Polynomials

- **Polynomial:** A function p(x) of the form $p(x) = a_0 + a_1x + ... + a_nx^n$, where $a_0, a_1, ...a_n$ are real numbers and 'n' is a non-negative (positive) integer is called a polynomial.

- **Note:** $a_0, a_1, ...a_n$ are called the coefficients of the polynomial.
- If the coefficients of a polynomial are integers, then it is called a polynomial over integers.
- If the coefficients of a polynomial are rational numbers, then it is called a polynomial over rational.
- If the coefficients of a polynomial are real numbers, then it is called a polynomial over real numbers.
- A function $p(x) = a_0 + a_1x, ... + a_nx^n$ is not a polynomial if the power of the variable is either negative or a fractional number.
- **Standard form:** A polynomial is said to be in a standard form if it is written either in the ascending or descending powers of the variable.
- **Degree of a polynomial:** If p(x) is a polynomial, then the highest power of x in p(x) is called the degree of the polynomial.
- **Types of polynomials:**

  (a) A polynomial of degree 1 is called a **linear polynomial**.

  (b) A polynomial of degree 2 is called a **quadratic polynomial**.

  (c) A polynomial of degree 3 is called a **cubic polynomial**.

  (d) A polynomial of degree 4 is called a **biquadratic polynomial** or a **quadratic polynomial**.

Polynomial	General form	Coefficients
Linear polynomial	$ax + b$	$a, b \in R, a \neq 0$
Quadratic polynomial	$ax^2 + bx + c$	$a, b, c \in R . a \neq 0$
Cubic polynomial	$ax^3 + bx^2 + cx + d$	$a, b, c, d \in R, a \neq 0$
Quadratic polynomial	$ax^4 + bx^3 + cx^2 + dx + e$	$a, b, c, d, e \in R, a \neq 0$

- Value of a polynomial: If p(x) is a polynomial in x. and if 'a' is any real number, then the value obtained by replacing 'x' by 'a' in p(x), denoted by p(a) is called the value of p(x) at x = a.
- **Zero of a polynomial:** Areal number 'a' for which the value of the polynomial p(x) is zero, is called the zero of the polynomial.

  In other words, a real number 'a' is called a zero of a polynomial p(x) if p(a) =0.

Note: The number zero is known as zero polynomial and its degree is not defined.

- **Geometric meaning of the zero of a polynomial:**

(a) The graph of a linear equation of the form $y = ax + b, a \neq 0$ is a straight line which intersects the X-axis at $\left(\frac{-b}{a}, 0\right)$.

Zero of the polynomial ax + b is the x-coordinate of the point of intersection of the graph with X-axis. Thus, the zero of $y = ax + b$ is $\left(\frac{-b}{a}\right)$

Note: A linear polynomial ax + b, a -f- 0 has exactly one zero, i.e., $\left(\frac{-b}{a}\right)$

(b) The graph of a quadratic equation $y = ax^2 + bx + c, a \neq 0$ is a curve called parabola that either opens upwards like $\cup$ when the coefficient of x^2 is positive or opens downwards like $\cap$ when the coefficient of x^2 is negative.

The zeros of a quadratic polynomial $ax^2 + bx + c$ are the x-coordinates of the points where the parabola represented by $y = ax^2 + bx + c$ intersects the X-axis.

(i) Case 1: if $b^2 - 4ac > 0$ and $a \neq 0$, then the graph of $y = ax^2 + bx + c$ intersects the X-axis at two distinct points. The x-coordinates of the two points are the zeros of the quadratic polynomial $ax^2 + bx + c$.

(ii) Case2: If $\mathrm{b}^2 - 4\mathrm{ac} = 0$, and $a \neq 0$,then the graph of $y = ax^2 + bx + c$ intersects the X-axis at exactly one point (in fact at two coincident points).The x-coordinate of this point is the zero of the quadratic polynomial $ax^2 + bx + c$.

(iii) Case 3: If $\mathrm{b}^2 - 4\mathrm{ac} < 0$, and $a \neq 0$, then the graph of $y = ax^2 + bx + c$ does not intersect the X-axis at any point. The graph is completely above or completely below the X-axis. So, the quadratic polynomial $ax^2 + bx + c$ has no zero.

Thus, a quadratic polynomial can have either two distinct zeros, two equal zeros (i.e., one zero) or no zeros. Hence a polynomial of degree 2 has at most 2 zeros.

Note: For the parabola $ax^2 + bx + c$,

(i) Vertex $= \left(\frac{-b}{2a}, -\frac{D}{4a}\right)$ where $D = b^2 - 4ac$.

(ii) Axis of symmetry, $x = \frac{-b}{2a}$ parallel to Y-axis.

(iii) Zeros are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$.

(c) The graph of a cubic polynomial intersects the X-axis at three points, whose x- coordinates are the zeros of the cubic polynomial.

(d) In general, the graph of a polynomial y = p(x) passes through at most 'n' points on the X-axis. Thus, a polynomial p(x) of degree 'n' has at most 'n' zeros.

- **Relationship between zeros and coefficients of a polynomial:**

Type of Polynomial	**General Form**	**Number of Zeros**	**Relationship between zeros and coefficients**	
			Sum of zeros	Product of zeros
Linear polynomial	$ax+b,\ a\neq 0$	1	$k=\frac{-b}{a}$ i.e., $k=\frac{-(constant\ term)}{(coefficient\ of\ x)}$	
Quadratic polynomial	$ax^2+bx+c \quad a\neq 0$	2	$\frac{-(coefficient\ of\ x)}{(cofficient\ of\ x^2)}=\frac{-b}{a}$	$\frac{constant\ term}{coefficient\ of\ x^2}=\frac{c}{a}$
Cubic polynomial	$ax^3+bx^2+cx+d,\ a\neq 0$	3	$\frac{-(coefficient\ of\ x^2)}{(coefficient\ of\ x^2)}=\frac{-b}{a}$	$\frac{constant\ term}{coefficient\ of\ x^2}=\frac{d}{a}$
			Sum of the product of roots taken at a time $\frac{coefficient\ of\ x}{coefficient\ of\ x^2}=\frac{c}{a}$	

- **To form a quadratic polynomial with the given zeros:** If α and β are the zeros of a quadratic polynomial, then the quadratic polynomial is obtained by expanding $(x-\alpha)(x-\beta)$.

  i.e., x^2 - (sum of the zeros) x + product of zeros.
- **To form a cubic polynomial with the given zeros:** If a, p and y are the zeros of a polynomial, then the cubic polynomial is obtained by expanding $(x-\alpha)(x-\beta)(x-\gamma)$
- **Division algorithm of polynomials:** If p(x) and g(x) are any two polynomials with $g(x)\neq 0$, then we can find polynomials q(x) and r(x) such that $p(x)=g(x)\times q(x)+r(x)$, where $r(x)=0$ or degree $r(x)<$ degree of $g(x)$.
- If $(x-a)$ is a factor of polynomial p(x) of degree n > 0, then 'a' is the zero of the polynomial.