Relations and Functions

- **Cartesian product of two sets:** Two non-empty sets P and Q are given. The Cartesian product $P \times Q$ is the set of all ordered pairs of elements from P and Q, i.e.,

$P \times Q = \{(p, q) : p \in P \text{ and } q \in Q\}$

Example: If $P = \{x, y\}$ and $Q = \{-1, 1, 0\}$, then $P \times Q = \{(x, -1), (x, 1), (x, 0), (y, -1), (y, 1), (y, 0)\}$

If either P or Q is a null set, then $P \times Q$ will also be a null set, i.e., $P \times Q = \varphi$.

In general, if A is any set, then $A \times \varphi = \varphi$.

- **Property of Cartesian product of two sets:**
  - If $n(A) = p$, $n(B) = q$, then $n(A \times B) = pq$.
  - If A and B are non-empty sets and either A or B is an infinite set, then so is the case with $A \times B$
  - .$A \times A \times A = \{(a, b, c) : a, b, c \in A\}$. Here, (a, b, c) is called an ordered triplet.
  - $A \times (B \cap C) = (A \times B) \cap (A \times C)$
  - $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Two ordered pairs are equal if and only if the corresponding first elements are equal and the second elements are also equal. In other words, if $(a, b) = (x, y)$, then $a = x$ and $b = y$.

Example: Show that there does not exist $x, y \in R$ if $(x - y + 1, 4x - 2y - 6) = (y - x - 4, 7x - 5y - 2)$.
Solution: It is given that
$(x - y + 1, 4x - 2y - 6) = (y - x - 4, 7x - 5y - 2)$.
$\Rightarrow x - y + 1 = y - x - 4$ and $4x - 2y - 6 = 7x - 5y - 2$
$\Rightarrow 2x - 2y + 5 = 0 \qquad \ldots (1)$

And $-3x + 3y - 4 = 0$ … (2)

Now,

$$\frac{2}{-3} = -\frac{2}{3}, \frac{-2}{3} = -\frac{2}{3} \text{ and } \frac{5}{-4} = -\frac{5}{4}$$

Since $\frac{2}{-3} = \frac{-2}{3} \neq \frac{5}{-4}$, equations (1) and (2) have no solutions. This shows that there does not exist $x, y \in$ R if $(x - y + 1, 4x - 2y - 6) = (y - x - 4, 7x - 5y - 2)$.

In general, for any two sets A and B, A × B ≠ B × A.

- **Relation:** A relation *R* from a set A to a set B is a subset of the Cartesian product A × B, obtained by describing a relationship between the first element *x* and the second element *y* of the ordered pairs (x, y) in A × B.
- The image of an element *x* under a relation *R* is *y*, where $(x, y) \in$ R
- **Domain:** The set of all the first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R.
- **Range and Co-domain:** The set of all the second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the co-domain of the relation *R*. Range ⊆Co-domain

Example: In the relation X from **W** to **R**, given by X = $\{(x, y): y = 2x + 1; x \in W, y \in R\}$, we obtain X = {(0, 1), (1, 3), (2, 5), (3, 7) …}. In this relation X, domain is the set of all whole numbers, i.e., domain = {0, 1, 2, 3 …}; range is the set of all positive odd integers, i.e., range = {1, 3, 5, 7 …}; and the co-domain is the set of all real numbers. In this relation, 1, 3, 5 and 7 are called the images of 0, 1, 2 and 3 respectively.

- The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A × B.

If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and the total number of relations is 2^{pq}.

- A relation R from a set A to a set B is said to be a **function** if for every a in A, there is a unique b in B such that $(a, b) \in R$.
- If R is a function from A to B and $(a, b) \in R$, then b is called the **image** of a under the relation R and a is called the **preimage** of b under R.
- For a function R from set A to set B, set A is the **domain** of the function; the images of the elements in set A or the second elements in the ordered pairs form the **range**,while the whole of set B is the **codomain** of the function.

For example, in relation $f = \{(-1,3)(0,2),(1,3),(2,6),(3,11)\}$ since each element in A has a unique image, therefore f is a function.

Each image in B is 2 more than the square of pre-image.

Hence, the formula for f is $f(x) = x^2 + 2$ Or $f : x \to x^2 + 2$

Domain = $\{-1, 0, 1, 2, 3\}$

Co-domain = $\{2, 3, 6, 11, 13\}$

Range = $\{2, 6, 3, 11\}$

- **Real-valued Function:** A function having either R (real numbers) or one of its subsets as its range is called a real-valued function. Further, if its domain is also either R or *a* subset of R, it is called a real function.

Types of functions:

- **Identity function:** The function f: R → R defined by $y = f(x) = x$, for each $x \in$ R, is called the identity function.

Here, R is the domain and range of f.

- **Constant function:** The function f: R → R defined by $y = f(x) = x$, for each $x \in$ R, where c is a constant, is a constant function.

Here, the domain of f is R and its range is $\{c\}$.

- **Polynomial function:** A function f: R → R is said to be a polynomial function if for each $x \in$ R, $y = f(x) = a^0 + a_1 x + \_\_\_ + a_n x^n a$ where n is a non-negative integer and $a_0, a_1, \ldots.., a_n \in$ R .

- **Rational function:** The functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$, are called rational functions.
- **Modulus function:** The function f: R → R^+ defined by $f(x) = |x|$, for each $x \in$ R, is called the modulus function.

In other words, $f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$

- **Signum function:** The function f: R → R defined by

$$f(x) = \begin{Bmatrix} 1, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -1, \text{if } x < 0 \end{Bmatrix}$$

is called the signum function. Its domain is R and its range is the set {–1, 0, 1}.

- **Greatest Integer function**: The function f: R → R defined by $f(x) = [x]$, $x \in$ R, assuming the value of the greatest integer less than or equal to x, is called the greatest integer function.

Example: [–2.7] = –3, [2.7] = 2, [2] = 2

- **Linear function:** The function f defined by $f(x) = mx + c$, for $x \in$ R, where m and c are constants, is called the linear function. Here, R is the domain and range of f.

- **Addition and Subtraction of functions:** For functions f: X → R and g: X → R, we define

  - Addition of Functions

$(f + g)$: X → R by $(f + g)(x) = f(x) + g(x)$, $x \in$ X

  - Subtraction of Functions

$(f - g)$: X → R by $(f - g)(x) = f(x) - g(x)$, $x \in X$

Example: Let $f(x) = 2x – 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then

$(f + g)(x) = f(x) + g(x)$
$= (2x – 3) + (x^2 + 3x + 2)$
$= x^2 + 5x - 1$
$(f – g) = f(x) – g(x)$

- **Multiplication of real functions:** For functions f: X → R and g: X → R, we define$= (2x – 3) – (x^2 + 3x + 2)$
$= – x^2 – x – 5$

    - Multiplication of two real functions

    (fg): X → R by $(fg)(x) = f(x).\, g(x)$ $x \in X$

    - Multiplication of a function by a scalar

    (af): X → R by $(a f)(x) = af(x)$ $x \in X$ and a is a real number

Example: Let $f(x) = 2x – 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then

$(fg)(x) = f(x) \times g(x)$
$= (2x – 3) \times (x^2 + 3x + 2)$
$= 2x^3 + 3x^2 – 5x – 6$

- **Addition and Subtraction of functions:** For functions f: X → R and g: X → R, we define

$(2f)(x) = 2.f(x)$
$= 2\times(2x – 3)$
$= 4x – 6$

Addition of Functions

$(f + g)$: X → R by $(f + g)(x) = f(x) + g(x)$, $x \in X$

- Subtraction of Functions

$(f - g): X \rightarrow R$ by $(f - g)(x) = f(x) - g(x), \; x \in X$

Example: Let $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then

$$(f + g)(x) = f(x) + g(x)$$

$$= (2x - 3) + (x^2 + 3x + 2)$$

$$= x^2 + 5x - 1$$

$$(f - g) = f(x) - g(x)$$

$$= (2x - 3) - (x^2 + 3x + 2)$$

$$= - x^2 - x - 5$$