^2$

$= \left(\vec{a} + \vec{b}\right) \cdot \left(\vec{a} + \vec{b}\right)$

$= \vec{a} \cdot \left(\vec{a} + \vec{b}\right) + \vec{b}\left(\vec{a} + \vec{b}\right)$

= $\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$

= $\vec{a}\cdot\vec{a}+0+0+\vec{b}\cdot\vec{b}$ ....[By (1)]

= $|\vec{a}|^2+\left|\vec{b}\right|^2$

RHS = $\left(\vec{a}-\vec{b}\right)^2$

= $\left(\vec{a}-\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right)$

= $\vec{a}\cdot\left(\vec{a}-\vec{b}\right)+\vec{b}\left(\vec{a}-\vec{b}\right)$

= $\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$

= $\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}$ ....[By (i)]

= $|\vec{a}|^2+\left|\vec{b}\right|^2$

∴ LHS = RHS

Hence, $\left(\vec{a}+\vec{b}\right)^2=\left(\vec{a}-\vec{b}\right)^2$

Q24. Integrate the following functions w.r.t. x :

$$\frac{1+x}{x.\sin(x+\log x)}$$

Solution

Let I = $\int\frac{1+x}{x.\sin(x+\log x)}.dx$

= $\int\frac{1}{\sin(x+\log x)}.\left(\frac{1+x}{x}\right).dx$

= $\int\frac{1}{\sin(x+\log x)}.\left(\frac{1}{x}+1\right).dx$

Put x + log x = t

∴ $\left(1+\frac{1}{x}\right).dx$ = dt

∴ I = $\int\frac{1}{\sin t}dt=\int\text{cosec }t\,dt$

= log |cosec t – cot t| + c

= log |cosec(x + log x) – cot(x + logx)| + c.

Q25. Find the Cartesian co-ordinates of the point whose polar co-ordinates are:

$$\left(\frac{3}{4}, \frac{3\pi}{4}\right)$$

Solution

Here, r = $\frac{3}{4}$ and θ = $\frac{3\pi}{4}$

Let the cartesian coordinates be (x, y)

Then,

$$x = r\cos\theta = \frac{3}{4}\cos\frac{3\pi}{4} = \frac{3}{4}\cos\left(\pi - \frac{\pi}{4}\right)$$

$$= -\frac{3}{4}\cos\frac{\pi}{4} = -\frac{3}{4} \times \frac{1}{\sqrt{2}} = -\frac{3}{4\sqrt{2}}$$

$$y = r\sin\theta = \frac{3}{4}\sin\frac{3\pi}{4} = \frac{3}{4}\sin\left(\pi - \frac{\pi}{4}\right)$$

$$= \frac{3}{4}\sin\frac{\pi}{4} = \frac{3}{4} \times \frac{1}{\sqrt{2}} = \frac{3}{4\sqrt{2}}$$

∴ The cartesian coordinates of the given point are $\left(-\frac{3}{4\sqrt{2}}, \frac{3}{4\sqrt{2}}\right)$.

Q26. Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as number greater than 4 appears on at least one die.

Solution 1

When a die is tossed twice, the sample space S has 6 × 6 = 36 sample points.

∴ n(S) = 36

Trial will be a success if the number on at least one die is 5 or 6.

Let X denote the number of dice on which 5 or 6 appears.

Then X can take values 0, 1, 2

When X = 0 i.e., 5 or 6 do not appear on any of the dice, then

X = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)}.

∴ n(X) = 16

∴ P(X = 0) = $\frac{n(X)}{n(S)} = \frac{16}{36} = \frac{4}{9}$

When X = 1, i.e. 5 or 6 appear on exactly one of the dice, then

X = {(1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (6, 1), (6, 2), (6, 3), (6, 4)}

∴ n(X) = 16

∴ P(X = 1) = $\frac{n(X)}{n(S)} = \frac{16}{36} = \frac{4}{9}$

When X = 2, i.e. 5 or 6 appear on both of the dice, then

X = {(5, 5), (5, 6), (6, 5), (6, 6)}

∴ n(X) = 4

∴ P(X = 2) = $\frac{n(X)}{n(S)} = \frac{4}{36} = \frac{1}{9}$

∴ The required probability distribution is

X	0	1	2
P(X = x)	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

Solution 2

Success is defined as a number greater than 4 appears on at least one die

Let X denote the number of successes.

∴ Possible values of X and 0, 1, 2.

Let P(getting a number greater than 4) = p

$= \frac{2}{6}$

$= \frac{1}{3}$

∴ q = 1 – p

$= 1 - \frac{1}{3}$

$= \frac{2}{3}$

∴ P(X = 0) = P(no success)

= qq

$= q^2$

$= \frac{4}{9}$

P(X = 1) = P(one success)

= qp + pq = 2pq

$= 2 \times \frac{1}{3} \times \frac{2}{3}$

$= \frac{4}{9}$

P(X = 2) = P(two successes)

= pp

$= p^2$

$= \frac{1}{9}$

∴ Probability distribution of X is as follows:

X	0	1	2
P(X = x)	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

SECTION – D

Attempt any FIVE of the following questions:

Q27. Solve the following:

Find the maximum and minimum values of the function $f(x) = \cos^2 x + \sin x$.

Solution

$f(x) = \cos^2 x + \sin x$

$\therefore f'(x) = \frac{d}{dx}\left(\cos^2 x + \sin x\right)$

$= 2\cos x . \frac{d}{dx}(\cos x) + \cos x$

$= 2 \cos x (-\sin x) + \cos x$

$= -\sin 2x + \cos x$

and $f''(x) = \frac{d}{dx}(-\sin 2x) + \cos x$

$= -\cos 2x . \frac{d}{dx}(2x) - \sin x$

$= -\cos 2x \times 2 - \sin x$

$= -2 \cos 2x - \sin x$

For extreme values of f(x), $f'(x) = 0$

$\therefore -\sin 2x + \cos x = 0$

$\therefore -2 \sin x \cos x + \cos x = 0$

$\therefore \cos x (-2\sin x + 1) = 0$

$\therefore \cos x = 0$ or $-2 \sin x + 1 = 0$

$\therefore \cos x = \cos \frac{\pi}{2}$ or $\sin x = \frac{1}{2} = \sin \frac{\pi}{6}$

$\therefore x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$

(i) $f\left(\frac{\pi}{2}\right) = 2\cos\pi - \sin\frac{\pi}{2}$

$= -2(-1) - 1 = 1 > 0$

∴ By the second derivative test, f is minimum at x = $\frac{\pi}{2}$ and minimum value of f at x = $\frac{\pi}{2}$

= $f\left(\frac{\pi}{2}\right) = \cos^2 \frac{\pi}{2} + \sin \frac{\pi}{2} = 0 + 1 = 1$

(ii) $f\left(\frac{\pi}{6}\right) = -2 \cos \frac{\pi}{3} - \sin \frac{\pi}{6}$

= $-2\left(\frac{1}{2}\right) - \frac{1}{2}$

= $-\frac{3}{2} < 0$

∴ By the second derivative test, f is maximum at x = $\frac{\pi}{6}$ and maxmuum value of f at x = $\frac{\pi}{6}$ is

= $f\left(\frac{\pi}{6}\right) = \cos^2 \frac{\pi}{6} + \sin \frac{\pi}{6}$

= $\left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4}$

Hence, the maximum and minimum values of the 5 function f(x) are $\frac{5}{4}$ and 1 respectively.

Q28.

If y = $\log\left(x + \sqrt{x^2 + a^2}\right)^m$, show that $\left(x^2 + a^2\right) \frac{d^2y}{dx^2} + x \frac{\text{d}}{\text{dx}}$ = 0.

Solution

y = $\log\left(x + \sqrt{x^2 + a^2}\right)^m$

= $m \log\left(x + \sqrt{x^2 + a^2}\right)$

∴ $\frac{\text{dy}}{\text{dx}} = m \frac{\text{d}}{\text{dx}} \left[\log\left(x + \sqrt{x^2 + a^2}\right)\right]$

= $m \times \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \frac{\text{d}}{\text{dx}} \left(x + \sqrt{x^2 + a^2}\right)$

$$= \frac{m}{x+\sqrt{x^2+a^2}} \times \left[1 + \frac{1}{2\sqrt{x^2+a^2}} \cdot \frac{d}{dx}(x^2+a^2)\right]$$

$$= \frac{m}{x+\sqrt{x^2+a^2}} \times \left[1 + \frac{1}{2\sqrt{x^2+a^2}} \cdot (2x+0)\right]$$

$$= \frac{m}{x+\sqrt{x^2+a^2}} \times \frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}$$

$$\therefore \frac{dy}{dx} = \frac{m}{\sqrt{x^2+a^2}}$$

$$\therefore \sqrt{x^2+a^2}\frac{dy}{dx} = m$$

$$\therefore (x^2+a^2)\left(\frac{dy}{dx}\right)^2 = m^2$$

Differentiating both sides w.r.t. x, we get

$$(x^2+a^2) . \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2 . \frac{d}{dx}(x^2+a^2) = \frac{d}{dx}(m^2)$$

$$\therefore (x^2+a^2) \times 2\frac{dy}{dx} . \frac{d}{dx}\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2 \times (2x+0) = 0$$

$$\therefore (x^2+a^2) . 2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 = 0$$

Cancelling $2\frac{dy}{dx}$ throughtout, we get

$$(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

Q29. Solve the following differential equation:

$$\frac{\cos^2 y}{x}dy + \frac{\cos^2 x}{y}dx = 0$$

Solution

$$\frac{\cos^2 y}{x}dy + \frac{\cos^2 x}{y}dx = 0$$

$\therefore$ y $\cos^2 y$ dy + x $\cos^2$ x dx = 0

$\therefore x\left(\frac{1+\cos 2x}{2}\right)dx + y\left(\frac{1+\cos 2y}{2}\right)dy = 0$

∴ x(1 + cos 2x) dx + y(1 + cos 2y)dy = 0

∴ x dx + x cos 2x dx + y dy + y cos 2y dy = 0

Integrating both sides, we get

$$\int x dx + \int y dy + \int x \cos 2x dx + \int y \cos 2y dy = c_1 \quad .....(i)$$

Using integration by parts

$$\int x \cos 2x dx = x \int \cos 2x dx - \int \left[\frac{d}{dx}(x) \int \cos 2x dx\right] dx$$

$= x\left(\frac{\sin 2x}{2}\right) - \int 1. \frac{\sin 2x}{2} dx$

$= \frac{x \sin 2x}{2} + \frac{1}{2} . \frac{\cos 2x}{2}$

$= \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}$

Similarly, $\int y \cos 2y dy = \frac{y \sin 2y}{2} + \frac{\cos 2y}{4}$

∴ From equation (i), we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + \frac{y \sin 2y}{2} + \frac{\cos 2y}{4} = c_1$$

Multiplying throughout by 4, this becomes

$2x^2 + 2y^2 + 2x \sin 2x + \cos 2x + 2y \sin 2y + \cos 2y = 4c_1$

$\therefore 2(x^2 + y^2) + 2(x \sin 2x + y \sin 2y) + \cos 2y + \cos 2x + c = 0$

where $c = -4c_1$

This is the general solution.

Q30. In ΔABC, if cot A, cot B, cot C are in A.P. then show that a^2, b^2, c^2 are also in A.P.

Solution

By the sine rule,

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c} = k$$

∴ sin A = ka, sin B = kb, sin C = kc ...(i)

Now, cot A, cot B, cot C are in A.P.

∴ cot C – cot B = cot B – cot A

∴ cot A + cot C = 2 cot B

$$\therefore \frac{\cos A}{\sin A}+\frac{\cos C}{\sin C} = 2\cot B$$

$$\therefore \frac{\sin C\cos A+\sin A\cos C}{\sin A.\sin C} = 2\cot B$$

$$\therefore \frac{\sin(A+C)}{\sin A.\sin C} = 2\cot B$$

$$\therefore \frac{\sin(\pi-B)}{\sin A.\sin C} = 2\cot B \quad ...[\because A+B+C=\pi]$$

$$\therefore \frac{\sin B}{\sin A.\sin C}=\frac{2\cos B}{\sin B}$$

$$\therefore \frac{\sin^2 B}{\sin A.\sin C} = 2\cos B$$

$$\therefore \frac{k^2b^2}{(ka)(kc)}=2\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$$\therefore \frac{b^2}{ac}=\frac{a^2+c^2-b^2}{ac}$$

∴ $b^2 = a^2 + c^2 - b^2$

∴ $2b^2 = a^2 + c^2$

Hence, a^2 b^2, c^2 are in A.P.

Q31. Integrate the following w.r.t. x :

$$\frac{(3\sin-2)\cdot\cos x}{5-4\sin x-\cos^2 x}$$

Solution

$$\text{Let I} = \int \frac{(3\sin-2)\cdot\cos x}{5-4\sin x-\cos^2 x}\cdot dx$$

$$= \int \frac{(3\sin x - 2)\cdot \cos x}{5 - \left(1 - \sin^2 x\right) - 4\sin x}\cdot dx$$

$$= \int \frac{(3\sin x - 2)\cdot \cos x}{5 - 1 + \sin^2 x - 4\sin x}\cdot dx$$

$$= \int \frac{(3\sin x - 2)\cdot \cos x}{\sin^2 x - 4\sin x + 4}\cdot dx$$

Put sin x = t

∴ cos x dx = dt

∴ I = $\int \frac{3t-2}{t^2 - 4t + 4}\cdot dt$

$= \int \frac{3t-2}{(t-2)^2}\cdot dt$

Let $\frac{3t-2}{(t-2)^2} = \frac{A}{t-2} + \frac{B}{(t-2)^2}$

∴ 3t – 2 = A(t – 2) + B

Put t – 2 = 0, i.e. t = 2, we get

4 = A(0) + B

∴ B = 4

Put t = 0, we get

– 2 = A(– 2) + B

∴ – 2 = – 2A + 4

∴ 2A = 6

∴ A = 3

∴ $\frac{3t-2}{(t-2)^2} = \frac{3}{t-2} + \frac{4}{(t-2)^2}$

∴ I = $\int \left[\frac{3}{t-2} + \frac{4}{(t-2)^2}\right]\cdot dt$

$$= 3\int \frac{1}{t-2} \cdot dt + 4\int (t-2)^{-2} \cdot dt$$

$$= 3\log|t-2| + 4 \cdot \frac{(t-2)^{-1}}{-1} \cdot \frac{1}{1} + c$$

$$= 3\log|t-2| - \frac{4}{(t-2)} + c$$

$$= 3\log|\sin x - 2| - \frac{4}{(\sin x - 2)} + c.$$

Q32. Using the rules in logic, write the negation of the following:

(p ∨ q) ∧ (q ∨ ~r)

Solution

The negation of (p ∨ q) ∧ (q ∨ ~r) is
~ [(p ∨ q) ∧ (q ∨ ~r)]

≡ ~(p ∨ q) ∨ ~(q ∨ ~r) .....(Negation of conjunction)

≡ (~p ∧ ~q) ∨ [~q ∧ ~(~r)] ....(Negation of disjunction

≡ (~p ∧ ~q) ∨ (~q ∧ r) ....(Negation of negation)

≡ (~q ∧ ~p) ∨ (~q ∧ r) ....(Commutative law)

≡ (~q) ∧ (~p ∨ r) ....(Distributive Law)

Q33. Find the co-ordinates of the foot of the perpendicular drawn from the point $2\hat{i} - \hat{j} + 5\hat{k}$ to the line $\bar{r} = \left(11\hat{i} - 2\hat{j} - 8\hat{k}\right) + \lambda\left(10\hat{i} - 4\hat{j} - 11\hat{k}\right)$.

Also find the length of the perpendicular.

Solution

Let M be the foot of perpendicular drawn from the point $P\left(2\hat{i} - \hat{j} + 5\hat{k}\right)$ on the line.

$$\bar{r} = \left(11\hat{i} - 2\hat{j} - 8\hat{k}\right) + \lambda\left(10\hat{i} - 4\hat{j} - 11\hat{k}\right)$$

Let the position vector of the point M be

$\left(11\hat{i}-2\hat{j}-8\widehat{k}\right)+\lambda\left(10\hat{i}-4\hat{j}-11\widehat{k}\right)$

= $(11+10\lambda)\hat{i}+(-2-4\lambda)\hat{j}+(-8-11\lambda)\widehat{k}$.

Then PM = Position vector of M – Position vector of P

= $\left[(11+10\lambda)\hat{i}+(-2-4\lambda)\hat{j}+(-8-11\lambda)\widehat{k}\right]-\left(2\hat{i}-\hat{j}+5\widehat{k}\right)$

= $(9+10\lambda)\hat{i}+(-1-4\lambda)\hat{j}+(-13-11\lambda)\widehat{k}$

Since PM is perpendicular to the given line which is parallel to $\bar{b}=10\hat{i}-4\hat{j}-11\widehat{k}$,

PM ⊥ $\bar{b}$

∴ PM.$\bar{b}$ = 0

∴ $\left[(9+10\lambda)\hat{i}+(-1-4\lambda)-11(-13-11\lambda)\widehat{k}\right].\left(10\hat{i}-4\hat{j}-11\widehat{k}\right)$ = 0

∴ 10(9 + 10λ) – 4(–1 – 4λ) – 11(13 – 11λ) = 0

∴ 90 + 100λ + 4 + 16λ + 143 + 121λ = 0

∴ 237λ + 237 = 0

∴ λ = – 1

Putting this value of λ, we get the position vector of M as $\hat{i}+2\hat{j}+3\widehat{k}$.

∴ Coordinates of the foot of perpendicular M are (1, 2, 3).

Now, PM = $\left(\hat{i}+2\hat{j}+3\widehat{k}\right)-\left(2\hat{i}-\hat{j}+5\widehat{k}\right)$

= $-\hat{i}+3\hat{j}-2\widehat{k}$

∴ |PM| = $\sqrt{(-1)^2+(3)^2+(-2)^2}$

= $\sqrt{1+9+4}$

= $\sqrt{14}$

Hence, the coordinates of the foot of perpendicular are (1, 2, 3) and length of perpendicular = $\sqrt{14}$ units.

Q34.

If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = \hat{j} - \hat{k}$. find a vector $\vec{b}$ satisfying $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a} \cdot \vec{b} = 3$.

Solution

Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{c} = \hat{j} - \hat{k}$

Let $\vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\vec{a} \cdot \vec{b} = 3$ gives

$$\left(\hat{i} + \hat{j} + \hat{k}\right) . \left(x\hat{i} + y\hat{j} + z\hat{k}\right) = 3$$

$\therefore (1)(x) + (1)(y) + (1)(z) = 3$

Also, $x + y + z = 3$ ...(1)

Also, $\vec{c} = \hat{a} \times \widehat{b}$

$$\hat{j} - \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix}$$

$= (z - y)\hat{i} - (z - x)\hat{j} + (y - x)\hat{k}$

$= (z - y)\hat{i} + (x - z)\hat{j} + (y - x)\hat{k}$

By equality of vectors,

$z - y = 0$ ...(2)

$x - z = 1$ ...(3)

$y - x = - 1$ ...(4)

From (2), $y = z$

From (3), $x = 1 + z$

Substituting these values of x and y in (1), we get

1 + z + z + z = 3

$z = \frac{2}{3}$

$y = z = \frac{2}{3}$

$x = 1 + z = 1 + \frac{2}{3} = \frac{5}{3}$

$\vec{b}\frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

i.e, $\vec{b} = \frac{1}{3}\left(5\hat{i} + 2\hat{j} + 2\hat{k}\right)$