Exercise 6.5

Answer 1E.

The growth rate of population of country P is $k = 0.7944$ per member per day.

Let t be the time in days.

The value $t = 0$ represents the day zero.

The value $P(0)$ represents population on day zero.

That is $P(0) = 2$

The objective is to find the population size after six days.

Use the following formula:

The population size be denoted by $P(t) = P(0)e^{kt}$.

Substitute $P(0) = 2, k = 0.7944$ in $P(t) = P(0)e^{kt}$.

$$P(t) = 2e^{(0.7944)t}.$$

Find $P(t)$ when $t = 6$.

$$\begin{aligned} P(6) &= 2e^{(0.7944)(6)} \\ &= 2e^{4.7664} \\ &= 234.99 \\ &\approx 235 \end{aligned}$$

Therefore, the population size after six days is $\boxed{235}$.

Answer 2E.

In general, $y(t)$ is the size of the population at a time *t*. and *k* is a probability constant.

Thus,

$$\frac{dy}{dt} = ky$$

And from known theorem,

The solution of the differential equation $\frac{dy}{dt} = ky$ is the exponential function

$$y(t) = y_0 e^{kt}$$

Since here dealing with population growth, so use the following theorem:

$$y(t) = y_0 e^{kt}$$

(a)

To find the relative growth rate, solve for *k*.

To do so, plug into the equation.

The initial population $y_0 = 60$ cells

After 20 minutes each cell divided into two cells.

That is

$$\begin{aligned} y(20) &= 2 \times 60 \\ &= 120 \end{aligned}$$

Change the time into hours.

$$\begin{aligned} t &= 20 \text{ min} \\ &= 20 \times \frac{1}{60} \text{ hours} \\ &= \frac{1}{3} \text{ hour} \end{aligned}$$

Thus,

$$y\left(\frac{1}{3}\right) = 120$$

Substitute these values in $y(t) = y_0 e^{kt}$.

$$120 = 60 e^{\frac{1}{3}k}$$

Solve for *k*.

$$\begin{aligned} 120 &= 60 e^{\frac{1}{3}k} \\ 2 &= e^{\frac{1}{3}k} \end{aligned}$$

In order to get rid of e and to bring down the exponent, take the natural log of both sides.

$$\ln(2) = \ln\left(e^{\frac{1}{3}k}\right)$$

$$\begin{aligned} \ln(2) &= \frac{1}{3} k \ln(e) \\ 3\ln(2) &= k \\ k &= \ln(2^3) \\ &= \boxed{\ln(8)} \end{aligned}$$

(b)

Since already found *k* in part (a), simply plug into the theorem.

After *t* hours, the expression is

$y(t) = y_0 e^{kt}$ Substitute $k = \ln(8)$

$y(t) = 60e^{\ln(8)t}$

Since e and $\ln$ terms are inverses of each other, the expression becomes:

$$\begin{aligned} y(t) &= 60e^{t\ln(8)} \\ &= 60e^{\ln(8^t)} \\ &= 60 \times 8^t \end{aligned}$$

Therefore,

After *t* hours, the expression for the number of cells is $\boxed{y(t) = 60 \times 8^t}$.

(c)

To find the number of cells after 8 hours, simply plug in 8 for *t* into the expression found in part (b).

$$\begin{aligned} y(t) &= 60 \times 8^t \\ y(8) &= 60 \times 8^8 \\ &= \boxed{1{,}006{,}632{,}960} \end{aligned}$$

(d)

To find the rate of growth after 8 hours, use the equation:

$$\frac{dy(t)}{dt} = k \cdot y(t)$$

Substitute $t = 8$.

$$\frac{dy}{dt} = k \cdot y(8)$$

$$\begin{aligned} \frac{dy}{dt} &= \ln(8) \times (60 \times 8^8) \\ &= \boxed{209.32 \text{ billion cells per hour}} \end{aligned}$$ Since $y(8) = 60 \times 8^8$

(e)

To find when the population will reach 20,000 cells, simply set the expression found in part (b) to 20,000 and solve for *t*.

$y(t) = 60 \times 8^t$ Substitute $y(t) = 20{,}000$

$$20{,}000 = 60 \times 8^t$$

$$333.33 = 8^t$$

To take *t* out of the exponent, take the natural log of both sides of the equation.

$$\begin{aligned} \ln(333.33) &= \ln(8^t) \\ \ln(333.33) &= t\ln(8) \\ t &= \frac{\ln(333.33)}{\ln(8)} \\ &= \boxed{2.79 \text{ hours}} \end{aligned}$$

Answer 3E.

The only solution of the differential equation $\frac{dy}{dt}=ky$, is the exponential function

$y(t)=Ce^{kt}$. ...... (1)

Here, C is the initial value of the population.

(a)

The bacteria culture initially contains 100 cells.

That is, $C=100$

After 1 hour, the population increased to 420.

That is, $y(1)=420$.

Now, substitute $t=1$ and $C=100$ in equation (1).

$$y(1)=100e^{k(1)}$$
$$420=100e^{k}$$

Solve for k.

$$\frac{420}{100}=e^{k}$$
$$4.2=e^{k}$$
$$\ln(4.2)=k$$

Substitute the values of k and C in the equation (1):

$$\begin{aligned}y(t)&=Ce^{kt}\\&=100e^{\ln(4.2)t}\\&=100(4.2)^{t}\end{aligned}$$

Therefore, the expression after *t* hours is $\boxed{y(t)=100(4.2)^{t}}$.

(b)

To find the number of bacteria after 3 hours, substitute $t=3$ in the above expression.

$$y(t)=100(4.2)^{t}$$
$$\begin{aligned}y(3)&=100\times(4.2)^{3}\\&\approx 100\times(74.09)\\&\approx 7409\end{aligned}$$

Therefore, the number of bacteria after 3 hours is $\boxed{7409}$.

(c)

To find the rate of growth after 3 hours, use the equation:

$$\frac{dy}{dt}=ky(t)$$

Substitute $t=3$ in the above equation.

$$\frac{dy}{dt}=k\times y(3)$$

Substitute $y(3)=7409$, $k=\ln(4.2)$

$$=\ln(4.2)\times(7409)$$
$$=10,632 \text{ bacteria/h}$$

Therefore, the growth rate after 3 hours is $\boxed{10,632 \text{ bacteria/h}}$.

(d)

To find the value of t, when the population will reach 10,000

$$\begin{aligned} y(t) &= 100(4.2)^t \\ 10{,}000 &= 100 \times (4.2)^t \\ 100 &= (4.2)^t \end{aligned}$$

Take the natural log of both sides of the equation.

$$\begin{aligned} \ln(100) &= \ln(4.2)^t \\ \ln(100) &= t\ln(4.2) \\ t &= \frac{\ln(100)}{\ln(4.2)} \\ &\approx 3.2\text{hours} \end{aligned}$$

Therefore, the population will reach 10,000 at $\boxed{3.2\text{hours}}$.

Answer 4E.

(a)

Let $B(t)$ represent the bacteria count after *t* hours. From the given information, we know that $B(2)=400$ and $B(6)=25{,}600$. Since the bacteria grows with constant relative growth rate, $dB/dt=kB$.

The only solutions of the differential equation $dB/dt=kB$ are the exponential functions $B(t)=B(0)e^{kt}$. Write two equations for $B(2)=400$ and $B(6)=25{,}600$.

$$B(2)=B(0)e^{2k}=400$$

$$B(6)=B(0)e^{6k}=25{,}600$$

The relative growth rate is $k=\frac{1}{B}\frac{dB}{dt}$.

To find *k*, solve the first equation for $B(0)$.

$$\begin{aligned} B(0)e^{2k} &= 400 \\ B(0) &= 400e^{-2k} \end{aligned}$$

Then substitute this into the second equation and solve for *k*.

$$\begin{aligned} B(0)e^{6k} &= 25{,}600 \\ \left(400e^{-2k}\right)e^{6k} &= 25{,}600 \\ e^{4k} &= 64 \\ \ln\left(e^{4k}\right) &= \ln 64 \end{aligned}$$

$$\begin{aligned} 4k &= 2\ln 8 \\ k &= \frac{\ln 8}{2} \end{aligned}$$

Thus, the relative growth rate is $\boxed{\frac{\ln 8}{2} \text{ or about } 1.039721}$.

(b)

The initial size of the culture is $B(0)$.

To find this value, substitute *k* into the first equation above and solve for $B(0)$.

$$\begin{aligned} B(0)e^{2k} &= 400 \\ B(0)e^{\ln 8} &= 400 \\ B(0)8 &= 400 \\ B(0) &= \frac{400}{8} \\ B(0) &\approx 50 \end{aligned}$$

Thus, the initial size is $\boxed{50 \text{ bacteria}}$.

(c)

The number of bacteria after *t* hours is given by the function

$$B(t) = B(0)e^{kt} = \boxed{50e^{t\ln 8/2}}$$

(d)

The number of bacteria after 4.5 hours is given by $B(4.5)$.

$$\begin{aligned} B(4.5) &= 50e^{4.5\ln 8/2} \\ &\approx \boxed{5382 \text{ bacteria}} \end{aligned}$$

(e)

The rate of growth after *t* hours is

$$\begin{aligned} \frac{dB}{dt} &= kB \\ &= \left(\frac{\ln 8}{2}\right)50e^{t\ln 8/2} \\ &= 25\ln 8 \cdot e^{t\ln 8/2} \end{aligned}$$

Find the rate of growth after 4.5 hours.

$$\begin{aligned} \frac{dB}{dt}(4.5) &= 25\ln 8 \cdot e^{4.5\ln 8/2} \\ &\approx \boxed{5595.5 \text{ bacteria per hour}} \end{aligned}$$

(f)

Use the equation for *B* to calculate when $B(t) = 50{,}000$.

$$\begin{aligned} 50e^{t\ln 8/2} &= 50{,}000 \\ e^{t\ln 8/2} &= 1000 \\ t\ln 8/2 &= \ln 1000 \\ t &\approx 6.644 \end{aligned}$$

Thus, the population reaches 50,000 after about $\boxed{6.644 \text{ hours}}$.

Answer 5E.

Consider the table

Year	Population
1750	790
1800	980
1850	1260
1900	1650
1950	2560
2000	6080

In general,

$y(t)$ is the size of the population at a time *t*. and *k* is a probability constant.

Thus,

$$\frac{dy}{dt} = ky$$

And from known theorem,

The only solution of the differential equation $\frac{dy}{dt} = ky$ is the exponential function

$$y(t) = y_0 e^{kt}$$

Since here dealing with population growth, so use the following equation:

$$y(t) = y_0 e^{kt}$$

a)

For this problem, since 1750 is the first year, set the year 1750 as $t = 0$.

So, the year1800 as $t = 50$

Thus,

$y_0 = 790$ From the table

$$y(50) = 980$$

From $y(t) = y_0 e^{kt}$

$$y(50) = y_0 e^{k\times 50}$$

$$980 = 790e^{50k}$$

Solve for *k*.

$$\frac{980}{790} = e^{50k}$$

$$1.2405 = e^{50k}$$

In order to get rid of e and to bring down the exponent, take the natural log of both sides

$$\ln(1.2405) = 50k$$
$$\frac{\ln(1.2405)}{50} = k$$

Thus, the equation:

$$y(t) = 790e^{\frac{\ln(1.2405)}{50}t}$$

Predict the world population in 1900, set $t = 150$:

$$y(150) = 790e^{\frac{\ln(1.2405)}{50}*150}$$
$$\approx \boxed{1508\text{million}}$$

From the table, it is clear that at 1900 the population is 1650 million.

But from the exponential model, the population is 1508 million.

Predict the world population in 1900, set $t = 150$:

$$y(200) = 790e^{\frac{\ln(1.2405)}{50}*200}$$
$$\boxed{\approx 1871\text{million}}$$

From the table, it is clear that at 1950 the population is 2560 million.

But from the exponential model, the population is 1871 million.

In order to get rid of e and to bring down the exponent, take the natural log of both sides

$$\ln(1.5515) = 50k$$
$$\frac{\ln(1.5515)}{50} = k$$

Thus, the equation:

$$y(t) = 1650e^{\frac{\ln(1.5515)}{50}t}$$

Predict the world population in 1000, set $t = 100$:

$$y(t) = 1650e^{\frac{\ln(1.5515)}{50}t}$$
$$y(100) = 1650e^{\frac{\ln(1.5515)}{50}100}$$
$$\approx \boxed{3972\text{million}}$$

From the table, it is clear that at 2000 the population is 6080 million.

But from the exponential model, the population is 2161 million.

There is such a discrepancy because wars in the first half of the century increased life expectancy in the second half of the century.

b)

For this problem, use 1850 is the first year, set the year 1850 as $t=0$.

So, the year1900 as $t=50$

Thus,

$y_0 = 1260$ From the table

$y(50) = 1650$

From $y(t) = y_0 e^{kt}$

$y(50) = y_0 e^{k \times 50}$

$1650 = 1260 e^{50k}$

Solve for *k*.

$\frac{1650}{1260} = e^{50k}$

$1.3095 = e^{50k}$

In order to get rid of e and to bring down the exponent, take the natural log of both sides

$$\ln(1.3095) = 50k$$
$$\frac{\ln(1.3095)}{50} = k$$

Thus, the equation:

$$y(t) = 1260 e^{\frac{\ln(1.3095)}{50}t}$$

Predict the world population in 1950, set $t = 100$:

$$y(t) = 1260 e^{\frac{\ln(1.3095)}{50}t}$$
$$y(100) = 1260 e^{\frac{\ln(1.3095)}{50}100}$$
$$\approx \boxed{2161 \text{million}}$$

From the table, it is clear that at 1950 the population is 2560 million.

But from the exponential model, the population is 2161 million.

c)

For this problem, use 1900 is the first year, set the year 1900 as $t=0$.

So, the year1950 as $t=50$

Thus,

$y_0 = 1650$ From the table

$y(50) = 2560$

From $y(t) = y_0 e^{kt}$

$y(50) = y_0 e^{k \times 50}$

$2560 = 1650 e^{50k}$

Solve for *k*.

$\frac{2560}{1650} = e^{50k}$

$1.5515 = e^{50k}$

Answer 6E.

Consider the table

Year	Population
1951	361
1961	439
1971	548
1981	683
1991	846
2001	1029

In general,

$y(t)$ is the size of the population at a time *t*. and *k* is a probability constant.

Thus,

$$\frac{dy}{dt} = ky$$

And from known theorem,

The only solution of the differential equation $\frac{dy}{dt} = ky$ is the exponential function

$$y(t) = y_0 e^{kt}$$

Since here dealing with population growth, so use the following equation:

$$y(t) = y_0 e^{kt}$$

a)

For this problem, since 1951 is the first year, set the year1951as $t = 0$.

So, the year1961 as $t = 10$

Thus,

$y_0 = 361$ From the table

$$y(10) = 439$$

From $y(t) = y_0 e^{kt}$

$$y(10) = y_0 e^{k \times 10}$$

$$439 = 361 e^{10k}$$

Solve for *k*.

$$\frac{439}{361} = e^{10k}$$

$$1.216 = e^{10k}$$

In order to get rid of e and to bring down the exponent, take the natural log of both sides

$$\ln(1.216) = 10k$$
$$\frac{\ln(1.216)}{10} = k$$
$$k = 0.0196$$

Thus, the equation:

$$y(t) = 361e^{0.0196t}$$

Predict the world population in 2001, set $t = 50$:

$$y(50) = 361e^{0.0196 \cdot 50}$$
$$\approx \boxed{961\text{million}}$$

From the table, it is clear that at 1951 the population is 1029 million.

But from the exponential model, the population is 961 million.

b)

For this problem, use 1961 is the first year, set the year1961as $t = 0$.

So, the year1981 as $t = 20$

Thus,

$y_0 = 439$ From the table

$$y(20) = 683$$

From $y(t) = y_0 e^{kt}$

$$y(20) = y_0 e^{k \times 20}$$
$$683 = 439e^{20k}$$

Solve for *k*.

$$\frac{683}{439} = e^{20k}$$
$$1.556 = e^{20k}$$

In order to get rid of e and to bring down the exponent, take the natural log of both sides

$$\ln(1.556) = 20k$$
$$\frac{\ln(1.556)}{20} = k$$
$$k = 0.0221$$

Thus, the equation:

$$y(t) = 439e^{0.0221t}$$

Predict the world population in 2001, set $t = 40$:

$$y(40) = 439e^{0.0221 \cdot 40}$$
$$\approx \boxed{1062\text{million}}$$

From the table, it is clear that at 1961 the population is 1029 million.

But from the exponential model, the population is 1062 million.

To predict the world population in 2010, set $t = 49$:

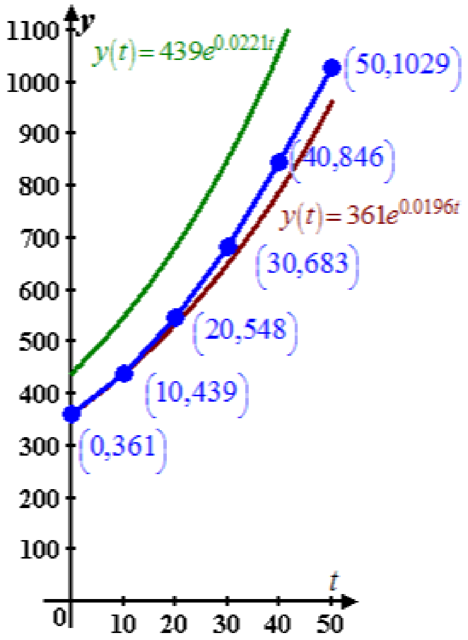
$$y(t) = 439e^{0.0221t}$$
$$y(49) = 439e^{0.0221 \cdot 49}$$
$$\approx \boxed{1296\text{million}}$$

To predict the world population in 2020, set $t = 59$

$$y(t) = 439e^{0.0221t}$$
$$y(59) = 439e^{0.0221 \cdot 59}$$
$$\approx \boxed{1617\text{million}}$$

c)

Plot $y(t) = 361e^{0.0196t}$ and $y(t) = 439e^{0.0221t}$, and table values in the same coordinate axis.



The exponential model $y(t) = 439e^{0.0221t}$ and given data are reasonable.

Answer 7E.

Consider the rate of di-nitrogen pentoxide be proportional to its concentration. Mathematically stated as follows:

$$-\frac{d[N_2O_5]}{dt} = 0.0005[N_2O_5].$$

(a)

Find an expression for the concentration of $[N_2O_5]$ after *t* seconds if the initial concentration is C.

Let, the expression for the concentration of $[N_2O_5]$ after *t* seconds be denoted as follows:

$$P(t) = P(0)e^{kt}.$$

Here $P(0)$, is the initial concentration of $[N_2O_5]$, *k* is the constant, a negative number because the concentration $[N_2O_5]$ reduces over the time , *t* is the time in seconds.

Now ,substitute $P(0) = C, k = -0.0005$ in $P(t) = P(0)e^{kt}$.

Therefore, the expression for the concentration $[N_2O_5]$ after *t* seconds is as follows:

$$\boxed{P(t) = Ce^{-0.0005t}}.$$

(b)

Now, find how long the reaction takes to reduce the concentration of $[N_2O_5]$ to 90% of its original value.

The expression for the concentration $[N_2O_5]$ after *t* seconds is, $P(t) = Ce^{-0.0005t}$.

Now, substitute $P(t) = 90\% \text{ of } C$ in $P(t) = Ce^{-0.0005t}$.

Then, proceed as follows:

$$\begin{aligned} 90\% \text{ of } C &= Ce^{-0.0005t} \\ \frac{90}{100}C &= Ce^{-0.0005t} \\ 0.9C &= Ce^{-0.0005t} \\ e^{-0.0005t} &= \frac{0.9C}{C} \end{aligned}$$

Further simplify as follows:

$$\begin{aligned} e^{-0.0005t} &= 0.9 \\ -0.0005t &= \ln 0.9 \qquad \text{Take natural logarithms on both sides} \\ t &= \frac{\ln 0.9}{-0.0005} \\ &= -2000 \ln 0.9 \\ &\approx 211 \qquad \text{Simplify} \end{aligned}$$

Therefore, the time taken to reduce the concentration $[N_2O_5]$ to 90% of its original value is $\boxed{211\text{s}}$.

Answer 8E.

(a)

The half-life of strontium-90 is 28 days

Let $m(t)$ be the mass of strontium -90(in milligrams) remains after *t* days.

Then

$$\begin{aligned} \frac{dm}{dt} &= km \\ \frac{dm}{m} &= kdt \\ \int \frac{dm}{m} &= \int kdt \\ \ln m &= kt + C \\ m(t) &= Ce^{kt} \quad \cdots\cdots(1) \end{aligned}$$

Since a sample has a mass of 50 mg initially.

That is $m(0) = 50$

Use (1), we obtain that

$$\begin{aligned} m(0) &= Ce^{k(0)} \\ 50 &= C \end{aligned}$$

Therefore, from (1)

$$m(t) = 50e^{kt}$$

In order to determine the value of k we use the fact

When $t = 28, m(t) = \dfrac{m(0)}{2}$

That is

$$m(28) = \frac{1}{2}(50)$$

$$= 25$$

Thus

$$50e^{28k} = 25$$

$$e^{28k} = \frac{1}{2}$$

$$28k = \ln\left(\frac{1}{2}\right)$$

$$28k = -0.693$$

$$k = \frac{-0.693}{28}$$

$$k = -0.0247$$

Thus mass remaining after *t* days

$$\boxed{m(t) = 50e^{-0.0247t}}.$$

(b)

The mass after 40 days is

$$m(40) = 50e^{-0.0247\times 40}$$

$$= 50 \times \frac{1}{e^{0.99}}$$

$$\approx 18.5788$$

Thus, the mass after 40 days is $\boxed{18.5788 \text{ mg}}$.

(c)

Suppose after time *t* the mass is 2 mg.

Then

$$2 = 50e^{-0.0247t}$$

$$\frac{1}{25} = e^{-0.0247t}$$

Taking logarithm on both sides

$$\ln(1) - \ln 25 = -0.0247t$$

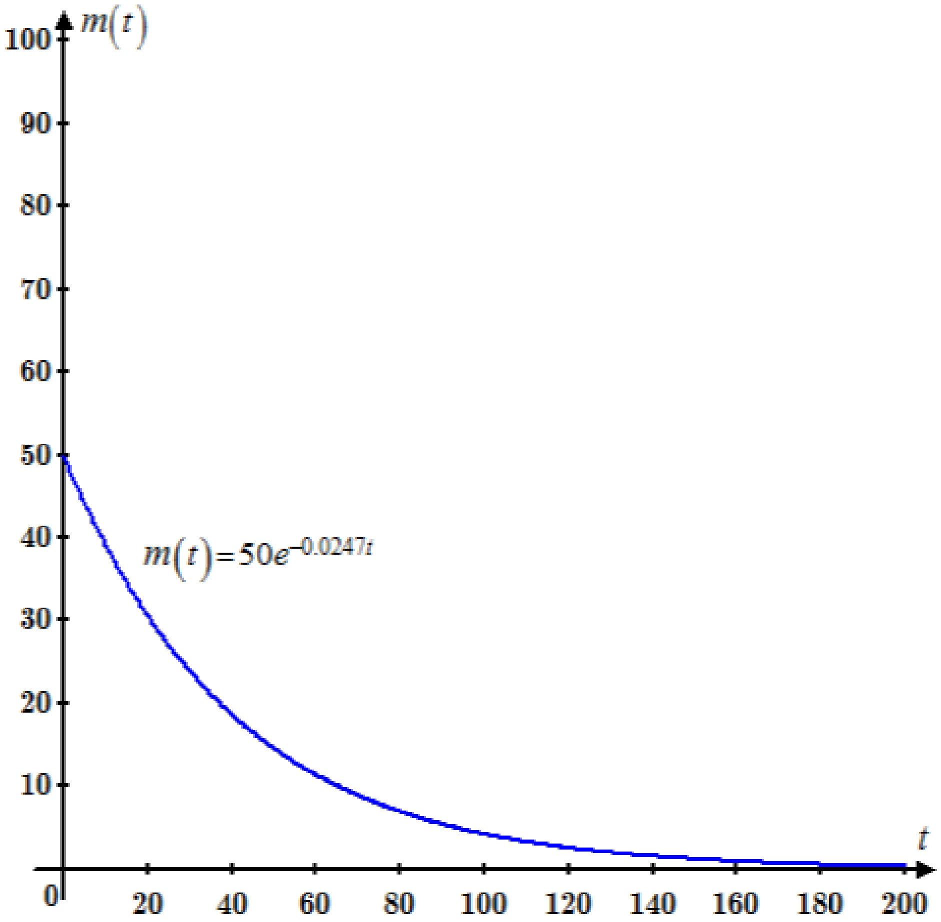
$$0 - 3.218 = -0.0247t$$

$$t = \frac{3.218}{0.0247}$$

$$t = \boxed{130.318 \text{ days}}.$$

(d)

The graph of the mass function is shown below:



Answer 9E.

(a)

Suppose the sample of cesium-137 is 100mg initially.

The half-life of cesium-137 is 30 years

Let $m(t)$ be the mass of cesium-137 that remains after t years.

Then

$$\frac{dm}{dt} = km$$

And $y(0) = 100$

Consider,.

$$\frac{dm}{m} = kdt$$

Integrate both side of the equation,

$$\int \frac{dm}{m} = \int kdt$$

$$\ln m = kt + \ln C$$

$$m(t) = Ce^{kt}$$

At $t=0$, $m(0)=C$

Then $m(t)=m(0)e^{kt}$

Here, the initial mass of cesium-137 is 100mg.

That is, $m(0)=100\,\text{mg}$

Therefore, $m(t)=100e^{kt}$.

Now find *k* value.

The half-life of cesium-137 is 30 years

Then,

$$\begin{aligned} m(30) &= \frac{1}{2}(m(0)) \\ &= \frac{1}{2}(100) \\ &= 50 \end{aligned}$$

Substitute $t=30$ in $m(t)=100e^{kt}$.

Then $m(30)=100e^{30k}$

So,

$$\begin{aligned} 100e^{30k} &= 50 \\ e^{30k} &= \frac{50}{100} \\ e^{30k} &= \frac{1}{2} \\ 30k &= \ln\frac{1}{2} \end{aligned}$$

$$k=\frac{-\ln 2}{30}$$

Substitute k value in $m(t)=100e^{kt}$.

Then,

$$\begin{aligned} m(t) &= 100e^{t\left(\frac{-\ln 2}{30}\right)} \\ &= 100e^{(\ln 2)\left(\frac{-t}{30}\right)} \\ &= 100\times 2^{\frac{-t}{30}} \end{aligned}$$

Thus, the mass of cesium-137 that remains after *t* years is, $\boxed{m(t)=100\times 2^{\left(\frac{-t}{30}\right)}}$.

(b)

The mass of cesium-137 that remains after t years is, $m(t)=100\times 2^{\left(\frac{-t}{30}\right)}$.

Substitute $t=100$ in $m(t)=100\times 2^{\left(\frac{-t}{30}\right)}$.

Then

$$\begin{aligned} m(100) &= 100\times 2^{\left(\frac{-100}{30}\right)} \\ &\approx 9.92 \end{aligned}$$

Therefore, the sample that remains after 100 years is $\boxed{9.92\text{ mg}}$.

(c)

Find the value of t such that $m(t)=1$.

So,

$$100\times e^{-(\ln 2)t/30}=1$$
$$e^{-(\ln 2)t/30}=0.01$$
$$\frac{-(\ln 2)t}{30}=\ln 0.01$$
$$t=\frac{-30\ln 0.01}{\ln 2}$$

$$=199.3$$

Therefore, after $\boxed{199.3 \text{ years}}$, the sample of cesium-137 that remains 1 mg.

Answer 10E.

Let $m(t)$ be the mass of tritum-3 remains after t-years. Then

$$\frac{dm}{dt}=km \qquad \text{......(1)}$$

The above equation can be written as $\frac{dm}{dt}-km=0$

It is in the form of $\frac{dm}{dt}+pm=q$

Where $p=-k$ and $q=0$

Integration factor: $e^{\int pdt}=e^{\int -kdt}$
$=e^{-kt}$

Therefore, $m(t).e^{-kt}=m(0)$

Then the solution of (1) is $m(t)=m(0)e^{kt}$

Given that $m(1)=94.5\%$
$=0.945$

When $m(0)=1$

Then $0.945=1\cdot e^{k}$

$\Rightarrow \quad e^{k}=0.945$

$\Rightarrow \quad k=\ln(0.945)$
$=-0.0566$

Then $m(t)=e^{-0.0566t}$

(a) Let half-life of tritum-3 is t then

$$\frac{m(0)}{2}=m(0)e^{-0.0566t}$$

$$\Rightarrow \frac{1}{2}=e^{-0.0566t}$$

$$\Rightarrow -0.0566t=\ln\left(\frac{1}{2}\right)$$
$$=-0.931$$

$$\Rightarrow \boxed{t\approx 12.2464 \text{ years}}$$

The half-life of triturm-3 is 12.2464 years.

(b) Let t-time (years) required to decay to 20% of its original amount $m(0)$. Then

$$(20\%)m(0) = m(0)e^{-0.0566t}$$

$$\Rightarrow \frac{20}{100} = e^{-0.0566t}$$

$$\Rightarrow \frac{1}{5} = e^{-0.0566t}$$

$$\Rightarrow \ln\left(\frac{1}{5}\right) = -0.0566t$$

$$\Rightarrow -1.6094 = -0.0566t$$
$$\Rightarrow t = 28.4353$$

$$\boxed{t \approx 28.4353 \text{ years}}$$

Answer 11E.

Consider ^{14}C with a half- life about5730 years.

A parchment fragment was discovered that had about 74% as much ^{14}C radioactivity as does plant material on the earth today

Estimate the age of the parchment.

First of all find the expression for the mass of ^{14}C after t years using half-life.

Let, the expression mass of ^{14}C after t years is denoted by

$$m(t) = m(0)e^{kt}.$$

Where $m(0)$, is the initial mass of ^{14}C, k is the decay constant, t is the time in years.

Now substitute $m(t) = \frac{m(0)}{2}, t = 5730$ in $m(t) = m(0)e^{kt}$.

Then,

$$\frac{m(0)}{2} = m(0)e^{k(5730)}$$

$$e^{5730k} = \frac{m(0)}{2m(0)}$$

$$e^{5730k} = \frac{1}{2}$$

$$5730k = \ln\frac{1}{2}$$

$$k = \frac{\ln\frac{1}{2}}{5730}$$
$$\approx -0.00121 \quad \text{Use CAS}$$

Therefore, the expression mass of ^{14}C after t years is denoted by

$$m(t) = m(0)e^{-0.000121t}.$$

Now find the time to remain 74% of ${}^{14}C$ using the expression $m(t)=m(0)e^{-0.000121t}$.

Substitute $m(t)=74\%$ of $\frac{m(0)}{2}$, $k=-0.00121$.

Then,

$$\frac{74}{100}m(0)=m(0)e^{-0.000121t}$$
$$e^{-0.000121t}=\frac{74m(0)}{100m(0)}$$
$$e^{-0.000121t}=0.74$$
$$-0.000121t=\ln(0.74)$$
$$t=\frac{\ln(0.74)}{-0.000121}$$
$$\approx 2500$$

Therefore, the age of the parchment is about $\boxed{2500 \text{ years}}$.

Answer 12E.

Let $y=f(x)$ be the equation of the curve

Then $y'=\frac{dy}{dx}$

$=f'(x)$

i.e., slope $y'=f'(x)$

Given that slope of the curve at any point $p(x,y)$ is twice the $y-$coordinate of P,

i.e., $y'=2y$

$$\Rightarrow \frac{dy}{dx}=2y$$
$$\Rightarrow \frac{dy}{y}=2dx$$

On integration

$\ln y=2x+c$

$\Rightarrow y=e^{2x+c}$ ......(1)

Given that the curve passes through the point $(0,5)$.

Hence $5=e^{2(0)+c}$

$\Rightarrow \quad 5=e^{c}$

$\Rightarrow \quad c=\ln 5$

Plug in c value in (1)

$y=e^{2x+c}$

$y=e^{2x+\ln 5}$

$y=e^{2x}.e^{\ln 5}$

$y=5e^{2x}$

Therefore, the equation of the curve is $\boxed{y=5e^{2x}}$

Answer 13E.

(a)

The temperature of the turkey is $185°F$.

Let $T(t)$ is the temperature of the roast turkey after *t* minutes.

The surrounding temperature is $T_s = 75°F$.

From Newton's Law of Cooling,

$$\frac{dT}{dt} = k(T - 75)$$

Let, $y = T - 75$, then

$$\begin{aligned} y(0) &= T(0) - 75 \\ &= 185 - 75 \\ &= 110 \end{aligned}$$

So, *y* satisfies

$$\frac{dy}{dt} = ky$$
$$y(0) = 110$$

Then,

$$\frac{dy}{y} = kdt$$
$$\int \frac{dy}{y} = \int kdt$$
$$\ln y = kt + \ln C$$
$$y(t) = Ce^{kt}$$

At $t = 0$, $y(0) = C$

Then,

$$\begin{aligned} y(t) &= y(0)e^{kt} \\ &= 110e^{kt} \end{aligned}$$

After half an hour, the temperature of the turkey is $150°F$.

That is, $T(30) = 150°F$

Consider,.

$$\begin{aligned} y(30) &= T(30) - 75 \\ &= 150 - 75 \\ &= 75 \end{aligned}$$

So,

$$110e^{k30} = 75$$
$$e^{k30} = \frac{75}{110}$$
$$e^{k30} = \frac{15}{22}$$
$$30k = \ln\frac{15}{22}$$
$$\begin{aligned} k &= \frac{\ln\left(\frac{15}{22}\right)}{30} \\ &\approx -0.0128 \end{aligned}$$

Therefore, $y(t)=110e^{(-0.0128)t}$

Then,

$$T(t)=75+110e^{(-0.0128)t}$$

After 45 minutes,

$$\begin{aligned}T(45)&=75+110e^{(-0.0128)45}\\&\approx 136.83\end{aligned}$$

Therefore, after 45 minutes, the temperature of the turkey is around $\boxed{137°\text{F}}$.

(b)

Let

$T(t)=100$.

Then,

$$\begin{aligned}-0.128t&=\ln\left(\frac{25}{110}\right)\\t&=\frac{\ln\left(\frac{25}{110}\right)}{-0.0128}\\&\approx 115.75\end{aligned}$$

Therefore, the temperature of the turkey reaches $100°\text{F}$ after 116 minutes.

Answer 14E.

Let $T(t)$ be the temperature of the corpse *t* minutes after the murder. The surrounding temperature is $T_s=20$, so Newton's Law of Cooling states that

$$\frac{dT}{dt}=k(T-20)$$

the above differential equation that governs the temperature *T* of the body at time *t* hours with $T_s=20.0\,^0\text{C}$ being the constant surrounding temperature and *k* a constant that depends on the conditions (numbers) given in the problem.

Let $y(t)=T(t)-T_s$

With this substitution the differential equation takes the form $\frac{dy}{dt}=ky$.

Now for the initial condition,

$$\begin{aligned}y(0)&=T(0)-20\\&=37-20\\&=17\end{aligned}$$

The differential equation has solution $y(t)=y(0)e^{kt}$ hence plug in 17 we have

$$y(t)=17e^{kt}$$

Let us assume that the murder occurred n minutes before 1:30 PM.

Thus, $T(n)=32.5$ and $T(n+60)=30.3$.

Write in terms of *y*, then

$y(n)=32.5-20=12.5$ and $y(n+60)=30.3-20=10.3$.

Use this to write two equations.

$$\begin{aligned}y(n)&=12.5 & y(n+60)&=10.3\\17e^{nk}&=12.5 & 17e^{(n+60)k}&=10.3\\e^{nk}&=\frac{12.5}{17} & e^{nk+60k}&=\frac{10.3}{17}\\nk&=\ln\frac{12.5}{17} & nk+60k&=\ln\frac{10.3}{17}\end{aligned}$$

Solve the equations for k.

$$\begin{aligned} nk+60k &= \ln\frac{10.3}{17} \\ \ln\frac{12.5}{17}+60k &= \ln\frac{10.3}{17} \\ 60k &= \ln\frac{10.3}{17}-\ln\frac{12.5}{17} \\ k &\approx -0.003226 \end{aligned}$$

Use this value to solve the first equation for n.

$$\begin{aligned} nk &= \ln\frac{12.5}{17} \\ -0.003226n &= \ln\frac{12.5}{17} \\ n &\approx 95 \end{aligned}$$

Thus, the murder occurred about 95 minutes before 1:30 PM i.e. at about $\boxed{11:55 \text{ AM}}$.

Answer 15E.

Newton's law of cooling:

Let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then a differential equation

$$\frac{dT}{dt}=k(T-T_s) \quad \text{...... (1)}$$

where k is a constant

So we make the change of variable $y(t)=T(t)-T_s(t)$.

Because T_s is constant, we have $y'(t)=T'(t)$ and so the equation becomes

$$\frac{dy}{dt}=ky \quad \text{...... (2)}$$

We can then use (2) to find an expression for y, from which we can find T.

(a)

Let $T(t)$ be the temperature of the drink after t minutes. The surrounding temperature is $T_s=20°C$, so Newton's Law of Cooling states that

$$\frac{dT}{dt}=k(T-T_s)$$

$$\frac{dT}{dt}=k(T-20)$$

If we let $y=T-20$, then

$$\begin{aligned} y(0) &= T(0)-20 \\ &= 5-20 \\ &= -15 \end{aligned}$$

Thus, y must satisfy the equations $\frac{dy}{dt}=ky$ and $y(0)=-15$.

Recall that the only solutions of the differential equation $\frac{dy}{dt}=ky$ are the exponential functions

$y(t)=y(0)e^{kt}$.

As a result, $y(t)=-15e^{kt}$

We are given that $T(25)=10$, so

$$\begin{aligned} y(25) &= 10-20 \\ &= -10 \end{aligned}$$

Use this to write an equation and solve for *k*.

$$\begin{aligned} -15e^{kt} &= y(t) \\ -15e^{25k} &= y(25) \\ -15e^{25k} &= -10 \\ e^{25k} &= \frac{2}{3} \\ 25k &= \ln\frac{2}{3} \\ k &\approx -0.016219 \end{aligned}$$

Use this to find $T(t)$.

$$\begin{aligned} y(t) &= -15e^{kt} \\ y(t) &= -15e^{-0.016219t} \\ T(t)-20 &= -15e^{-0.016219t} \\ T(t) &= 20-15e^{-0.016219t} \end{aligned}$$

The temperature of the drink after 50 minutes is $T(50)$.

$$\begin{aligned} T(50) &= 20-15e^{-0.016219(50)} \\ &\approx 13.\overline{3} \end{aligned}$$

Thus, the drink is about $\boxed{13.\overline{3}°C}$ after 50 minutes.

(b)

The drink is 15C when $T(t)=15$. Solve this equation for *t*.

$$\begin{aligned} 20-15e^{-0.016219t} &= 15 \\ e^{-0.016219t} &= \frac{1}{3} \\ -0.016219t &= \ln\frac{1}{3} \\ t &\approx 67.74 \end{aligned}$$

Thus, the drink is 15C after about $\boxed{67.74 \text{ minutes}}$.

Answer 16E.

Consider $T(t)$ is the temperature of the coffee after *t* minutes.

The surrounding temperature is $T_s = 20$.

From the Newton's Law of Cooling law, the equation can be written as,

$\frac{dT}{dt} = k(T-T_s)$, where k is a constant.

Substitute the value of T_s in the above equation.

Since the coffee is cooling at a rate of $1^0 C$ per minute when its temperature is $70^0 C$

$$\begin{aligned} -1 &= k(70-20) \\ k &= -0.02 \end{aligned}$$

Let $y = T-20$.

Substitute $t=0$ in $y=T-20$

$$\begin{aligned} y(0) &= T(0)-20 \\ &= 95-20 \\ &= 75 \end{aligned}$$

Here, y must satisfy the equations $\frac{dy}{dt} = ky$ and $y(0) = 75$.

The only solutions of the differential equation $\frac{dy}{dt} = ky$ are the exponential functions

$y(t) = Ce^{kt}$.

Here C is the initial temperature.

That is, $C = y(0) = 75$.

Now substitute the values of C and k in the $y(t) = Ce^{kt}$

$y(t) = 75e^{-0.02t}$

Find the value of $T(t)$.

$$\begin{aligned} y(t) &= 75e^{-0.02t} \\ T(t) - 20 &= 75e^{-0.02t} \\ T(t) &= 20 + 75e^{-0.02t} \end{aligned}$$

Now find the value of t, when $T(t) = 70$.

$$20 + 75e^{-0.02t} = 70$$
$$75e^{-0.02t} = 70 - 20$$
$$75e^{-0.02t} = 50$$
$$e^{-0.02t} = \frac{50}{75}$$
$$e^{-0.02t} = \frac{2}{3}$$
$$-0.02t = \ln\frac{2}{3}$$
$$-0.02t \approx (-0.4)$$
$$t \approx \frac{0.41}{0.02}$$
$$t \approx 20.5$$

Thus, the coffee is $70^0 C$ after about $\boxed{20.5 \text{ minutes}}$.

Answer 17E.

Population growth:

Let $P(t)$ be the size of a population at time t, then

$$\frac{dP}{dt} = kP$$

where k is the proportionality constant

(a)

Since the rate of change of atmospheric pressure P with respect to altitude h is proportional to P, we have

$$\frac{dP}{dh} = kP$$

Recall that the only solutions of the differential equation $\frac{dP}{dh} = kP$ are the exponential functions $P(h) = P(0)e^{kh}$.

Write two equations for $P(0) = 101.3$ and $P(1000) = 87.14$.

From the first equation, $P(0) = 101.3$. Use this to solve the second equation for k.

$$P(1000) = P(0)e^{1000k}$$

$$\begin{aligned} 101.3e^{1000k} &= 87.14 \\ e^{1000k} &= \frac{87.14}{101.3} \\ 1000k &= \ln\frac{87.14}{101.3} \\ &\approx -0.1506 \end{aligned}$$

$$k \approx -0.0001506$$

Thus, the atmospheric pressure at 15°C is $P(h) = 101.3e^{-0.0001506h}$ at h meters above sea level.

The pressure at an altitude of 3000 m is $P(3000)$.

$$\begin{aligned} P(3000) &= 101.3e^{-0.0001506\times 3000} \\ &= 101.3e^{-0.4518} \\ &\approx 64.5 \end{aligned}$$

Therefore, the pressure at an altitude of 3000 m is

$\boxed{64.5 \text{ kPa}}$.

(b)

The pressure at an altitude of 6187 m is $P(6187)$.

$$\begin{aligned} P(6187) &= 101.3e^{-0.0001506\times 6187} \\ &= 101.3e^{-0.9317622} \\ &\approx 39.9 \end{aligned}$$

Therefore, the pressure at the top of Mount McKinley, at an altitude of 6187 m is

$\boxed{39.9 \text{ kPa}}$.

Answer 18E.

(a)

Recollect the formula:

If an amount $A0$ is invested at an annual interest rate r compounded n times per year, then after t years, the investment is worth

$$A(t) = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

(i)

If \$1000 is borrowed at 8% interest compounded annually, then after 3 years,

$$\begin{aligned} A(3) &= 1000\left(1 + \frac{0.08}{1}\right)^{1\cdot 3} \\ &= 1000(1.08)^3 \\ &\approx \boxed{\$1259.71} \end{aligned}$$

Here interest $8\% = \frac{8}{100} = 0.08$

(ii)

If it is compounded quarterly, then $n=4$

$$\begin{aligned} A(3) &= 1000\left(1+\frac{0.08}{4}\right)^{4\cdot 3} \\ &= 1000(1.02)^{12} \\ &\approx \boxed{\$1268.24} \end{aligned}$$

(iii)

If it is compounded monthly, then $n=12$.

$$\begin{aligned} A(3) &= 1000\left(1+\frac{0.08}{12}\right)^{12\cdot 3} \\ &= 1000(1.00\overline{6})^{36} \\ &\approx \boxed{\$1270.24} \end{aligned}$$

(iv)

If it is compounded weekly, then $n=52$.

$$\begin{aligned} A(3) &= 1000\left(1+\frac{0.08}{52}\right)^{52\cdot 3} \\ &= 1000(1+0.00154)^{156} \\ &= 1000(1.27132) \\ &\approx \boxed{\$1271.01} \end{aligned}$$

(v)

If it is compounded daily, then $n=365$.

$$\begin{aligned} A(3) &= 1000\left(1+\frac{0.08}{365}\right)^{365\cdot 3} \\ &= 1000(1.00022)^{1095} \\ &= 1000(1.27236) \\ &\approx \boxed{\$1272.36} \end{aligned}$$

(vi)

If it is compounded hourly, then $n=365\cdot 24$.

$$\begin{aligned} A(3) &= 1000\left(1+\frac{0.08}{365\cdot 24}\right)^{365\cdot 24\cdot 3} \\ &= 1000(1+0.000009)^{26280} \\ &= 1000(1.26683) \\ &\approx \boxed{\$1266.83} \end{aligned}$$

(vii)

With continuous compounding of interest at interest rate *r*, the amount after *t* years is

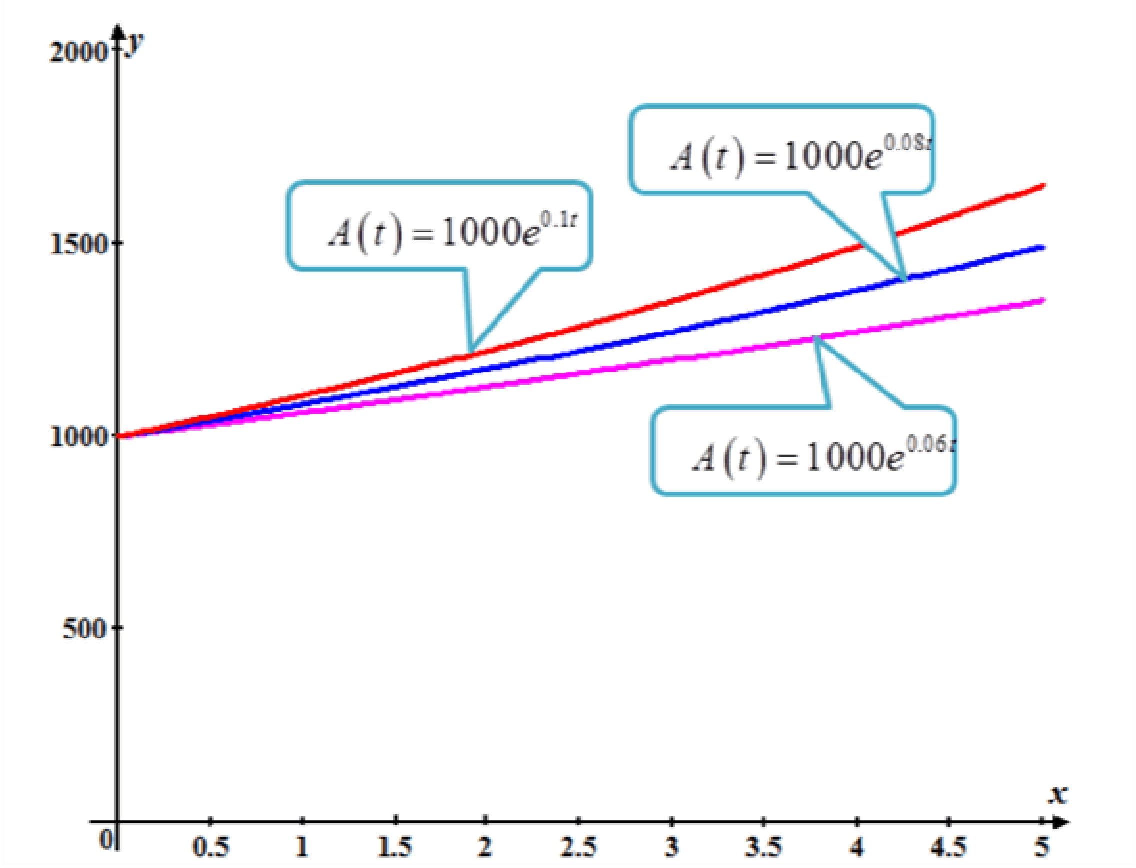
$$A(t)=A_0e^{rt}$$

Use this to find the amount of money after 3 years at 8% interest compounded continuously.

$$\begin{aligned} A(3) &= 1000e^{0.08\cdot 3} \\ &\approx \boxed{\$1271.25} \end{aligned}$$

(b)

Graph the functions $A(t)=1000e^{0.06t}$, $A(t)=1000e^{0.08t}$, and $A(t)=1000e^{0.1t}$ for $0\le t\le 3$.



Answer 19E.

Continuously compound interest:

If an amount *A0* is invested at an annual interest rate *r* compounded *n* times per year, then after *t* years, the investment is worth

$$A(t)=A_0\left(1+\frac{r}{n}\right)^{nt}$$

If $n\to\infty$, then continuous compounding of interest at interest rate *r*, the amount after *t* years is

$$A(t)=A_0e^{rt}$$

Here $A_0=\$3000, r=5\%, t=5\text{ years}$.

(a)

If it is compounded annually, then $n=1$. So,

$$A(t)=A_0\left(1+\frac{r}{n}\right)^{nt}$$

$$\begin{aligned}A(5)&=3000\left(1+\frac{0.05}{1}\right)^{1\times 5}\\&=3000(1+0.05)^5\\&=3000(1.05)^5\\&\approx\boxed{\$3828.84}\end{aligned}$$

(b)

If it is compounded semiannually, then $n = 2$. So,

$$A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}$$

$$\begin{aligned} A(5) &= 3000\left(1+\frac{0.05}{2}\right)^{2\times5} \\ &= 3000(1+0.025)^{10} \\ &= 3000(1.025)^{10} \\ &\approx \boxed{\$3840.25} \end{aligned}$$

(c)

If it is compounded monthly, then $n = 12$. So,

$$A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}$$

$$\begin{aligned} A(5) &= 3000\left(1+\frac{0.05}{12}\right)^{12\times5} \\ &= 3000(1+0.00417)^{60} \\ &= 3000(1.00417)^{60} \\ &\approx \boxed{\$3850.08} \end{aligned}$$

(d)

If it is compounded weekly, then $n = 52$. So,

$$A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}$$

$$\begin{aligned} A(5) &= 3000\left(1+\frac{0.05}{52}\right)^{52\times5} \\ &= 3000(1+0.0009)^{260} \\ &= 3000(1.0009)^{260} \\ &\approx \boxed{\$3851.61} \end{aligned}$$

(e)

If it is compounded daily, then $n = 365$. So,

$$A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}$$

$$\begin{aligned} A(5) &= 3000\left(1+\frac{0.05}{365}\right)^{365\times5} \\ &= 3000(1+0.00013)^{1825} \\ &= 3000(1.00013)^{1825} \\ &\approx \boxed{\$3852.01} \end{aligned}$$

(f)

The amount of money after 5 years at 5% interest compounded continuously is

$$A(t) = A_0e^{rt}$$

$$\begin{aligned} A(5) &= 3000e^{0.05\times5} \\ &= 3000e^{0.25} \\ &\approx \boxed{\$3852.08} \end{aligned}$$

Answer 20E.

(a) Let A_0 be the amount inverted at an interest rate r,
With continues compounding interest, the amount after t years is

$$A(t) = A_0 e^{rt}$$

Given that $r = 0.06$
Suppose the amount A_0 is double after t years, then

$$2A_0 = A_0 e^{rt}$$
$$\Rightarrow 2 = e^{0.06t}$$
$$\Rightarrow \ln 2 = 0.06t$$
$$\Rightarrow t = \frac{\ln 2}{0.06}$$
$$= 11.5525 \text{ years}$$

i.e., After 11.55 yeras the amount will be double with rate of interest 6%

(b) If the interest rate is annual, then the amount A_0 after t years is

$$A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}$$

Where n = number of times in a year
= 1

Hence $A(t) = A_0(1+r)^t$
Suppose after t-years the amount becomes double.
Then $A(t) = 2A_0$ after t-years.

Hence $2A_0 = A_0(1+r)^t$

$$\Rightarrow \quad 2 = (1+r)^{11.55}$$
$$\Rightarrow \quad \ln 2 = (11.55)\ln(1+r)$$
$$\Rightarrow \quad \ln(1+r) = 0.06$$
$$\Rightarrow \quad r = 1.0619 - 1$$
$$\Rightarrow \quad r = 0.0619$$

i.e., $r \approx 0.062$

i.e., $r = 6.2\%$

i.e., The principal becomes double in 11.55 years with the rate of interest 6.2%