Exercise 14.7

Answer 1E.

(A)

$$f_{xx}(1,1)=4,\ f_{xy}(1,1)=1,\ f_{yy}(1,1,)=2$$

Then $D=f_{xx}(1,1)f_{yy}(1,1)-[f_{xy}(1,1)]^2$

$=(4)(2)-(1)^2$

$=8-1=7>0$

Now $D>0$ and $f_{xx}(1,1)=4>0$

Then f(1,1) is a local minimum

(B)

$$f_{xx}(1,1)=4,\ f_{xy}(1,1)=3,\ f_{yy}(1,1)=2$$

Then $D=f_{xx}(1,1)f_{yy}(1,1)-[f_{xy}(1,1)]^2$

$=4(2)-9$

$=8-9$

$=-1<0$

Now $D<0$, then (1, 1) is a saddle point of f

Answer 2E.

(A) $g_{xx}(0,2)=-1,\ g_{xy}(0,2)=6,\ g_{yy}(0,2)=1$

Then $D=g_{xx}(0,2)g_{yy}(0,2)-[g_{xy}(0,2)]^2$

$=(-1)(1)-(6)^2$

$=-1-36$

$=-37<0$

Now D < 0, then (0, 2) is a saddle point of g

(B) $g_{xx}(0,2)=-1,\ g_{xy}(0,2)=2,\ g_{yy}(0,2)=-8$

Then $D = g_{xx}(0,2)\, g_{yy}(0,2) - \left[g_{xy}(0,2)\right]^2$

$= (-1)(-8) - (2)^2$

$= 8 - 4$

$= 4 > 0$

Now D > 0 and $g_{xx}(0,2) < 0$, then $g(0,2)$ is a local maximum

(C) $g_{xx}(0,2) = 4$, $g_{xy}(0,2) = 6$, $g_{yy}(0,0) = 9$

Then $D = g_{xx}(0,2)\, g_{yy}(0,2) - \left[g_{xy}(0,2)\right]^2$

$= (4)(9) - (6)^2$

$= 36 - 36$

Now D = 0, then the second derivative test fails here and gives no information about g(0, 2)

Answer 3E.

$$f(x,y) = 4 + x^3 + y^3 - 3xy$$

First we locate the critical points

$f_x = 3x^2 - 3y$

$f_y = 3y^2 - 3x$

Setting these partial derivatives equal to zero, we obtain

$3x^2 - 3y = 0$

And $3y^2 - 3x = 0$

Or $x^2 - y = 0$ and $y^2 - x = 0$

To solve these equations

$y^4 - y = 0$

$y(y^3 - 1) = 0$

i.e. $y = 0, 1$

Then $x = 0, 1$

Thus the critical points are (0, 0) and (1, 1)

Now the contour map of the function f is given
The level curves near (1, 1) are in oval shape and indicate that as we move away from (1, 1) in any direction the values of f are increasing thus f has a minima at $(1,1)$
Also the level curves near (0, 0) resemble hyperbola
They reveal that we move away from origin, the values of f decrease in some directions but increase in other directions, thus f has a saddle point at (0, 0)

Now $f_{xx} = 6x$, $f_{yy} = 6y$

$f_{xy} = -3$

Then $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2$

$= 36xy - 9$

At (1, 1):

$$D(1,1) = 36 - 9$$
$$= 27 > 0$$

Since D > 0 and $f_{xx} = 6 > 0$ f has a local minima at (1, 1)

At (0, 0):

$$D(0,0) = 36(0) - 9$$
$$= -9 < 0$$

Since D < 0, f has a saddle point at (0, 0)

Answer 4E.

$$f(x,y) = 3x - x^3 - 2y^2 + y^4$$

First we locate the critical points

$$f_x = 3 - 3x^2$$
$$f_y = -4y + 4y^3$$

Setting these partial derivatives equal to zero, we obtain

$3 - 3x^2 = 0$

Or $3(1 - x^2) = 0$

And $-4y + 4y^3 = 0$

Or $-4y(1 - y^2) = 0$

On solving these equations we obtain

$x = \pm 1$, $y = 0, \pm 1$

Then the critical points are: $(1,-1)$, $(-1,1)$, $(1,1)$, $(1,0)$, $(-1,0)$ and $(-1,-1)$

Now the contour map of the function f is given
The level curves near (1, -1), (1, 1) and (-1, 0) resemble hyperbola thus f has saddle points at (-1, 0), (1, -1) and (1, 1)
Also the level curves near (-1, 1), (-1, -1) and (1, 0) are in oval shape. As we move away from (-1 1) and (-1, -1) in any direction, the values of f are increasing thus f has a minima at (-1, 1) and (-1, -1)
But as we move away from (1, 0) in any direction, the values of f are decreasing, thus f has a maxima at (1, 0)

Now $f_{xx} = -6x$

$f_{yy} = -4 + 12y^2$

$f_{xy} = 0$

Now $D = f_{xx} f_{yy} - f_{xy}^2$

$= 24x(1 - 3y^2)$

At (1, -1): $D = 24(-2)$

$= -48 < 0$

That is (1, -1) is a saddle point

At (-1, 1); $D = -24(-2)$

$= 48 > 0$

And $f_{xx}(-1,1) = 6 > 0$

That is (-1, 1) is a local minima

At (1, 1); $D = 24(1-3)$

$= 24(-2)$

$= -48 < 0$

That is (1, 1) is a saddle point

At (1, 0); $D = 24 > 0$

And $f_{xx}(1,0) = -6 < 0$

That is (1, 0) is a local maxima

At (-1, 0); $D = -24 < 0$

That is (-1, 0) is a saddle point

At (-1, -1); $D = -24(-2)$

$= 48 > 0$

And $f_{xx}(-1, -1) = 6 > 0$

That is (-1, -1) is a local minima

Hence f has:

Local minima at (-1, 1), (-1, -1)

Local maxima at (1, 0)

Saddle point at (1, -1), (1, 1), (-1, 0)

Answer 5E.

We have $f(x, y) = x^2 + xy + y^2 + y$.

Find $f_x(x, y), f_y(x, y), f_{xx}(x, y), f_{yy}(x, y)$, and $f_{xy}(x, y)$.

$$f_x(x, y) = 2x + y$$

$$f_{xx}(x, y) = 2$$

$$f_y(x, y) = 2y + x + 1$$

$$f_{yy}(x, y) = 2$$

$$f_{xy}(x, y) = 1$$

Equate $f_x(x, y)$ to 0 and $f_y(x, y)$ to 0.

$$2x + y = 0$$

$$y = -2x$$

$$2y + x + 1 = 0$$

Replace y with $-2x$ in $2y + x + 1 = 0$ and solve for x.

$$2(-2x) + x + 1 = 0$$

$$-3x + 1 = 0$$

$$x = \frac{1}{3}$$

Substitute $\frac{1}{3}$ for x in $y=-2x$.

$$y = -2\left(\frac{1}{3}\right)$$

$$y = -\frac{2}{3}$$

Now, evaluate D given by $f_{xx}(a,b)f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$.

$$D = (2)(2)-[1]^2$$
$$= 3$$

Since $D>0, f_{xx}(x,y)>0$, the given function has a relative minimum at $\left(\frac{1}{3},-\frac{2}{3}\right)$.

Replace x with $\frac{1}{3}$ and y with $-\frac{2}{3}$ and find the value of $f\left(\frac{1}{3},-\frac{2}{3}\right)$.

$$f\left(\frac{1}{3},-\frac{2}{3}\right) = \left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)+\left(-\frac{2}{3}\right)^2+\left(-\frac{2}{3}\right)$$
$$= \frac{1}{9}-\frac{2}{9}+\frac{4}{9}-\frac{2}{3}$$
$$= -\frac{1}{3}$$

Thus, the given function has a relative minimum $\boxed{f\left(\frac{1}{3},-\frac{2}{3}\right) = -\frac{1}{3}}$.

Answer 6E.

By solving $f_x=0, f_y=0$,

$y-2x-2=0$ ...... (1)

$x-2y-2=0$ ...... (2)

Multiply (2) by 2, and add to (1),

$$-3y-6=0$$
$$y=-2$$

By (2),

$$x-2(-2)-2=0$$
$$x=-2$$

Therefore the critical point of f is $(-2,-2)$

Next calculate the second partial derivatives and $D(x,y)$:

$$\begin{aligned} f_{xx} &= \frac{\partial f_x}{\partial x} \\ &= \frac{\partial}{\partial x}(y-2x-2) \\ &= 0-2-0 \\ &= -2 \end{aligned}$$

$$\begin{aligned} f_{xy} &= \frac{\partial f_x}{\partial y} \\ &= \frac{\partial}{\partial y}(y-2x-2) \\ &= 1 \end{aligned}$$

And,

$$\begin{aligned} f_{yy} &= \frac{\partial f_y}{\partial y} \\ &= \frac{\partial}{\partial y}(x-2y-2) \\ &= 0-2-0 \\ &= -2 \end{aligned}$$

Also,

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy} - f_{xy}^2 \\ &= (-2)(-2)-(1)^2 \\ &= 3 \end{aligned}$$

Then,

$$D(-2,-2) = 3 > 0$$

Now,

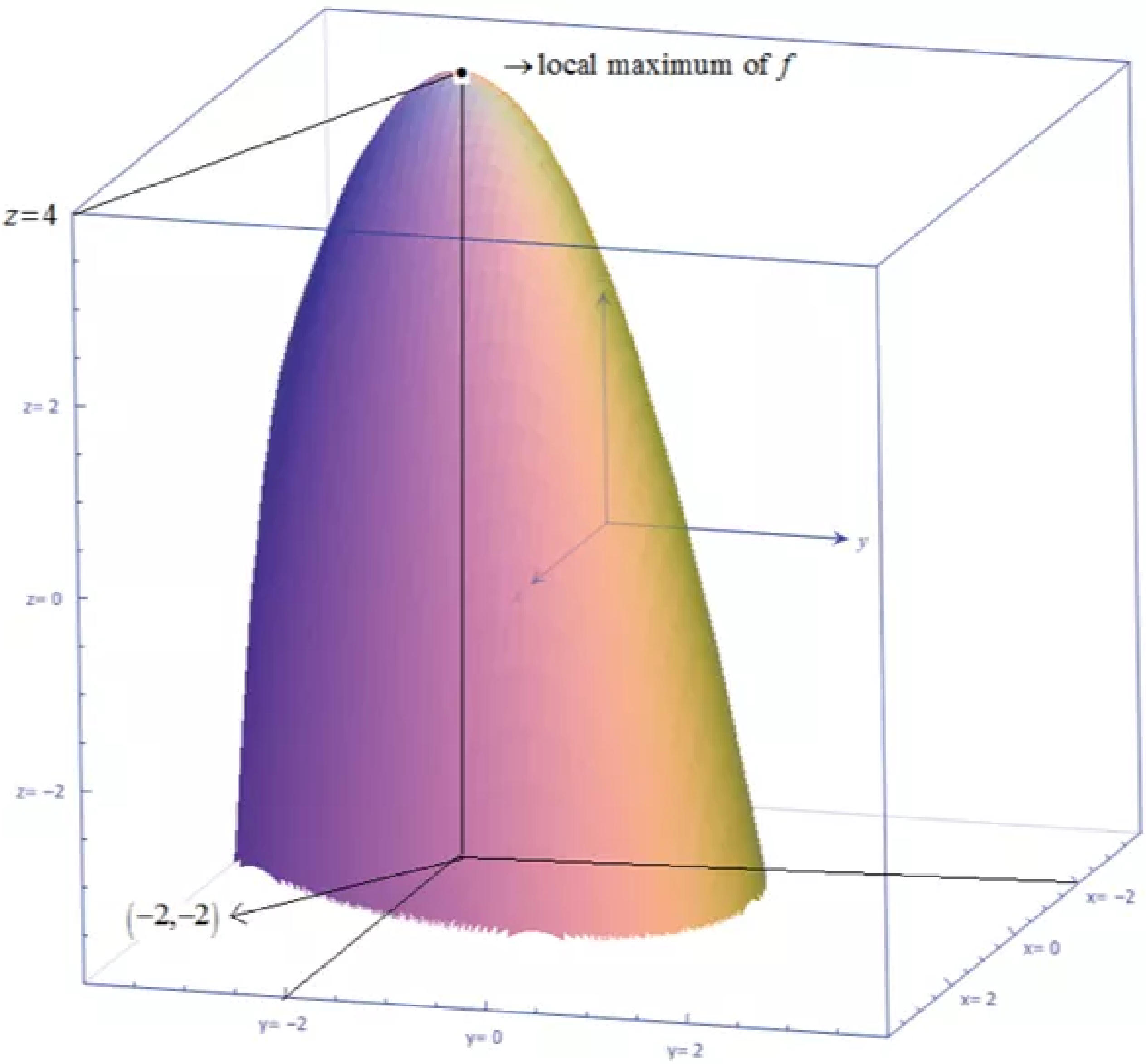
$$f_{xx}(-2,-2) = -2 < 0$$

So, by the Second Derivative Test, the local maximum of f is,

$$\begin{aligned} f(-2,-2) &= (-2)(-2)-2(-2)-2(-2)-(-2)^2-(-2)^2 \\ &= 4+4+4-4-4 \\ &= \boxed{4} \end{aligned}$$

As $D \not< 0$, from the Second Derivative Test, there are no saddle points of f.

The graph of the given function is shown below:



Answer 7E.

Consider the following function:

$$f(x,y)=(x-y)(1-xy)$$

Rewrite the given function as,

$$\begin{aligned} f(x,y) &= (x-y)(1-xy) \\ &= x-x^2y-y+xy^2 \end{aligned}$$

Find the local maximum and minimum values and saddle points of the given function.

Recall that the second derivative test as,

Suppose the second partial derivatives of f are continuous on a disk with center (a,b).

suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$.

Here (a,b) is a critical point of f.

Let $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2$.

If $D>0$ and $f_{xx}(a,b)>0$ then $f(a,b)$ is a local minimum.

If $D>0$ and $f_{xx}(a,b)<0$ then $f(a,b)$ is a local maximum.

If $D<0$ then $f(a,b)$ is not a local maximum or minimum.

First partial derivative of $f(x,y)$ with respect to x.

$$\begin{aligned}
f_x(x,y) &= \frac{\partial}{\partial x}f(x,y) \\
&= \frac{\partial}{\partial x}\left(x-x^2y-y+xy^2\right) \\
&= 1-(2x)y-0+(1)y^2 \\
&= 1-2xy+y^2
\end{aligned}$$

Equate the expression $f_x(x,y)$ to zero.

$$\begin{aligned}
f_x(x,y) &= 0 \\
1-2xy+y^2 &= 0
\end{aligned}$$

First partial derivative of $f(x,y)$ with respect to y.

$$\begin{aligned}
f_y(x,y) &= \frac{\partial}{\partial y}f(x,y) \\
&= \frac{\partial}{\partial y}\left(x-x^2y-y+xy^2\right) \\
&= 0-x^2(1)-1+x(2y) \\
&= -x^2-1+2xy
\end{aligned}$$

Equate the expression $f_y(x,y)$ to zero.

$$\begin{aligned}
f_y(x,y) &= 0 \\
-x^2-1+2xy &= 0
\end{aligned}$$

Solve the two equations $1-2xy-y^2=0$ and $-x^2-1+2xy=0$.

Add two equations as,

$$\begin{array}{r} 1-2xy+y^2=0 \\ \underline{-1+2xy-x^2=0} \\ y^2-x^2=0 \\ y^2=x^2 \end{array}$$

So, $y=\pm x.$

Substitute x for y in $1-2xy+y^2=0.$

$$\begin{aligned} 1-2x(x)+(x)^2&=0 \\ 1-2x^2+x^2&=0 \\ x^2&=1 \\ x&=\pm 1 \end{aligned}$$

So, $y=\pm 1.$

Thus, the points are $(1,1),(-1,-1).$

Substitute $-x$ for y in $1-2xy+y^2=0.$

$$\begin{aligned} 1-2x(-x)+(-x)^2&=0 \\ 1+2x^2+x^2&=0 \\ 3x^2&=-1 \end{aligned}$$

So, x have no real roots.

Thus, the critical points are $(1,1)\text{and}(-1,-1).$

Second partial derivative of $f(x,y)$ with respect to $x.$

$$\begin{aligned} f_{xx}(x,y)&=\frac{\partial}{\partial x}f_x(x,y) \\ &=\frac{\partial}{\partial x}\left(1-2xy+y^2\right) \\ &=0-2(1)y+0 \\ &=-2y \end{aligned}$$

Second partial derivative of $f(x,y)$ with respect to y.

$$\begin{aligned} f_{yy}(x,y) &= \frac{\partial}{\partial y} f_y(x,y) \\ &= \frac{\partial}{\partial y}\left(-x^2-1+2xy\right) \\ &= 0-0+2x(1) \\ &= 2x \end{aligned}$$

Partial derivative of $f_x(x,y)$ with respect to y.

$$\begin{aligned} f_{xy}(x,y) &= \frac{\partial}{\partial y} f_x(x,y) \\ &= \frac{\partial}{\partial y}\left(1-2xy+y^2\right) \\ &= 0-2x(1)+2y \\ &= -2x+2y \end{aligned}$$

The value of D is,

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy}-\left(f_{xy}\right)^2 \\ &= (-2y)(2x)-(-2x+2y)^2 \\ &= -4xy-\left(4x^2+4y^2-8xy\right) \\ &= -4x^2-4y^2+4xy \end{aligned}$$

At the critical point $(1,1)$.

$$\begin{aligned} D(1,1) &= -4(1)^2-4(1)^2+4(1)(1) \\ &= -4-4+4 \\ &= -8+4 \\ &= -4 \end{aligned}$$

At the critical point $(-1,-1)$.

$$D(-1,-1) = -4(-1)^2 - 4(-1)^2 + 4(-1)(-1)$$
$$= -4 - 4 + 4$$
$$= -8 + 4$$
$$= -4$$

$$f_{xx}(x,y) = -2y$$
$$f_{xx}(0,1) = -2$$

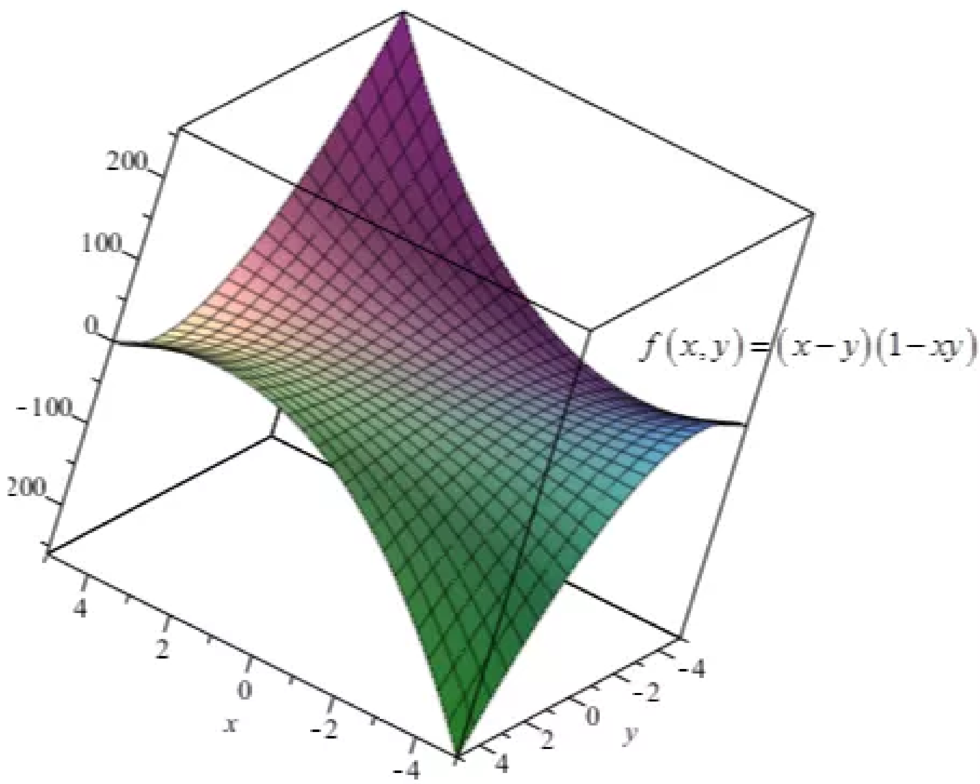
By the second derivative test

Since $D < 0$, the given function has no maximum and minimum at $(1,1)$ and $(-1,-1)$.

Therefore, the given function has $\boxed{\text{saddle points at } (1,1) \text{ and } (-1,-1)}$.

The graph of the function $f(x,y) = (x-y)(1-xy)$ is as shown below:

Use maple software.



Answer 8E.

We have $f(x, y) = xe^{-2x^2-2y^2}$

$$f_x(x, y) = e^{-2x^2-2y^2} - 4x^2e^{-2x^2-2y^2}$$
$$= e^{-2x^2-2y^2}\left(1-4x^2\right)$$
$$f_{xx}(x, y) = -12xe^{-2x^2-2y^2} + 16x^3e^{-2x^2-2y^2}$$
$$= 4xe^{-2x^2-2y^2}\left(-3+4x^2\right)$$

$$f_y(x, y) = -4xye^{-2x^2-2y^2}$$
$$f_{yy}(x, y) = -4xe^{-2x^2-2y^2} + 16xy^2e^{-2x^2-2y^2}$$
$$= 4xe^{-2x^2-2y^2}\left(-1+4y^2e^{-2x^2-2y^2}\right)$$
$$f_{xy}(x, y) = -4ye^{-2x^2-2y^2} + 16x^2ye^{-2x^2-2y^2}$$
$$= 4ye^{-2x^2-2y^2}\left(-1+4x^2\right)$$

We equate $f_x(x, y)$ to 0 and $f_y(x, y)$ to 0 to get
$e^{-2x^2-2y^2}\left(1-4x^2\right) = 0$ and $-4xye^{-2x^2-2y^2} = 0$.

On solving both the equations, we get $\left(\frac{1}{2}, 0\right)$ and $\left(-\frac{1}{2}, 0\right)$.

Find $f_{xx}\left(\frac{1}{2}, 0\right)$ by replacing x with $\frac{1}{2}$ and y with 0 in

$$f_{xx}(x, y) = 4xe^{-2x^2-2y^2}\left(-3+4x^2\right).$$
$$f_{xx}\left(\frac{1}{2}, 0\right) = 4\left(\frac{1}{2}\right)e^{-2\left(\frac{1}{2}\right)^2-2(0)^2}\left[-3+4\left(\frac{1}{2}\right)^2\right]$$
$$= -4e^{-\frac{1}{2}}$$

Similarly, we get

$$f_{xx}\left(-\frac{1}{2}, 0\right) = 4e^{-\frac{1}{2}},$$
$$f_{yy}\left(\frac{1}{2}, 0\right) = -2e^{-\frac{1}{2}},$$
$$f_{yy}\left(-\frac{1}{2}, 0\right) = 2e^{-\frac{1}{2}}$$
$$f_{xy}\left(\frac{1}{2}, 0\right) = 0, \text{ and } f_{xy}\left(-\frac{1}{2}, 0\right) = 0.$$

Now, evaluate $D\left(\frac{1}{2}, 0\right)$ and $D\left(-\frac{1}{2}, 0\right)$ given by

$D = f_{xx}(a, b) f_{yy}(a, b) - \left[f_{xy}(a, b)\right]^2$.

$$\begin{aligned} D\left(\frac{1}{2}, 0\right) &= f_{xx}\left(\frac{1}{2}, 0\right) f_{yy}\left(\frac{1}{2}, 0\right) - \left[f_{xy}\left(\frac{1}{2}, 0\right)\right]^2 \\ &= \left(-4e^{-\frac{1}{2}}\right)\left(-2e^{-\frac{1}{2}}\right) - [0]^2 \\ &= 8e^{-1} \end{aligned}$$

$$\begin{aligned} D\left(-\frac{1}{2}, 0\right) &= f_{xx}\left(-\frac{1}{2}, 0\right) f_{yy}\left(-\frac{1}{2}, 0\right) - \left[f_{xy}\left(-\frac{1}{2}, 0\right)\right]^2 \\ &= \left(4e^{-\frac{1}{2}}\right)\left(2e^{-\frac{1}{2}}\right) - [0]^2 \\ &= 8e^{-1} \end{aligned}$$

We note that $D\left(\frac{1}{2}, 0\right) = 8e^{-1} > 0$ and $f_{xx}\left(\frac{1}{2}, 0\right) = -4e^{-\frac{1}{2}} < 0$. This means that $f\left(\frac{1}{2}, 0\right)$ is a local maximum.

Replace x with $\frac{1}{2}$ and y with 0 in $f(x, y) = xe^{-2x^2-2y^2}$.

$$\begin{aligned} f\left(\frac{1}{2}, 0\right) &= \left(\frac{1}{2}\right) e^{-2\left(\frac{1}{2}\right)^2 - 2(0)^2} \\ &= \frac{1}{2} e^{-\frac{1}{2}} \end{aligned}$$

Thus, the given function has a relative maximum $\boxed{f\left(\frac{1}{2}, 0\right) = \frac{1}{2} e^{-\frac{1}{2}}}$.

Since $D\left(-\frac{1}{2}, 0\right) = 8e^{-1} > 0$ and $f_{xx}\left(-\frac{1}{2}, 0\right) = 4e^{-\frac{1}{2}} > 0$, we can say that $f\left(-\frac{1}{2}, 0\right)$ is a local minimum.

Substitute $\frac{1}{2}$ for x and 0 for y in $f(x, y) = xe^{-2x^2-2y^2}$.

$$f\left(-\frac{1}{2}, 0\right) = \left(-\frac{1}{2}\right)e^{-2\left(-\frac{1}{2}\right)^2-2(0)^2}$$
$$= -\frac{1}{2}e^{-\frac{1}{2}}$$

Thus, the given function has a relative minimum $\boxed{f\left(-\frac{1}{2}, 0\right) = -\frac{1}{2}e^{-\frac{1}{2}}}$.

Answer 9E.

Consider,

$$f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$$

To find the local maximum, local minimum values, and saddle points of this function by using the Second Derivative Test as follows.

Suppose the second partial derivatives of the function f are continuous on a disk with center (a,b), and suppose that $f_x(a,b) = 0, f_y(a,b) = 0$

That is, the point (a,b) is a critical point of f.

Let $D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$

1. If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.

2. If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.

3. If $D < 0$ then $f(a,b)$ is not a local maximum or local minimum.

In case (c), the point (a,b) is called a saddle point of f.

To find the critical points:

$$f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$$

$$f_x(x, y) = \frac{\partial f}{\partial x} = 6xy - 12x \qquad f_y(x, y) = \frac{\partial f}{\partial y} = 3y^2 + 3x^2 - 12y \qquad f_{xy}(x, y) = \frac{\partial f_x}{\partial y} = 6x$$

$$f_{xx}(x, y) = \frac{\partial f_x}{\partial x} = 6y - 12 \qquad f_{yy}(x, y) = \frac{\partial f_y}{\partial y} = 6y - 12$$

Solve the equation, $f_x = 0$

$$6xy - 12x = 0$$
$$x(6y-12) = 0$$
$$x = 0,\ 6y - 12 = 0$$
$$x = 0,\ y = 2$$

Substitute $y = 2$ in the equation, $f_y = 0$

$$3y^2 + 3x^2 - 12y = 0$$
$$3(2)^2 + 3x^2 - 12(2) = 0$$
$$12 + 3x^2 - 24 = 0$$
$$3x^2 = 12$$
$$x^2 = 4$$
$$x = \pm 2$$

Substitute $x = 0$ in the equation, $f_y = 0$

$$3y^2 + 3(0)^2 - 12y = 0$$
$$y(3y-12) = 0$$
$$y(3y-12) = 0$$
$$y = 0, y = 4$$

Thus, the critical points of f are $(2,2), (-2,2), (0,0), \text{and} (0,4)$.

Now,

$$D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

At the critical point $(2,2)$,

$$D(2,2) = f_{xx}(2,2) f_{yy}(2,2) - \left[f_{xy}(2,2)\right]^2$$
$$= (6(2)-12)(6(2)-12) - \left[6(2)\right]^2$$
$$= (0)(0) - [12]^2$$
$$= -144 < 0$$

As $D < 0$, there is no local maximum or local minimum of f at $(2,2)$

At the critical point $(-2,2)$,

$$D(-2,2) = f_{xx}(-2,2) f_{yy}(-2,2) - \left[f_{xy}(-2,2)\right]^2$$
$$= (6(2)-12)(6(2)-12) - \left[6(-2)\right]^2$$
$$= (0)(0) - [-12]^2$$
$$= -144 < 0$$

As $D < 0$, there is no local maximum or local minimum of f at $(-2,2)$

At the critical point $(0,0)$,

$$D(0,0) = f_{xx}(0,0) f_{yy}(0,0) - \left[f_{xy}(0,0)\right]^2$$
$$= (6(0)-12)(6(0)-12) - \left[6(0)\right]^2$$
$$= (-12)(-12) - [0]^2$$
$$= 144 > 0$$

So, $D > 0$

And,

$$f_{xx}(0,0) = 6(0) - 12$$
$$= -12 < 0$$

Therefore, by the Second Derivative Test, the local maximum of f is,

$$f(0,0) = 0^3 + 3(0)^2(0) - 6(0)^2 - 6(0)^2 + 2$$
$$= \boxed{2}$$

At the critical point $(0,4)$,

$$D(0,4) = f_{xx}(0,4) f_{yy}(0,4) - [f_{xy}(0,4)]^2$$
$$= (6(4)-12)(6(4)-12) - [6(0)]^2$$
$$= (12)(12) - [0]^2$$
$$= 144 > 0$$

So, $D > 0$

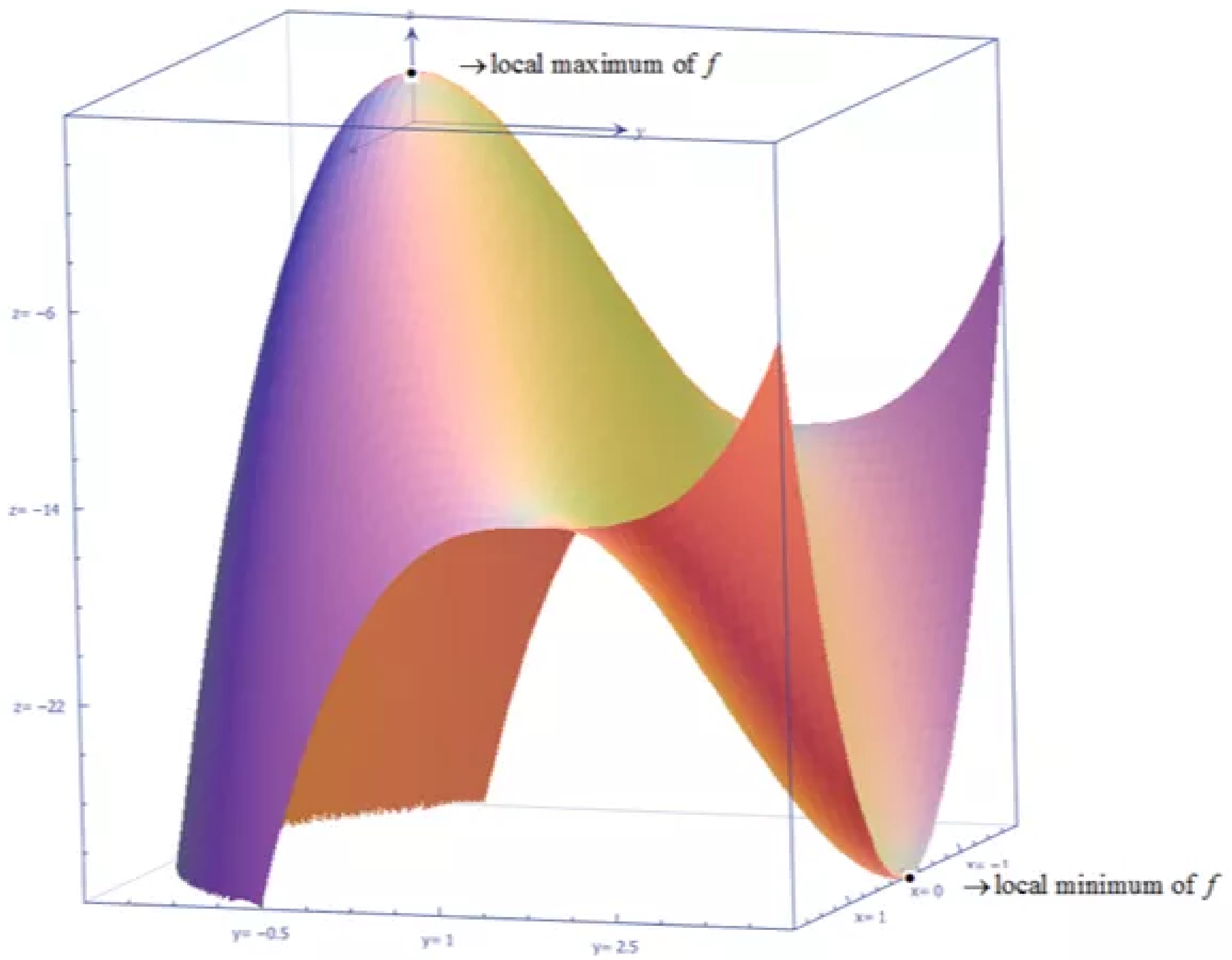
And,

$$f_{xx}(0,4) = 6(4) - 12$$
$$= 12 > 0$$

Therefore, by the Second Derivative Test, the local minimum of f is,

$$f(0,4) = 4^3 + 3(0)^2(4) - 6(0)^2 - 6(4)^2 + 2$$
$$= 64 - 96$$
$$= \boxed{-32}$$

The graph of the given function is shown below:



Answer 10E.

Consider,

$$f(x,y) = xy(1-x-y)$$

To find the local maximum, local minimum values, and saddle points of this function by using the Second Derivative Test as follows.

Suppose the second partial derivatives of the function f are continuous on a disk with center (a,b), and suppose that $f_x(a,b)=0, f_y(a,b)=0$

That is, the point (a,b) is a critical point of f.

Let $D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$

1. If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.

2. If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.

3. If $D<0$ then $f(a,b)$ is not a local maximum or local minimum.

In case (c), the point (a,b) is called a saddle point of f.

To find the critical points, solve $f_x=0, f_y=0$

Consider,

$f(x,y) = xy(1-x-y)$, or $f(x,y) = xy - x^2y - xy^2$.

$$\begin{aligned} f_x(x,y) &= \frac{\partial f}{\partial x} \\ &= y - 2xy - y^2 \\ &= y(1-2x-y) \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= \frac{\partial f}{\partial y} \\ &= x - x^2 - 2xy \\ &= x(1-x-2y) \end{aligned}$$

If $f_x = 0$, then

$$y(1-2x-y)=0$$

$y=0$ or $1-2x-y=0$ ...... (1)

If $f_y = 0$, then

$$x(1-x-2y)=0$$

$x=0$ or $1-x-2y=0$ ...... (2)

From the equations (1), and (2), $(0,0)$ is a critical point of f

Substitute $y=0$ from (1) in $1-x-2y=0$,

$$x=1$$

Therefore $(1,0)$ is a critical point of f

Substitute $x=0$ from (2) in $1-2x-y=0$,

$$y=1$$

Therefore $(0,1)$ is a critical point of f

Multiply the equation $1-2x-y=0$ by 2, and subtract from the equation $1-x-2y=0$,

$$\begin{array}{r} 1-x-2y=0 \\ \underline{2-4x-2y=0} \\ -1+3x=0 \end{array}$$

$$x=\frac{1}{3}$$

Then by (2),

$$1-x-2y=0$$

$$1-\frac{1}{3}-2y=0$$

$$2y=\frac{2}{3}$$

$$y=\frac{1}{3}$$

Therefore $\left(\frac{1}{3},\frac{1}{3}\right)$ is a critical point of f

Therefore all the critical points of f are $(0,0),(1,0),(0,1),\left(\frac{1}{3},\frac{1}{3}\right)$

Now find the second order partial derivatives:

$$f_x(x,y) = y - 2xy - y^2$$

$$\begin{aligned} f_{xx}(x,y) &= \frac{\partial f_x}{\partial x} \\ &= -2y \end{aligned}$$

$$f_y(x,y) = x - x^2 - 2xy$$

$$\begin{aligned} f_{yy}(x,y) &= \frac{\partial f_y}{\partial y} \\ &= -2x \end{aligned}$$

Also,

$$\begin{aligned} f_{xy}(x,y) &= \frac{\partial f_y}{\partial x} \\ &= \frac{\partial}{\partial x}\left(x - x^2 - 2xy\right) \\ &= 1 - 2x - 2y \end{aligned}$$

Now, find the values of all second order partial derivatives at the critical points,

$$(0,0),(1,0),(0,1),\left(\frac{1}{3},\frac{1}{3}\right)$$

At the critical point $(0,0)$.

$$\begin{aligned} f_{xx}(0,0) &= -2(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_{yy}(0,0) &= -2(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_{xy}(0,0) &= 1 - 2(0) - 2(0) \\ &= 1 \end{aligned}$$

And,

$$\begin{aligned} D(0,0) &= f_{xx}(0,0)\, f_{yy}(0,0) - \left[f_{xy}(0,0)\right]^2 \\ &= (0)(0) - [1]^2 \\ &= -1 \end{aligned}$$

As $D(0,0) < 0$, by the Second Derivative Test, there is no local maximum or minimum for f, and a saddle point of f is $(0,0)$

At the critical point $(1,0)$,

$$\begin{aligned} f_{xx}(1,0) &= -2(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_{yy}(1,0) &= -2(1) \\ &= -2 \end{aligned}$$

$$\begin{aligned} f_{xy}(1,0) &= 1-2(1)-2(0) \\ &= -1 \end{aligned}$$

And,

$$\begin{aligned} D(1,0) &= f_{xx}(1,0)\,f_{yy}(1,0)-\left[f_{xy}(1,0)\right]^2 \\ &= (0)(-2)-[-1]^2 \\ &= -1 \end{aligned}$$

As $D(1,0)<0$, by the Second Derivative Test, there is no local maximum or minimum for f, and a saddle point of f is $(1,0)$

At the critical point $(0,1)$,

$$\begin{aligned} f_{xx}(0,1) &= -2(1) \\ &= -2 \end{aligned}$$

$$\begin{aligned} f_{yy}(0,1) &= -2(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_{xy}(0,1) &= 1-2(0)-2(1) \\ &= -1 \end{aligned}$$

And,

$$\begin{aligned} D(0,1) &= f_{xx}(0,1)\,f_{yy}(0,1)-\left[f_{xy}(0,1)\right]^2 \\ &= (-2)(0)-[-1]^2 \\ &= -1 \end{aligned}$$

As $D(0,1)<0$, by the Second Derivative Test, there is no local maximum or minimum for f, and a saddle point of f is $(0,1)$

At the critical point $\left(\frac{1}{3},\frac{1}{3}\right)$,

$$f_{xx}\left(\frac{1}{3},\frac{1}{3}\right)=-2\left(\frac{1}{3}\right)$$
$$=-\frac{2}{3}<0$$

$$f_{yy}\left(\frac{1}{3},\frac{1}{3}\right)=-2\left(\frac{1}{3}\right)$$
$$=-\frac{2}{3}$$

$$f_{xy}\left(\frac{1}{3},\frac{1}{3}\right)=1-2\left(\frac{1}{3}\right)-2\left(\frac{1}{3}\right)$$
$$=-\frac{1}{3}$$

And,

$$D\left(\frac{1}{3},\frac{1}{3}\right)=f_{xx}\left(\frac{1}{3},\frac{1}{3}\right)f_{yy}\left(\frac{1}{3},\frac{1}{3}\right)-\left[f_{xy}\left(\frac{1}{3},\frac{1}{3}\right)\right]^2$$
$$=\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)-\left[-\frac{1}{3}\right]^2$$
$$=\frac{4}{9}-\frac{1}{9}$$
$$=\frac{1}{3}$$

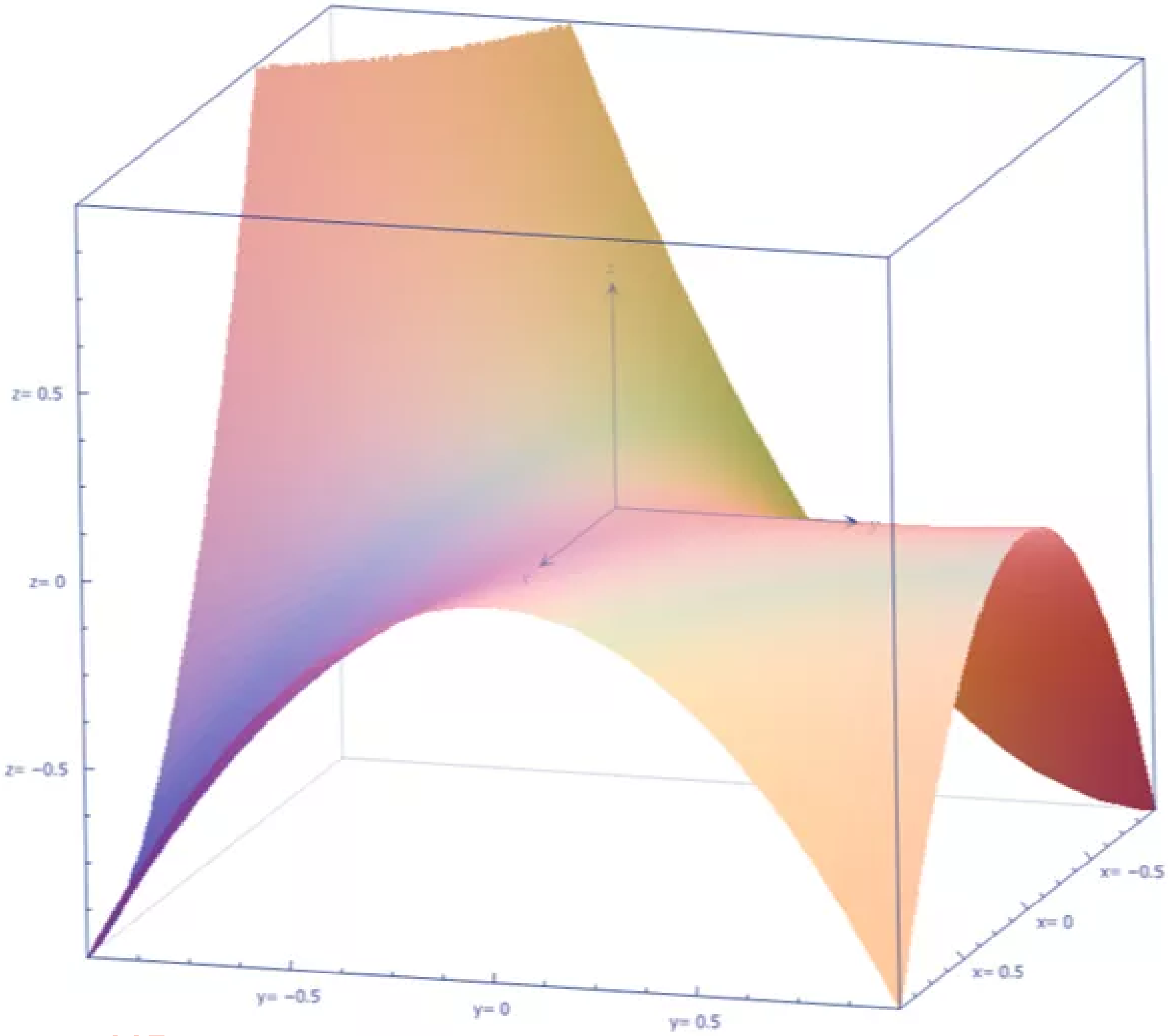
As $D(0,1)>0$, and $f_{xx}\left(\frac{1}{3},\frac{1}{3}\right)<0$ by the Second Derivative Test, the local maximum of f is $f\left(\frac{1}{3},\frac{1}{3}\right)$

So, the local maximum of f is,

$$f\left(\frac{1}{3},\frac{1}{3}\right)=\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(1-\frac{1}{3}-\frac{1}{3}\right)\quad \left(\text{Since } f(x,y)=xy(1-x-y)\right)$$
$$=\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$$
$$=\frac{1}{27}$$
$$=0.037037$$
$$\approx \boxed{0.04}$$

And, the saddle points of f are $\boxed{(0,0),(1,0),(0,1)}$

The sketch of the surface $f(x,y)=xy(1-x-y)$ is shown below:



Answer 11E.

Consider the function $f(x,y)=x^3-12xy+8y^3$.

Find the local maximum and minimum values and saddle points of the given function.

First locate the critical points:

Differentiate the function $f(x,y)$ with respect to x, we get

$$f_x(x,y)=3x^2-12y$$

Differentiate the function $f(x,y)$ with respect to y, we get

$$f_y(x,y)=-12x+24y^2.$$

Set these partial derivatives f_x, f_y equal to 0, we obtain the equations

$3x^2-12y=0$ ...... (1)

$-12x+24y^2=0$ ...... (2)

From equation (1), we have

$$
\begin{aligned}
3x^2 - 12y &= 0 \\
12y &= 3x^2 \\
y &= \frac{3x^2}{12} \\
&= \frac{x^2}{4}
\end{aligned}
$$

Substitute $y = \frac{x^2}{4}$ in the equation (2), get

$$
\begin{aligned}
-12x + 24y^2 &= 0 \\
-12x + 24\left(\frac{x^2}{4}\right)^2 &= 0 \\
-12x + \frac{24x^4}{16} &= 0 \\
-12x\left(1 - \frac{2x^3}{16}\right) &= 0
\end{aligned}
$$

$$x\left(1 - \frac{2x^3}{16}\right) = 0$$

$$
\begin{aligned}
&\text{either } x = 0 \text{ or } 1 - \frac{2x^3}{16} = 0 \\
&\text{either } x = 0 \text{ or } 1 - \frac{x^3}{8} = 0 \\
&\text{either } x = 0 \text{ or } x^3 = 8 \\
&\text{either } x = 0 \text{ or } x = 2
\end{aligned}
$$

Find the value of y corresponding to $x = 0.$

$$
\begin{aligned}
y &= \frac{x^2}{4} \\
&= \frac{0^2}{4} \\
&= 0
\end{aligned}
$$

The corresponding critical points is $(0,0).$

Find the value of y corresponding to $x = 2$.

$$y = \frac{x^2}{4}$$
$$= \frac{2^2}{4}$$
$$= 1$$

The corresponding critical points is $(2,1)$.

Now calculate the second partial derivatives and $D(x,y)$:

Differentiate the function $f_x(x,y)$ with respect to x, we get $f_{xx}(x,y) = 6x$.

Differentiate the function $f_x(x,y)$ with respect to y, we get $f_{xy}(x,y) = -12$.

Differentiate the function $f_y(x,y)$ with respect to y, we get $f_{yy}(x,y) = 48y$.

Now find $D(x,y)$:

$$D(x,y) = f_{xx} f_{yy} - f_{xy}^2$$
$$= (6x)(48y) - (-12)^2$$
$$= 288xy - 144$$

Therefore, $D(x,y) = 288xy - 144$.

At $(0,0)$:

$$D(x,y) = 288xy - 144$$
$$D(0,0) = -144$$

Since $D(0,0) = -144 < 0$, by second derivative test we conclude that $f(x,y)$ has neither a local maximum nor a local minimum at $(0,0)$.

The critical point $(0,0)$ is a saddle point.

At $(2,1)$:

$$D(x,y)=288xy-144$$
$$D(2,1)=288(2)(1)-144$$
$$=432$$
$$f_{xx}(x,y)=6x$$
$$f_{xx}(2,1)=6(2)$$
$$=12$$

Since $D(2,1)>0$ and $f_{xx}(2,1)>0$, by second derivative test we conclude that $f(x,y)$ has a local minimum at $(2,1)$.

Find the function value at $(2,1)$.

$$f(x,y)=x^3-12xy+8y^3$$
$$f(2,1)=2^3-12(2)(1)+8(1)^3$$
$$=8-24+8$$
$$=-8$$

Therefore, the local minimum value is $(2,1,-8)$.

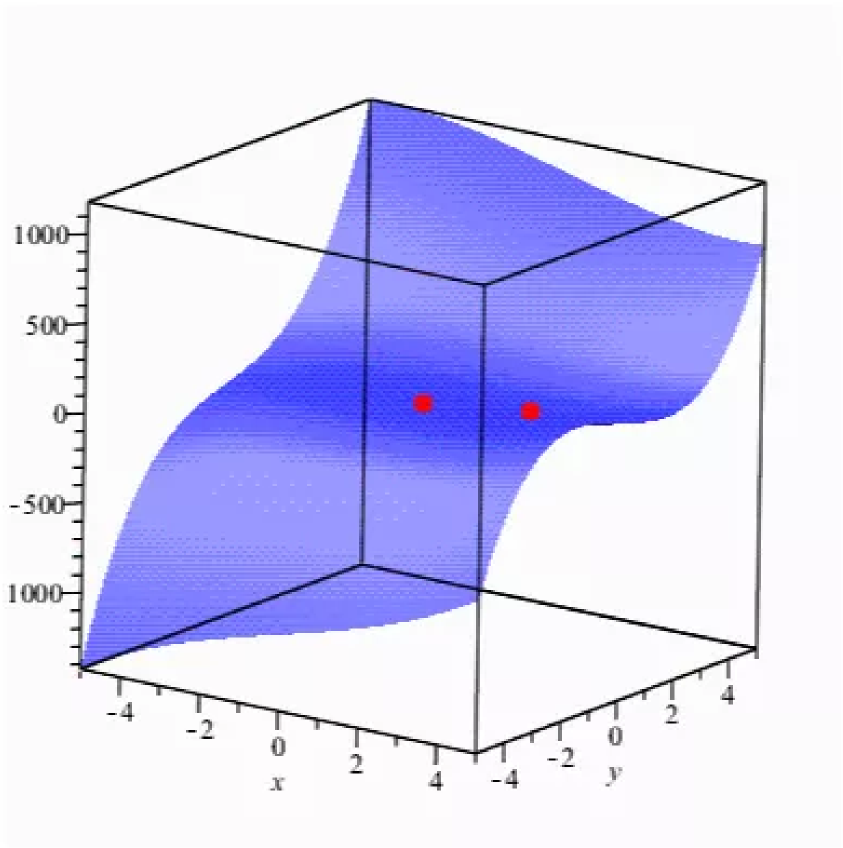
Use Maple software to graph the function $f(x,y)=x^3-12xy+8y^3$.

Follow the commands given below:

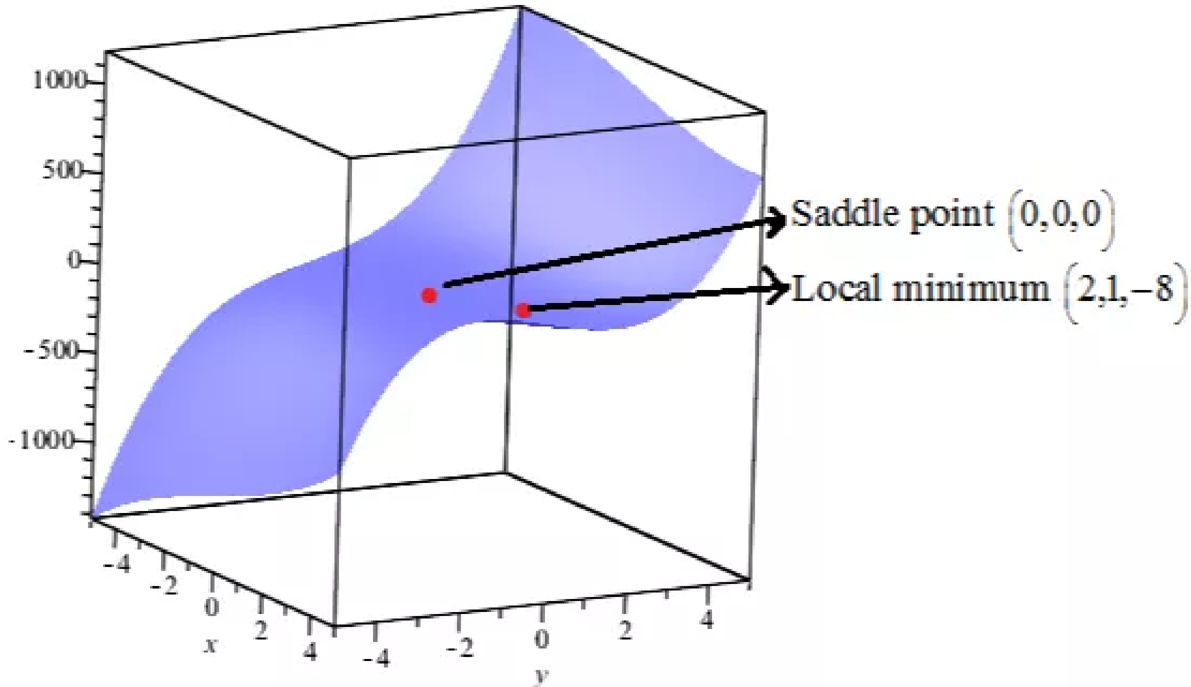
```
with(plots):
a := pointplot3d([[2, 1, -8], [0, 0, 0]], color = [red], axes = normal, symbol = box)
b := plot3d(x^3+8*y^3-12*x*y, x = -5 .. 5, y = -5 .. 5)
display(a, b, axes = boxed, style = surface, color = blue, transparency = .5)
```

Maple output:

```
> with(plots) :
> a := pointplot3d([[2, 1,-8], [0, 0, 0]], color = [red], axes = normal, symbol = box)
a := PLOT3D(...)
> b := plot3d(x^3 − 12·x·y + 8·y^3, x =-5 ..5, y =-5 ..5) :
> display(a, b, axes = boxed, style = surface, color = blue, transparency = 0.5);
```



Saddle point and the local minimum value are located on the graph $f(x,y) = x^3 - 12xy + 8y^3$ shown below:



Answer 12E.

Consider the function $f(x,y)=xy+\frac{1}{x}+\frac{1}{y}$.

Find the local maximum and minimum values and saddle points of the given function.

First locate the critical points:

Differentiate the function $f(x,y)$ with respect to x, we get

$$f_x(x,y)=y-\frac{1}{x^2}$$

Differentiate the function $f(x,y)$ with respect to y, we get

$$f_y(x,y)=x-\frac{1}{y^2}$$

Set these partial derivatives f_x, f_y equal to 0, we obtain the equations

$$y-\frac{1}{x^2}=0 \quad \ldots\ldots (1)$$

$$x-\frac{1}{y^2}=0 \quad \ldots\ldots (2)$$

From equation (2), we have

$$x=\frac{1}{y^2}$$

Substitute $x=\frac{1}{y^2}$ in the equation (1), get

$$y-\frac{1}{x^2}=0$$

$$y-\frac{1}{\left(\frac{1}{y^2}\right)^2}=0$$

$$y-y^4=0$$

$$y(1-y^3)=0$$

$$y=0 \text{ or } y^3=1$$

$$y=0 \text{ or } y=1$$

Take only $y=1$ since $y=0$ is not in the domain of the function $f(x,y)$.

Find the value of x corresponding to $y=1$.

$$x=\frac{1}{y^2}$$

$$=\frac{1}{1}$$

$$=1$$

The corresponding point is $(1,1)$.

Therefore, the only critical point is $(1,1)$.

Now calculate the second partial derivatives and $D(x,y)$:

Differentiate the function $f_x(x,y)$ with respect to x, we get $f_{xx}=\frac{2}{x^3}$.

Differentiate the function $f_x(x,y)$ with respect to y, we get $f_{xy}=1$.

Differentiate the function $f_y(x,y)$ with respect to y, we get $f_{yy}=\frac{2}{y^3}$.

Now find $D(x,y)$:

$$D(x,y)=f_{xx}f_{yy}-f_{xy}^2$$
$$=\left(\frac{2}{x^3}\right)\left(\frac{2}{y^3}\right)-(1)^2$$
$$=\frac{4}{x^3y^3}-1$$

Therefore, $D(x,y)=\frac{4}{x^3y^3}-1$.

At $(1,1)$:

$$D(x,y)=\frac{4}{x^3y^3}-1$$
$$D(1,1)=\frac{4}{(1^3)(1^3)}-1$$
$$=4-1$$
$$=3$$

$$f_{xx}(x,y)=\frac{2}{x^3}$$
$$f_{xx}(1,1)=\frac{2}{1^3}$$
$$=2$$

Since $D(1,1) = 3 > 0$ and $f_{xx}(3,11) = 2 > 0$, by second derivative test we conclude that $f(x, y)$ has a local minimum at $(1,1)$.

And the local minimum value of *f* is

$$f(x,y) = xy + \frac{1}{x} + \frac{1}{y}$$

$$\begin{aligned} f(1,1) &= (1)(1) + \frac{1}{1} + \frac{1}{1} \\ &= 1+1+1 \\ &= 3 \end{aligned}$$

Therefore, local minimum value of $f(x,y)$ is $\boxed{3}$ obtained at $(1,1)$.

Graph the surface and locate the minimum value on it.

Use Maple to graph the surface.

Maple commands:

with(plots);

a := implicitplot3d(z = x*y+1/x+1/y, x = -5 .. 5, y = -5 .. 5, z = 0 .. 5, style = surface, transparency = .5, color = gold);

b := pointplot3d({[1, 1, 3]}, color = red);

display(a, b, axes = boxed);

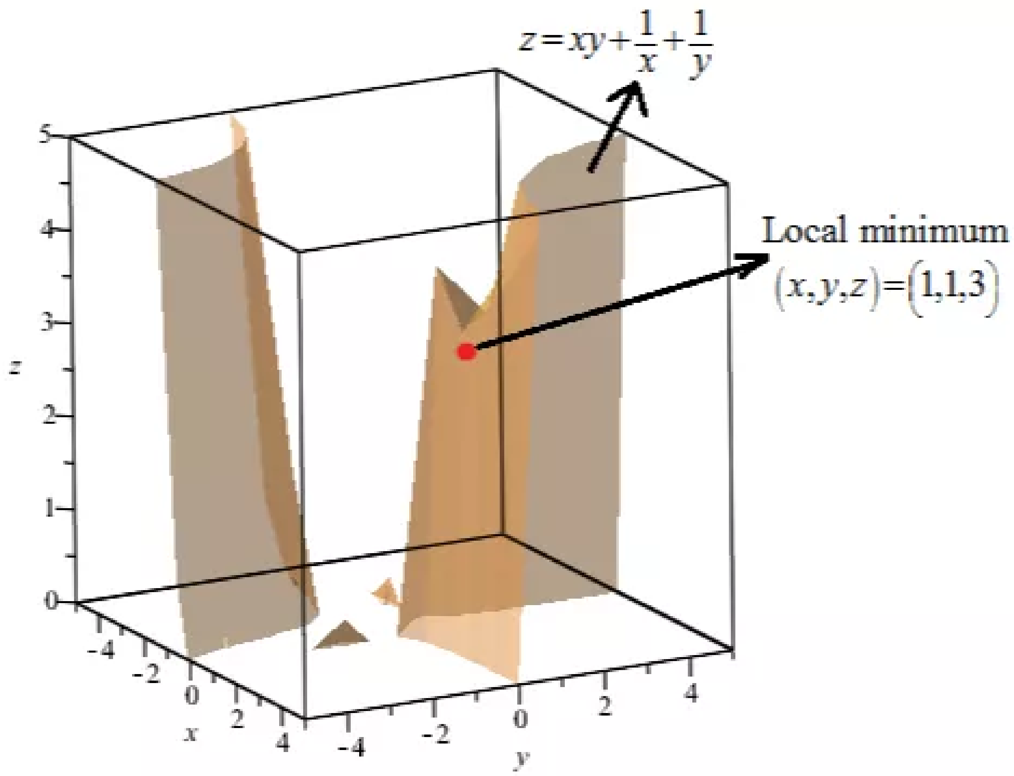
Maple output:

> *with*(*plots*) :

> $a := implicitplot3d\left(z = x \cdot y + \frac{1}{x} + \frac{1}{y}, x = -5..5, y = -5..5, z = 0..5, style = surface, transparency = 0.5, color = gold\right)$:

> $b := pointplot3d(\{[1, 1, 3]\}, color = red)$:

> display(a, b, axes = boxed);



Answer 13E.

$$f(x, y) = e^x \cos y$$

Then $f_x(x, y) = e^x \cos y$

$f_y(x, y) = -e^x \sin y$

First we find the critical points by setting $f_x = 0$, $f_y = 0$

i.e. $e^x \cos y = 0$

And $-e^x \sin y = 0$

But these two equations cannot be solved for x and y. This gives that f has no critical point.

Answer 14E.

Consider the function

$$f(x,y) = y\cos x$$

The partial derivative of $f(x,y)$ with respect to x,

$$f(x,y) = y\cos x$$
$$f_x(x,y) = -y\sin x$$

Equating this expression to zero,

$$f_x(x,y) = 0$$

$$-y\sin x = 0 \quad \text{......} \ (1)$$

$$y = 0 \quad \text{or} \quad \sin x = 0$$

$$y = 0 \quad \text{or} \quad x = n\pi \text{ for any integer } n$$

The partial derivative of $f(x,y)$ with respect to y,

$$f(x,y) = y\cos x$$
$$f_y(x,y) = \cos x$$

Equating this expression to zero,

$$f_y(x,y) = 0$$
$$\cos x = 0$$
$$x = \frac{(2n+1)\pi}{2} \text{ for any integer } n \quad \text{......} \ (2)$$

For any integer *n*, x cannot satisfy (1) and (2) simultaneously.

So, the only possibility is $y = 0$.

Hence, the critical points of $f(x,y) = y\cos x$ are $\boxed{\left(\frac{(2n+1)\pi}{2}, 0\right)}$ where *n* is any integer.

It appears that *f* had infinitely many critical points.

The second partial derivatives of $f(x,y)$ with respect to x,

$$\begin{aligned} f_{xx}(x,y) &= \frac{\partial}{\partial x}[f_x(x,y)] \\ &= \frac{\partial}{\partial x}[-y\sin x] \\ &= -y\frac{\partial}{\partial x}(\sin x) \\ &= -y\cos x \end{aligned}$$

The second partial derivatives of $f(x,y)$ with respect to y,

$$\begin{aligned} f_{yy}(x,y) &= \frac{\partial}{\partial y}[f_y(x,y)] \\ &= \frac{\partial}{\partial y}[\cos x] \\ &= 0 \end{aligned}$$

The partial derivative of $f_x(x,y)$ with respect to y,

$$\begin{aligned} f_{xy}(x,y) &= \frac{\partial}{\partial y}[f_x(x,y)] \\ &= \frac{\partial}{\partial y}[-y\sin x] \\ &= -\frac{\partial}{\partial y}(y)\sin x \\ &= -\sin x \end{aligned}$$

Now,

$$\begin{aligned} D(x,y) &= f_{xx}(x,y)f_{yy}(x,y)-(f_{xy}(x,y))^2 \\ &= (-y\cos x)(0)-(-\sin x)^2 \\ &= -\sin^2 x \end{aligned}$$

Second derivative test:

Suppose the second partial derivatives of f are continuous on a disk with center (a,b) and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$ [(a,b) is a critical point of f].

Let $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left(f_{xy}(a,b)\right)^2$.

(a) If $D>0$ and $f_{xx}(a,b)>0$ then $f(a,b)$ is a local minimum.

(b) If $D>0$ and $f_{xx}(a,b)<0$ then $f(a,b)$ is a local maximum.

(c) If $D<0$ then $f(a,b)$ is not a local maximum or local minimum.

The value of D at the critical point $\left(\frac{(2n+1)\pi}{2},0\right)$:

$$D(x,y)=\sin^2 x$$

$$\begin{aligned} D\left(\frac{(2n+1)\pi}{2},0\right) &= -\sin^2\left(\frac{(2n+1)\pi}{2}\right) \\ &= -1 \\ &< 0 \end{aligned}$$

Since $D\left(\frac{(2n+1)\pi}{2},0\right)<0$, it follows from case (c) of the second derivative test that $\left(\frac{(2n+1)\pi}{2},0\right)$ is a saddle point; that is, f has no local maximum or minimum at $\left(\frac{(2n+1)\pi}{2},0\right)$.

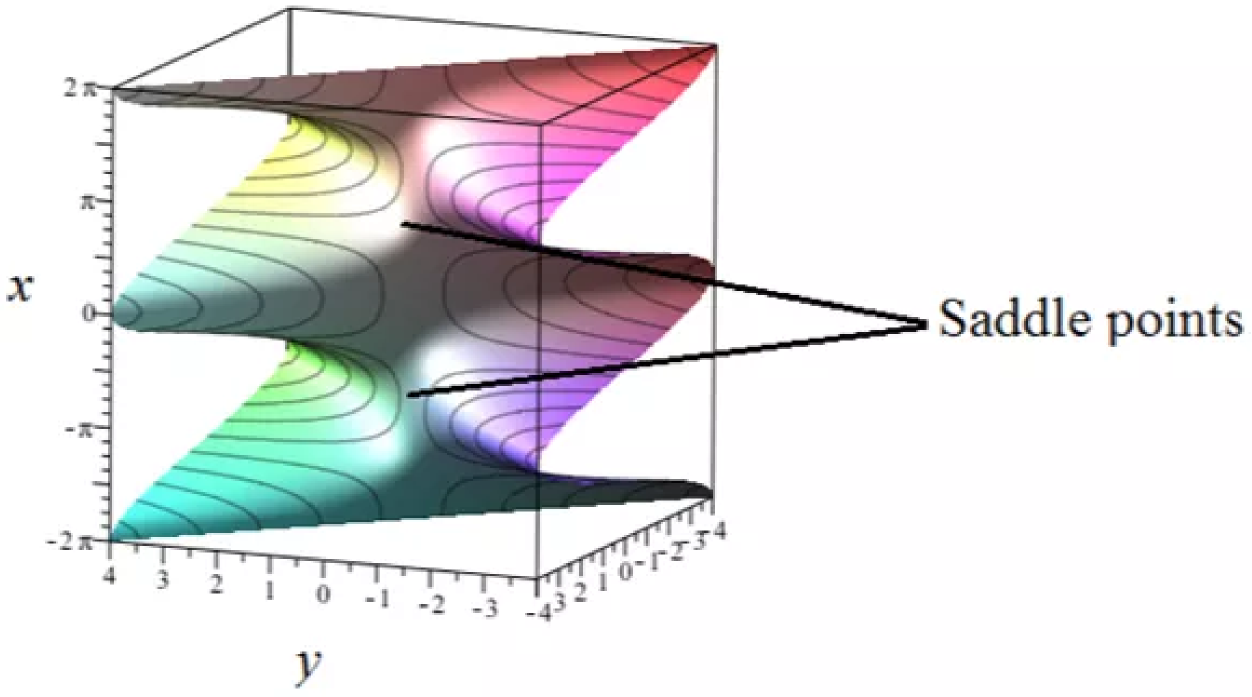
Sketch the graph $f(x,y)=y\cos x$ using computer algebraic system:

First load the package, $with(plots);$

After that use the following command, to plot the function

```
> plot3d(y·cos(x), x=-2π..2π, y=-4..4)
```

The graph of $f(x,y)=y\cos x$ as shown below:



Answer 15E.

Consider the function $f(x,y)=(x^2+y^2)e^{y^2-x^2}$.

Find the local maximum, minimum values and saddle points of the function.

The partial derivative of $f(x,y)$ with respect to x,

$$f(x,y)=(x^2+y^2)e^{y^2-x^2}$$

$$\begin{aligned}f_x(x,y)&=(x^2+y^2)\frac{d}{dx}\left(e^{y^2-x^2}\right)+e^{y^2-x^2}\frac{d}{dx}(x^2+y^2)\\&=(x^2+y^2)e^{y^2-x^2}(-2x)+e^{y^2-x^2}(2x)\\&=2xe^{y^2-x^2}(-x^2-y^2+1)\end{aligned}$$

Equate this expression to zero,

$$f_x(x,y)=0$$

$$2xe^{y^2-x^2}(-x^2-y^2+1)=0$$

$$x=0 \text{ or} -x^2-y^2+1=0$$

$$x=0 \text{ or} \qquad x^2=1-y^2 \quad \text{......} \ (1)$$

The partial derivative of $f(x,y)$ with respect to y,

$$f(x,y)=\left(x^2+y^2\right)e^{y^2-x^2}$$

$$f_y(x,y)=\left(x^2+y^2\right)\frac{d}{dy}\left(e^{y^2-x^2}\right)+e^{y^2-x^2}\frac{d}{dy}\left(x^2+y^2\right)$$

$$=\left(x^2+y^2\right)e^{y^2-x^2}(2y)+e^{y^2-x^2}(2y)$$

$$=2ye^{y^2-x^2}\left(x^2+y^2+1\right)$$

Equate this expression to zero,

$$f_y(x,y)=0$$

$$2ye^{y^2-x^2}\left(x^2+y^2+1\right)=0$$

$$y=0 \quad \text{since } e^{y^2-x^2}\neq 0, x^2+y^2+1\neq 0 \text{ for all values of } x,y$$

Substitute $y=0$ in $x^2=1-y^2$ (From (1)).

$$x^2=1-(0)^2$$

$$x^2=1$$

$$x=\pm 1$$

Hence, the critical points are $(0,0),(1,0)$ and $(-1,0)$.

The second partial derivatives of $f(x,y)$ are,

Differential two times with respect to x,

$$f(x,y)=\left(x^2+y^2\right)e^{y^2-x^2}$$

$$f_x(x,y)=2xe^{y^2-x^2}\left(-x^2-y^2+1\right)$$

$$=2e^{y^2-x^2}\left(-x^3-xy^2+x\right)$$

$$f_{xx}(x,y)=\left(-x^3-xy^2+x\right)e^{y^2-x^2}(-2x)+2e^{y^2-x^2}\left(-3x^2-y^2+1\right)$$

$$=\left(2x^4+2x^2y^2-2x^2-6x^2-2y^2+2\right)e^{y^2-x^2}$$

$$=\left(2x^4+2x^2y^2-8x^2-2y^2+2\right)e^{y^2-x^2}$$

The value of $f_{xy}(x,y)$ is,

$$f(x,y)=\left(x^2+y^2\right)e^{y^2-x^2}$$

$$f_x(x,y)=2xe^{y^2-x^2}\left(-x^2-y^2+1\right)$$

$$=2e^{y^2-x^2}\left(-x^3-xy^2+x\right)$$

$$f_{xy}(x,y)=2e^{y^2-x^2}(-2xy)+2\left(-x^3-xy^2+x\right)e^{y^2-x^2}(2y)$$

$$=\left(-4xy-4x^3y-4xy^3+4xy\right)e^{y^2-x^2}$$

$$=\left(-4x^3y-4xy^3\right)e^{y^2-x^2}$$

Differential two times with respect to y,

$$f(x,y)=(x^2+y^2)e^{y^2-x^2}$$
$$f_y(x,y)=2ye^{y^2-x^2}(x^2+y^2+1)$$
$$=2e^{y^2-x^2}(x^2y+y^3+y)$$

$$f_{yy}(x,y)=2e^{y^2-x^2}(x^2+3y^2+1)+2(x^2y+y^3+y)e^{y^2-x^2}(2y)$$
$$=e^{y^2-x^2}(2x^2+6y^2+2+4x^2y^2+4y^4+4y^2)$$
$$=e^{y^2-x^2}(4x^2y^2+2x^2+4y^4+10y^2+2)$$

The value of f_{xx} at critical points $(0,0),(1,0)$ and $(-1,0)$ is

$$f_{xx}(x,y)=(2x^4+2x^2y^2-8x^2-2y^2+2)e^{y^2-x^2}$$
$$f_{xx}(0,0)=(0+0-0-0+2)e^0$$
$$=2$$
$$f_{xx}(1,0)=(2+0-8-0+2)e^{0-1}$$
$$=-\frac{4}{e}$$

$$f_{xx}(-1,0)=(2+0-8-0+2)e^{0-1}$$
$$=-\frac{4}{e}$$

The value of f_{xy} at critical points $(0,0),(1,0)$ and $(-1,0)$ is,

$$f_{xy}(x,y)=(-4x^3y-4xy^3)e^{y^2-x^2}$$
$$f_{xy}(0,0)=(0-0)e^{0-0}$$
$$=0$$
$$f_{xy}(1,0)=(0-0)e^{0-1}$$
$$=0$$
$$f_{xy}(-1,0)=(0-0)e^{0-1}$$
$$=0$$

The value of f_{yy} at critical points $(0,0),(1,0)$ and $(-1,0)$ is,

$$f_{yy}(x,y)=e^{y^2-x^2}(4x^2y^2+2x^2+4y^4+10y^2+2)$$
$$f_{yy}(0,0)=e^{0^2-0^2}(0+0+0+0+2)$$
$$=2$$
$$f_{yy}(1,0)=e^{0^2-1^2}(0+2+0+0+2)$$
$$=\frac{4}{e}$$
$$f_{yy}(-1,0)=e^{0^2-1^2}(0+2+0+0+2)$$
$$=\frac{4}{e}$$

Recall the second derivative test

Suppose the second partial derivatives of f are continuous on a disk with center (a,b) and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$ [(a,b) is a critical point of f].

Let $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2$.

(a) If $D>0$ and $f_{xx}(a,b)>0$ then $f(a,b)$ is a local minimum.

(b) If $D>0$ and $f_{xx}(a,b)<0$ then $f(a,b)$ is a local maximum.

(c) If $D<0$ then $f(a,b)$ is not a local maximum or local minimum.

At critical point $(0,0)$, the value of D is,

$$\begin{aligned} D(a,b) &= f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2 \\ D(0,0) &= (2)(2)-[0]^2 \\ &= 4 \end{aligned}$$

Since $D>0$ and $f_{xx}>0$, the critical point $(0,0)$ is a local minimum.

The local minimum values is

$$\begin{aligned} f(0,0) &= (0^2+0^2)e^{0-0} \\ &= \boxed{0} \end{aligned}$$

At critical point $(1,0)$, the value of D is,

$$\begin{aligned} D(a,b) &= f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2 \\ D(1,0) &= (-4e^{-1})(4e^{-1})-[0]^2 \\ &= -16e^{-2} \end{aligned}$$

Since $D<0$, thus, the point $(1,0)$ is a saddle point.

At critical point $(-1,0)$, the value of D is,

$$\begin{aligned} D(a,b) &= f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2 \\ D(-1,0) &= (-4e^{-1})(4e^{-1})-[0]^2 \\ &= -16e^{-2} \end{aligned}$$

Since $D<0$, thus, the point $(-1,0)$ is a saddle point.

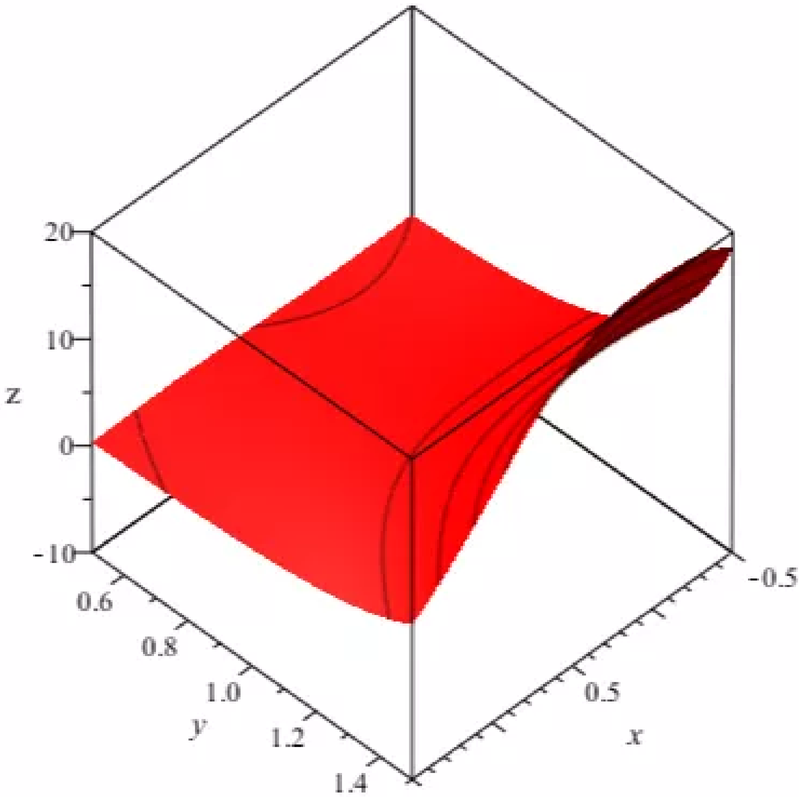
The graph of $f(x,y)$ is shown below,

Use the Maple software, give the command as shown in the following, and then press the ENTER button, get required graph.

> $f := (x, y) \rightarrow (x^2 + y^2) \cdot \exp(y^2 - x^2)$

$f := (x, y) \rightarrow (x^2 + y^2)\, e^{y^2 - x^2}$

> *plot3d*$(f(x, y), x = -0.5..1.5, y = 0.5..1.5,$ *style* = *patchcontour*,
color = *red*, *axes* = *boxed*, *view* = −10..20,
lightmodel = *light4*);



Answer 16E.

Consider the function $f(x,y)=e^y(y^2-x^2)$.

Find the local maximum and minimum values and saddle points of the given function.

First locate the critical points:

Differentiate the function $f(x,y)$ with respect to x, we get

$$f_x(x,y)=-2xe^y$$

Differentiate the function $f(x,y)$ with respect to y, we get

$$f_y(x,y)=e^y(2y-x^2+y^2)$$

Set these partial derivatives f_x, f_y equal to 0, we obtain the equations

$-2xe^y=0$ ...... (1)

$e^y(2y-x^2+y^2)=0$ ...... (2)

From equation (1), get $x=0$ since exponential function never be zero.

Also from equation (2) get $2y-x^2+y^2=0$.

Substitute $x=0$ in the equation $2y-x^2+y^2=0$.

$$\begin{aligned} 2y-x^2+y^2&=0 \\ 2y-0^2+y^2&=0 && [\text{Substitute } x=0.] \\ y(2+y)&=0 \\ y&=0,-2 \end{aligned}$$

Therefore, the critical points are $(0,0),(0,-2)$.

Now calculate the second partial derivatives and $D(x,y)$:

Differentiate the function $f_x(x,y)$ with respect to x, we get $f_{xx} = -2e^y$.

Differentiate the function $f_x(x,y)$ with respect to y, we get $f_{xy} = -2xe^y$.

Differentiate the function $f_y(x,y)$ with respect to y, we get $f_{yy} = e^y\left(2+4y-x^2+y^2\right)$.

Now find $D(x,y)$:

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2$$
$$= \left(-2e^y\right)e^y\left(2+4y-x^2+y^2\right) - \left(-2xe^y\right)^2$$
$$= \left(-2e^{2y}\right)\left(2+4y-x^2+y^2\right) - 4x^2e^{2y}$$

Therefore, $D(x,y) = \left(-2e^{2y}\right)\left(2+4y-x^2+y^2\right) - 4x^2e^{2y}$.

Find the value of $D(x,y)$ at $(0,0)$:

$$D(x,y) = \left(-2e^{2y}\right)\left(2+4y-x^2+y^2\right) - 4x^2e^{2y}$$
$$D(0,0) = \left(-2e^{2(0)}\right)\left(2+4(0)-0^2+0^2\right) - 4(0)^2e^{2(0)}$$
$$= \left(-2(1)\right)(2)$$
$$= -4$$

$$f_{xx}(x,y) = -2e^y$$
$$f_{xx}(0,0) = -2e^0$$
$$= -2$$

Since $D(0,0) = -4 < 0$ and $f_{xx}(3,11) = -2 < 0$, by second derivative test we conclude that $f(x,y)$ has a saddle point at $(0,0)$.

And the value of $f(x,y)$ at $(0,0)$.

$$f(x,y) = e^y\left(y^2 - x^2\right)$$
$$f(0,0) = e^0\left(0^2 - 0^2\right)$$
$$= 0$$

Therefore, $(0,0,0)$ be the saddle point.

Find the value of $D(x,y)$ at $(0,-2)$:

$$D(x,y)=\left(-2e^{2y}\right)\left(2+4y-x^2+y^2\right)-4x^2e^{2y}$$

$$D(0,-2)=\left(-2e^{2(-2)}\right)\left(2+4(-2)-0^2+(-2)^2\right)-4(0)^2e^{2(-2)}$$

$$=\left(-2e^{-4}\right)(2-8+4)$$

$$=4e^{-4}$$

$$f_{xx}(x,y)=-2e^{y}$$

$$f_{xx}(0,-2)=-2e^{-2}$$

Since $D(0,0)=4e^{-4}>0$ and $f_{xx}(3,11)=-2e^{-2}<0$, by second derivative test we conclude that $f(x,y)$ has a local maximum value at $(0,-2)$.

The local maximum value of *f* is

$$f(x,y)=e^{y}\left(y^2-x^2\right)$$

$$f(0,-2)=e^{-2}\left((-2)^2-0^2\right)$$

$$=4e^{-2}$$

$$\approx 0.54136$$

Therefore, the local maximum value of $f(x,y)$ is $\boxed{4e^{-2}}$ obtained at $(0,-2)$.

Graph the surface and locate the maximum value on it.

Use Maple to graph the surface.

Maple commands:

```
with(plots);

a := implicitplot3d(z = exp(y)*(-x^2+y^2), x = -5 .. 5, y = -5 .. 5, z = 0 .. 1, style = surface,
transparency = .5, color = gold)

b := pointplot3d({[0, -2, .54136], [0, 0, 0]}, color = red);

display(a, b, axes = boxed);
```

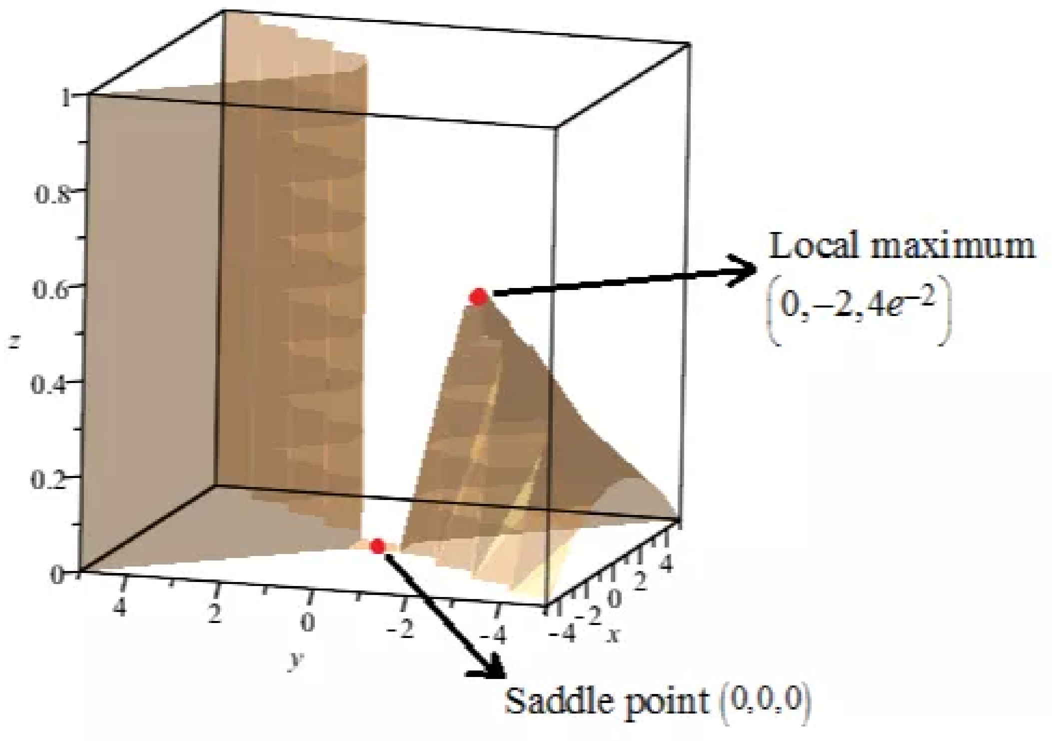
Maple output:

> $with(plots):$

> $a := implicitplot3d\left(z=e^{y}\cdot\left(y^2-x^2\right), x=-5..5, y=-5..5, z=0..1, style=surface, transparency=.5, color=gold\right);$

> $b := pointplot3d(\{[0,-2,0.54136],[0,0,0]\}, color=red);$

> $display(a,b,axes=boxed);$



Answer 17E.

f(x,y) = y2 - 2ycosx

To find the critical points for this function, you need to partially differentiate this function with respect to x and y. Then you equate the two equations to 0 and solve for x and y to get the critical points:

fx(x,y) = 2ysinx

fy(x,y) = 2y - 2cosx

Now to solve for them by equating them to 0:

fx(x,y) = 2ysinx = 0 (1)

fy(x,y) = 2y - 2cosx = 0 => y = cosx (2) plug this into (1)

fx(x,y) = 2cosxsinx = sin2x = 0 (by trig identities)

and you get the values x = 0, π/2, π, 3π/2, 2π in the range $1 \leq x \leq 7$

You plug these values back into (2) to solve for the corresponding y values:

x = 0, y = 1 (0,1)

x = π/2, y = 0 (π/2, 0)

x = π, y = -1 (π, -1)

x = 3π/2, y = 0 (3π/2, 0)

x = 2π, y = 1 (2π, 1)

Now you use the second derivative test to classify them:

fxx = 2ycosx

fyy = 2

fxy = 2sinx

D = fxx fyy - fxy2

x = 0, y = 1 D: (2)(2) - 0 = 4 > 0 fxx = 2 > 0 **local minimum**

x = π/2, y = 0 D: (0)(2) - 1 <0 **saddle point**

x = π, y = -1 D: (2)(2) - 0 > 0 fxx = 2 > 0 **local minimum**

x = 3π/2, y = 0 D: (0)(2) - (-1)2 = -1 < 0 **saddle point**

x = 2π, y = 1 D: (2)(2) - 0 = 4 > 0 fxx = 2 > 0 **local minimum**

f(0, 1) = 12 - 2cos0 = -1

f(π, -1) = (-1)2 + 2cosπ = -1

f(2π, 1) = 12 - 2cos2π = -1

Therefore, the local minima are: f(0,1) = f(π, -1) = f(2π, 1) = -1. The saddle points are f(π/2, 0) and f(3π/2, 0).

Answer 18E.

The second partial derivatives of $f(x,y)$ are calculated as follows:

Differential two times with respect to x,

$$f(x,y)=\sin x\sin y$$
$$f_x(x,y)=\cos x\sin y$$
$$f_{xx}(x,y)=-\sin x\sin y$$

Differential two times with respect to y, then the partial derivatives are,

$$f(x,y)=\sin x\sin y$$
$$f_x(x,y)=\cos x\sin y$$
$$f_{xy}(x,y)=\cos x\cos y$$

Differential two times with respect to y,

$$f(x,y)=\sin x\sin y$$
$$f_y(x,y)=\sin x\cos y$$
$$f_{yy}(x,y)=-\sin x\sin y$$

Recalls that, assume the second partial derivatives of f are continuous on a disk with center (a,b) and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$ [(a,b) is a critical point of f].

Let $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2$.

(a) If $D>0$ and $f_{xx}(a,b)>0$ then $f(a,b)$ is a local minimum.

(b) If $D>0$ and $f_{xx}(a,b)<0$ then $f(a,b)$ is a local maximum.

(c) If $D<0$ then $f(a,b)$ is not a local maximum or local minimum.

Substitute all the values in $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2$, then value of D is,

$$\begin{aligned}D(a,b)&=f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2\\ D&=(-\sin x\sin y)(-\sin x\sin y)-[\cos x\cos y]^2\\ &=(\sin x\sin y)^2-(\cos x\cos y)^2\\ &=\sin^2 x\sin^2 y-\cos^2 x\cos^2 y\end{aligned}$$

Since all the values are squared, we can find the value of $D\left(\pm\frac{\pi}{2},\pm\frac{\pi}{2}\right)$ in a single equation.

$$D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2$$
$$\begin{aligned}D\left(\pm\frac{\pi}{2},\pm\frac{\pi}{2}\right)&=(1)(1)-(0)(0)\\ &=2\\ &>0\end{aligned}$$

At the critical point $\left(-\frac{\pi}{2},-\frac{\pi}{2}\right)$, the second derivative is,

$$\begin{aligned} f_{xx}(x,y) &= -\sin x \sin y \\ &= -\sin\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) \\ &= -1 \end{aligned}$$

And

$$\begin{aligned} f(x,y) &= \sin x \sin y \\ f\left(-\frac{\pi}{2},-\frac{\pi}{2}\right) &= \sin\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) \\ &= 1 \end{aligned}$$

Since $D > 0$ and $f_{xx} < 0$, the function has local maximum at $\left(-\frac{\pi}{2},-\frac{\pi}{2}\right)$.

Substitute $(x,y)=\left(-\frac{\pi}{2},-\frac{\pi}{2}\right)$ in $f(x,y)=\sin x \sin y$, and then the maximum value is,

$$\begin{aligned} f\left(-\frac{\pi}{2},-\frac{\pi}{2}\right) &= \sin\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) \\ &= (-1)(-1) \\ &= 1 \end{aligned}$$

Therefore, the local maximum value of the function is,

$$f\left(-\frac{\pi}{2},-\frac{\pi}{2}\right) = \boxed{1}.$$

At the critical point $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, then

$$f_{xx}(x,y)=-\sin x\sin y$$

$$\begin{aligned}f_{xx}&=-\sin\left(-\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)\\&=1\\&>0\end{aligned}$$

And

$$f(x,y)=\sin x\sin y$$

$$\begin{aligned}f\left(-\frac{\pi}{2},\frac{\pi}{2}\right)&=\sin\left(-\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)\\&=-1\end{aligned}$$

Since $D>0$ and $f_{xx}>0$, the function has local minimum at $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Substitute $(x,y)=\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ in $f(x,y)=\sin x\sin y$, and then the minimum value is,

$$\begin{aligned}f\left(-\frac{\pi}{2},\frac{\pi}{2}\right)&=\sin\left(-\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)\\&=(-1)(1)\\&=-1\end{aligned}$$

Therefore, the local minimum value of the function is,

$$f\left(\frac{-\pi}{2},\frac{\pi}{2}\right)=\boxed{-1}.$$

At the critical point $\left(\frac{\pi}{2},-\frac{\pi}{2}\right)$, then

$$f_{xx}(x,y) = -\sin x \sin y$$

$$\begin{aligned} f_{xx} &= -\sin\left(\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) \\ &= 1 \\ &> 0 \end{aligned}$$

And

$$f(x,y) = \sin x \sin y$$

$$\begin{aligned} f\left(\frac{\pi}{2},\frac{-\pi}{2}\right) &= \sin\left(\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) \\ &= -1 \end{aligned}$$

Since $D>0$ and $f_{xx}>0$, the function has local minimum at $\left(\frac{\pi}{2},-\frac{\pi}{2}\right)$.

Substitute $(x,y)=\left(\frac{\pi}{2},-\frac{\pi}{2}\right)$ in $f(x,y)=\sin x \sin y$, and then the minimum value is,

$$\begin{aligned} f\left(\frac{\pi}{2},-\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{-\pi}{2}\right) \\ &= (1)(-1) \\ &= -1 \end{aligned}$$

Therefore, the local minimum value of the function is,

$$f\left(\frac{\pi}{2},-\frac{\pi}{2}\right) = \boxed{-1}.$$

At the critical point $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$, then

$$f_{xx}(x,y) = -\sin x \sin y$$

$$\begin{aligned} f_{xx} &= -\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) \\ &= -1 \\ &< 0 \end{aligned}$$

And,

$$f(x,y)=\sin x\sin y$$
$$f\left(\frac{\pi}{2},\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)$$
$$=1$$

Since $D>0$ and $f_{xx}<0$, the function has local maximum at $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$.

Substitute $(x,y)=\left(\frac{\pi}{2},\frac{\pi}{2}\right)$ in $f(x,y)=\sin x\sin y$, and then the maximum value is,

$$f\left(\frac{\pi}{2},\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)$$
$$=(1)(1)$$
$$=1$$

Therefore, the local maximum value of the function is,

$$f\left(\frac{\pi}{2},\frac{\pi}{2}\right)=\boxed{1}.$$

At the critical point $(0,0)$, then

$$D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left(f_{xy}(a,b)\right)^2$$
$$=(0)^2-(1)^2$$
$$=-1$$
$$<0$$

Since $D<0$, the function has saddle point at $\boxed{(0,0)}$.

$$f_{xx}(x,y)=-\sin x\sin y$$
$$f_{xx}=-\sin(0)\sin(0)$$
$$=0$$

Use maple software, the graph of $f(x,y)=\sin x \sin y$ is shown below.

Maple input command: f:=(x,y)→sin(x)*sin(y)

Maple Output:

$f := (x, y) \rightarrow \sin(x)\ \sin(y)$

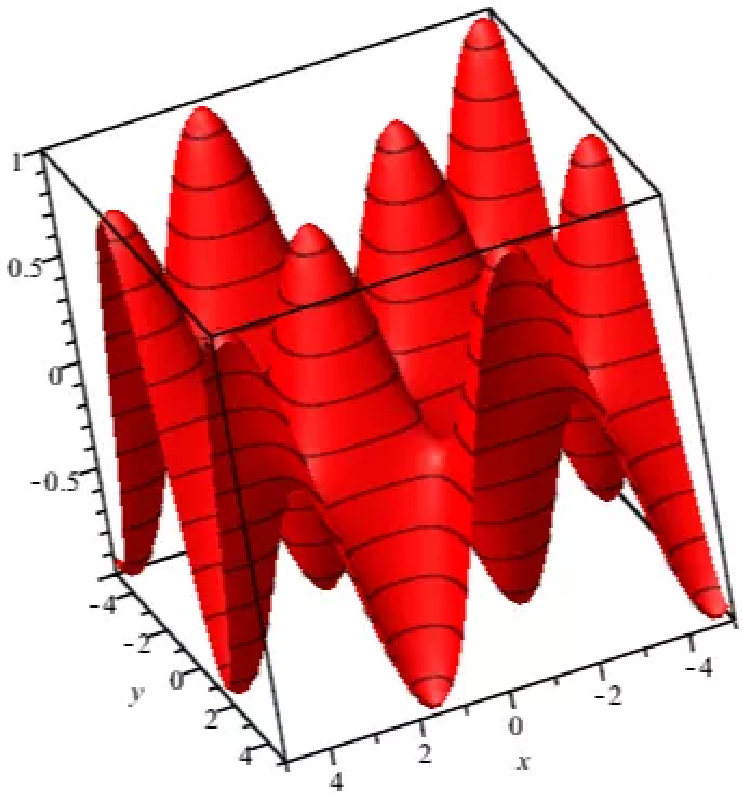
Maple input command:

plot3d(f(x, y), x = -5 .. 5, y = -5 .. 5, style = patchcontour, color = red, axes = boxed, view = -1 .. 1, lightmodel = light4);

Maple output:

> *plot3d(f(x, y), x = -5 ..5, y = -5 ..5, style = patchcontour, color = red, axes = boxed, view = -1 ..1, lightmodel = light4);*

The sketch of the function is shown below:



Answer 19E.

Consider the function.

$$f(x,y)=x^2+4y^2-4xy+2$$

Differentiate the function partially with respect to x and y to get the following:

$$\begin{aligned}f_x(x,y)&=\frac{\partial}{\partial x}\left[x^2+4y^2-4xy+2\right]\\&=\frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial x}\left(4y^2\right)-\frac{\partial}{\partial x}(4xy)+\frac{\partial}{\partial x}(2)\\&=2x+0-4y+0\\&=2x-4y\end{aligned}$$

And

$$\begin{aligned}f_y(x,y)&=\frac{\partial}{\partial y}\left[x^2+4y^2-4xy+2\right]\\&=\frac{\partial}{\partial y}\left(x^2\right)+\frac{\partial}{\partial y}\left(4y^2\right)-\frac{\partial}{\partial y}(4xy)+\frac{\partial}{\partial y}(2)\\&=0+8y-4x+0\\&=8y-4x\end{aligned}$$

Find the critical points, by setting $f_x=0$, and $f_y=0$.

$$\begin{aligned}&2x-4y=0\\&\Rightarrow x-2y=0\end{aligned}\quad \text{And} \quad \begin{aligned}&8y-4x=0\\&\Rightarrow x-2y=0\end{aligned}$$

Solve these two equations.

$$\begin{aligned}x-2y&=0\\ \underline{x-2y}&\underline{=0}\\ 0&=0 \quad \text{(Subtract)}\end{aligned}$$

Observe that the two equations are identical, so there are infinitely many solutions.

Since, $x-2y=0\Rightarrow x=2y$.

Let $y=k\in\mathbb{R}$, then $x=2k$.

Thus, the function $f(x,y)$ has an infinite number of critical points and the critical points are of the form $(2k,k)$, for all real numbers k.

Find the second order partial derivatives of $f(x,y)$.

$f_{xx}=2,\ \ f_{yy}=8$ and $f_{xy}=-4$

The variable *D* is given by the following:

$$D=D(a,b)=f_{xx}(a,b)\cdot f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$$

$$\begin{aligned}D(2k,k)&=(2)(8)-(-4)^2\\&=16-16\\&=0\end{aligned}$$

Therefore, at each critical point, the value is $\boxed{D=0}$.

Find the value of $f(x,y)$, at each critical point.

$$\begin{aligned}f(2k,k)&=(2k)^2+4(k)^2-4(2k)(k)+2\\&=4k^2+4k^2-8k^2+2\\&=2\end{aligned}$$

So, $f_{xx}=2>0$ and $f(x,y)=2$.

Thus by Second Derivative Test, *f* has a **local** (and **absolute**) **minimum** at each critical point and the minimum value is $\boxed{2}$.

Answer 20E.

Critical points occur when the partial derivatives f_x and f_y are either 0, or when one of them does not exist. We find these partial derivatives.

To find f_x, hold y constant and apply the Product Rule, which states that the derivative of a product of two terms is the derivative of the first term times the second plus the derivative of the second term times the first.

$$f(x,y)=x^2ye^{-x^2-y^2}$$

$$\begin{aligned}f_x(x,y)&=(2x)(ye^{-x^2-y^2})+(x^2)(-2xye^{-x^2-y^2})\\&=2xye^{-x^2-y^2}(1-x^2)\end{aligned}$$

There are no domain restrictions, so f_x always exists. We find where it equals 0:

$$\begin{aligned}f_x(x,y)&=2xye^{-x^2-y^2}(1-x^2)\\0&=2xye^{-x^2-y^2}(1-x^2)\end{aligned}$$

In order to satisfy this equation, we must have $x = 0$, $y = 0$ or $1 - x^2 = 0$, which would mean $x = \pm 1$.

To find f_y, hold x constant and apply the Product Rule.

$$f(x,y) = x^2 y e^{-x^2-y^2}$$

$$f_y(x,y) = (x^2)(e^{-x^2-y^2}) + (x^2 y)(-2ye^{-x^2-y^2})$$

$$= x^2 e^{-x^2-y^2}(1-2y^2)$$

There are no domain restrictions, so f_y always exists. We find where it equals 0:

$$f_y(x,y) = x^2 e^{-x^2-y^2}(1-2y^2)$$

$$0 = x^2 e^{-x^2-y^2}(1-2y^2)$$

In order to satisfy this equation, we must have $x = 0$ or $1 - 2y^2 = 0$, which would mean $y = \pm\frac{\sqrt{2}}{2}$.

Critical points happen when one of f_x and f_y does not exist (which never happens) or both are 0. When $x = 0$, both partial derivatives are 0. The other possibility to make f_y equal 0 is $y = \pm\frac{\sqrt{2}}{2}$, which is mutually exclusive with $y = 0$, so in order to make f_x zero as well we would have to set $x = \pm 1$. Therefore, the critical points occur at the following (x, y) points:

$(0, a)$

$\left(\pm 1, \pm\frac{\sqrt{2}}{2}\right)$

Where a is any real number.

To determine whether critical points are maxima or minima, we use the Second Derivative Test, which states that under most commonly encountered conditions, if a function f has a critical point at (a,b), then if we let

$$D = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

Then the following rules determine whether *f* has a local minimum or local maximum at (*a*, *b*):

(1) If $D > 0$ and $f_{xx}(a,b) > 0$, then *f* has a local minimum at (*a*, *b*).
(2) If $D > 0$ and $f_{xx}(a,b) < 0$, then *f* has a local maximum at (*a*, *b*).
(3) If $D < 0$, then (*a*, *b*) is a saddle point.
(4) If $D = 0$, then the test is inconclusive.

In order to apply the Second Derivative Test, we will need *D*, for which we will need to calculate all the second partial derivatives. Start with f_x and take another partial derivative in terms of x:

$$f_x(x,y) = 2xye^{-x^2-y^2}(1-x^2)$$

$$f_{xx}(x,y) = \left[(2y)(e^{-x^2-y^2}) + (2xy)(-2xe^{-x^2-y^2})\right](1-x^2) + \left(2xye^{-x^2-y^2}\right)(-2x)$$

$$= 2ye^{-x^2-y^2}\left[(1-2x^2)(1-x^2) - 2x^2\right]$$

$$= 2ye^{-x^2-y^2}\left(1-3x^2+2x^4-2x^2\right)$$

$$= 2ye^{-x^2-y^2}\left(1-5x^2+2x^4\right)$$

Now start with f_x and take a partial derivative in terms of y:

$$f_x(x,y) = 2xye^{-x^2-y^2}(1-x^2)$$

$$f_{xy}(x,y) = (2x)(e^{-x^2-y^2}(1-x^2)) + (2xy)(-2ye^{-x^2-y^2}(1-x^2))$$

$$= 2xe^{-x^2-y^2}(1-x^2)(1-2y^2)$$

Finally, start with f_y and take another partial derivative in terms of y:

$$f_y(x,y) = x^2e^{-x^2-y^2}(1-2y^2)$$

$$f_{yy}(x,y) = (-2yx^2e^{-x^2-y^2})(1-2y^2) + (x^2e^{-x^2-y^2})(-4y)$$

$$= -2yx^2e^{-x^2-y^2}(1-2y^2+2)$$

$$= -2yx^2e^{-x^2-y^2}(3-2y^2)$$

Now we calculate D for each of the four critical points $\left(\pm 1, \pm\frac{\sqrt{2}}{2}\right)$. Start with $\left(1, \frac{\sqrt{2}}{2}\right)$:

$$D = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

$$= \left[2ye^{-x^2-y^2}\left(1-5x^2+2x^4\right)\right]\left[-2yx^2e^{-x^2-y^2}(3-2y^2)\right] - \left[2xe^{-x^2-y^2}(1-x^2)(1-2y^2)\right]^2$$

$$= \left[2\left(\frac{\sqrt{2}}{2}\right)e^{-1^2-\left(\frac{\sqrt{2}}{2}\right)^2}\left(1-5(1)^2+2(1)^4\right)\right]\left[-2\left(\frac{\sqrt{2}}{2}\right)(1)^2e^{-(1)^2-\left(\frac{\sqrt{2}}{2}\right)^2}\left(3-2\left(\frac{\sqrt{2}}{2}\right)^2\right)\right]$$

$$-\left[2(1)e^{-(1)^2-\left(\frac{\sqrt{2}}{2}\right)^2}(1-(1)^2)\left(1-2\left(\frac{\sqrt{2}}{2}\right)^2\right)\right]^2$$

$$= \left[-2\sqrt{2}e^{-\frac{3}{2}}\right]\left[-2\sqrt{2}e^{-\frac{3}{2}}\right] - [0]^2$$

$$= 8e^{-3}$$

$$-\left[2(1)e^{-(1)^2-\left(\frac{\sqrt{2}}{2}\right)^2}(1-(1)^2)\left(1-2\left(\frac{\sqrt{2}}{2}\right)^2\right)\right]^2$$

$$= \left[-2\sqrt{2}e^{-\frac{3}{2}}\right]\left[-2\sqrt{2}e^{-\frac{3}{2}}\right] - [0]^2$$

$$= 8e^{-3}$$

So for $\left(1, \frac{\sqrt{2}}{2}\right)$, D has a positive value. In the course of finding D we also found f_{xx} to be $-2\sqrt{2}e^{-\frac{3}{2}}$, which is negative. Therefore, by the Second Derivative Test, f has a local maximum at $\left(1, \frac{\sqrt{2}}{2}\right)$.

Now apply the Second Derivative Test to $\left(-1, \frac{\sqrt{2}}{2}\right)$:

$$D = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

$$= \left[2ye^{-x^2-y^2}\left(1-5x^2+2x^4\right)\right]\left[-2yx^2e^{-x^2-y^2}(3-2y^2)\right] - \left[2xe^{-x^2-y^2}(1-x^2)(1-2y^2)\right]^2$$

$$= \left[2\left(\frac{\sqrt{2}}{2}\right)e^{-(-1)^2-\left(\frac{\sqrt{2}}{2}\right)^2}\left(1-5(-1)^2+2(-1)^4\right)\right]\left[-2\left(\frac{\sqrt{2}}{2}\right)(-1)^2e^{-(-1)^2-\left(\frac{\sqrt{2}}{2}\right)^2}\left(3-2\left(\frac{\sqrt{2}}{2}\right)^2\right)\right]$$

$$-\left[2(-1)e^{-(-1)^2-\left(\frac{\sqrt{2}}{2}\right)^2}\left(1-(-1)^2\right)\left(1-2\left(\frac{\sqrt{2}}{2}\right)^2\right)\right]^2$$

$$= \left[-2\sqrt{2}e^{-\frac{3}{2}}\right]\left[-2\sqrt{2}e^{-\frac{3}{2}}\right]-[0]^2$$

$$= 8e^{-3}$$

The values of D and f_{xx} for $\left(-1,\frac{\sqrt{2}}{2}\right)$ worked out to be the same as the values for $\left(1,\frac{\sqrt{2}}{2}\right)$. Once again, D is positive and f_{xx} is negative, so by the Second Derivative Test, **f has a local maximum at $\left(-1,\frac{\sqrt{2}}{2}\right)$**.

Now apply the Second Derivative Test to $\left(1,-\frac{\sqrt{2}}{2}\right)$:

$$D = f_{xx}(a,b)f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$$

$$= \left[2ye^{-x^2-y^2}\left(1-5x^2+2x^4\right)\right]\left[-2yx^2e^{-x^2-y^2}\left(3-2y^2\right)\right]-\left[2xe^{-x^2-y^2}\left(1-x^2\right)\left(1-2y^2\right)\right]^2$$

$$= \left[2\left(-\frac{\sqrt{2}}{2}\right)e^{-1^2-\left(-\frac{\sqrt{2}}{2}\right)^2}\left(1-5(1)^2+2(1)^4\right)\right]\left[-2\left(-\frac{\sqrt{2}}{2}\right)(1)^2e^{-(1)^2-\left(-\frac{\sqrt{2}}{2}\right)^2}\left(3-2\left(-\frac{\sqrt{2}}{2}\right)^2\right)\right]$$

$$-\left[2(1)e^{-(1)^2-\left(-\frac{\sqrt{2}}{2}\right)^2}\left(1-(1)^2\right)\left(1-2\left(-\frac{\sqrt{2}}{2}\right)^2\right)\right]^2$$

$$= \left[2\sqrt{2}e^{-\frac{3}{2}}\right]\left[2\sqrt{2}e^{-\frac{3}{2}}\right]-[0]^2$$

$$= 8e^{-3}$$

The values of D once again turn out to be $8e^{-3}$, a positive value, but this time in the course of finding it we found f_{xx} to be $2\sqrt{2}e^{-\frac{3}{2}}$, also a positive value. Therefore, by the Second Derivative Test, **f has a local minimum at $\left(1,-\frac{\sqrt{2}}{2}\right)$**.

Now apply the Second Derivative Test to $\left(-1,-\frac{\sqrt{2}}{2}\right)$:

$$D = f_{xx}(a,b)f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$$

$$=\left[2ye^{-x^2-y^2}\left(1-5x^2+2x^4\right)\right]\left[-2yx^2e^{-x^2-y^2}(3-2y^2)\right]-\left[2xe^{-x^2-y^2}(1-x^2)(1-2y^2)\right]^2$$

$$=\left[2\left(-\frac{\sqrt{2}}{2}\right)e^{-(-1)^2-\left(-\frac{\sqrt{2}}{2}\right)^2}\left(1-5(-1)^2+2(-1)^4\right)\right]\left[-2\left(-\frac{\sqrt{2}}{2}\right)(-1)^2e^{-(-1)^2-\left(-\frac{\sqrt{2}}{2}\right)^2}\left(3-2\left(-\frac{\sqrt{2}}{2}\right)^2\right)\right]$$

$$-\left[2(-1)e^{-(-1)^2-\left(-\frac{\sqrt{2}}{2}\right)^2}\left(1-(-1)^2\right)\left(1-2\left(-\frac{\sqrt{2}}{2}\right)^2\right)\right]^2$$

$$=\left[2\sqrt{2}e^{-\frac{3}{2}}\right]\left[2\sqrt{2}e^{-\frac{3}{2}}\right]-[0]^2$$

$$=8e^{-3}$$

Just as at $\left(1,-\frac{\sqrt{2}}{2}\right)$, the value of D is $8e^{-3}$, a positive value, and the value of f_{xx} was $2\sqrt{2}e^{-\frac{3}{2}}$, also a positive value. Therefore, by the Second Derivative Test, **f has a local minimum at $\left(-1,-\frac{\sqrt{2}}{2}\right)$**.

We have already shown that f has an infinite number of other critical points, since it also has a critical point at any $\boxed{(0, a)}$. We now calculate D for any of these critical points.

$$D = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$
$$= \left[2ye^{-x^2-y^2}\left(1-5x^2+2x^4\right)\right]\left[-2yx^2e^{-x^2-y^2}(3-2y^2)\right] - \left[2xe^{-x^2-y^2}(1-x^2)(1-2y^2)\right]^2$$
$$= \left[2ae^{-0^2-a^2}\left(1-5(0)^2+2(0)^4\right)\right]\left[-2a(0)^2e^{-(0)^2-a^2}(3-2a^2)\right]$$
$$-\left[2(0)e^{-(0)^2-a^2}(1-(0)^2)(1-2a^2)\right]^2$$
$$= \left[2ae^{-a^2}\right]\left[0\right] - \left[0\right]^2$$
$$= 0$$

So $\boxed{D = 0}$ at the critical points of the form $(0, a)$.

To figure out what is happening at the critical points of the form $(0, a)$, examine f. Since

$f(x,y) = x^2ye^{-x^2-y^2}$

The function value is 0 whenever x is 0. So at all of these critical points, which line the y-axis where $x = 0$, we have $f = 0$.

Let us see what happens when we stray off the y-axis slightly. Examine the positive y-axis first. If we stray slightly in either x-direction, making x slightly positive or slightly negative, the function value of f is positive: x^2 is a square so is positive, y is positive as we are on the positive y-axis, and powers of e are always positive. Since $f = 0$ along the positive y-axis but is greater than 0 if we stray slightly off the y-axis, the points $\boxed{(0,a) \text{ are minima for } y > 0}$. Visually, the positive y-axis forms a trough and increases to either side.

Next examine the negative y-axis. If we stray slightly in either x-direction, making x slightly positive or slightly negative, the function value of f is negative: x^2 is a square so is positive, y is negative as we are on the negative y-axis, and powers of e are always positive. Since $f = 0$ along the negative y-axis but is less than 0 if we stray slightly off the y-axis, the points $\boxed{(0, a) \text{ are maxima for } y < 0}$. Visually, the negative y-axis forms a ridge and decreases to either side.

Finally examine $y = 0$. Since we know that as y is a trough with minimum $f = 0$ on the positive y side and a ridge with maximum $f = 0$ on the negative y side, and since $f(0,0) = 0$ itself, we can visualize what is happening—f is always 0 along the y-axis but on the positive side starts to rise to either side of that line and to the negative side starts to fall to either side of that line.

With this setup, drawing a straight line in the xy-plane through the origin will never result in the function values along that straight line having a minimum at (0, 0) or a maximum at (0,0)—in all cases there will be an inflection point, as any straight line through the origin (other than the y-axis) will rise from the negative y-side and keep on rising on the positive y-side. Since a saddle point requires the cross-sections show both maxima and minima, this is not a saddle point. However, it is not a maximum or minimum either, as when we venture out in different y-directions the graph grows on one side and lessens on the other. So the point (0, 0) is not a minimum, maximum, or saddle point.

Answer 21E.

Consider the function,

$$f(x,y)=x^2+y^2+x^{-2}y^{-2}.$$

The object is to find the local maximum, minimum values and saddle points of the function.

Use maple software to estimate the local maximum, minimum and saddle points.

Maple input command: f:=(x,y)→x^2+y^2+x^(-2)*y^(-2).

Maple output:

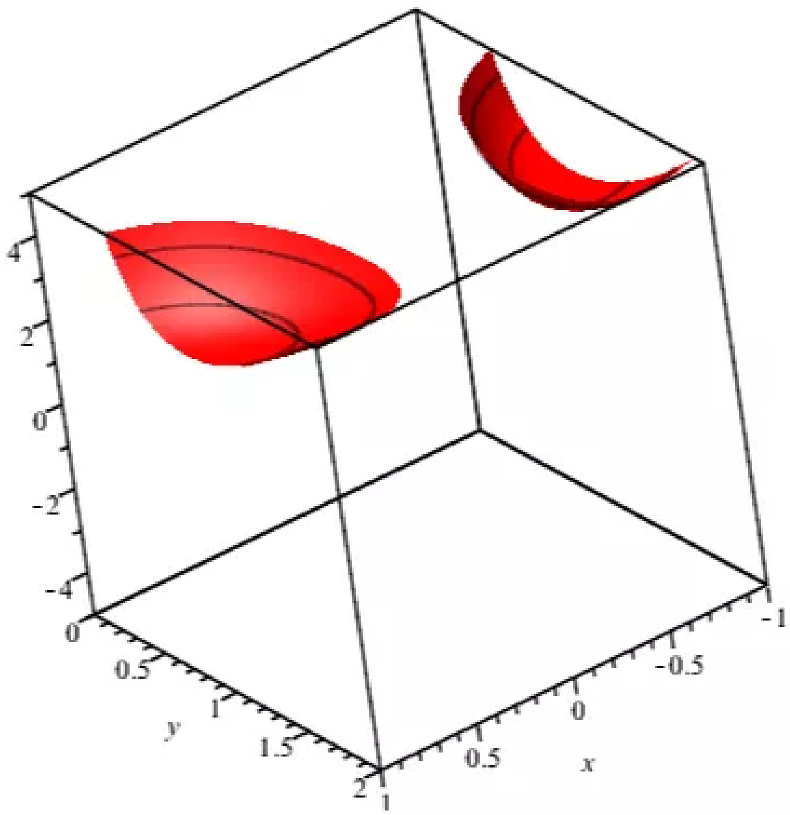
```
> f := (x, y)→x^2 + y^2 + x^(-2)*y^(-2)
```

$$(x,y)\rightarrow x^2+y^2+\frac{1}{x^2y^2}$$

Maple input command:

```
> plot3d(f(x,y), x = -1..1, y = 0..2, style = patchcontour, color = red, axes = boxed, view = -5..5,
      lightmodel = light4);
```

The graph is shown below:



Use maple software to find the level curves of the function as follows.

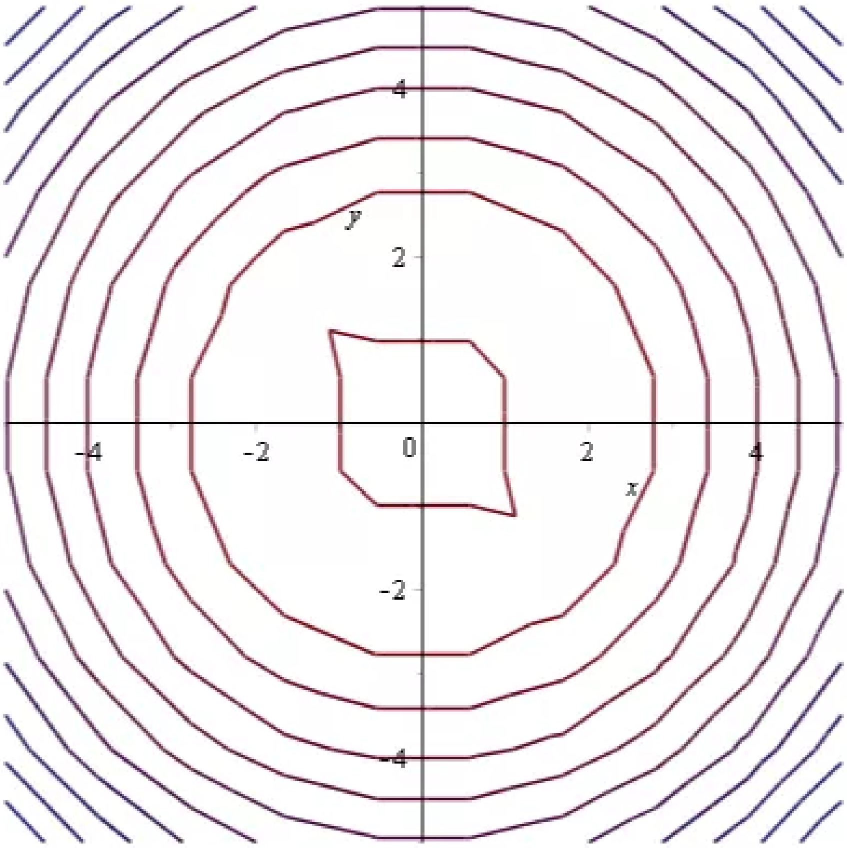
Maple input command:

```
plots[contourplot](f(x,y),x=-5..5,y=-5..5,grid=[10,10],contours=10);
```

Maple output:

```
> plots[contourplot](f(x, y), x = -5 ..5, y = -5 ..5, grid = [10, 10], contours = 10);
```

The contour sketch is shown below:



From the graph, it appears that f has a local minimum $f(\pm1,\pm1)\approx 3$ and the function has no saddle points.

Consider the function,

$$f(x,y)=x^2+y^2+x^{-2}y^{-2}.$$

The partial derivative of $f(x,y)=x^2+y^2+x^{-2}y^{-2}$ with respect to x are,

$$\begin{aligned}
f_x(x,y)&=\frac{\partial}{\partial x}\left(x^2+y^2+x^{-2}y^{-2}\right)\\
&=2x+y^{-2}\frac{\partial}{\partial x}\left(\frac{1}{x^2}\right)\\
&=2x+\frac{1}{y^2}\left(\frac{-2}{x^3}\right)\\
&=2x-\frac{2}{x^3y^2}
\end{aligned}$$

Equating this expression to zero,

$$f_x(x,y)=0$$

$$2x-\frac{2}{x^3y^2}=0$$

$$x=\frac{1}{x^3y^2}$$

$$x^4=\frac{1}{y^2}$$

$$y^2=\frac{1}{x^4}$$

$$y=\pm\frac{1}{x^2}$$

The partial derivative of $f(x,y)$ with respect to y,

$$f_y(x,y)=\frac{\partial}{\partial y}\left(x^2+y^2+x^{-2}y^{-2}\right)$$
$$=2y+x^{-2}\frac{\partial}{\partial y}\left(\frac{1}{y^2}\right)$$
$$=2y+x^{-2}\left(\frac{-2}{y^3}\right)$$
$$=2y-\frac{2}{y^3x^2}$$

Set $f_y(x,y)=0$ for critical points, then

$$2y-\frac{2}{y^3x^2}=0$$
$$y=\frac{1}{y^3x^2}$$
$$x^2=\frac{1}{y^4}$$
$$x=\pm\frac{1}{y^2}$$

$$y^2-y=0$$
$$y=0 \text{ or } y=\pm 1$$

So the point of intersection is,

$(0,0)$ and $(\pm 1,\pm 1)$.

Thus, the four critical points of the function are $(0,0),(-1,\pm 1)$ and $(1,\pm 1)$.

The second partial derivatives of $f(x, y)$ are,

Differential two times with respect to x,

$$f_x(x, y) = 2x - \frac{2}{x^3y^2}$$

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\left(2x - \frac{2}{x^3y^2}\right)$$

$$= 2 - \frac{2}{y^2}\frac{\partial}{\partial x}\left(\frac{1}{x^3}\right)$$

$$= 2 - \frac{2}{y^2}\left(\frac{-3}{x^4}\right)$$

$$= 2 + \frac{6}{x^4y^2}$$

Differential two times with respect to y,

$$f_x(x, y) = 2x - \frac{2}{x^3y^2}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\left(2x - \frac{2}{x^3y^2}\right)$$

$$= \frac{-2}{x^3}\frac{\partial}{\partial y}\left(\frac{1}{y^2}\right)$$

$$= \frac{-2}{x^3}\left(\frac{-2}{y^3}\right)$$

$$= \frac{4}{x^3y^3}$$

Differential two times with respect to y,

$$f_y(x, y) = 2y - \frac{2}{y^3x^2}$$

$$f_{yy}(x, y) = 2 - \frac{2}{x^2}\frac{\partial}{\partial y}\left(\frac{1}{y^3}\right)$$

$$= 2 - \frac{2}{x^2}\left(\frac{-3}{y^4}\right)$$

$$= 2 + \frac{6}{x^2y^4}$$

Suppose the second partial derivatives of f are continuous on a disk with center (a,b) and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$ [(a,b) is a critical point of f].

Let $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2$.

(a) If $D>0$ and $f_{xx}(a,b)>0$ then $f(a,b)$ is a local minimum.

(b) If $D>0$ and $f_{xx}(a,b)<0$ then $f(a,b)$ is a local maximum.

(c) If $D<0$ then $f(a,b)$ is not a local maximum or local minimum.

Plug all the values, the value of D is,

$$D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2$$

$$\begin{aligned} D &= \left(2+\frac{6}{x^4y^2}\right)\left(2+\frac{6}{x^2y^4}\right)-\left[\frac{4}{x^3y^3}\right]^2 \\ &= \frac{20}{x^6y^6}+\frac{12}{x^4y^2}+\frac{12}{x^2y^4}+4 \end{aligned}$$

At the critical point $(0,0)$, then the value of D is,

$$D=\infty(\text{not defined})$$

At the critical point $(1,1),(-1,1),(1,-1)$, and $(-1,-1)$, then the value of D is,

$$D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2$$

$$\begin{aligned} D(\pm1,\pm1) &= \left(2+\frac{6}{x^4y^2}\right)\left(2+\frac{6}{x^2y^4}\right)-\left[\frac{4}{x^3y^3}\right]^2 \\ &= 4 \\ &> 0 \end{aligned}$$

And

$$f_{xx}(x,y)=2+\frac{6}{x^4y^2}$$

$$\begin{aligned} f_{xx}(1,1) &= 8 \\ &> 0 \end{aligned}$$

$$f_{xx}(0,0)=\infty$$

$$\begin{aligned} f_{xx}(1,1) &= 2+\frac{6}{(1)^4(1)^2} \\ &= 8 \\ &> 0 \end{aligned}$$

$$f_{xx}(1,-1) = 2 + \frac{6}{(1)^4(-1)^2}$$
$$= 8 > 0$$
$$f_{xx}(-1,1) = 2 + \frac{6}{(-1)^4(1)^2}$$
$$= 8 > 0$$

Since $D =$ not defined, so the test is in conclusive at the point $(0,0)$.

Since $D > 0$ and $f_{xx} > 0$, the point $(1,1),(-1,-1),(1,-1)$, and $(-1,1)$ is a local minimum.

Therefore, the local minimum values of the function are,

$(1,\pm1)$ and $(-1,\pm1)$.

At the critical point $(1,1)$ the minimum value is,

$$f(x,y) = x^2 + y^2 + x^{-2}y^{-2}$$
$$f(1,\pm1) = 1^2 + (\pm1)^2 + (1)^{-2}(\pm1)^{-2}$$
$$= 3$$
$$f(-1,\pm1) = (-1)^2 + (\pm1)^2 + (-1)^{-2}(\pm1)^{-2}$$
$$= 3$$

Therefore, the local minimum value of the function is 3 at the points $(1,\pm1)$ and $(-1,\pm1)$.

Answer 22E.

$$f(x,y) = xye^{-x^2-y^2}$$

Then $f_x = ye^{-x^2-y^2} - 2x^2ye^{-x^2-y^2}$

$f_y = xe^{-x^2-y^2} - 2xy^2e^{-x^2-y^2}$

First we find the critical points by setting $f_x = 0$, $f_y = 0$

$$(y - 2x^2y)e^{-x^2-y^2} = 0$$
$$(2 - xy^2)e^{-x^2-y^2} = 0$$

i.e. $y(1-2x^2) = 0$

$x(1-2y^2) = 0$

On solving these equations we find the critical points

$$(0,0),\ \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\ \left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right),\ \left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$$

Now $\quad f_{xx} = -2xye^{-x^2-y^2} - 4xye^{-x^2-y^2} + 4x^3ye^{-x^2-y^2}$

$$= 2xy\left(1+2-2x^2\right)e^{-x^2-y^2}$$

$$= -2xy\left(3-2x^2\right)e^{-x^2-y^2}$$

$$f_{yy} = -2xye^{-x^2-y^2} - 4xye^{-x^2-y^2} + 4xy^3e^{-x^2-y^2}$$

$$= -2xy\left(3-2y^2\right)e^{-x^2-y^2}$$

$$f_{xy} = e^{-x^2-y^2} - 2y^2e^{-x^2-y^2} - 2x^2e^{-x^2-y^2} + 4x^2y^2e^{-x^2-y^2}$$

$$= \left(1-2y^2-2x^2+4x^2y^2\right)e^{-x^2-y^2}$$

Then $\quad D = f_{xx}f_{yy} - f_{xy}^2$

$$= \left[4x^2y^2\left(3-2x^2\right)\left(3-2y^2\right) - \left(1-2x^2-2y^2+4x^2y^2\right)\right]e^{-2\left(x^2+y^2\right)}$$

At (0, 0); $\quad D = \left[0-(1-0)\right]e^0$

$$= -1 < 0$$

That is f has a saddle point at (0, 0)

At $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$; $D = \left[4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(3-1)(3-1) - (1-1-1+1)\right]e^{-2}$

$$= [4-0]e^{-2}$$

$$= 4e^{-2} > 0$$

$$f_{xx}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -2\left(\frac{1}{2}\right)(3-1)e^{-1}$$

$$= -2e^{-1} < 0$$

That is f has local maximum at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

At $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$; $D = 4e^{-2} > 0$

And $\quad f_{xx}\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = -2e^{-1} < 0$

That is f has local maximum at $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

And maximum value is $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) - f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{1}{2}e^{-1}$

At $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$; $D = 4e^{-2} > 0$

And $\quad f_{xx}\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = 2\left(\frac{1}{2}\right)(3-1)e^{-1}$

$$= 2e^{-1} > 0$$

That is f has local minimum at $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

At $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$; $D = 4e^{-2} > 0$

And $\quad f_{xx}\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2\left(\frac{1}{2}\right)(3-1)e^{-1}$

$$= 2e^{-1} > 0$$

That is f has local minimum at $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

And the minimum value is:

$$f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{-1}{2}e^{-1}$$

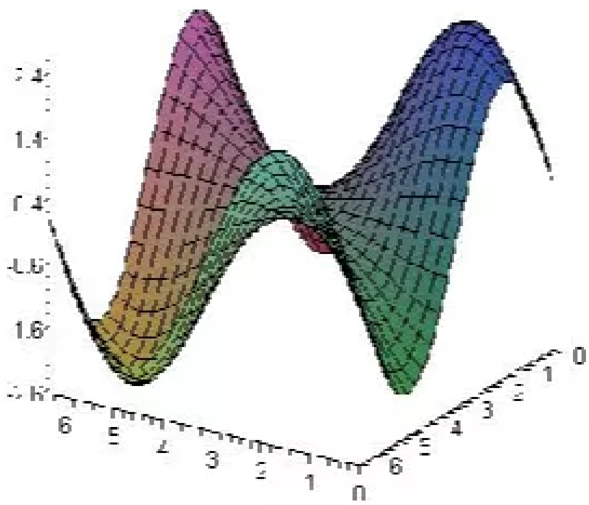
Hence f has saddle point at $\boxed{(0,\ 0)}$, local maximum at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

With maximum value $\boxed{1/2e}$

And it has local minimum at $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ with minimum value $\boxed{-1/2e}$

Answer 23E.

$$f(x,y) = \sin x + \sin y + \sin(x+y)$$
$$0 \le x \le 2\pi, \quad 0 \le y \le 2\pi$$



Then $f_x = \cos x + \cos(x+y)$

$f_y = \cos y + \cos(x+y)$

First we locate critical points by setting $f_x = 0$, $f_y = 0$

i.e. $\cos x + \cos(x+y) = 0$

And $\cos y + \cos(x+y) = 0$

i.e. $\cos x = \cos y$

i.e. $x = y$

Then only values of x and y $\in [0, 2\pi]$ and satisfying equations $f_x = 0$, $f_y = 0$ are

$$x = y = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

Then the critical points are $(\pi, \pi), \left(\frac{\pi}{3}, \frac{\pi}{3}\right), \left(\frac{5\pi}{3}, \frac{5\pi}{3}\right)$

Now $f_{xx} = -\sin x - \sin(x+y)$

$f_{yy} = -\sin y - \sin(x+y)$

$f_{xy} = -\sin(x+y)$

Then $D = f_{xx} f_{yy} - f_{xy}^2$

$$= [\sin x + \sin(x+y)][\sin y + \sin(x+y)] - \sin^2(x+y)$$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$; $D = \left[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right]\left[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right] - \frac{3}{4}$

$$= 3\frac{3}{4} = \frac{9}{4} > 0$$

And $f_{xx}\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = -\sin\frac{\pi}{3} - \sin\left(\frac{2\pi}{3}\right)$

$$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$$

$$= -\sqrt{3} < 0$$

That is f has locate maximum at $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$

With the maximum value; $f\left(\frac{\pi}{3},\frac{\pi}{3}\right)=\sin\frac{\pi}{3}+\sin\frac{\pi}{3}+\sin\left(\frac{2\pi}{3}\right)$

$$=\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}$$

$$=\boxed{\frac{3\sqrt{3}}{2}}$$

At $\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)$; $D=\left[-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right]\left[-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right]-\frac{3}{4}$

$$=3-\frac{3}{4}=\frac{9}{4}>0$$

And $f_{xx}\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)=-\sin\frac{5\pi}{3}-\sin\left(\frac{10\pi}{3}\right)$

$$=\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}$$

$$=\sqrt{3}>0$$

That is f has locate minimum at $\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)$

With minimum value $f\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)$

Answer 24E.

$$f(x,y)=\sin x+\sin y+\cos(x+y)$$

$$0\le x\le\frac{\pi}{4},\ 0\le y\le\frac{\pi}{3}$$

Then $f_x=\cos x-\sin(x+y)$

$$f_y=\cos y-\sin(x+y)$$

First we find the critical point by setting $f_x = 0$, $f_y = 0$

i.e. $\cos x - \sin(x+y) = 0$

And $\cos y - \sin(x+y) = 0$

i.e. $\cos x = \cos y$

i.e. $x = y$

The only values of $x, y \in \left[0, \frac{\pi}{4}\right]$ satisfying $f_x = 0$ and $f_y = 0$ are $x = y = \frac{\pi}{6}$

Then the critical point is $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$

Now $f_{xx} = -\sin x - \cos(x+y)$

$f_{yy} = -\sin y - \cos(x+y)$

$f_{xy} = -\cos(x+y)$

Then $D = f_{xx} f_{yy} - f_{xy}^2$

$= \left[\sin x + \cos(x+y)\right]\left[\sin y + \cos(x+y)\right] - \cos^2(x+y)$

At $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$; $D = \left[\frac{1}{2} + \frac{1}{2}\right]\left[\frac{1}{2} + \frac{1}{2}\right] - \frac{1}{4}$

$= 1 - \frac{1}{4}$

$= \frac{3}{4} > 0$

And $f_{xx}\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = -\sin\frac{\pi}{6} - \cos\frac{\pi}{3}$

$f_{xx}\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = -\frac{1}{2} - \frac{1}{2}$

$= -1 < 0$

That is f has local maximum at $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$

With maximum value

$$f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \sin\frac{\pi}{6} + \sin\frac{\pi}{6} + \cos\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$= \boxed{\frac{3}{2}}$$

Answer 25E.

When one of the variables is associated with another in an equation, the function is called the implicit function.

The partial derivative of a function in two or more variables with respect to a variable is determined by keeping the other variables as a constant in the function.

Consider the function:

$$f(x,y)=x^4+y^4-4x^2y+2y$$

Determine the partial derivatives as shown below:

$$\begin{aligned} f_x(x,y) &= 4x^3-8xy \\ &= x(4x^2-8y) \end{aligned}$$

$$f_{xx}(x,y)=12x^2-8y$$

$$f_{xy}(x,y)=-8x$$

Also, the other partial derivatives are:

$$f_y(x,y)=4y^3-4x^2+2$$

$$f_{yy}(x,y)=12y^2$$

Equate $f_x(x,y)$ and $f_y(x,y)$ to 0.

$$x(4x^2-8y)=0$$

$$4y^3-4x^2+2=0$$

Solve the obtained equations to get the critical points.

The critical points are:

$$\boxed{\begin{array}{l}(0,-0.794),(1.592,1.267),(-1.592,1.267),\\(0.720,0.259)\text{ and }(-0.720,0.259)\end{array}}$$

Evaluate $D(0,-0.794)$ as shown below:

$$D = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

$$\begin{aligned} D(0,-0.794) &= f_{xx}(0,-0.794)f_{yy}(0,-0.794) - \left[f_{xy}(0,-0.794)\right]^2 \\ &= (6.352)(7.565) - [0]^2 \\ &= 48.053 \end{aligned}$$

Similarly obtain the values for the other coordinates:

$$D(1.592,1.267) = 228.411$$
$$D(-1.592,1.267) = 228.411$$
$$D(0.720,0.259) = -29.838$$
$$D(-0.720,0.259) = -29.838$$

Observe that $D(0,-0.794) > 0$ and $f_{xx}(0,-0.794) > 0$.

So, the point $f(0,-0.794)$ is a local minimum.

Replace *x* with 0 and *y* with -0.794 to obtain $f(0,-0.794) \approx -1.191$.

Similarly, observe that:

$$D(1.592,1.267) > 0 \text{ and } D(-1.592,1.267) > 0$$
$$f_{xx}(1.592,1.267) > 0 \text{ and } f_{xx}(-1.592,1.267) > 0$$

So, the points *f*(1.592, 1.267) and *f*(−1.592, 1.267) are also the local minimum.

Substitute the known values in the original function:

$$f(x,y) = x^4 + y^4 - 4x^2y + 2y$$
$$f(\pm 1.592, 1.267) \approx -1.310$$

Since, the points $D(0.720,0.259) < 0$ and $D(-0.720,0.259) < 0$, so the function has saddle point at $f(\pm 0.720, 0.259)$.

Compare the values of *f* at its local minimum points, so that the absolute minimum value of *f* is $f(\pm 1.592, 1.267) \approx -1.310$.

Hence, the absolute minimum point is $\boxed{-1.310}$.

Answer 26E.

Find the critical points by examining where the partial derivatives both equal 0, or where one of them doesn't exist. Then use the Second Derivative Test to classify them, and compare the extrema to find the absolute maximum and absolute minimum.

First find the critical points. For that we need the partial derivatives.
Differentiate in terms of *x*, holding *y* constant:

$f(x,y) = y^6 - 2y^4 + x^2 - y^2 + y$

$f_x(x,y) = 2x$

The partial derivative f_x always exists and equals 0 when $x = 0$.

Differentiate in terms of *y*, holding *x* constant:

$f(x,y) = y^6 - 2y^4 + x^2 - y^2 + y$

$f_y(x,y) = 6y^5 - 8y^3 - 2y + 1$

The partial derivative f_y always exists. Find where it equals 0:

$0 = 6y^5 - 8y^3 - 2y + 1$

The problem indicates that a calculator should be used to solve this. Plugging it in, we get the following approximate solutions for *y*:

$y \approx -1.345, 0.347, 1.211$

The (*x*, *y*) points that make both f_x and f_y equal 0 are:

$$\boxed{\begin{matrix}(0,-1.345)\\(0,0.347)\\(0,1.211)\end{matrix}}$$

These are the critical points of *f*.
Use the Second Derivative Test to see whether the critical points are maxima or minima.
The Second Derivative Test: Let (*a*, *b*) be a critical point of function *f*, and let $D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$. Then:

If $D > 0$ and $f_{xx} > 0$, then $f(a,b)$ is a local minimum.
If $D > 0$ and $f_{xx} < 0$, then $f(a,b)$ is a local maximum.
If $D < 0$, then (a,b) is a saddle point.
If $D = 0$, then the test is inconclusive.

In order to find D for the Second Derivative Test, we must find the second partial derivatives.

Find f_{xx}:

$f_x(x,y) = 2x$

$f_{xx}(x,y) = 2$ ...... (1)

Find f_{xy}:

$f_x(x,y) = 2x$

$f_{xy}(x,y) = 0$

Find f_{yy}:

$f_y(x,y) = 6y^5 - 8y^3 - 2y + 1$

$f_{yy}(x,y) = 30y^4 - 24y^2 - 2$

Note from (1) that $f_{xx} = 2$ always, so is always positive. This will be important when applying the Second Derivative Test.

Find D and test the critical points:

$$\begin{aligned} D &= f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2 \\ &= (2)(30y^4 - 24y^2 - 2) - (0)^2 \\ &= 60y^4 - 48y^2 - 4 \end{aligned}$$

For critical point (0,-1.345),

$$\begin{aligned} D &= 60(-1.345)^4 - 48(-1.345)^2 - 4 \\ &\approx 105.521 \end{aligned}$$

Since $D > 0$ and $f_{xx} > 0$, by the Second Derivative Test *f* has a local minimum at (0,-1.345).

For critical point (0, 0.347),

$$\begin{aligned} D &= 60(0.347)^4 - 48(0.347)^2 - 4 \\ &\approx -8.910 \end{aligned}$$

Since $D < 0$, by the Second Derivative Test *f* has a saddle point at (0, 0.347).

For the critical point (0, 1.211),

$$\begin{aligned} D &= 60(1.211)^4 - 48(1.211)^2 - 4 \\ &\approx 54.648 \end{aligned}$$

Since $D > 0$ and $f_{xx} > 0$, by the Second Derivative Test *f* has a local minimum at (0, 1.211).

Find the function values at the critical points:

$f(x,y) = y^6 - 2y^4 + x^2 - y^2 + y$

$$\begin{aligned} f(0,-1.345) &= (-1.345)^6 - 2(-1.345)^4 + (0)^2 - (-1.345)^2 + (-1.345) \\ &\approx -3.779 \end{aligned}$$

$$f(0,0.347) = (0.347)^6 - 2(0.347)^4 + (0)^2 - (0.347)^2 + (0.347)$$
$$\approx 0.199$$
$$f(0,1.211) = (1.211)^6 - 2(1.211)^4 + (0)^2 - (1.211)^2 + (1.211)$$
$$\approx -1.403$$

The full classification of the critical points is therefore:

Local minima:
$f(0,-1.345) \approx -3.779$
$f(0,1.211) \approx -1.403$

Saddle point:
$(0,0.347)$ with function value approximately 0.199.

The limit of the function as x and y both go to infinity is infinity, so the function has no absolute maximum (also, it can have no absolute maximum as it has no local maximum). Since y^6 dominates the y-terms as y increases and x^2 is the only x term, and since both have even exponents and are therefore positive, as the function moves sufficiently out from the origin it will always be positive. Its polynomial natures mean it is continuous, so since it eventually increases in all directions, it must have an absolute minimum. The absolute minimum will be the least of the local minima, or -3.779.

Therefore, the function has the following absolute extrema: lowest point -3.779 at (0,-1.345), and no highest point.

Answer 27E.

We have $f(x,y) = x^4 + y^3 - 3x^2 + y^2 + x - 2y + 1$.

Finding we have

$f_x(x,y), f_y(x,y), f_{xx}(x,y), f_{yy}(x,y)$ and $f_{xy}(x,y)$.

$$f_x(x, y) = 4x^3 - 6x + 1$$
$$f_{xx}(x, y) = 12x^2 - 6$$
$$f_y(x, y) = 3y^2 + 2y - 2$$
$$f_{yy}(x, y) = 6y + 2$$
$$f_{xy}(x, y) = 0$$

We equate $f_x(x, y)$ to 0 and $f_y(x, y)$ to 0 to get $4x^3 - 6x + 1 = 0$ and $3y^2 + 2y - 2 = 0$.

Using a graphing device, solve the obtained equations.

We get the critical points as

$(0.170, -1.215), (-1.301, 0.549), (1.131, 0.549), (-1.301, -1.215)$

$(0.170, 0.549), (1.131, -1.215)$

Evaluating $D(0.170, -1.25)$ given by $D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

$$\begin{aligned} D(0.170, -1.215) &= f_{xx}(0.170, -1.215) f_{yy}(0.170, -1.215) - [f_{xy}(0.170, -1.215)]^2 \\ &= (-5.65)(-5.29) - [0]^2 \\ &= 29.89 \end{aligned}$$

Similarly, we get

$D(-1.301, 0.549) = 75.76$

$D(1.131, 0.549) = 49.49$

$D(-1.301, -1.215) = -75.71$

$D(0.170, 0.549) = -29.93$

$D(1.131, -1.215) = -49.46$

We note that

$D(0.170, -1.215) > 0$ and $f_{xx}(0.170, -1.215) < 0$.

This means that $f(0.170, -1.215)$ is a local maximum.

On replacing x with 0.170 and y with -1.215, we get $f(0.170, -1.215) \approx 3.197$.

Similarly, $D(-1.301, 0.549) > 0$ and $D(1.131, 0.549) > 0$.

Also $f_{xx}(-1.301, 0.549) > 0$ and $f_{xx}(1.131, 0.549) > 0$.

Then, $f(-1.301, 0.549)$ and $f(1.131, 0.549)$ are the local minimum.

We substitute the known values in $f(x, y) = x^4 + y^3 - 3x^2 + y^2 + x - 2y + 1$ to get $f(-1.301, 0.549) \approx -3.145$ and $f(1.131, 0.549) \approx --0.701$.

Since

$D(-1.301, -1.215) < 0, D(0.170, 0.549) < 0, D(1.131, -1.215) < 0$

we can say that the given function has saddle point at the corresponding points.

Answer 28E.

Find the critical points by examining where the partial derivatives both equal 0, or where one of them doesn't exist. Then use the Second Derivative Test to classify them, and compare the extrema and the points on the boundary to find the absolute maximum and absolute minimum.

First find the critical points. For that we need the partial derivatives.
Differentiate in terms of x, holding y constant:

$$f(x,y) = 20e^{-x^2-y^2}\sin 3x\cos 3y$$

$$f_x(x,y) = 20e^{-x^2-y^2}(-2x)(\sin 3x\cos 3y) + 20e^{-x^2-y^2}(3\cos 3x\cos 3y) \quad \ldots\ldots (1)$$

$$= 20e^{-x^2-y^2}\cos 3y(3\cos 3x - 2x\sin 3x)$$

The partial derivative f_x always exists.
Differentiate in terms of y, holding x constant:

$$f(x,y) = 20e^{-x^2-y^2}\sin 3x\cos 3y$$

$$f_y(x,y) = 20e^{-x^2-y^2}(-2y)(\sin 3x\cos 3y) + 20e^{-x^2-y^2}(-3\sin 3x\sin 3y) \quad \ldots\ldots (2)$$

$$= -20e^{-x^2-y^2}\sin 3x(2y\cos 3y + 3\sin 3y)$$

The partial derivative f_y always exists.

Find where both partial derivatives equal 0. Since $20e^{-x^2-y^2}$ is always positive, equation (1) equals 0 when $\cos 3y = 0$ or when $3\cos 3x - 2x\sin 3x = 0$. Likewise, equation (2) equals 0 when $\sin 3x = 0$ or when $2y\cos 3y + 3\sin 3y = 0$. Solve these equations:

$$\cos 3y = 0$$

$$3y = \frac{\pi}{2} + k\pi$$

$$y = \frac{\pi}{6} + \frac{k\pi}{3}$$

Since $|y| \le 1$, this gives the solutions $y = \pm\frac{\pi}{6}$.

As calculator assistance is indicated for this problem, we solve $3\cos 3x - 2x\sin 3x = 0$ using a calculator to get $x = \pm 0.430$ as the solutions between -1 and 1.

Therefore, to make f_x zero, we must have $x = \pm 0.430$ or $y = \pm\frac{\pi}{6}$.

Solve the equations that make equation (2) equal zero:

$$\sin 3x = 0$$

$$3x = 0 + k\pi$$

$$x = \frac{k\pi}{3}$$

Since $|y| \le 1$, this gives the solution $x = 0$.

As calculator assistance is indicated for this problem, we solve $2y\cos 3y + 3\sin 3y = 0$ using a calculator to get $y = 0, \pm 0.872$ as the solutions between -1 and 1.

Therefore, to make f_y zero, we must have $x = 0$, $y = 0$ or $y = \pm 0.872$.

Since both f_x and f_y must be zero at the critical points, we mix and match to find all possible (*x*, *y*) matches that make both partial derivatives zero. Do this by matching each *x* that makes f_x zero with a *y* that makes f_y zero and then vice versa to get:

$$\begin{array}{|l|}\hline (.430, 0) \\ (-.430, 0) \\ (.430, .872) \\ (-.430, .872) \\ (.430, -.872) \\ (-.430, -.872) \\ (0, \frac{\pi}{6}) \\ (0, -\frac{\pi}{6}) \\ \hline \end{array}$$

These are the critical points for *f*.

Use the Second Derivative Test to see whether the critical points are maxima or minima.

The Second Derivative Test: Let (*a*, *b*) be a critical point of function *f*, and let $D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$. Then:

- If $D > 0$ and $f_{xx} > 0$, then $f(a,b)$ is a local minimum.
- If $D > 0$ and $f_{xx} < 0$, then $f(a,b)$ is a local maximum.
- If $D < 0$, then (a,b) is a saddle point.
- If $D = 0$, then the test is inconclusive.

In order to find D for the Second Derivative Test, we must find the second partial derivatives.

Find f_{xx}:

$$f_x(x,y) = 20e^{-x^2-y^2}\cos 3y(3\cos 3x - 2x\sin 3x)$$

$$\begin{aligned} f_{xx}(x,y) &= 20e^{-x^2-y^2}\cos 3y(-2x)(3\cos 3x - 2x\sin 3x) \\ &\quad + 20e^{-x^2-y^2}\cos 3y(-9\sin 3x - (2\sin 3x + 6x\cos 3x)) \qquad \ldots\ldots (3) \\ &= 20e^{-x^2-y^2}\cos 3y(-6x\cos 3x + 4x^2\sin 3x - 11\sin 3x - 6x\cos 3x) \\ &= 20e^{-x^2-y^2}\cos 3y(4x^2\sin 3x - 11\sin 3x - 12x\cos 3x) \end{aligned}$$

Find f_{xy}:

$$f_x(x,y) = 20e^{-x^2-y^2}\cos 3y(3\cos 3x - 2x\sin 3x)$$

$$f_{xy}(x,y) = (20e^{-x^2-y^2}(-2y)(\cos 3y) - 20e^{-x^2-y^2}(3)\sin 3y)(3\cos 3x - 2x\sin 3x)$$
$$+20e^{-x^2-y^2}\cos 3y(0)$$
$$= -20e^{-x^2-y^2}(2y\cos 3y + 3\sin 3y)(3\cos 3x - 2x\sin 3x)$$

Find f_{yy}:

$$f_y(x,y) = -20e^{-x^2-y^2}\sin 3x(2y\cos 3y + 3\sin 3y)$$

$$f_{yy}(x,y) = -20e^{-x^2-y^2}(-2y)\sin 3x(2y\cos 3y + 3\sin 3y)$$
$$-20e^{-x^2-y^2}\sin 3x(2\cos 3y - 3(2)y\sin 3y + 9\cos 3y)$$
$$= -20e^{-x^2-y^2}\sin 3x(-4y^2\cos 3y - 6y\sin 3y + 11\cos 3y - 6y\sin 3y)$$
$$= -20e^{-x^2-y^2}\sin 3x(-4y^2\cos 3y + 11\cos 3y - 12y\sin 3y)$$

Find D:

$$D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$
$$= [20e^{-x^2-y^2}\cos 3y(4x^2\sin 3x - 11\sin 3x - 12x\cos 3x)]$$
$$[-20e^{-x^2-y^2}\sin 3x(-4y^2\cos 3y + 11\cos 3y - 12y\sin 3y)]$$
$$-(-20e^{-x^2-y^2}(2y\cos 3y + 3\sin 3y)(3\cos 3x - 2x\sin 3x))^2$$

Test the critical points in D and in f_{xx} (equation (3)), then apply the Second Derivative Test to classify the critical points. As this is a problem with calculator assistance, plug into a calculator to find the values.

For critical point $(.430,0)$, $D = 32971.256$ and $f_{xx} = -187.657$. As $D > 0$ and $f_{xx} < 0$, there is a local maximum at this critical point.

For critical point $(-.430,0)$, $D = 32971.256$ and $f_{xx} = 187.657$. As $D > 0$ and $f_{xx} > 0$, there is a local minimum at this critical point.

For critical point $(.430,.872)$, $D = 9598.098$ and $f_{xx} = 75.886$. As $D > 0$ and $f_{xx} > 0$, there is a local minimum at this critical point.

For critical point $(-.430,.872)$, $D = -1796.205$ and $f_{xx} = -75.886$. As $D < 0$, this critical point is a saddle point.

For critical point $(.430,-.872)$, $D = -1796.205$ and $f_{xx} = 75.886$. As $D < 0$, this critical point is a saddle point.

For critical point $(-.430,-.872)$, $D = 9598.098$ and $f_{xx} = -75.886$. As $D > 0$ and $f_{xx} < 0$, there is a local maximum at this critical point.

For critical point $(0, \pi/6)$, $D = -18724.767$ and $f_{xx} = 0$. As $D < 0$, this critical point is a saddle point.
For critical point $(0, -\pi/6)$, $D = -18724.767$ and $f_{xx} = 0$. As $D < 0$, this critical point is a saddle point.

For the points that are local extrema, we find the values of those extrema, i.e., the function values at those critical points.
$f(.430, 0) = 15.973$
$f(-.430, 0) = -15.973$
$f(.430, .872) = -6.459$
$f(-.430, -.872) = 6.459$

The full classification of the critical points is therefore:

Local minima:
$f(-.430, 0) = -15.973$
$f(.430, .872) = -6.459$

Local maxima:
$f(.430, 0) = 15.973$
$f(-.430, -.872) = 6.459$

Saddle points:
$(-.430, .872)$
$(.430, -.872)$
$(0, \pi/6)$
$(0, -\pi/6)$

Now we find absolute extrema. We're looking within $|x| \leq 1$ and $|y| \leq 1$, which a closed and bounded region, so the function does have an absolute maximum and an absolute minimum. The possible absolute extrema are either the most extreme of the local extrema or any point on the boundary. We find upper and lower bounds on the boundary of the function.
On the $x = 1$ boundary,

$$f(x, y) = 20e^{-x^2-y^2} \sin 3x \cos 3y$$
$$= 20e^{-1^2-y^2} \sin 3(1) \cos 3y$$
$$= 20e^{-1-y^2} \sin 3 \cos 3y$$

Since y^2 is always positive, any increase in the absolute value of y makes the exponent more negative and therefore decreases $20e^{-1-y^2}$, so $20e^{-1-y^2}$ is maximal at $y=0$ and minimal at $y=\pm 1$. The maximum and minimum values of $\cos 3y$ are 1 and -1 (though they do not necessarily correspond with the maximum and minimum values of $20e^{-1-y^2}$).
At $y=0, 20e^{-1-y^2}=7.357$, and the maximum value of $\cos 3y$ anywhere is 1, so we know that on the $x=1$ boundary,

$$f(x,y)=20e^{-1-y^2}\sin 3\cos 3y$$
$$<(7.357)(\sin 3)(1)$$
$$f(x,y)<1.038$$

Which is already less than any of the local maxima.

The minimum value of $\cos 3y$ anywhere is -1, and $\cos 3y$ being negative is the only way to make the function negative as $20e^{-1-y^2}$ and $\sin 3$ are both positive. The lower boundary on the function will be the *most* negative value we can possibly get, so we still use the maximum value for $20e^{-1-y^2}$ to calculate the lower bound on the function, since the minimum value for $20e^{-1-y^2}$ will not make the function as negative.
So we know that on the $x=1$ boundary:

$$f(x,y)=20e^{-1-y^2}\sin 3(-1)$$
$$>(7.357)(\sin 3)(-1)$$
$$f(x,y)>-1.038$$

Which is already greater than any of the local minima.
On the $x=-1$ boundary, we do the same thing:

$$f(x,y)=20e^{-x^2-y^2}\sin 3x\cos 3y$$
$$=20e^{-1^2-y^2}\sin 3(-1)\cos 3y$$
$$=-20e^{-1-y^2}\sin 3\cos 3y$$

Notice that this is the opposite of the function when $x=1$. Therefore, its lower bound will be the opposite of the upper bound on the $x=1$ boundary and its upper bound will be the opposite of the lower bound on the $x=1$ boundary. Since both upper and lower bounds on the $x=1$ boundary had the same absolute value, switching them changes nothing, and the bounds on the $x=-1$ boundary are exactly the same, and as such are less extreme than the local maxima and minima.

On the $y=1$ boundary, we have:

$$f(x,y)=20e^{-x^2-y^2}\sin 3x\cos 3y$$
$$=20e^{-x^2-1^2}\sin 3x\cos 3(1)$$
$$=20e^{-x^2-1}\sin 3x\cos 3$$

Here $\cos 3$ is a negative constant, so the upper bound on this function will be when $\sin 3x = -1$ and $20e^{-x^2-1}$ is maximal. Since the x and y are symmetric in the $20e^{-x^2-y^2}$ expression, its maximum will be the same as its maximum value on the x boundaries, or 7.357. The lower bound on this function will be when $\sin 3x = 1$ and $20e^{-x^2-1}$ still has its maximum value of 7.357, as that will make the function as negative as possible.

The upper bound:

$$f(x,y) = 20e^{-1-y^2} \sin 3(-1)$$
$$< (7.357)(-1)(\cos 3)$$
$$f(x,y) < 7.283$$

Which is less than the greatest of the local maxima.

The lower bound:

$$f(x,y) = 20e^{-1-y^2} \sin 3(-1)$$
$$> (7.357)(1)(\cos 3)$$
$$f(x,y) > -7.283$$

This is greater than the least of the local minima.

As y^2 and $\cos 3y$ are both even functions, $f(x,y)$ is symmetric in y, and therefore the upper and lower bounds are the same on the $y = -1$ boundary as they were on the $y = 1$ boundary and are still not as extreme as the most extreme of the local extrema.

Since none of the points on the boundary can exceed or be less than the most extreme of the local maxima and minima, we choose the greatest local maximum and the least local minimum and know that they are the highest and lowest points of the function in the given region. Therefore, the function has the following absolute extrema:

Lowest point -15.973 at (-.430, 0

Highest point 15.973 at (.430, 0)

Answer 29E.

Consider the function:

$$f(x,y)=x^2+y^2-2x$$

Here D is the closed triangular region with vertices $(2,0),(0,2)$ and $(0,-2)$.

The objective is to find the absolute maximum and minimum values.

Since f is a polynomial, it is continuous on the bounded triangular region D.

Find partial derivatives of $f(x,y)$ with respect to x and y

$$f(x,y)=x^2+y^2-2x$$
$$f_x(x,y)=2x-2 \qquad \text{......(1)}$$

And

$$f_y(x,y)=2y \qquad \text{......(2)}$$

Now, set $f_x=0$ and $f_y=0$ to find the critical points.

So,

$$f_x(x,y)=0$$
$$2x-2=0$$
$$x=1$$

And

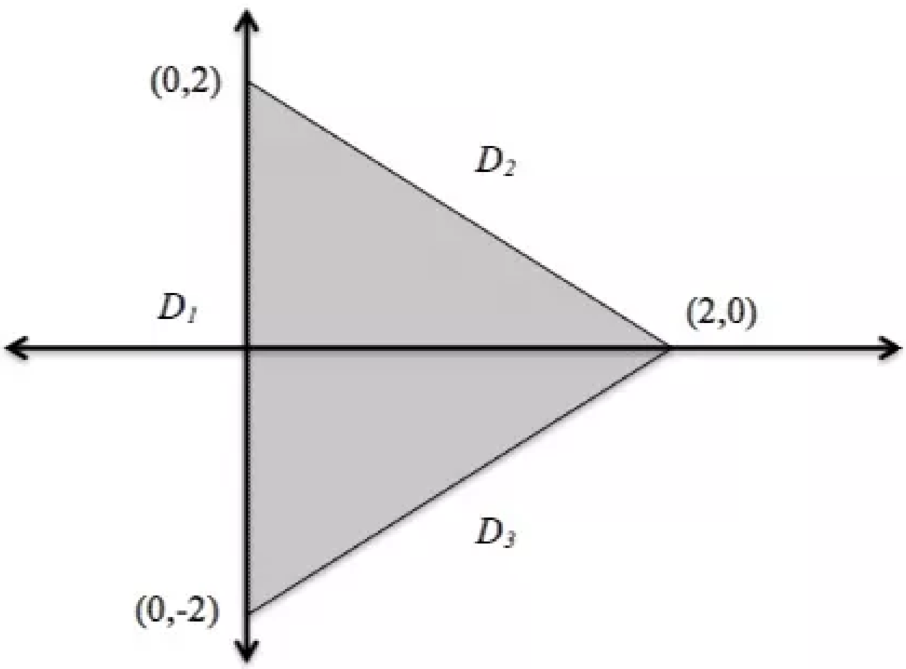
$$f_y(x,y)=0$$
$$2y=0$$
$$y=0$$

Hence, the critical point is $(1,0)$.

The value of the function f at $(1,0)$ is,

$$f(x,y)=x^2+y^2-2x$$
$$f(1,0)=1^2+0^2-2(1)$$
$$=-1$$

The closed triangular region D is drawn below.



Now evaluate the values of the function on the boundary of D.

Along line D_1, since $x=0$.

$$f(x,y)=x^2+y^2-2x$$
$$f(0,y)=0^2+y^2-2(0)$$
$$=y^2$$

This is an increasing function of f, the maximum value lies at $y=-2$ and $y=2$

$$f(0,2)=4$$
$$f(0,-2)=4$$

Along line D_2, $y = 2 - x$

$$f(x,2-x) = x^2 + (2-x)^2 - 2x$$
$$= x^2 + 4 + x^2 - 4x - 2x$$
$$= 2x^2 - 6x + 4$$

Along line D_2, $0 \le x \le 2$.

The value of the function at these endpoints is,

$$f(0,2) = 4$$
$$f(2,0) = 2(2)^2 - 6(2) + 4$$
$$= 8 - 12 + 4$$
$$= 0$$

Find the partial derivative with respect to *x* of $f(x,2-x)$ and set it zero to find the critical points.

$$f_x(x,2-x) = 4x - 6$$
$$4x - 6 = 0$$
$$x = \frac{3}{2}$$

Now plug this value of *x*, into the relation $y = 2 - x$.

$$y = 2 - x$$
$$= 2 - \frac{3}{2}$$
$$= \frac{1}{2}$$

The value of f at the critical point $\left(\frac{3}{2}, \frac{1}{2}\right)$ is,

$$f(x,y) = x^2 + y^2 - 2x$$
$$= \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - 2\left(\frac{3}{2}\right)$$
$$= \frac{9}{4} + \frac{1}{4} - 3$$
$$= \frac{-2}{4}$$
$$\approx -0.5$$

Hence, along D_2 the maximum value is $f(0,2) = 4$ and the minimum value is $f(2,0) = 0$.

Along line D_3, $y = x - 2$

$$f(x, x-2) = x^2 + (x-2)^2 - 2x$$
$$= x^2 + 4 + x^2 - 4x - 2x$$
$$= 2x^2 - 6x + 4$$

Since this function is the same as the one we found for D_2, the maximum and minimum values along D_3 are the same.

Hence, along D_3 the maximum value is $f(0,2) = 4$ and the minimum value is $f(2,0) = 0$.

Compare the maximum and minimum values inside D and along the boundaries D_1, D_2 and D_3.

Hence, the absolute maximum at $\boxed{f(0,\pm 2) = 4}$ and absolute minimum at $\boxed{f(1,0) = -1}$.

Answer 30E.

Consider the following function on the set D, where D is the closed triangular region with vertices $(0,0),(0,2),(4,0)$

$$f(x,y) = x + y - xy$$

To find the absolute maximum and absolute minimum of f on the set D, use the following procedure.

1. Find the values of f at the critical points of f in D

2. Find the extreme values of f on the boundary of D

3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Step1:

Consider,

$$f(x,y) = x + y - xy$$

$$f_x = 1 - y, f_y = 1 - x$$

To find the critical points, solve the equations $f_x = 0, f_y = 0$

$$1 - y = 0$$
$$y = 1$$
$$1 - x = 0$$
$$x = 1$$

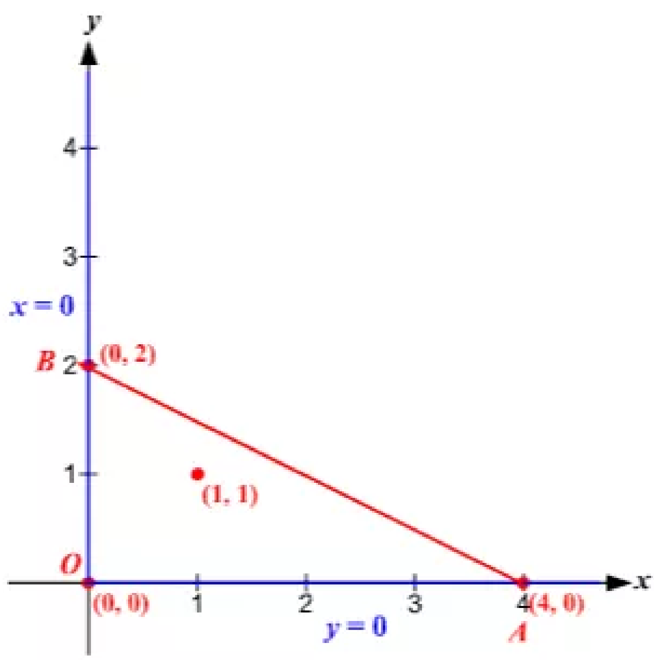
Thus, the critical point is obtained as $(1,1)$.

The values of f at the critical point $(1,1)$ is,

$$f(1,1) = 1 + 1 - 1 \cdot 1$$
$$= 1$$

Step 2:

The sketch of the region D is shown below:



The equation of the boundary $\overline{BA}$ (in intercept form) is,

$$\frac{x}{4}+\frac{y}{2}=1$$

$$y=\frac{4-x}{2}$$

Along $\overline{BA}$,

$$f\left(x,\frac{4-x}{2}\right)=x+\frac{4-x}{2}-x\left(\frac{4-x}{2}\right)$$

$$=\frac{2x+4-x-4x+x^2}{2}$$

$$=\frac{x^2-3x+4}{2}$$

$$f'\left(x,\frac{4-x}{2}\right)=\frac{2x-3}{2}$$

$$f'\left(x,\frac{4-x}{2}\right)=0$$

$$x=\frac{3}{2}$$

$$f''\left(x,\frac{4-x}{2}\right)=1$$

At $x=\frac{3}{2}$, $f''\left(x,\frac{4-x}{2}\right)=1>0$

So, by the Second Derivative Test, the minimum value of f is,

$$f\left(\frac{3}{2},\frac{4-\frac{3}{2}}{2}\right)=f\left(\frac{3}{2},\frac{5}{4}\right)$$

$$=\frac{3}{2}+\frac{5}{4}-\frac{3}{2}\cdot\frac{5}{4}$$

$$=\frac{12+10-15}{8}$$

$$=\frac{7}{8}$$

The equation of $\overline{AO}$ is, $y = 0$

Along $\overline{AO}$,

$$f(x,0) = x + 0 - x(0)$$
$$= x, \qquad \text{where } 0 \le x \le 4$$

Therefore the minimum of f is $f(0,0) = 0$, and the maximum of f is $f(4,0) = 4$

The equation of $\overline{OB}$ is, $x = 0$

Along $\overline{OB}$,

$$f(0,y) = 0 + y - (0)y$$
$$= y, \qquad \text{where } 0 \le y \le 2$$

Therefore the minimum of f is $f(0,0) = 0$, and the maximum of f is $f(0,2) = 2$

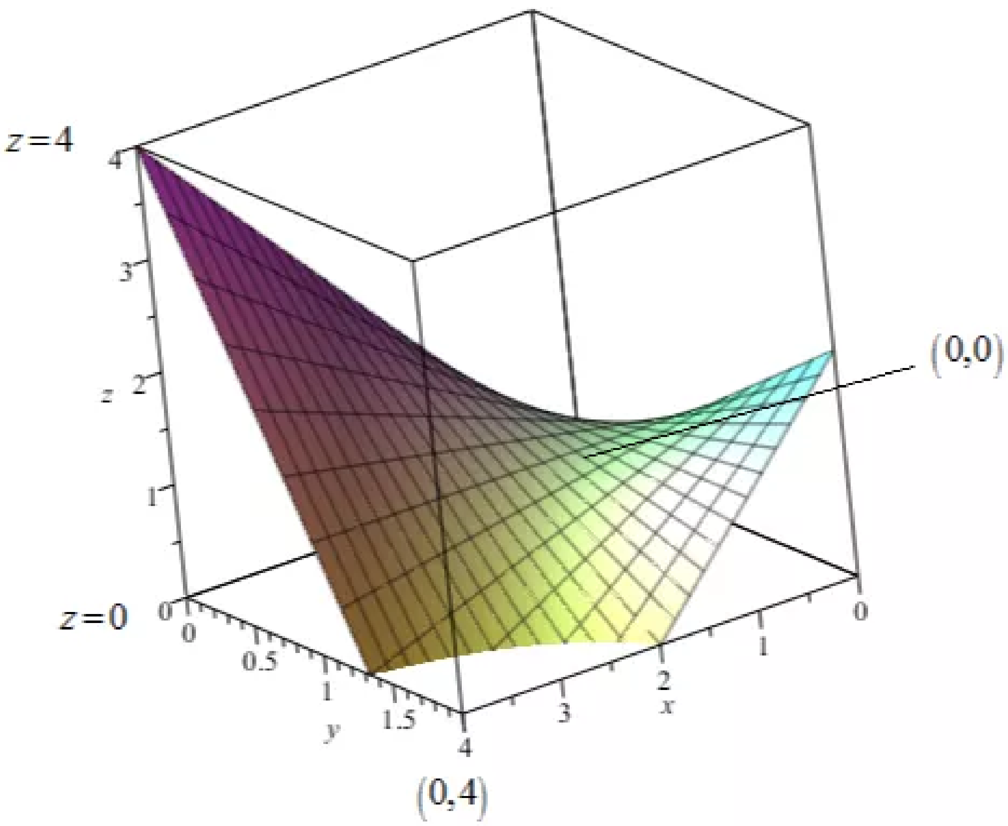
Step 3:

The smallest value of f from the steps 1 and 2 is 0

And the largest value of f from the steps 1 and 2 is 4

So, the absolute minimum of f is $f(0,0) = \boxed{0}$, and absolute maximum of f is $f(0,4) = \boxed{4}$

The absolute minimum and absolute maximum of f are as shown below:



Answer 31E.

Consider the function $f(x,y)=x^2+y^2+x^2y+4.$

And the domain $D=\{(x,y)\,|\,|x|\le 1,|y|\le 1\}.$

The objective is to find the absolute maximum and minimum of f on the set D.

The partial derivative of $f(x,y)$ with respect to x to get,

$$f_x = 2x+2xy$$

The partial derivatives of $f(x,y)$ with respect to y to get,

$$f_y = 2y+x^2$$

To find the critical points, set $f_x = 0$ and $f_y = 0$.

$$2x + 2xy = 0$$
$$2x(1+y) = 0 \qquad \text{and} \quad 2y + x^2 = 0$$
$$x = 0 \text{ or } y = -1$$

Substitute $x = 0$ in $2y + x^2 = 0$ to get,

$$2y + (0)^2 = 0$$
$$2y = 0$$
$$y = 0$$

So, $(0,0)$ is a critical point.

Substitute $y = -1$ in $2y + x^2 = 0$ to get,

$$2(-1) + x^2 = 0$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}$$

So, $(\pm\sqrt{2}, -1)$ are two critical points.

Hence, the critical points are $(0,0)$ and $(\pm\sqrt{2}, -1)$.

The value of f at $(0,0)$ is,

$$f(x,y) = x^2 + y^2 + x^2y + 4$$
$$f(0,0) = 0 + 0 - (0)(0) + 4$$
$$= 4$$

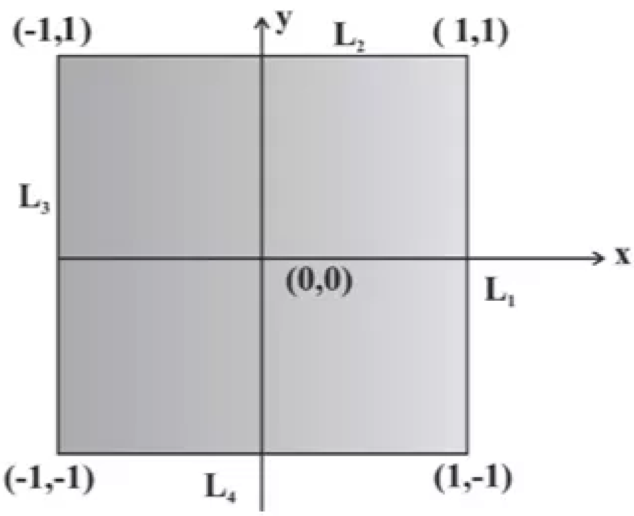
The value of f at $(\pm\sqrt{2}, -1)$ is,

$$f(x,y) = x^2 + y^2 + x^2y + 4$$
$$f(\pm\sqrt{2}, -1) = (\pm\sqrt{2})^2 + (-1)^2 - (\pm\sqrt{2})^2(-1) + 4$$
$$= 5$$

Observe the values of *f* on the boundary of $D = \{(x, y): |x| \le 1,\ |y| \le 1\}$.

So, $D = \{(x, y): -1 \le x \le 1, -1 \le y \le 1\}$

The set *D* consists of line segments $L_1, L_2, L_3, \text{and } L_4$, as shown in the figure below.



Evaluate the values of f on the boundary of *D*.

Along line L_1, $x = 1$, the function *f* becomes,

$$f(1, y) = 1 + y^2 + y + 4$$
$$= y^2 + y + 5 \qquad -1 \le y \le 1$$

This is an increasing function of f, the maximum value lies at $y = 1$.

$$f(1, y) = y^2 + y + 5$$
$$f(1,1) = 1 + 1 + 5$$
$$= 7$$

To find the critical point of $f(1, y)$, find its partial derivative with respect to y.

$$f_y(1, y) = 2y + 1$$

For critical points, set $f_y(1, y) = 0$.

$$2y + 1 = 0$$
$$y = -\frac{1}{2}$$

Hence, the critical point of the function is $\left(1, \frac{-1}{2}\right)$.

So, the value of the function at critical point $\left(1, \frac{-1}{2}\right)$ is,

$$f(x, y) = x^2 + y^2 + x^2 y + 4$$
$$f\left(1, \frac{-1}{2}\right) = 1^2 + \left(\frac{-1}{2}\right)^2 + (1)^2\left(\frac{-1}{2}\right) + 4$$
$$= 1 + \frac{1}{4} - \frac{1}{2} + 4$$
$$= \frac{19}{4}$$

On line L_2, $y=1$, the function becomes,

$$f(1,y)=x^2+1+x^2+4$$
$$=2x^2+5 \qquad -1\le x\le 1$$

This is an increasing function of f, the maximum value lies at $x=-1$ and $x=1$

$$f(\pm1,1)=2(\pm1)^2+5$$
$$=2+5$$
$$=7$$

So, *f* has a maximum value $f(1,1)=f(-1,1)=7$

To find the critical point of $f(x,1)$, find its partial derivative with respect to y.

$$f_x(x,1)=4x+0$$

For critical points, set $f_x(x,1)=0.$

$$4x=0$$
$$x=0$$

Hence, the critical point of the function is $(0,1)$.

The value of *f* at $(0,1)$ is:

$$f(0,1)=2(0)^2+5$$
$$=0+5$$
$$=5$$

And a minimum value $f(0,1)=5.$

Along line L_3, $x=-1$, the function *f* becomes,

$$f(x,y)=x^2+y^2+x^2y+4$$
$$f(-1,y)=(-1)^2+y^2+(-1)^2\cdot y+4$$
$$=y^2+y+5 \qquad -1<y<1$$

To find the critical point of $f(-1,y)$, find its partial derivative with respect to y.

$$f_y(-1,y)=2y+1$$

For critical points, set $f_y(-1, y) = 0.$

$$2y + 1 = 0$$

$$y = -\frac{1}{2}$$

Hence, the critical point of the function is $\left(-1, \frac{-1}{2}\right)$.

This is a parabolic function with maximum value $f(-1,1) = 7$ and minimum value $f\left(-1, \frac{-1}{2}\right) = \frac{19}{4}$.

Along line L_4, $y = -1$, the function *f* becomes,

$$f(x, y) = x^2 + y^2 + x^2 y + 4$$

$$f(x, -1) = x^2 + (-1)^2 + x^2 \cdot (-1) + 4$$

$$= 5$$

Which is a constant function with $f(-1,-1) = f(1,-1) = 5.$

Thus, by comparing these values with the value of *f* obtained at critical points to see that the absolute maximum of *f* is $\boxed{7}$ and the absolute minimum is $\boxed{4}$.

Answer 32E.

Consider the function

$$f(x, y) = 4x + 6y - x^2 - y^2 \text{ ...... (1)}$$

And the domain $D = \{(x, y) \mid 0 \le x \le 3, 0 \le y \le 2\}$

Find the absolute maximum and minimum of the function.

First find the critical points.

To find the critical points find the first derivative partially with respect to *x* and *y*. and equate them to zero.

Differentiate (1) with respect to *x*.

$$f_x = 4 - 2x$$

Differentiate (1) with respect to *y*.

$$f_y = 6 - 2y$$

Locate the critical points by setting $f_x = 0$, $f_y = 0$

$$4 - 2x = 0$$
$$2(2 - x) = 0$$
$$x = 2$$

And

$$6 - 2y = 0$$
$$2(3 - y) = 0$$
$$y = 3$$

On solving these equations, the critical points $x = 2, y = 3$

Substitute the critical points in (1).

$f(2,3) = 4(2) + 6(3) - (2)^2 - (3)^2$ Use (1)

$= 8 + 18 - 4 - 9$

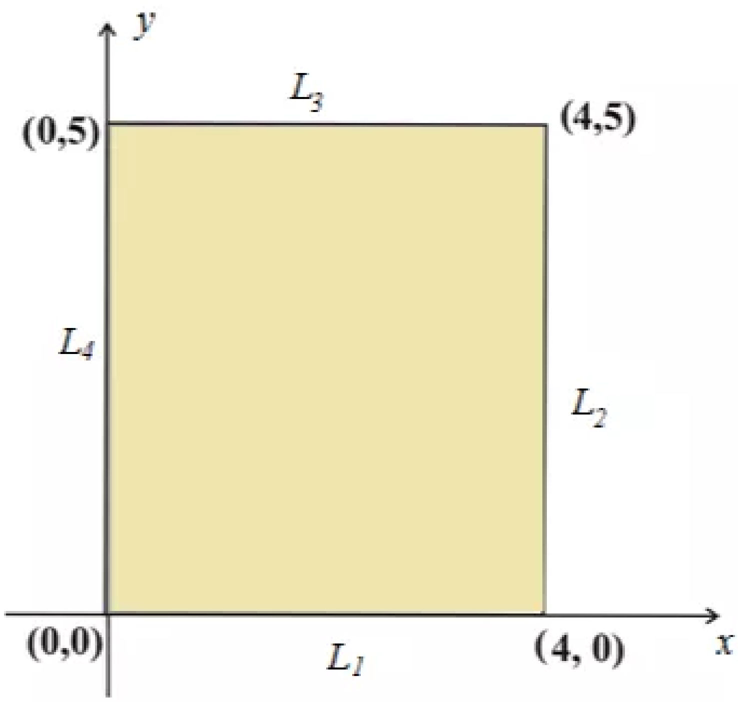
$= 13$

Find the extreme value of *f* on the boundary.

Observe the value of "*f*" on the boundary of

$D = \{(x, y) : 0 \le x \le 4, 0 \le y \le 5\}$

The boundary *D* constant of the line segments L_1, L_2, L_3 and L_4 is as shown in the figure below.



On line $L1$, $y = 0$

So,

$f(x,0) = 4x - x^2$, $0 \le x \le 4$

This is a parabolic function of x.

Find $f(0,0), f(4,0)$ to find the maximum and minimum.

$$f(0,0) = 4 \cdot 0 - (0)^2$$
$$= 0$$

$$f(4,0) = 4 \cdot 4 - (4)^2$$
$$= 16 - 16$$
$$= 0$$

At the critical point $x = 2$

$$f(x,0) = 4 \cdot 2 - 2^2$$
$$= 8 - 4$$
$$= 4$$

So, the minimum value on the line L_1 is 0 and the maximum is 4.

On line $L2$, $x = 4$

So,

$f(4,y) = 16 + 6y - 16 - y^2$, $0 \le y \le 5$

$$= -y^2 + 6y$$

This is a parabolic function of y.

Find $f(4,0), f(4,5)$ to find the maximum and minimum.

$$f(4,0) = -0^2 + 6 \cdot 0$$
$$= 0$$

$$f(4,5) = -5^2 + 6 \cdot 5$$
$$= -25 + 30$$
$$= 5$$

At the critical point $x = 2$

$$f(4,3) = -3^2 + 6 \cdot 3$$
$$= -9 + 18$$
$$= 9$$

So, the minimum value on the line L_2 is 0 and the maximum is 9.

On line $L3$, $y=5$ and

$$f(x,5)=4x+30-x^2-25,\ 0\le x\le 4$$

$$=-x^2+4x+5$$

This is a parabolic function of *x*.

Find $f(0,5), f(4,5)$ to find the maximum and minimum.

$$f(0,5)=-0^2+4\cdot 0+5$$
$$=5$$

$$f(4,5)=-4^2+4\cdot 4+5$$
$$=-16+16+5$$
$$=5$$

At the critical point $x=2$

$$f(2,5)=-2^2+4\cdot 2+5$$
$$=-4+8+5$$
$$=9$$

So, the minimum value on the line L_3 is 5 and the maximum is 9.

On line $L4$, $x=0$

$$f(0,y)=6y-y^2,\ 0\le y\le 5$$

This is a parabolic function of *y*.

Find $f(0,0), f(0,5)$ to find the maximum and minimum.

$$f(0,0)=6\cdot 0-0^2$$
$$=0$$

$$f(0,5)=6\cdot 5-5^2$$
$$=30-25$$
$$=5$$

At the critical point $y=3$

$$f(0,3)=6\cdot 3-3^2$$
$$=18-9$$
$$=9$$

So, the minimum value on the line L_3 is 0 and the maximum is 9.

On comparing these values with the value of the polynomial function f obtained at critical point, see that the absolute maximum of f is 13 and absolute minimum is 0.

Answer 33E.

$$f(x,y) = x^4 + y^4 - 4xy + 2$$

Then $f_x = 4x^3 - 4y$

$f_y = 4y^3 - 4x$

First we find the critical points by setting $f_x = 0$, $f_y = 0$

i.e. $4(x^3 - y) = 0$

And $4(y^3 - x) = 0$

On solving these equations we find the critical points (0, 0), (1, 1) and (-1, -1)

But (-1, -1) does not lie in region D
We consider critical points (0, 0) and (1, 1)

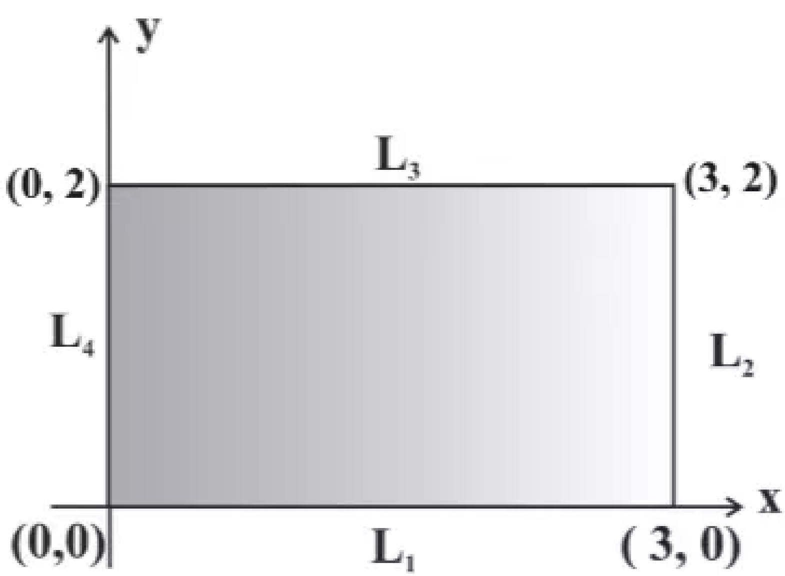
Now $f(0,0) = 2$

$f(1,1) = 0$

Now we observe the values of "f" on the boundary of

$$D = \{(x,y):\ 0 \le x \le 3,\ 0 \le y \le 2\}$$

Where D consists of the line segments L_1, L_2, L_3 and L_4 as shown in the figure below



On line L_1, $y = 0$ and

$$f(x,0) = x^4 + 2, \qquad 0 \le x \le 3$$

Which has minimum value $f(0, 0) = 2$ and maximum value $f(3, 0) = 83$

On line L_2, $x = 3$, and

$$f(3,y) = 81 + y^4 - 12y + 2, \quad 0 \le y \le 2$$
$$= y^4 - 12y + 83$$

Which has maximum value $f(3, 0) = 83$ and minimum value $f\left(3, 3^{1/3}\right) \approx 70$

On line L_3, $y = 2$ and

$$f(x,2) = x^4 + 16 - 8x + 2, \qquad 0 \le x \le 3$$
$$= x^4 - 8x + 18$$

Which has maximum value $f(3, 2) = 75$ and minimum value $f\left(\sqrt[3]{2}, 2\right) = 10.44$

On line L_4, $x = 0$ and

$$f(0,y) = y^4 + 2, \qquad 0 \le y \le 2$$

Which has maximum value $f(0, 2) = 18$ and minimum value $f(0, 0) = 2$

On comparing all these values with the value of "f" obtained at the critical points, we see that the absolute maximum of "f" is $f(3, 0) = \boxed{83}$ and absolute minimum is $f(1, 1) = \boxed{0}$

Answer 34E.

Consider the following function:

$$f(x,y) = xy^2 \quad \cdots\cdots(1)$$

Find the absolute maximum and minimum of the function *f* on the domain.

$$D = \{(x,y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}.$$

First find the critical points.

To find the critical points, find the first derivative partially with respect to *x* and *y* and equate them to zero.

Differentiate (1) with respect to *x* as follows:

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}\left[xy^2\right] \\ &= y^2 \end{aligned}$$

Differentiate (1) with respect to *y* as follows:

$$\begin{aligned} f_y &= \frac{\partial}{\partial y}\left[xy^2\right] \\ &= 2xy \end{aligned}$$

Set the equation $f_x(x,y)$ equal to zero as follows:

$$\begin{aligned} y^2 &= 0 \\ y &= 0 \end{aligned}$$

Set the equation $f_y(x,y)$equal to zero as follows:

$$\begin{aligned} 2xy &= 0 \\ x = 0 \quad &\text{or} \quad y = 0. \end{aligned}$$

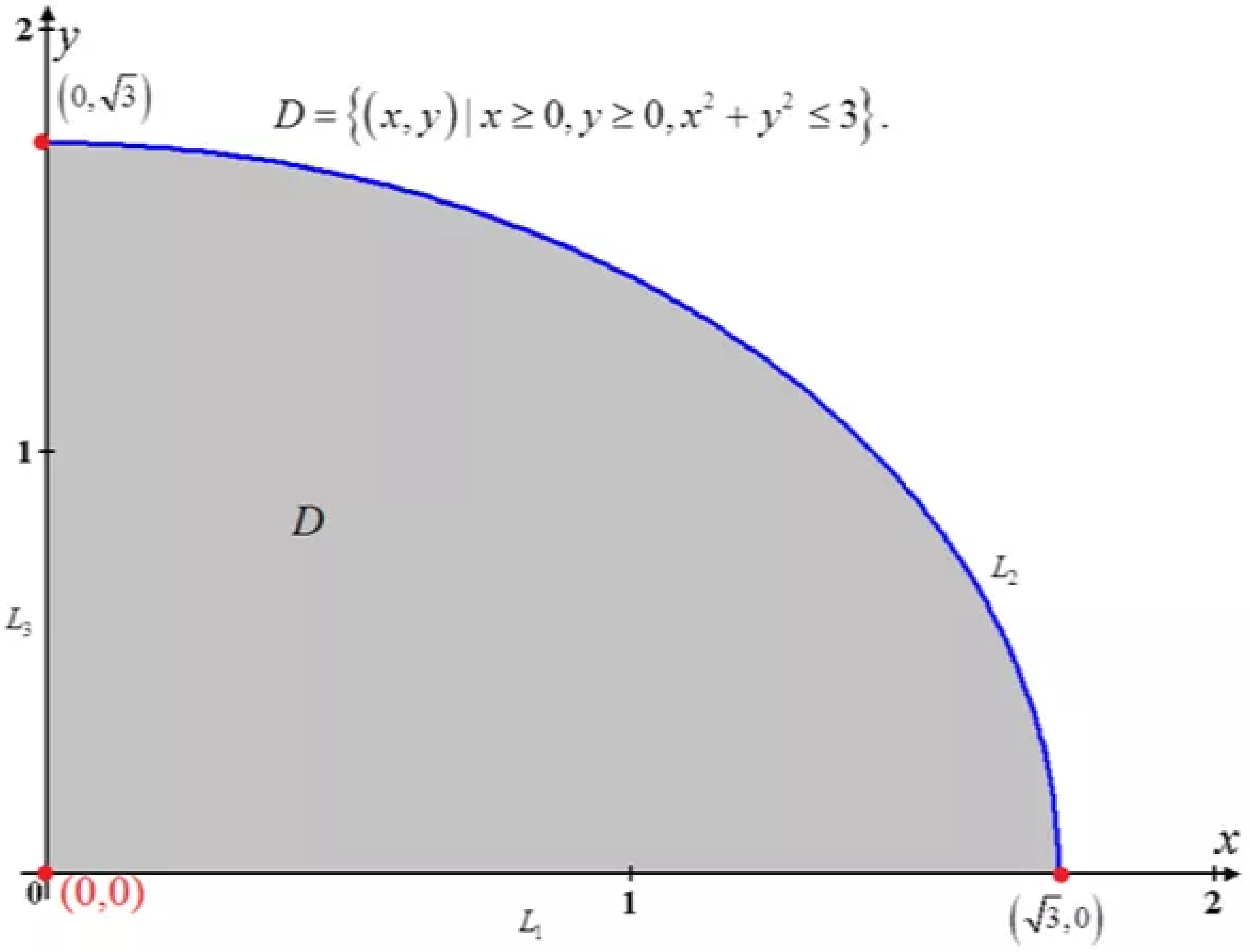
Therefore, the only critical point of the function $f(x,y) = xy^2$ is $(0,0)$.

Substitute the critical point $(0,0)$ in $f(x,y) = xy^2$, to get the following:

$$\begin{aligned} f(0,0) &= (0)(0)^2 \\ &= 0 \end{aligned}$$

Next, find the values of *f* on the boundary of *D*:

The boundary of *D* is shown below:



From the figure observe that, the boundary *D* consist of the line segments L_1, L_2, L_3.

Look at the values of *f* on the boundary of *D*.

The boundary of the region is $x^2 + y^2 = 3$.

$$x^2 + y^2 = 3$$
$$y^2 = 3 - x^2, \qquad 0 \le x \le \sqrt{3}$$

Substitute the above value in equation (1) to get,

$$f(x,y) = xy^2$$
$$f(x,x) = x(3 - x^2)$$
$$= 3x - x^3$$

Let $f(x,x) = g(x) = 3x - x^3$

The partial derivative of g with respect to x is,

$$\begin{aligned} g_x(x) &= \frac{\partial}{\partial x}(g(x)) \\ &= \frac{\partial}{\partial x}(3x - x^3) \\ &= 3 - 3x^2 \end{aligned}$$

To find critical points of g, set $g_x(x) = 0$

$$\begin{aligned} 3 - 3x^2 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

But, the value of x ranges from 0 to $\sqrt{3}$.

So, the value $x = -1$ will be omitted.

Hence, $x = 1$.

When $x = 1$ find y:

$$\begin{aligned} y^2 &= 3 - x^2 \\ y^2 &= 3 - (1)^2 \\ &= 2 \\ y &= \pm\sqrt{2} \end{aligned}$$

The critical points are $(1, -\sqrt{2})$ and $(1, \sqrt{2})$

Evaluate the values of f at these points:

$$f(x,y) = xy^2$$

$$f\left(1,-\sqrt{2}\right) = (1)\left(-\sqrt{2}\right)^2$$
$$= 2$$

$$f(x,y) = xy^2$$

$$f\left(1,\sqrt{2}\right) = (1)\left(\sqrt{2}\right)^2$$
$$= 2$$

From the figure observe that, the boundary points of *D* are $(0,0),\left(0,\sqrt{3}\right),\left(\sqrt{3},0\right)$.

Evaluate *f* at these boundary points:

Evaluate *f* at $(0,0)$.

$$f(0,0) = (0)(0)^2$$
$$= 0$$

Evaluate *f* at $\left(0,\sqrt{3}\right)$.

$$f\left(0,\sqrt{3}\right) = (0)\left(\sqrt{3}\right)^2$$
$$= 0$$

Evaluate *f* at $\left(\sqrt{3},0\right)$.

$$f\left(\sqrt{3},0\right) = \left(\sqrt{3}\right)(0)^2$$
$$= 0$$

On comparing these values with the value of the polynomial function f obtained at critical point, see that the absolute maximum of f is 2 and absolute minimum is 0.

Answer 35E.

Consider the following function and the domain:

$$f(x,y)=2x^3+y^4,\quad D=\{(x,y)|x^2+y^2\le 1\}$$

The objective is to find the absolute maximum and minimum values of f on the set D.

Since f is a polynomial, it is continuous on the closed, bounded rectangle, *D*,

Recall the extreme value theorem,

If f is continuous on a closed and bounded set *D*, then f attains an absolute maximum and an absolute minimum in *D*.

As the function f is closed bounded on *D*, f attains an absolute maximum and an absolute minimum in *D*.

To find the absolute maximum and minimum values of a continuous function *f* on a closed and bounded set *D*:

1. Find the values of f at the critical points of f in *D*

2. Find the extreme values of f on the boundary of *D*

3. The largest of the values from step1 and step2 is the absolute maximum value; the smallest value is the absolute minimum value.

The partial derivatives of f with respect to *x* and *y* is:

$$\begin{aligned}f_x(x,y)&=\frac{\partial}{\partial x}(f(x,y))\\&=\frac{\partial}{\partial x}(2x^3+y^4)\\&=6x^2\end{aligned}$$

And

$$\begin{aligned}f_y(x,y)&=\frac{\partial}{\partial y}(f(x,y))\\&=\frac{\partial}{\partial y}(2x^3+y^4)\\&=4y^3\end{aligned}$$

According to step1, first find the critical points.

To find the critical point of f, set $f_x(x,y)=0$ and $f_y(x,y)=0$ and solve the resulting equations.

$$f_x = 0$$
$$6x^2 = 0$$
$$x = 0$$

And

$$f_y = 0$$
$$4y^3 = 0$$
$$y = 0$$

Thus, there is only one critical point $(0,0)$ inside the circle $x^2+y^2=1$.

The value of f at $(0,0)$ is:

$$f(x,y) = 2x^3 + y^4$$
$$f(0,0) = 2(0)^3 + (0)^4$$
$$f(0,0) = 0$$

In step 2; look at the values of f on the boundary of *D*.

The boundary of the region is $x^2+y^2=1$.

$$x^2 + y^2 = 1$$
$$y^2 = 1 - x^2, \qquad -1 \le x \le 1$$

Substitute the above value in equation (1),

$$f(x,y) = 2x^3 + y^4$$
$$f(x,y) = 2x^3 + (y^2)^2$$
$$f(x,x) = 2x^3 + (1-x^2)^2$$
$$= 2x^3 + 1 - 2x^2 + x^4$$
$$= x^4 + 2x^3 - 2x^2 + 1$$

Let $f(x,x)=g(x)$.

The partial derivative of *g* with respect to *x* is,

$$\begin{aligned} g_x(x) &= \frac{\partial}{\partial x}(g(x)) \\ &= \frac{\partial}{\partial x}\left(x^4+2x^3-2x^2+1\right) \\ &= 4x^3+6x^2-4x \end{aligned}$$

To find critical points of *g*, set $g_x(x)=0$

$4x^3+6x^2-4x=0$

$2x\left(2x^2+3x-2\right)=0$ Factor out the common term, $2x$

$2x(x+2)(2x-1)=0$ Factor the expression inside the parentheses

$x=0,\frac{1}{2},-2$ Set each factor equal to zero and solve.

Here, $x=-2$ does not lie in $-1\le x\le 1$

So, -2 is not a critical point of $g(x)$.

Thus, the critical points occur at $x=0$ and $\frac{1}{2}$.

When $x=0$,

$$\begin{aligned} y^2 &= 1-x^2 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

When $x=\frac{1}{2}$,

$$\begin{aligned} y^2 &= 1-x^2 \\ y^2 &= 1-\frac{1}{4} \\ y^2 &= \frac{3}{4} \\ y &= \pm\frac{\sqrt{3}}{2} \end{aligned}$$

The critical points are $(0,\pm 1)$ and $\left(\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)$.

Evaluate the values of f at these points:

$$f(x,y)=2x^3+y^4$$
$$f(0,\pm1)=2\times0^3+(\pm1)^4$$
$$=0+1$$
$$=1$$

And

$$f\left(\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)=2x^3+y^4$$
$$=2\left(\frac{1}{2}\right)^3+\left(\pm\frac{\sqrt{3}}{2}\right)^4$$
$$=2\left(\frac{1}{8}\right)+\frac{9}{16}$$
$$=\frac{2}{8}+\frac{9}{16}$$
$$=\frac{13}{16}$$

The values of f at the end points of the interval are

$$f(x,y)=2x^3+y^4$$
$$f(1,0)=2(1)^3+0^4$$
$$=2+0$$
$$=2$$

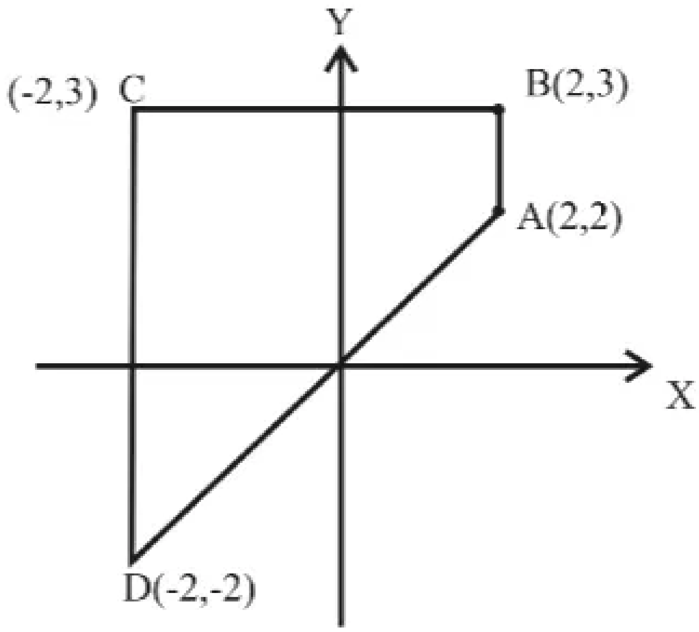
And

$$f(x,y)=2x^3+y^4$$
$$f(-1,0)=2(-1)^3+0^4$$
$$=-2+0$$
$$=-2$$

In step3, comparing these values with the value $f(0,0)=0$ at the critical point

Thus, conclude that the absolute maximum value of f on D is $f(1,0)=\boxed{2}$ and the absolute minimum value of f on D is $f(-1,0)=\boxed{-2}$

Answer 36E.



Given $f(x,y) = x^3 - 3x - y^3 + 12y$ and D is the quadrilateral ABCD.

Differentiating f partially with respect to x,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^3 - 3x - y^3 + 12y\right)$$
$$= 3x^2 - 3$$

Differentiating f partially with respect to y,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^3 - 3x - y^3 + 12y\right)$$
$$= -3y^2 + 12$$

For f to have maximum and minimum values

$\frac{\partial f}{\partial x} = 0$ And $\frac{\partial f}{\partial y} = 0$ give critical points.

Now, $\frac{\partial f}{\partial x} = 0$ gives $3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = 1, -1.$

And $\frac{\partial f}{\partial y} = 0$ gives $-3y^2 + 12 = 0 \Rightarrow y^2 = 4 \Rightarrow y = 2, -2$

Therefore, the critical points are $(1,2)$, $(1,-2)$, $(-1,2)$, $(-1,-2)$

Now the points $(1,-2)$ and $(-1,-2)$ do not lie in D.

Value of f at $(1,2) = x^3 - 3x - y^3 + 12y$
$$= (1)^3 - 3\times1 - (2)^3 + 12\times2$$
$$= 14$$

Value of f at $(-1,2) = x^3 - 3x - y^3 + 12y$
$$= (-1)^3 - 3\times-1 - (2)^3 + 12\times2$$
$$= -1 + 3 - 8 + 24$$
$$= 18.$$

Let us check the maximum and minimum values of f at the points which lie on the boundary of the region D.

Now, along the boundary line AB, $x = 2,\ 2 \le y \le 3$

Therefore,

$$f = x^3 - 3x - y^3 + 12y$$
$$= (2)^3 - 3\times 2 - y^3 + 12y$$
$$= 2 - y^3 + 12y$$

Differentiating f with respect to y.

$$\frac{df}{dy} = -3y^2 + 12$$

For f to be maximum or minimum.

$$\frac{df}{dy} = 0 \Rightarrow \quad -3y^2 + 12 = 0$$
$$\Rightarrow \quad y = 2, -2$$

$y = -2$ is not in $2 \le y \le 3$ so $y \ne -2$

Also $\frac{d^2 f}{dy^2} = -6y$

$\frac{d^2 f}{dy^2}$ at $y = 2$ is $= -6\times 2$

$= -12$ (Negative.)

$\Rightarrow$ f has maximum value at (2,2) along the line AB.

Value of f at $(2,2) = x^3 - 3x - y^3 + 12y$

$$= (2)^3 - 3\times 2 - (2)^3 + 12\times 2$$
$$= 18$$

Since f has maximum value at $y = 2$ and along the line AB x is constant and y is increasing. So f will have minimum value at (2, 3) in the line AB.

Therefore,

Value of f at $(2,3) = x^3 - 3x - y^3 + 12y$

$$= (2)^3 - 3\times 2 - (3)^3 + 12\times 3$$
$$= 8 - 6 - 27 + 36$$
$$= 11$$

Now, along the line BC, $y = 3,\ -2 \le x \le 2$

Therefore, $f = x^3 - 3x - y^3 + 12y$

$$= x^3 - 3x - (3)^3 + 12 \times 3$$

$$= x^3 - 3x + 9$$

Differentiating f with respect to x,

$$\frac{df}{dx} = \frac{d}{dx}\left(x^3 - 3x + 9\right)$$

$$= 3x^2 - 3$$

For f to have maximum and minimum value $\frac{df}{dx} = 0 \Rightarrow \quad 3x^2 - 3 = 0$

$\Rightarrow \quad x^2 = 1$

$\Rightarrow \quad x = 1, -1$

Also $\frac{d^2 f}{dx^2} = 6x$

$\frac{d^2 f}{dx^2}$ at x = 1 is $= 6 \times 1 = 6$ (Positive)

$\frac{d^2 f}{dx^2}$ at x = −1 is $= 6 \times -1 = -6$ (Negative)

Therefore, f has local maximum at (−1, 3) and local minimum at (1, 3) on the line BC.

Therefore, $f = x^3 - 3x - y^3 + 12y$

$$= x^3 - 3x - (3)^3 + 12 \times 3$$

$$= x^3 - 3x + 9$$

Differentiating f with respect to x,

$$\frac{df}{dx} = \frac{d}{dx}\left(x^3 - 3x + 9\right)$$

$$= 3x^2 - 3$$

For f to have maximum and minimum value $\frac{df}{dx} = 0 \Rightarrow \quad 3x^2 - 3 = 0$

$\Rightarrow \quad x^2 = 1$

$\Rightarrow \quad x = 1, -1$

Also $\frac{d^2 f}{dx^2} = 6x$

$\frac{d^2 f}{dx^2}$ at x = 1 is $= 6 \times 1 = 6$ (Positive)

$\frac{d^2 f}{dx^2}$ at x = −1 is $= 6 \times -1 = -6$ (Negative)

Therefore, f has local maximum at (−1, 3) and local minimum at (1, 3) on the line BC.

Now, Value of f at $(-1,3) = x^3 - 3x - y^3 + 12y$

$$= (-1)^3 - 3\times -1 - (3)^3 + 12\times 3$$
$$= -1 + 3 + (-27) + 36$$
$$= 11$$

Value of f at $(1,3) = x^3 - 3x - y^3 + 12y$

$$= (1)^3 - 3\times 1 - (3)^3 + 12\times 3$$
$$= 1 - 3 - 27 + 36$$
$$= 7$$

Value of f at $(-2,3) = x^3 - 3x - y^3 + 12y$

$$= (-2)^3 - 3\times -2 - (3)^3 + 12\times 3$$
$$= -8 + 6 - 27 + 36$$
$$= 7$$

Value of f at $(2,3) = x^3 - 3x - y^3 + 12y$

$$= (2)^3 - 3\times 2 - (3)^3 + 12\times 3$$
$$= 8 - 6 - 27 + 36$$
$$= 11$$

Thus maximum value of f along BC is 11 and minimum value is 7.

Also along the boundary CD, $x = -2, -2 \le y \le 3$

Therefore, $f = x^3 - 3x - y^3 + 12y$

$$= (-2)^3 - 3\times -2 - y^3 + 12y$$
$$= -8 + 6 - y^3 + 12y$$
$$= -y^3 + 12y - 2$$

Differentiating f with respect to y.

$$\frac{df}{dy} = -3y^2 + 12$$

For f to have maximum and minimum values

$$\frac{df}{dy} = 0 \Rightarrow \quad -3y^2 + 12 = 0$$

$\Rightarrow \quad y^2 = 4$

$\Rightarrow \quad y = -2, 2$

Also,

$$\frac{d^2 f}{dy^2} = -6y$$

$\frac{d^2 f}{dy^2}$ at y = −2 is $= -6\times -2 = 12$ (Positive)

$\frac{d^2 f}{dy^2}$ at y = 2 is $= -6\times 2 = -12$ (Negative)

Thus f has local maxima at (−2, 2) and local minima at (−2,−2) along the boundary CD.

Value of f at $(-2,2) = x^3 - 3x - y^3 + 12y$

$$= (-2)^3 - 3\times -2 - (2)^3 + 12\times 2$$
$$= -8 + 6 - 8 + 24$$
$$= 14$$

Value of f at $(-2,-2) = x^3 - 3x - y^3 + 12y$

$$= (-2)^3 - 3\times -2 - (-2)^3 + 12\times -2$$
$$= -8 + 6 + 8 - 24$$
$$= -18$$

Value of f at $(-2,3) = 7$

Thus the maximum value of f is 14 and minimum value of f is -18 along the boundary CD.

Equation of line AD is $y = x,\ -2 \le x \le 2$

Therefore, along the boundary DA

$$f = x^3 - 3x - y^3 + 12y$$
$$= x^3 - 3x - x^3 + 12x$$
$$= 9x$$

$$\frac{df}{dx} = 9 \text{ (Positive)}$$

This tells us that f is increasing along the line DA.

Therefore, maximum value of f along DA will be at (2, 2) and minimum at (–2 ,–2).

Value of f at $(2,2) = x^3 - 3x - y^3 + 12y$

$$= (2)^3 - 3\times 2 - (2)^3 + 12\times 2$$
$$= 8 - 6 - 8 + 24$$
$$= 18$$

Value of f at $(-2,-2) = x^3 - 3x - y^3 + 12y$

$$= (-2)^3 - 3\times -2 - (-2)^3 + 12\times -2$$
$$= -8 + 6 + 8 - 24$$
$$= -18$$

Thus maximum value of f is 18 and minimum value of f is –18 along the boundary.

From all the observations we found that the absolute maximum value of f is 18 and absolute minimum value of f is –18

Hence,

Absolute maximum value of f is 18 which is at $(2,2)$ and
Absolute minimum value of f is -18 which is at $(-2,-2)$

Answer 37E.

Consider the function

$$f(x,y) = -(x^2-1)^2 - (x^2y-x-1)^2$$

First simplify the function as follows.

$$\begin{aligned} f(x,y) &= -(x^2-1)^2 - (x^2y-x-1)^2 \\ &= -(x^4-2x^2+1) - x^4y^2 + 2(x+1)x^2y - (x+1)^2 \end{aligned}$$

$$= -x^4 + 2x^2 - 1 - x^4y^2 + 2x^3y + 2x^2y - x^2 - 2x - 1$$

$$= -x^4 - x^4y^2 + 2x^3y + 2x^2y + x^2 - 2x - 2$$

$$= -x^4(1+y^2) + 2x^3y + 2x^2y + x^2 - 2x - 2$$

It is required to find the critical points of *f* and prove that the function has local maxima at these two critical points.

To find the critical points find the partial derivatives of f independently with respect to *x* and *y* and equate them to zero.

First find the partial derivatives.

$$f_x(x,y) = \frac{\partial}{\partial x}\left[-x^4(1+y^2) + 2x^3y + 2x^2y + x^2 - 2x - 2\right]$$

$$= \frac{\partial}{\partial x}\left[-x^4(1+y^2)\right] + \frac{\partial}{\partial x}\left[2x^3y\right] + \frac{\partial}{\partial x}\left[2x^2y\right] + \frac{\partial}{\partial x}\left[x^2\right] + \frac{\partial}{\partial x}[-2x] + \frac{\partial}{\partial x}[-2]$$

Use sum rule

$$= -(1+y^2)\frac{\partial}{\partial x}\left[x^4\right] + 2y\frac{\partial}{\partial x}\left[x^3\right] + 2y\frac{\partial}{\partial x}\left[x^2\right] + \frac{\partial}{\partial x}\left[x^2\right] - 2\frac{\partial}{\partial x}[x] + \frac{\partial}{\partial x}[-2]$$

Use constant multiple rule

$$= -(1+y^2)(4x^3) + 2y(3x^2) + 2y(2x) + 2x - 2 + 0$$

Use power rule and $\frac{\partial}{\partial x}(k) = 0$

$$= -(1+y^2)(4x^3) + 6x^2y + 4xy + 2x - 2 \text{ ...... (1)}$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left[-x^4(1+y^2) + 2x^3y + 2x^2y + x^2 - 2x - 2\right]$$

$$= \frac{\partial}{\partial y}\left[-x^4(1+y^2)\right] + \frac{\partial}{\partial y}\left[2x^3y\right] + \frac{\partial}{\partial y}\left[2x^2y\right] + \frac{\partial}{\partial y}\left[x^2\right] + \frac{\partial}{\partial y}[-2x] + \frac{\partial}{\partial y}[-2]$$

Use sum rule

$$= -x^4\frac{\partial}{\partial y}\left[(1+y^2)\right] + 2x^3\frac{\partial}{\partial y}[y] + 2x^2\frac{\partial}{\partial y}[y] + \frac{\partial}{\partial y}\left[x^2\right] + \frac{\partial}{\partial y}[-2x] + \frac{\partial}{\partial y}[-2]$$

Use constant multiple rule

$= -x^4(2y) + 2x^3 + 2x^2$ Use power rule and $\frac{\partial}{\partial x}(k) = 0$

$$= -2x^4y + 2x^3 + 2x^2 \text{ ...... (2)}$$

Set the derivatives equal to zero.

$$f_x(x,y) = 0$$

$$-(1+y^2)(4x^3) + 6x^2y + 4xy + 2x - 2 = 0$$

$$f_y(x,y) = 0$$

$$-2x^4y + 2x^3 + 2x^2 = 0$$

$$x^2(-2x^2y + 2x + 2) = 0$$

Solve the equations using maple.

First enter command as follows.

```
solve ({-(1+y^2)*(4*x^3)+6*x^2*y+4*x*y+2*x-2=0,x^2*(-2*x^2*y+2*x+2=0},{x,y});
```

The maple output is as follows.

> $solve(\{-4(x^2-1)x - 2(x^2y - x - 1)(2xy - 1) = 0, -2\cdot(x^2\cdot y - x - 1)\cdot x^2 = 0\}, \{x, y\});$

$\{x = 1, y = 2\}, \{x = -1, y = 0\}$

Therefore, the only possible critical points are $\boxed{(1,2)}$ and $\boxed{(-1,0)}$.

Classify the behaviour of the critical points of f.

Recall the second derivative test,

A function *f* has continuous partial derivatives on disk (a,b) and $f_x(a,b)=0, f_y(a,b)=0$.

Let

$$D=D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$$

a. If $D>0$ and $f_{xx}(a,b)>0$ then $f(a,b)$ is a local minimum

b. If $D>0$ and $f_{xx}(a,b)<0$ then $f(a,b)$ is a local maximum

c. If $D<0$ then $f(a,b)$ is not a local minimum or local maximum.

To find $D(x,y)$, find $f_{xx}(x,y), f_{yy}(x,y)$ and $f_{xy}(x,y)$.

$$f_{xx}=\frac{\partial}{\partial x}\left(-\left(1+y^2\right)\left(4x^3\right)+6x^2y+4xy+2x-2\right)$$

Differentiate (1) with respect to *x*

$$=\frac{\partial}{\partial x}\left(-\left(1+y^2\right)\left(4x^3\right)\right)+\frac{\partial}{\partial x}\left(6x^2y\right)+\frac{\partial}{\partial x}(4xy)+\frac{\partial}{\partial x}(2x)\frac{\partial}{\partial x}(-2)$$

Use sum rule

$$=-12x^2-12x^2y^2+12xy+4y+2$$

$$f_{yy}=\frac{\partial}{\partial y}\left(-2x^4y+2x^3+2x^2\right)$$ Differentiate (2) with respect to *y*

$$=\frac{\partial}{\partial y}\left(-2x^4y\right)+\frac{\partial}{\partial y}\left(2x^3\right)+\frac{\partial}{\partial y}\left(2x^2\right)$$ Use sum rule

$$=-2x^4$$

$$f_{xy}=\frac{\partial}{\partial y}\left(-\left(1+y^2\right)\left(4x^3\right)+6x^2y+4xy+2x-2\right)$$

$$=-8x^3y+6x^2+4x$$

Substitute f_{xx}, f_{yy} and f_{xy} in *D*.

$$D=f_{xx}(x,y)f_{yy}(x,y)-\left[f_{xy}(x,y)\right]^2$$

$$=\left[-12x^2+4-2(2xy-1)^2-4\left(x^2y-x-1\right)y\right]\left[-2x^4\right]-\left[-8x^3y+6x^2+4x\right]^2$$

$$=2\left(12x^2-12x^2y^2+12xy+4y+2\right)x^4-4x^2\left(4x^2y-3x-2\right)^2$$

$$=24x^6+24x^6y^2-24x^5y-8x^4y-4x^4-4x^2\left(4x^2y-3x-2\right)^2$$

$$=24x^6+24x^6y^2-24x^5y-8x^4y-4x^4-4x^2\left(16x^4y^2+9x^2+4-24x^3y+12x-16x^2y\right)$$

$$=24x^6+24x^6y^2-24x^5y-8x^4y-4x^4-64x^6y^2-36x^4-16x^2+96x^5y-48x^3+64x^4y$$

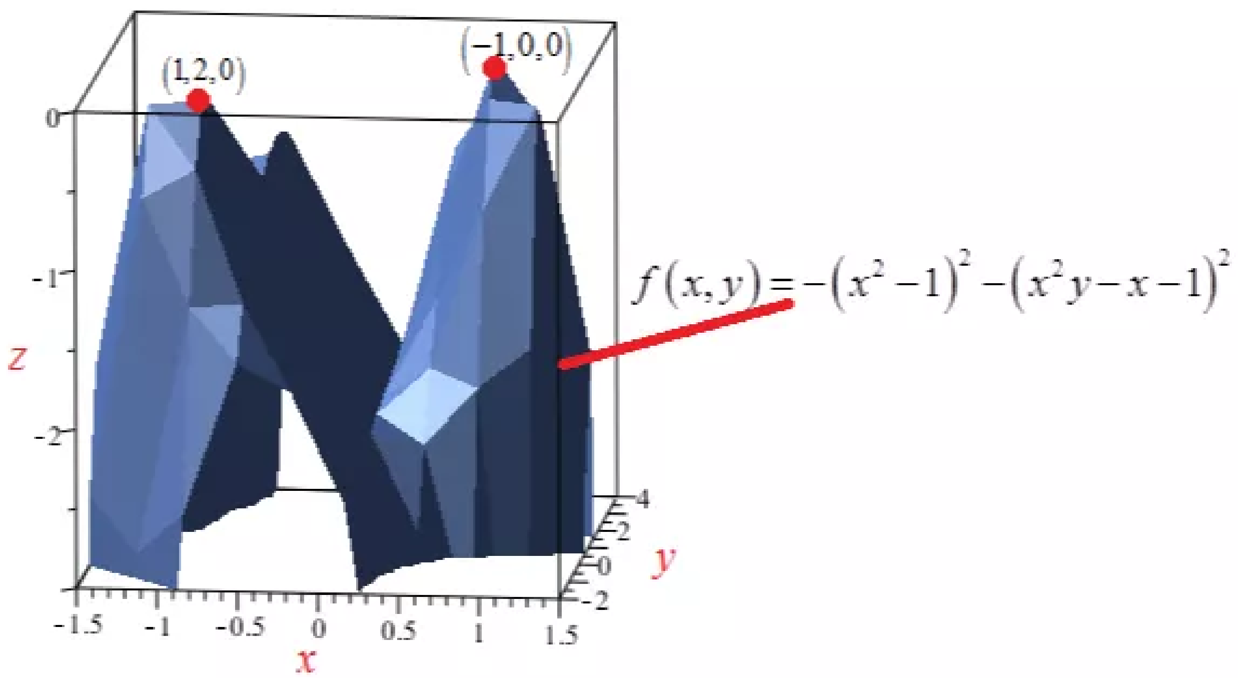
$$=24x^6-40x^6y^2+72x^5y+56x^4y-40x^4-48x^3-16x^2$$

Find D, f_{xx} and behaviour for the three critical points and tabulate them as follows.

Critical point	Value of f	f_{xx}	D	Conclusion
$(1,2)$	$f(1,2)=0$	$f_{xx}(1,2)=-26 < 0$	$16>0$	Local maximum
$(-1,0)$	$f(-1,0)=0$	$f_{xx}(-1,0)=-10 < 0$	$16>0$	Local maximum

Therefore, conclude that the local maxima exist at only two critical point and value of f is $\boxed{f(1,2)=f(-1,0)=0}$.

Sketch the graph of the function by choosing a viewing rectangle that displays the critical points exactly.



Answer 38E.

Consider the following function:

$$f(x,y)=3xe^y-x^3-e^{3y}$$

It is required to find the critical points of f and prove that the function has local maxima at these two critical points.

To find the critical points, find the partial derivatives of f independently with respect to x and y equate them to zero.

Differentiate $f(x,y)$ with respect to x.

$$\begin{aligned}f_x(x,y)&=\frac{\partial}{\partial x}\left(3xe^y-x^3-e^{3y}\right)\\&=\frac{\partial}{\partial x}\left(3xe^y\right)-\frac{\partial}{\partial x}\left(x^3\right)-\frac{\partial}{\partial x}\left(e^{3y}\right)\\&=3e^y-3x^2-0\\&=3e^y-3x^2\end{aligned}$$

Differentiate $f(x,y)$ with respect to y.

$$\begin{aligned}f_y(x,y)&=\frac{\partial}{\partial y}\left(3xe^y-x^3-e^{3y}\right)\\&=\frac{\partial}{\partial y}\left(3xe^y\right)-\frac{\partial}{\partial y}\left(x^3\right)-\frac{\partial}{\partial y}\left(e^{3y}\right)\\&=3xe^y-0-3e^{3y}\\&=3xe^y-3e^{3y}\end{aligned}$$

Set the equation $f_x(x,y)$ equal to zero.

$$\begin{aligned}3e^y-3x^2&=0\\3e^y&=3x^2\\e^y&=x^2\end{aligned}$$

Set the equation $f_y(x,y)$ equal to zero.

$$3xe^y - 3e^{3y} = 0$$

$$3x(x^2) - 3(x^2)^3 = 0 \qquad \text{Substitute } e^y = x^2.$$

$$3x^3 - 3x^6 = 0$$

$$3x^3(1 - x^3) = 0$$

$$x^3 = 0 \quad \text{or } 1 - x^3 = 0$$

$$x = 0 \quad \text{or } (1-x)(1+x+x^2) = 0$$

$$x = 0 \quad \text{or} \quad x = 1$$

Solve the equation $e^y = x^2$ for y.

$$e^y = x^2$$

$$y = \ln x^2$$

$$= 2\ln x$$

When $x = 0$:

$$y = 2\ln 0$$

$$= \text{undefined}$$

When $x = 1$:

$$y = 2\ln 1$$

$$= 0$$

Therefore, the only possible critical point is $\boxed{(1,0)}$.

Classify the behavior of the critical points of f.

Recall the second derivative test as follows:

Function f has continuous partial derivatives on disk (a,b) and $f_x(a,b) = 0, f_y(a,b) = 0.$

Let $D \approx D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2.$

If $D > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ is a local minimum.

If $D > 0$ and $f_{xx}(a,b) < 0$ then $f(a,b)$ is a local maximum.

If $D < 0$ then $f(a,b)$ is not a local minimum or local maximum.

To find $D(x,y)$, calculate $f_{xx}(x,y), f_{yy}(x,y)$ and $f_{xy}(x,y)$.

Differentiate $f_x(x,y)$ with respect to x.

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}\left(3e^y - 3x^2\right) \\ &= \frac{\partial}{\partial x}\left(3e^y\right) - \frac{\partial}{\partial x}\left(3x^2\right) \\ &= 0 - 6x \\ &= -6x \end{aligned}$$

Differentiate $f_y(x,y)$ with respect to y.

$$\begin{aligned} f_{yy} &= \frac{\partial}{\partial y}\left(3xe^y - 3e^{3y}\right) \\ &= \frac{\partial}{\partial y}\left(3xe^y\right) - \frac{\partial}{\partial y}\left(3e^{3y}\right) \\ &= 3xe^y - 9e^{3y} \end{aligned}$$

Differentiate $f_x(x,y)$ with respect to y.

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}\left(3e^y - 3x^2\right) \\ &= \frac{\partial}{\partial y}\left(3e^y\right) - \frac{\partial}{\partial y}\left(3x^2\right) \\ &= 3e^y - 0 \\ &= 3e^y \end{aligned}$$

Substitute the values of f_{xx}, f_{yy} **and** f_{xy} in D.

$$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2$$
$$= (-6x)(3xe^y - 9e^{3y}) - (3e^y)^2$$
$$= -18x^2e^y + 54xe^{3y} - 9e^{2y}$$

Find D, f_{xx} and behavior for the three critical points and tabulate them as follows:

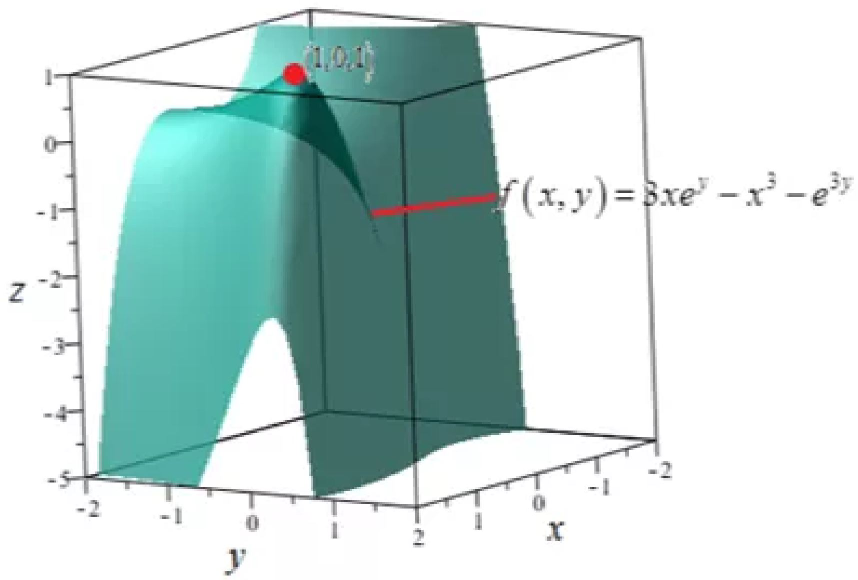
Critical point	Value of f	f_{xx}	D	Conclusion
$(1,0)$	$f(1,0)=1$	$f_{xx}(1,0)=-6$ <0	$D(1,0)=-18+54-9$ $=27$ >0	Local maximum

So, conclude that the local maxima exist at only one critical point and value of f is,

$\boxed{f(1,0)=1}$.

Therefore, the highest point of the graph is $\boxed{(1,0,1)}$.

Sketch the graph of the function by choosing a viewing rectangle that displays the critical points exactly as shown below:



Answer 39E.

The distance between the point (x, y, z) and $(2, 0, -3)$ is given by $d = \sqrt{(x-2)^2 + y^2 + (z+3)^2}$. We have $x + y + z = 1$ or $z = 1 - x - y$.

Replace z with $1 - x - y$.

$$\begin{aligned} d &= \sqrt{(x-2)^2 + y^2 + (-x-y+4)^2} \\ &= \sqrt{x^2 - 4x + 4 + y^2 + x^2 + 2xy - 8x + y^2 - 8y + 16} \\ &= \sqrt{2x^2 - 12x + 20 + 2y^2 + 2xy - 8y} \\ d^2 &= 2x^2 - 12x + 20 + 2y^2 + 2xy - 8y \end{aligned}$$

Let $d^2 = f(x, y)$.

Find $f_x(x, y), f_y(x, y), f_{xx}(x, y), f_{yy}(x, y)$, and $f_{xy}(x, y)$.

$$f_x = 4x - 12 + 2y$$
$$f_{xx} = 4$$

$$f_y = 2x + 4y - 8$$
$$f_{yy} = 4$$

$$f_{xy} = 2$$

Equate $f_x(x, y)$ to 0 and $f_y(x, y)$ to 0.

$$\begin{aligned} 4x - 12 + 2y &= 0 \\ 2x + y &= 6 \\ y &= 6 - 2x \end{aligned}$$

$$\begin{aligned} 2x + 4y - 8 &= 0 \\ x + 2y &= 4 \end{aligned}$$

Replace y with $6 - 2x$ in $x + 2y = 4$ and solve for x.

$$\begin{aligned} x + 2(6 - 2x) &= 4 \\ x + 12 - 4x &= 4 \\ -3x &= -8 \\ x &= \frac{8}{3} \end{aligned}$$

On replacing x with $\frac{8}{3}$ in $y = 6 - 2x$, we get $y = \frac{2}{3}$.

Now, evaluate $D\left(\frac{8}{3}, \frac{2}{3}\right)$ given by $f_{xx}(a, b) f_{yy}(a, b) - \left[f_{xy}(a, b)\right]^2$.

$$\begin{aligned} D\left(\frac{8}{3}, \frac{2}{3}\right) &= (4)(4) - (2)^2 \\ &= 16 - 4 \\ &= 12 \end{aligned}$$

Since $D > 0$ and $f_{xx} > 0$, the given function has local minimum at $\left(\frac{8}{3}, \frac{2}{3}\right)$.

Substitute $\frac{8}{3}$ for x and $\frac{2}{3}$ for y in $d = \sqrt{2x^2 - 12x + 20 + 2y^2 + 2xy - 8y}$.

$$\begin{aligned} d &= \sqrt{2\left(\frac{8}{3}\right)^2 - 12\left(\frac{8}{3}\right) + 20 + 2\left(\frac{2}{3}\right)^2 + 2\left(\frac{8}{3}\right)\left(\frac{2}{3}\right) - 8\left(\frac{2}{3}\right)} \\ &= \sqrt{\frac{128}{9} - \frac{96}{3} + 20 + \frac{8}{9} + \frac{32}{9} - \frac{16}{3}} \\ &= \sqrt{\frac{4}{3}} \\ &= \frac{2}{\sqrt{3}} \end{aligned}$$

Thus, the shortest distance between the point $(2, 0, -3)$ and the given plane is obtained as $\boxed{\frac{2}{\sqrt{3}}}$.

Answer 40E.

The distance between the point (x, y, z) and $(0, 1, 1)$ is given by $d = \sqrt{x^2 + (y-1)^2 + (z-1)^2}$. We have $x - 2y + 3z = 6$ or $z = \dfrac{6 - x + 2y}{3}$.

Replace z with $\dfrac{6 - x + 2y}{3}$.

$$d = \sqrt{x^2 + (y-1)^2 + \left(\frac{6 - x + 2y}{3} - 1\right)^2}$$

$$= \sqrt{x^2 + y^2 - 2y + 1 + 1 - \frac{2x}{3} + \frac{4y}{3} + \frac{x^2}{9} - \frac{4xy}{9} + \frac{4y^2}{9}}$$

$$= \sqrt{\frac{10}{9}x^2 + \frac{13}{9}y^2 - \frac{2}{3}y - \frac{2}{3}x - \frac{4}{9}xy + 2}$$

$$d^2 = \frac{10}{9}x^2 + \frac{13}{9}y^2 - \frac{2}{3}y - \frac{2}{3}x - \frac{4}{9}xy + 2$$

Let $d^2 = f(x, y)$.

Find $f_x(x, y)$, $f_y(x, y)$, $f_{xx}(x, y)$, $f_{yy}(x, y)$, and $f_{xy}(x, y)$.

$$f_x = \frac{20}{9}x - \frac{4}{9}y - \frac{2}{3}$$

$$f_{xx} = \frac{20}{9}$$

$$f_y = \frac{26}{9}y - \frac{4}{9}x - \frac{2}{3}$$

$$f_{yy} = \frac{26}{9}$$

$$f_{xy} = -\frac{4}{9}$$

Equate $f_x(x, y)$ to 0 and $f_y(x, y)$ to 0.

$$\frac{20}{9}x - \frac{4}{9}y - \frac{2}{3} = 0$$
$$\frac{20x - 4y - 6}{9} = 0$$
$$y = 5x - \frac{3}{2}$$

$$\frac{26}{9}y - \frac{4}{9}x - \frac{2}{3} = 0$$
$$\frac{26y - 4x - 6}{9} = 0$$
$$13y - 2x - 3 = 0$$

Replace y with $5x - \frac{3}{2}$ in $13y - 2x - 3 = 0$ and solve for x.

$$13\left(5x - \frac{3}{2}\right) - 2x - 3 = 0$$
$$65x - \frac{39}{2} - 2x - 3 = 0$$
$$63x - \frac{45}{2} = 0$$
$$x = \frac{5}{14}$$

On replacing x with $\frac{5}{14}$ in $y = 5x - \frac{3}{2}$, we get $y = \frac{2}{7}$.

Now, evaluate $D\left(\frac{5}{14}, \frac{2}{7}\right)$ given by $f_{xx}(a, b)f_{yy}(a, b) - \left[f_{xy}(a, b)\right]^2$.

$$D\left(\frac{5}{14}, \frac{2}{7}\right) = \left(\frac{20}{9}\right)\left(\frac{26}{9}\right) - \left(-\frac{4}{9}\right)^2$$
$$= \frac{520}{81} - \frac{16}{81}$$
$$= \frac{56}{9}$$

Since $D > 0$ and $f_{xx} > 0$, the given function has local minimum at $\left(\frac{5}{14}, \frac{2}{7}\right)$.

Substitute $\frac{5}{14}$ for x and $\frac{2}{7}$ for y in $z = \frac{6 - x + 2y}{3}$.

$$z = \frac{6 - \left(\frac{5}{14}\right) + 2\left(\frac{2}{7}\right)}{3}$$
$$= \frac{84 - 5 + 8}{42}$$
$$= \frac{29}{14}$$

Thus, the closest point to (0, 1, 1) is $\boxed{\left(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}\right)}$.

Answer 41E.

Consider the point $(4,2,0)$ and the cone $z^2 = x^2 + y^2$.

Find the point on the plane that is closest to the point.

The distance from point (x,y,z) on the cone to the point $(4,2,0)$ is

$$d = \sqrt{(x-4)^2 + (y-2)^2 + (z-0)^2}$$
$$d^2 = (x-4)^2 + (y-2)^2 + z^2 \qquad \text{......(1)}$$

The point (x,y,z) lies on the cone $z^2 = x^2 + y^2$.

So,

$$z^2 = x^2 + y^2 \qquad \text{......(2)}$$

We need to minimize the distance.

To find the minimum distance, we consider d^2 as a function of (x, y).

So,

$$f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$$

Plugging the value of $z^2 = x^2 + y^2$

$$f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$$
$$f(x,y,z) = (x-4)^2 + (y-2)^2 + (x^2 + y^2)$$

Take partial derivative of $f(x, y)$ with respect to x,

$$\begin{aligned} f_x(x,y) &= (x-4)^2 + (y-2)^2 + (x^2 + y^2) \\ &= 2(x-4) + 2x \\ &= 4x - 8 \end{aligned}$$

Taking partial derivative of $f(x, y)$ with respect to y,

$$\begin{aligned} f_y(x,y) &= (x-4)^2 + (y-2)^2 + (x^2 + y^2) \\ &= 2(y-2) + 2y \\ &= 4y - 4 \end{aligned}$$

Equating this equation to zero,

$$\begin{aligned} f_x(x,y) &= 0 \\ 4x - 8 &= 0 \\ x &= \frac{8}{4} \\ x &= 2 \end{aligned}$$

Equating $f_y(x, y)$ to zero,

$$\begin{aligned} 4y - 4 &= 0 \\ y &= \frac{4}{4} \\ y &= 1 \end{aligned}$$

Hence, critical point of the function is $(2,1)$.

The second partial derivatives of $f(x,y)$ are,

Differential two times with respect to x,

$$f(x,y)=(x-4)^2+(y-2)^2+(x^2+y^2)$$
$$f_x(x,y)=4x-8$$
$$f_{xx}(x,y)=4$$

Differential second time with respect to y,

$$f(x,y)=(x-4)^2+(y-2)^2+(x^2+y^2)$$
$$f_x(x,y)=4x-8$$
$$f_{xy}(x,y)=0$$

Differential two times with respect to y,

$$f(x,y)=(x-4)^2+(y-2)^2+(x^2+y^2)$$
$$f_y(x,y)=2(y-2)+2y$$
$$f_{yy}(x,y)=4$$

Suppose the second partial derivatives of f are continuous on a disk with center (a,b) and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$ [(a,b) is a critical point of f].

Let $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-(f_{xy}(a,b))^2$.

(a) If $D>0$ and $f_{xx}(a,b)>0$ then $f(a,b)$ is a local minimum.

(b) If $D>0$ and $f_{xx}(a,b)<0$ then $f(a,b)$ is a local maximum.

(c) If $D<0$ then $f(a,b)$ is not a local maximum or local minimum.

The value of D at the critical point $(2,1)$ is,

$$D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - (f_{xy}(a,b))^2$$
$$D(2,1) = (4)(4) - [0]^2$$
$$= 16 - 0$$
$$= 16$$

Since $D > 0$ and $f_{yy} > 0$, the point $(2,1)$ is a local minimum.

The value of z is,

$$z = x^2 + y^2$$
$$z^2 = 2^2 + 1^2$$
$$= 5$$
$$z = \pm\sqrt{5}$$

Thus, the points on the cone closest to $(4,2,0)$ are $\boxed{(2,1,\sqrt{5}) \text{ and } (2,1,-\sqrt{5})}$.

Answer 42E.

Consider the origin and the surface,

$y^2 = 9 + xz$.

The objective is to determine the point on the plane that is closest to the origin.

The distance from point (x,y,z) on the cone to the point $(0,0,0)$ is,

$$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$d^2 = (x)^2 + (y)^2 + z^2$$

The point (x,y,z) lies on the cone $y^2 = 9 + xz$, so that

$y^2 = 9 + xz$ ...... (1)

To find the minimum distance, need to consider d^2 as a function of x, y.

So,

$f(x,y,z) = (x)^2 + (y)^2 + z^2$ ...... (2)

Plug in the equation (1) into the equation (2), to get:

$f(x,y,z) = x^2 + 9 + xz + z^2$.

The first partial derivatives of *f* are,

$f_x(x,z) = 2x + z$ and $f_z(x,z) = 2z + x$

Equate these equations to zero, to get:

$$f_z(x,z) = 0 \qquad 2x + z = 0$$
$$2z + x = 0 \quad \text{and} \quad 2x - \frac{x}{2} = 0$$
$$z = \frac{-x}{2} \qquad x = 0$$

Plug in the value of $x = 0$ into $z = \frac{-x}{2}$, to obtain:

$$z = \frac{-x}{2}$$
$$z = \frac{-0}{2}$$
$$z = 0$$

So, the critical point of the function is $(0,0)$.

The second partial derivatives of *f* are,

$$f_x(x,z)=2x+z \quad f_x(x,z)=2x+z \text{ and } f_z(x,z)=2z+x$$
$$f_{xx}(x,z)=2, \quad f_{xz}(x,z)=1, \quad f_{zz}(x,z)=2$$

Use the second derivative test, a function *f* has continuous partial derivatives on disk (a,b) and $f_x(a,b)=0, f_z(a,b)=0$.

Let

$$D=D(a,b)=f_{xx}(a,b)f_{zz}(a,b)-\left[f_{xz}(a,b)\right]^2$$

a. If $D>0$ and $f_{xx}(a,b)>0$ then $f(a,b)$ is a local minimum

b. If $D>0$ and $f_{xx}(a,b)<0$ then $f(a,b)$ is a local maximum

c. If $D<0$ then $f(a,b)$ is not a local minimum or local maximum. It is called saddle point.

d. If $D=0$, we cannot say anything with this test.

At the critical point $(0,0)$,

$$D(0,0)=(2)(2)-(1)^2=3>0$$

Since $D>0$ and $f_{xx}>0$, the point $(0,0)$ is a local minimum.

The value of *y* at that local minimum is,

$$\begin{aligned} y^2 &= 9+xz \\ y^2 &= 9+(0)(0) \\ &= 9 \\ y &= \pm 3 \end{aligned}$$

Thus, the points on the surface $y^2=9+xz$ that are closest to the origin are,

$\boxed{(0,-3,0), \text{ and } (0,3,0)}$.

Answer 43E.

Let the three positive numbers be x, y, z

Then the sum is $x + y + z = 100$ ...... (1)

Product of numbers be xyz

From (1) rewrite

$$z = 100 - x - y$$

Substitute the value of z in the product

$$xy\,(100 - x - y)$$

Take $f(x,y) = xy(100-x-y)$

$$f(x,y) = 100xy - x^2y - xy^2$$

Differentiate the function $f(x,y)$ with respective x and y

Then

$$f_x = 100y - 2xy - y^2$$

$$f_y = 100x - x^2 - 2xy$$

First find the critical points by setting $f_x = 0$, $f_y = 0$

That is $y(100-2x-y) = 0$ And $x(100-x-2y) = 0$ ...... (2)

On solving (2)

$$100 - 2x - y$$

$$2(100 - x - 2y) \quad \text{Multiply with 2}$$

$$-100 + 3y = 0 \quad \text{On subtraction}$$

$$-100 = -3y$$

$$y = \frac{100}{3}$$

Substitute the value of y in any one of the above equation get the value $x = \frac{100}{3}$

So the critical points are (0, 0) and $\left(\frac{100}{3}, \frac{100}{3}\right)$

[we cannot take (0, 0) to be critical point as it is not feasible since it is gives $xyz = 0$]

Now $f_{xx} = -2y$ and $f_{yy} = -2x$

Similarly

$$f_x = 100y - 2xy - y^2$$
$$f_{xy} = 100 - 2x - 2y$$

By second derivative test:

Suppose the second partial derivatives of f are continues on a disk with center (a,b), and suppose that $f_x(a,b) = 0$ and $f_y(a,b) = 0$ [that is (a,b) is a critical point of f]

Let $D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$

If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum

Then $D = f_{xx} f_{yy} - f_{xy}^2$

$$D = 4xy - (100 - 2x - 2y)^2 \qquad \ldots\ldots (3)$$

Substitute the $\left(\frac{100}{3}, \frac{100}{3}\right)$ in the equation (3)

$$D = \frac{40000}{9} - \left(100 - \frac{200}{3} - \frac{200}{3}\right)^2$$
$$= \frac{40000}{9} - \left(\frac{300 - 200 - 200}{3}\right)^2$$
$$= \frac{40000}{9} - \left(\frac{-100}{3}\right)^2$$

By continue the above

$$D = \frac{40000}{9} - \frac{10000}{9}$$
$$= \frac{30000}{9}$$
$$D = \frac{10000}{3} > 0$$

And at $f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right)$

The equation (3) becomes

$$f_{xx} = -2y$$
$$f_{xx} = \frac{-200}{3} < 0$$

So $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum

From the above results at $\left(\frac{100}{3},\frac{100}{3}\right)$ a point of maxima

That is when $x=\frac{100}{3}$, $y=\frac{100}{3}$, "f" will have maximum value

The product of the three numbers x, y, z will be maximum

Required numbers are

$$x=\frac{100}{3},\ y=\frac{100}{3}$$

And

$$\begin{aligned} z &= 100-\frac{100}{3}-\frac{100}{3} \\ &= \frac{300-100-100}{3} \\ &= \frac{300-200}{3} \\ z &= \frac{100}{3} \end{aligned}$$

Therefore;

$$\boxed{x=\frac{100}{3}, y=\frac{100}{3}, z=\frac{100}{3}}$$

Answer 45E.

Consider the rectangular box is to be inscribed in a sphere of radius *r*.

For convenience assume that the sphere is centered at the origin.

This sphere has equation $x^2+y^2+z^2=r^2$.

By symmetry the box must also be centered at the origin. If not, then at least one corner will not lie on the surface of the sphere. It is convenient to assume that the box is oriented along the axes.

Let the point (x,y,z) represent the corner of the box in the first quadrant that lies on the sphere.

The volume of the box is:

$$\begin{aligned} V &= \text{length}\times\text{breadth}\times\text{height} \\ &= (2x)(2y)(2z) \\ &= 8xyz \end{aligned}$$

The goal is to maximize *V*, subject to the constraint $x^2+y^2+z^2=r^2$.

Let $f(x,y,z) = V = 8xyz$, then

$$\nabla f = \langle 8yz, 8xz, 8xy \rangle$$

And, the constraint is:

$$g(x,y,z) = x^2 + y^2 + z^2 - r^2$$

$$\nabla g = \langle 2x, 2y, 2z \rangle$$

Therefore, Using Lagrange's multipliers,

$$\nabla f = \lambda \nabla g$$

$$\langle 8yz, 8xz, 8xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

$$4yz = \lambda x$$

$$\Rightarrow \lambda = \frac{4yz}{x} \qquad \text{......}(1)$$

$$4xz = \lambda y$$

$$\Rightarrow \lambda = \frac{4xz}{y} \qquad \text{......}(2)$$

$$4xy = \lambda z$$

$$\Rightarrow \lambda = \frac{4xy}{z} \qquad \text{......}(3)$$

From equations (1) and (2),

$$\frac{4yz}{x} = \frac{4xz}{y}$$

$$4y^2 z = 4x^2 z$$

$$\Rightarrow y^2 = x^2 \qquad \text{......}(4)$$

Similarly using equations (1) and (3),

$$\frac{4yz}{x} = \frac{4xy}{z}$$

$$4yz^2 = 4x^2 y$$

$$\Rightarrow z^2 = x^2 \qquad \text{......}(5)$$

From (4) and (5), we conclude that,

$$x^2 = y^2 = z^2$$

Substitute $x = y = z$ in the constraint equation, we get

$$x^2 + y^2 + z^2 = r^2$$
$$3x^2 = r^2$$
$$x^2 = \frac{r^2}{3}$$
$$\Rightarrow x = \frac{r}{\sqrt{3}} \qquad \text{Since } x > 0$$

Therefore, $x = y = z = \dfrac{r}{\sqrt{3}}$

The maximum value of the rectangular box is,

$$V = 8xyz$$
$$= 8\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right)$$
$$= \frac{8r^3}{3\sqrt{3}}$$

Hence, the maximum volume of a rectangular box is $\boxed{V = \dfrac{8r^3}{3\sqrt{3}}}$.

Answer 46E.

Find an expression for surface area for the given volume, and find the minimum of this surface area using partial derivatives.

Let the three dimensions of the box be $x, y,$ and z. The volume is 1000 cm^3:

$$V = xyz$$
$$1000 = xyz \qquad \ldots\ldots (1)$$

The surface area is the function we wish to minimize:

$$A = 2xy + 2xz + 2yz$$

Use equation (1) to substitute in for z in A:

$$z = \frac{1000}{xy}$$

$$A = 2xy + 2xz + 2yz$$
$$= 2xy + 2x\left(\frac{1000}{xy}\right) + 2y\left(\frac{1000}{xy}\right)$$
$$= 2xy + \frac{2000}{y} + \frac{2000}{x}$$

To find the (x, y) point that minimizes A, find the critical points of A. Critical points occur when the partial derivatives A_x and A_y are either zero or when one of them does not exist. We find these partial derivatives.

To find A_x, hold y constant and take the derivative in terms of x.

$$A = 2xy + \frac{2000}{y} + \frac{2000}{x}$$

$$A_x = 2y - \frac{2000}{x^2}$$

To find A_y, hold x constant and take the derivative in terms of y.

$$A = 2xy + \frac{2000}{y} + \frac{2000}{x}$$

$$A_y = 2x - \frac{2000}{y^2}$$

If $x = 0$ or $y = 0$, at least one of these partial derivatives does not exist. However, that would make one dimension of the box 0, which would give it a zero volume, and therefore would not satisfy the real-life constraints of the problem. Therefore, we discard the critical points resulting from $x = 0$ and $y = 0$.

We find the critical points that occur when A_x and A_y both equal 0:

$$0 = 2y - \frac{2000}{x^2}$$

$$0 = 2x - \frac{2000}{y^2}$$

Multiply through by the denominators to simplify:

$$0 = 2yx^2 - 2000$$

$$0 = 2xy^2 - 2000$$

$$yx^2 = 1000$$

$$xy^2 = 1000 \quad \ldots\ldots (2)$$

Solve this system by substitution. Solve the first equation in (2) for y:

$$y = \frac{1000}{x^2}$$

Plug into the second equation in (2):

$$x\left(\frac{1000}{x^2}\right)^2 = 1000$$

$$x\left(\frac{1000^2}{x^4}\right) = 1000$$

$$\frac{1000}{x^3} = 1$$

$$x^3 = 1000$$

$$x = 10$$

Back-substitute to solve for y:

$$y = \frac{1000}{(10)^2}$$

$$y = 10$$

We now have (10, 10) as the only viable critical point for A.

It remains to show that the critical point (10, 10) gives minimal surface area. To show it is a minimum, we use the Second Derivative Test, which states that under most commonly encountered conditions, if a function f has a critical point at (a,b), then if we let

$$D = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

Then the following rules determine whether f has a local minimum or local maximum at (a, b):

(1) If $D > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum at (a, b).

(2) If $D > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at (a, b).

(3) If $D < 0$, then (a, b) is a saddle point.

(4) If $D = 0$, then the test is inconclusive.

In order to apply the Second Derivative Test, we will need D, for which we will need to calculate all the second partial derivatives. Start with A_x and take another partial derivative in terms of x:

$$A_x = 2y - \frac{2000}{x^2}$$

$$A_{xx} = (2)\frac{2000}{x^3}$$

$$= \frac{4000}{x^3}$$

Now start with A_x and take a partial derivative in terms of y:

$$A_x = 2y - \frac{2000}{x^2}$$

$$A_{xy} = 2$$

Finally, start with A_y and take another partial derivative in terms of y:

$$A_y = 2x - \frac{2000}{y^2}$$

$$A_{yy} = (2)\frac{2000}{y^3}$$

$$= \frac{4000}{y^3}$$

Now we calculate D for the critical point (10, 10):

$$D = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b) \right]^2$$

$$= \left(\frac{4000}{x^3}\right)\left(\frac{4000}{y^3}\right) - 2$$

$$= \left(\frac{4000}{10^3}\right)\left(\frac{4000}{10^3}\right) - 2$$

$$= (4)(4) - 2$$

$$= 14$$

So for (10, 10), $D = 14$, a positive value. In the course of calculating D, we found A_{xx} to be 4, which is also positive. Since both are positive, by the Second Derivative Test the function A must have a minimum here. Since minimum surface area is what we were looking for, $(x, y) = (10, 10)$ indeed fits as a solution.

Plugging $x = 10$ and $y = 10$ into the original volume equation in (1) gives

$$1000 = (10)(10)z$$

$$z = 10$$

So dimensions of the box with minimal surface area and a volume of 1000 cm^3 are **10 by 10 by 10**.

Answer 47E.

Consider a rectangular box in the first octant with three faces in the coordinate planes and one vertex of the rectangular box is in the plane $x+2y+3z=6.$

The objective is to determine the volume of the largest rectangular box.

It means that, one needs to find the maximum volume of the rectangular box in the first octant.

Here, $x, y, z \geq 0$ because, the rectangular box is lies in the first octant.

Let (x, y, z) be the vertex on the plane $x+2y+3z=6,$ and *x*, *y*, and *z* are the length, width, and height of the box.

The volume of a rectangular solid is $V=(\text{length})(\text{width})(\text{height}).$

$$V=(\text{length})(\text{width})(\text{height})$$
$$=xyz$$

Next, solve the plane equation $x+2y+3z=6$ for z as follows:

$$x+2y+3z=6$$
$$3z=6-x-2y$$
$$z=\frac{6-x-2y}{3}$$

Substitute $z=\dfrac{6-x-2y}{3}$ in $V=xyz$.

$$V(x,y,z)=xyz$$
$$=xy\left(\frac{6-x-2y}{3}\right)$$
$$V(x,y)=\frac{1}{3}\left(6xy-x^2y-2xy^2\right)$$

To find the critical points, one needs to find the partial derivatives of $V(x,y)$ independently with respect to *x* and *y* and equate them to zero.

Find $V_x(x,y)$ as follows:

Differentiate $V(x,y)$ with respect to *x* by assuming *y* as constant:

$$V_x(x,y)=\frac{\partial}{\partial x}\left[\frac{1}{3}\left(6xy-x^2y-2xy^2\right)\right]$$
$$=\frac{1}{3}\left(\frac{\partial}{\partial x}[6xy]+\frac{\partial}{\partial x}\left[-x^2y\right]+\frac{\partial}{\partial x}\left[-2xy^2\right]\right)$$

Use the formula $\dfrac{d}{dx}x^n=nx^{n-1}$

$$=\frac{1}{3}\left(6y\frac{\partial}{\partial x}[x]-y\frac{\partial}{\partial x}\left[x^2\right]-2y^2\frac{\partial}{\partial x}[x]\right)$$
$$=\frac{1}{3}\left(6y(1)-y(2x)-2y^2(1)\right)$$
$$V_x(x,y)=\frac{1}{3}\left(6y-2xy-2y^2\right) \quad \text{....... (1)}$$

Find $V_y(x,y)$ as follows:

Differentiate $V(x,y)$ with respect to *y* by assuming *x* as constant:

$$V_y(x,y) = \frac{\partial}{\partial y}\left[\frac{1}{3}\left(6xy - x^2y - 2xy^2\right)\right]$$

$$= \frac{1}{3}\left(\frac{\partial}{\partial y}[6xy] + \frac{\partial}{\partial y}\left[-x^2y\right] + \frac{\partial}{\partial y}\left[-2xy^2\right]\right)$$

Use, $\frac{d}{dx}x^n = nx^{n-1}$

$$= \frac{1}{3}\left(6x\frac{\partial}{\partial y}[y] - x^2\frac{\partial}{\partial y}[y] - 2x\frac{\partial}{\partial y}\left[y^2\right]\right)$$

$$= \frac{1}{3}\left(6x(1) - x^2(1) - 2x(2y)\right)$$

$$V_y(x,y) = \frac{1}{3}\left(6x - x^2 - 4xy\right) \text{ ...... (3)}$$

Set the equation (1) equal to zero as follows:

$$\frac{1}{3}\left(6y - 2xy - 2y^2\right) = 0$$

$$6y - 2xy - 2y^2 = 0$$

$$2y(3 - x - y) = 0$$

$$y = 0 \quad \text{and} \quad 3 - x - y = 0$$

$$y = 0 \quad \text{and} \quad 3 = x + y$$

Set the equation (2) equal to zero as follows:

$$\frac{1}{3}\left(6x - x^2 - 4xy\right) = 0$$

$$6x - x^2 - 4xy = 0$$

$$x(6 - x - 4y) = 0$$

$$x = 0 \quad \text{and} \quad 6 - x - 4y = 0$$

$$x = 0 \quad \text{and} \quad 6 = x + 4y$$

Find *x* when $y = 0$:

To solve for *x*, substitute $y = 0$ in $x + 4y = 6$ as follows:

$$x + 4y = 6$$

$$x + 4(0) = 6$$

$$x = 6$$

This implies that, $(6,0)$ is the critical point.

Find *y* when $x = 0$:

To solve for *y*, substitute $x = 0$ in $x + y = 3$ as follows:

$$x + y = 3$$
$$0 + y = 3$$
$$y = 3$$

This implies that, $(0,3)$ is another critical point.

Solve $x + y = 3$ and $x + 4y = 6$ for *x* and *y*:

First, solve $x + y = 3$ for *x*:

$$x + y = 3$$
$$x = 3 - y$$

To solve for *y*, substitute $x = 3 - y$ in $x + 4y = 6$ as follows:

$$3 - y + 4y = 6$$
$$3 + 3y = 6$$
$$3y = 3$$
$$y = 1$$

Substitute $y = 1$ in $x = 3 - y$, to solve for *x*:

$$x = 3 - 1$$
$$= 2$$

Therefore, the critical points of the function are $(0,0),(0,3),(6,0)$, and $(2,1)$.

It can be observed that, when $x = 0$, and $y = 0$ the volume *V* will be zero, which is absolute minimum.

Use the second derivative test, a function *f* has continuous partial derivatives on disk (a,b) and $f_x(a,b) = 0, f_y(a,b) = 0$.

Let

$$D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

a. If $D > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ is a local minimum

b. If $D > 0$ and $f_{xx}(a,b) < 0$ then $f(a,b)$ is a local maximum

c. If $D < 0$ then $f(a,b)$ is not a local minimum or local maximum.

Find second order partial derivatives as follows:

To find $D(x,y)$, find $V_{xx}(x,y), V_{yy}(x,y)$ and $V_{xy}(x,y)$.

Find $V_{xx}(x,y)$ as follows:

$$\begin{aligned}
V_{xx}(x,y) &= \frac{\partial}{\partial x}\left(\frac{1}{3}\left(6y-2xy-2y^2\right)\right) \\
&= \frac{1}{3}\left[6y\frac{\partial}{\partial x}(1)-2y\frac{\partial}{\partial x}(x)-2y^2\frac{\partial}{\partial x}(1)\right] \\
&= \frac{1}{3}\left[6y(0)-2y(1)-2y^2(0)\right] \\
&= \frac{1}{3}[-2y] \\
V_{xx}(x,y) &= -\frac{2}{3}y
\end{aligned}$$

Find $V_{yy}(x,y)$ as follows:

$$\begin{aligned}
V_{yy}(x,y) &= \frac{\partial}{\partial y}\left(\frac{1}{3}\left(6x-x^2-4xy\right)\right) \\
&= \frac{1}{3}\left[6x\frac{\partial}{\partial y}(1)-x^2\frac{\partial}{\partial y}(1)-4x\frac{\partial}{\partial y}(y)\right] \\
&= \frac{1}{3}\left[6x(0)-x^2(0)-4x(1)\right] \\
&= -\frac{4}{3}x
\end{aligned}$$

Find $V_{xy}(x,y)$ as follows:

$$\begin{aligned}
V_{xy}(x,y) &= \frac{\partial}{\partial x}\left(\frac{1}{3}\left(6x-x^2-4xy\right)\right) \\
&= \frac{1}{3}\left[6\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial x}\left(x^2\right)-4y\frac{\partial}{\partial x}(x)\right] \\
&= \frac{1}{3}\left[6(1)-2x-4y(1)\right] \\
&= -\frac{2x+4y-6}{3}
\end{aligned}$$

Substitute V_{xx}, V_{yy} and V_{xy} in D.

$$D = V_{xx}(x,y)V_{yy}(x,y) - \left[V_{xy}(x,y)\right]^2$$
$$= \left(-\frac{2}{3}y\right) - \frac{4}{3}x - \left(-\frac{2x+4y-6}{3}\right)^2$$
$$= \left(\frac{8}{9}xy\right) - \frac{(2x+4y-6)^2}{9}$$
$$= \frac{8xy - 4x^2 - 16xy - 16y^2 + 24x + 48y - 36}{9}$$
$$= \frac{-4x^2 - 8xy - 16y^2 + 24x + 48y - 36}{9}$$

Find D, f_{xx} and behaviour for the three critical points and tabulate them as follows.

Critical point	Value of $V(x,y)$	$V_{xx}(x,y) = -\frac{2}{3}y$	D	Conclusion
$(0,3)$	$V(0,3) = 0$	$V_{xx}(0,3) = -2 < 0$	$-4 < 0$	Neither local maximum nor minimum
$(6,0)$	$V(6,0) = 0$	$V_{xx}(6,0) = 0$	$-4 < 0$	Neither local maximum nor minimum
$(2,1)$	$V(2,1) = \frac{4}{3}$	$V_{xx}(2,1) = -\frac{2}{3} < 0$	$\frac{4}{3} > 0$	Local maximum

From the above table, it is concluded that the local maxima exist at only one critical point $(2,1)$, and the value of f is $\boxed{f(2,1) = \frac{4}{3}}$.

Thus, volume of the largest rectangular box in the first octant is $\boxed{\frac{4}{3}}$.

Answer 48E.

Suppose that the dimensions of the rectangular box are $x, y,$ and z.

The volume of the rectangular box is $v = xyz$.

The surface area of the rectangular box is $2(xy + yz + zx)$.

Given that the surface area of rectangular box is 64cm2.

That is,

$$2(xy + yz + zx) = 64$$

$$xy + yz + zx = \frac{64}{2}$$ Divide by 2 on both sides

$$xy + yz + zx = 32$$

$$xy + (y + x)z = 32$$

$$(y + x)z = 32 - xy$$ Subtract xy from both sides

$$z = \frac{32 - xy}{y + x}$$ Divide by $y + x$ on both sides

Substitute $z = \frac{32 - xy}{y + x}$ in $v = xyz$, obtain

$$v = xy\frac{(32 - xy)}{x + y}$$

$$= \frac{32xy}{x + y} - \frac{x^2y^2}{x + y}$$

Take $v = f(x, y) = \frac{32xy}{x + y} - \frac{x^2y^2}{x + y}$

Differentiate $f(x, y)$ partially with respect to x.

$$f_x = \frac{y^2(32 - x^2 - 2xy)}{(x + y)^2}$$

Differentiate $f(x, y)$ partially with respect to x.

$$f_y = \frac{x^2(32 - y^2 - 2xy)}{(x + y)^2}$$

Now find the critical points by setting $f_x = 0$, $f_y = 0$

That is,

$$\frac{y^2\left(32 - x^2 - 2xy\right)}{\left(x+y\right)^2} = 0$$

$$32 - x^2 - 2xy = 0$$

And

$$\frac{x^2\left(32 - y^2 - 2xy\right)}{\left(x+y\right)^2} = 0$$

$$32 - y^2 - 2xy = 0$$

Solve these two equations, get $x^2 = y^2$ or $x = y$.

Substitute $x = y$ in $32 - x^2 - 2xy = 0$, get

$$32 - x^2 - 2x(x) = 0$$

$$32 - x^2 - 2x^2 = 0$$

$$32 - 3x^2 = 0$$

$$3x^2 = 32$$

$$x^2 = \frac{32}{3}$$

$$x = \sqrt{\frac{32}{3}}$$

$$= \boxed{4\sqrt{\frac{2}{3}}}$$

$$= y$$

To find the value of z, substitute $x = 4\sqrt{\frac{2}{3}}, y = 4\sqrt{\frac{2}{3}}$ in $z = \frac{32 - xy}{y + x}$, obtain

$$z = \frac{32 - \left(4\sqrt{\frac{2}{3}}\right)\left(4\sqrt{\frac{2}{3}}\right)}{\left(4\sqrt{\frac{2}{3}}\right) + \left(4\sqrt{\frac{2}{3}}\right)}$$

$$= \frac{32 - \frac{32}{3}}{8\sqrt{\frac{2}{3}}}$$

$$z = \boxed{4\sqrt{\frac{2}{3}}}$$

Definition:

Suppose the second partial derivative of f are continuous on a disk with center (a,b) and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$ [that is (a,b) a critical point of f].

Let

$$D=D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$$

If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum

If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum

If $D<0$, then $f(a,b)$ is not a local maximum or local minimum

To find $f_{xx}(x,y)$ differentiate $f_x(x,y)$ partially with respect to *x*.

$$f_{xx}=\frac{-2y^2\left(xy^2+y^3+32x+32y\right)}{(x+y)^4}$$

$$=\frac{-2y^2(x+y)\left(32+y^2\right)}{(x+y)^4}$$

$$=\frac{-2y^2\left(32+y^2\right)}{(x+y)^3}$$

To find $f_{yy}(x,y)$ differentiate $f_y(x,y)$ partially with respect to *y*.

$$f_{yy}=\frac{-2x^2\left(32+x^2\right)}{(x+y)^3}$$

And

$$f_{xy}=\frac{-2x^4(x+y)+64xy(x+y)}{(x+y)^4}$$

$$=\frac{64xy-2x^4}{(x+y)^3}$$

Now,

$$D=f_{xx}f_{yy}-f_{xy}^2$$

$$=\frac{4x^2y^2\left(32+x^2\right)\left(32+y^2\right)-\left(64xy-2x^4\right)^2}{(x+y)^6}$$

At $\left(4\sqrt{\frac{2}{3}},4\sqrt{\frac{2}{3}}\right)$,

$$D\left(4\sqrt{\frac{2}{3}},4\sqrt{\frac{2}{3}}\right)=\frac{4\left(4\sqrt{\frac{2}{3}}\right)^2\left(4\sqrt{\frac{2}{3}}\right)^2\left(32+\left(4\sqrt{\frac{2}{3}}\right)^2\right)\left(32+\left(4\sqrt{\frac{2}{3}}\right)^2\right)-\left(64\left(4\sqrt{\frac{2}{3}}\right)\left(4\sqrt{\frac{2}{3}}\right)-2\left(4\sqrt{\frac{2}{3}}\right)^4\right)^2}{\left(\left(4\sqrt{\frac{2}{3}}\right)+\left(4\sqrt{\frac{2}{3}}\right)\right)^6}$$

$$=\frac{4(32)^2}{81}[128\times128-64\times64]$$

$$=\frac{4(32)^2(64)^2}{81}(3)$$

$$>0$$

And

$$f_{xx}\left(4\sqrt{\frac{2}{3}},4\sqrt{\frac{2}{3}}\right)=\frac{-2\left(\frac{32}{3}\right)\left(\frac{128}{3}\right)}{8\left(\frac{2}{3}\right)^{3/2}}$$

$$<0$$

Therefore,

From the definition $\left(4\sqrt{\frac{2}{3}},4\sqrt{\frac{2}{3}}\right)$ is a point of local maxima.

The volume of the box will be maximum and the maximum volume is

$$v = xyz$$

$$= \left(4\sqrt{\frac{2}{3}}\right)\left(4\sqrt{\frac{2}{3}}\right)\left(4\sqrt{\frac{2}{3}}\right)$$

$$= 16\left(\sqrt{\frac{2}{3}}\right)^2\left(4\sqrt{\frac{2}{3}}\right)$$

$$= 64\left(\frac{2}{3}\right)\sqrt{\frac{2}{3}}$$

$$= \boxed{\frac{128}{3}\sqrt{\frac{2}{3}}\text{cm}^2}$$

And the dimensions are,

$$4\sqrt{\frac{2}{3}},\ 4\sqrt{\frac{2}{3}},\ 4\sqrt{\frac{2}{3}}\text{ cm}$$

Answer 49E.

Let the dimensions of the rectangular box are x, y, z
Then the volume of the box is

$$v = xyz$$

It is given that $4x + 4y + 4z = c$ (constant)

i.e. $x + y + z = \frac{c}{4} = k$ (constant)

i.e. $z = k - x - y$

Then the volume is

$$v = xy(k - x - y)$$

Take $v = f(x, y) = kxy - x^2y - xy^2$

Then $f_x = ky - 2xy - y^2$

$$= y(k - 2x - y)$$

And $f_y = kx - x^2 - 2xy$

$$= x(k - x - 2y)$$

First we find critical points by setting $f_x = 0$, $f_y = 0$

i.e. $y(k-2x-y)=0$

And $x(k-x-y)=0$

i.e. $2x+y=k \quad \{x, y > 0\}$

And $x+2y=k$

On solving we find the critical point $\left(\frac{k}{3}, \frac{k}{3}\right)$

Now $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = k-2x-2y$

Then $D = f_{xx}f_{yy} - f_{xy}^2$

$= 4xy - (k-2x-2y)^2$

At $\left(\frac{k}{3}, \frac{k}{3}\right)$

$$D = \frac{4k^2}{9} - \left(k - \frac{2k}{3} - \frac{2k}{3}\right)^2$$

$$= \frac{4k^2}{9} - \frac{1k^2}{9}$$

$$= \frac{k^2}{3} > 0$$

And $f_{xx} = -\frac{2k}{3} < 0$

That is $\left(\frac{k}{3}, \frac{k}{3}\right)$, is the point of maximum value of "f"

That is when $x = \frac{k}{3}$, $y = \frac{k}{3}$, "f" will be maximum

That is when $x = \frac{k}{3}$, $y = \frac{k}{3}$ $z = \frac{k}{3}$ the volume will be maximum

Hence we see that the box is a $\boxed{\text{cube}}$ with dimensions $\boxed{\frac{c}{12}}$ (as $k = \frac{c}{4}$)

Answer 50E.

Let the dimensions of the base of the aquarium are x, y and its height be z

Then the volume v = xyz ---------- (1)

Let the cost of making the walls is 1 per unit area then the cost of making the base is 5 per unit area

Then the total cost of making aquarium is

$$c = 5xy + 2xz + 2yz$$

But $z = \frac{v}{xy}$ (from (1))

Then $c = 5xy + 2x\frac{v}{xy} + 2y\frac{v}{xy}$

$$= 5xy + \frac{2v}{y} + \frac{2v}{x}$$

Take $c = f(x,y) = 5xy + \frac{2v}{y} + \frac{2v}{x}$

Then $f_x = 5y - \frac{2v}{x^2}$

$$f_y = 5x - \frac{2v}{y^2}$$

First we find critical points by setting $f_x = 0$, $f_y = 0$

i.e. $5y - \frac{2v}{x^2} = 0$

And $5x - \frac{2v}{y^2} = 0$

On comparing we find x = y this gives

$$5x^3 = 2v$$

i.e. $x^3 = \frac{2v}{5}$

i.e. $x = y = \left(\frac{2v}{5}\right)^{1/3}$

Then the critical point is $\left(\left(\frac{2v}{5}\right)^{1/3}, \left(\frac{2v}{5}\right)^{1/3}\right)$

Now $f_{xx} = \frac{4v}{x^3}$, $f_{yy} = \frac{4v}{y^3}$, $f_{xy} = 5$

Then $D = f_{xx}f_{yy} - f_{xy}^2$

$$= \frac{16v^2}{x^3y^3} - 25$$

At $\left(\left(\frac{2v}{5}\right)^{1/3}, \left(\frac{2v}{5}\right)^{1/3}\right)$

$$D = \frac{16v^2}{4v^2} \times 25 - 25$$
$$= 75 > 0$$

And $f_{xx} = 10 > 0$

That is $\left(\left[\frac{2v}{5}\right]^{1/3}, \left[\frac{2v}{5}\right]^{1/3}\right)$, is a point of minima

That is at this point "f" will have minimum value

That is when $x = \left(\frac{2v}{5}\right)^{1/3}$, $y = \left(\frac{2v}{5}\right)^{1/3}$ the cost of the aquarium will be minimum

Hence the dimensions of the aquarium for the cost to be minimum are

$$\boxed{x = \left(\frac{2v}{5}\right)^{1/3},\ y = \left(\frac{2v}{5}\right)^{1/3},\ z = \frac{5}{2}\left(\frac{2v}{5}\right)^{1/3}}$$

Answer 51E.

Let the dimensions of the cardboard are x, y, z
Then volume = xyz = 32000 cm^3 ------------ (1)
The amount of cardboard will be minimum if the surface area of the box is minimum

The surface area (when x, y are dimensions of base z is height)

$$A = 2xz + 2yz + xy$$

But $z = \frac{32000}{xy}$ {from (1)}

$$= \frac{k}{xy}, \text{ where } 32000 = k \text{ (say)}$$

Then $A = xy + \frac{2kx}{xy} + \frac{2ky}{xy}$

$$= xy + \frac{2k}{y} + \frac{2k}{x}$$

Take $A = f(x,y) = xy + \dfrac{2k}{y} + \dfrac{2k}{x}$

Then $f_x = y - \dfrac{2k}{x^2}$

$f_y = x - \dfrac{2k}{y^2}$

First we find the critical points by setting $f_x = 0$, $f_y = 0$

i.e. $y - \dfrac{2k}{x^2} = 0$

And $x - \dfrac{2k}{y^2} = 0$

On solving we find $x = y = (2k)^{1/3}$

That is the critical point is $\left((2k)^{1/3}, (2k)^{1/3}\right)$

Now $f_{xx} = \dfrac{4k}{x^3}$, $f_{yy} = \dfrac{4k}{y^3}$, $f_{xy} = 1\sqrt{a^2 + b^2}$

Then $D = f_{xx} f_{yy} - f_{xy}^2$

$= \dfrac{16k^2}{x^3 y^3} - 1$

At $\left(\sqrt[3]{2k}, \sqrt[3]{2k}\right)$; $D = \dfrac{16k^2}{4k^2} - 1$

$= 3 > 0$

And $f_{xx} = \dfrac{4k}{(2k)^{1/3}} > 0$

That is $\left(\sqrt[3]{2k}, \sqrt[3]{2k}\right)$, is the point of minimum value of "f"

That is when $x = (2k)^{1/3}$, $y = (2k)^{1/3}$ the surface area of the box will be minimum

Hence the required dimensions of the box are

$$x = (2k)^{1/3},\ y = (2k)^{1/3},\ z = \frac{k}{(2k)^{2/3}}$$

As $k = 32000 \text{ cm}^3$

Then $\boxed{x = 40 \text{ cm},\ y = 40\text{cm},\ z = 20 \text{ cm}}$

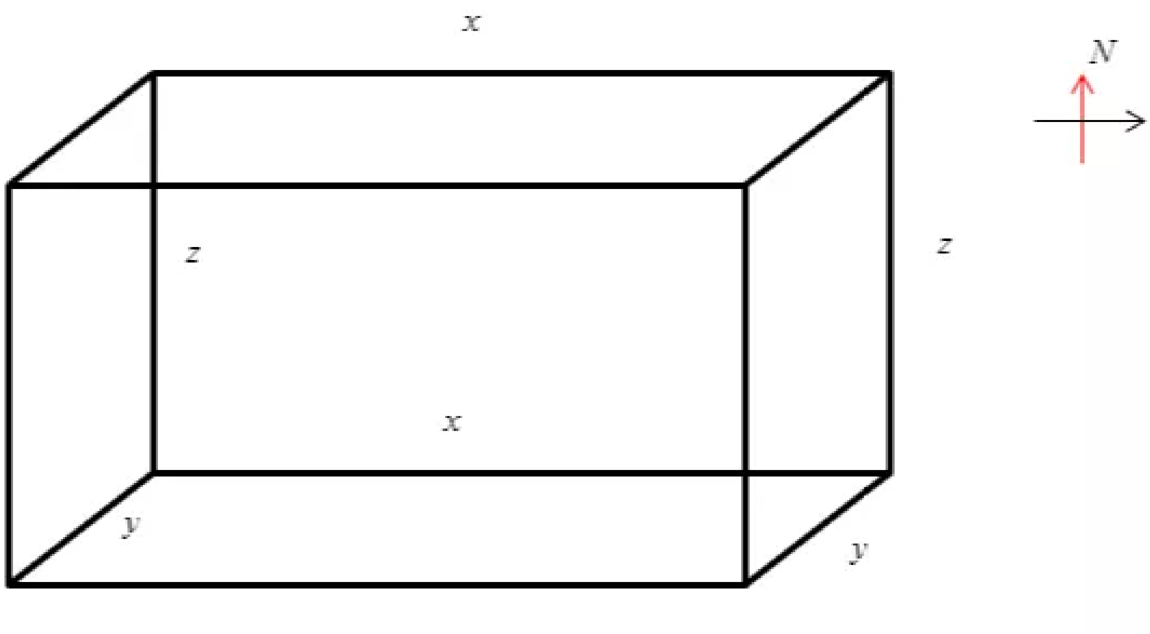
Answer 52E.

(a)

Consider the rectangular building that is given in the problem, which is being designed to minimize the heat loss.

Let x be the length of the walls on the north and south sides, y the length of the walls on the east and west sides, and z the height of the building.

The following shows the rectangular building.



The rectangular building is being designed to minimize the heat loss. The east and west wall lose heat at rate of 10 units/ m2 per day, the north and south wall at a rate of 8 units/ m2 per day, the floor at a rate of 1 unit/m2 per day, and the roof at a rate of 5 units/m2 per day.

The objective is to find and sketch the domain of the heat loss as function of the lengths of the sides.

The heat loss,

$$h_0(x,y,z) = 10(2yz) + 8(2xz) + 1xy + 5xy$$
$$= 6xy + 16xz + 20yz$$

The volume of the rectangular box is $xyz = 4000$.

$$xyz = 4000$$
$$z = \frac{4000}{xy}$$

Substitute $z = \frac{4000}{xy}$ into the heat function h_0 and get the following function.

$$h(x,y) = 6xy + 16x\left(\frac{4000}{xy}\right) + 20y\left(\frac{4000}{xy}\right)$$
$$= 6xy + \frac{64000}{y} + \frac{80000}{x}$$

It is also given that each wall must be at least 30 m long, and the height must be at least 4 m.

So, the constraints on the dimensions says

$$x, y \geq 30,\ z \geq 4, \text{and } xyz = 4000$$

$$x, y \geq 30,\ z \geq 4, \text{and } z = \frac{4000}{xy}$$

$$x, y \geq 30,\ \frac{4000}{xy} \geq 4$$

$$x, y \geq 30,\ \frac{1000}{x} \geq y$$

Thus, $x \geq 30$ and $30 \leq y \leq \frac{1000}{x}$.

Therefore, the domain of the heat function is $D = \left\{(x,y) \in \mathbb{R}^2 \mid x \geq 30, 30 \leq y \leq \frac{1000}{x}\right\}$.

Sketch the domain of the heat function.

Set $h_y = 0$ and implies the following result.

$$6x - \frac{64000}{y^2} = 0$$

$$6x = \frac{64000}{y^2}$$

$$xy^2 = \frac{64000}{6}$$

$$xy^2 = \frac{32000}{3}$$

Substitute $y = \frac{40000}{3x^2}$ in the equation $xy^2 = \frac{32000}{3}$ and get the below result.

$$xy^2 = \frac{32000}{3}$$

$$x\left(\frac{40000}{3x^2}\right)^2 = \frac{32000}{3}$$

$$x^3 = \frac{50000}{3}$$

$$x = \sqrt[3]{\frac{50000}{3}} \cong 25.54$$

So, this cannot lead to a critical point in the domain. So the answer must lies on the boundary.

So on the horizontal line $y = 30$, we have $30 \le x \le \frac{100}{3}$. The restriction of h to the line gives a function $g_1(x) = h(x,30) = 180x + \frac{80000}{x} + \frac{6400}{3}$.

Find the derivative of g_1:

$$g_1'(x) = 180 - \frac{80000}{x^2}$$

Which is positive on the interval $\left[30, \frac{100}{3}\right]$, so no critical points.

So on the horizontal line $y=30$, we have $30 \le x \le \frac{100}{3}$. The restriction of h to the line gives a function $g_1(x)=h(x,30)=180x+\frac{80000}{x}+\frac{6400}{3}$.

Find the derivative of g_1:

$$g_1'(x)=180-\frac{80000}{x^2}$$

Which is positive on the interval $\left[30,\frac{100}{3}\right]$, so no critical points.

The end points yields $h(30,30)=g_1(30)=10,200$ and

$$\begin{aligned} h\left(\frac{100}{3},30\right) &= g_1\left(\frac{100}{3}\right) \\ &= 180\left(\frac{100}{3}\right)+\frac{80000}{\left(\frac{100}{3}\right)}+\frac{6400}{3} \\ &\cong 10,533. \end{aligned}$$

On the vertical segment with $x=30$ we obtain $g_2(y)=180y+\frac{64000}{y}+\frac{8000}{3}$, where $30 \le y \le \frac{100}{3}$.

Find the derivative of g_2:

$$g_2'(y)=180-\frac{64000}{y^2}$$

We check $g_2'(y)>0$ on the interval and the endpoints yield

$g_2(30)=h(30,30)=10,200$, which $g_2\left(\frac{100}{3}\right)=h\left(30,\frac{100}{3}\right)\approx 10587.$

Finally we consider the hyperbola with bound the third side, $y=\frac{100}{x}$. The function

$$g_3(x)=h\left(x,\frac{100}{x}\right)=6000+64x+\frac{80000}{x} \text{ for } 30\le x\le\frac{100}{3}.$$

Since $g_3'(x)=64-\frac{80000}{x^2}<0$ for $30\le x\le\frac{100}{3}$, so g_3 is decreasing on the interval

achieving a maximum at $g_3(30)\sim 10587$ and a minimum at $g_3\left(\frac{100}{3}\right)\sim 10533$.

Thus, the absolute minimum is $h(30,30)=10,200.$

(c)

From the above part, we discarded the critical point of $x=\sqrt[3]{\frac{50000}{3}}\approx 25.54$ and $y=\frac{80}{\sqrt[3]{60}}$,

and hence $z=\frac{40000}{xy}$, for approximate dimensions $x\approx 25.54, y\approx 20.43, \text{ and } z\approx 7.67$ which has the minimum heat loss.

Answer 53E.

Let the dimensions of the rectangular are x, y, z

Then the length of the diagonal is

$$\sqrt{x^2+y^2+z^2}=L \qquad \text{(Given)}$$

Or $\quad x^2+y^2+z^2=L^2 \qquad$ ------------- (1)

The volume of the box is

$$v=xyz$$

$$=xy\sqrt{L^2-x^2-y^2} \qquad \{\text{Using (1) as } z>0\}$$

Take $\quad v=f(x,y)=xy\sqrt{L^2-x^2-y^2}$

Then $\quad f_x=yL\sqrt{1^2-x^2-y^2}-\frac{x^2y}{\sqrt{1^2-x^2-y^2}}$

$$=\frac{y\left(L^2-x^2-y^2\right)-x^2y}{\sqrt{L^2-x^2-y^2}}$$

And $\quad f_y=\frac{x\left(L^2-x^2-y^2\right)-xy^2}{\sqrt{L^2-x^2-y^2}}$

First we find the critical points by setting $f_x = 0$, $f_y = 0$

i.e. $y(L^2 - x^2 - y^2) - x^2 y = 0$

And $x(L^2 - x^2 - y^2) - xy^2 = 0$

On comparing we find x = y

Then $L^2 - x^2 - y^2 = x^2$

i.e. $L^2 = 3x^2$

i.e. $x^2 = L^2/3$

i.e. $x = \frac{L}{\sqrt{3}} = y$ (x, y = 0)

Then the critical point is $\left(\frac{L}{\sqrt{3}}, \frac{L}{\sqrt{3}}\right)$

Now $f_{xx} = \frac{-x}{\sqrt{L^2 - x^2 - y^2}} - \frac{2xy}{\sqrt{L^2 - x^2 - y^2}} + \frac{x^3 y}{(L^2 - x^2 - y^2)^{3/2}}$

$$f_{yy} = \frac{-y}{\sqrt{L^2 - x^2 - y^2}} - \frac{2xy}{\sqrt{L^2 - x^2 - y^2}} + \frac{xy^3}{(L^2 - x^2 - y^2)^{3/2}}$$

$$f_{xy} = \sqrt{L^2 - x^2 - y^2} - \frac{y}{\sqrt{L^2 - x^2 - y^2}} - \frac{x^2}{\sqrt{L^2 - x^2 - y^2}} + \frac{x^2 y^2}{(L^2 - x^2 - y^2)^{3/2}}$$

Then $D = f_{xx} f_{yy} - f_{xy}^2$

At $\left(\frac{L}{\sqrt{3}}, \frac{L}{\sqrt{3}}\right)$

$$D = \left(1 + \frac{L}{\sqrt{3}}\right)\left(1 + \frac{L}{\sqrt{3}}\right) - \left(1 - \frac{L}{\sqrt{3}}\right)^2$$

$$= \left(1 + \frac{L}{\sqrt{3}}\right)^2 - \left(1 - \frac{L}{\sqrt{3}}\right)^2$$

$$= \frac{4L}{\sqrt{3}} > 0$$

And $f_{xx} = -\left(1 + \frac{L}{\sqrt{3}}\right) < 0$

That is $\left(\frac{L}{\sqrt{3}}, \frac{L}{\sqrt{3}}\right)$, is a point of maximum of "f"

That is when $x=\frac{L}{\sqrt{3}}$, $y=\frac{L}{\sqrt{3}}$ the function "f" will have maximum value

That is when $x=\frac{L}{\sqrt{3}}$, $y=\frac{L}{\sqrt{3}}$, $z=\frac{L}{\sqrt{3}}$ the volume of the box will be maximum

And the maximum volume is

$$\boxed{v=\left(\frac{L}{\sqrt{3}}\right)\left(\frac{L}{\sqrt{3}}\right)\left(\frac{L}{\sqrt{3}}\right)=\frac{L^3}{3\sqrt{3}}}$$

Answer 54E.

It is given that

$$P=2pq+2pr+2rq$$

Also $p+q+r=1$

i.e. $r=1-p-q$

Then
$$\begin{aligned}P&=2pq+2p(1-p-q)+2q(1-p-q)\\&=2pq+2p-2p^2-2pq+2q-2pq-2q^2\\&=2p+2q-2pq-2p^2-2q^2\end{aligned}$$

Now $\frac{\partial P}{\partial p}=2-2q-4p$

And $\frac{\partial P}{\partial q}=2-2p-4q$

We find critical point by setting $\frac{\partial P}{\partial p}=0$, $\frac{\partial P}{\partial q}=0$

i.e. $2(1-q-2p)=0$

And $2(1-p-2q)=0$

On solving these equations e find

$$p=q=\frac{1}{3}$$

Then the critical point is $\left(\frac{1}{3},\frac{1}{3}\right)$

Now $\frac{\partial^2 P}{\partial p^2}=-4$, $\frac{\partial^2 P}{\partial q^2}=-4$, $\frac{\partial^2 P}{\partial p\,\partial q}=-2$

Then
$$\begin{aligned}D&=\frac{\partial^2 P}{\partial p^2}\cdot\frac{\partial^2 P}{\partial q^2}-\left[\frac{\partial^2 P}{\partial p\,\partial q}\right]^2\\&=16-4\\&=12\end{aligned}$$

At $\left(\frac{1}{3}, \frac{1}{3}\right)$

$$D = 12 > 0$$

And $\quad \frac{\partial^2 P}{\partial p^2} = -4 < 0$

That is $\left(\frac{1}{3}, \frac{1}{3}\right)$, is a point of maximum value of P (by second derivative test)

That is when $p = \frac{1}{3}$, $q = \frac{1}{3}$ and $r = 1 - p - q$

$$= 1 - \frac{1}{3} - \frac{1}{3}$$

$$= \frac{1}{3}$$

P will have maximum value

And the maximum value of P is

$$P = 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$$

$$= \frac{2}{9} + \frac{2}{9} + \frac{2}{9}$$

$$= \frac{2}{3}$$

Hence P is at most $\frac{2}{3}$

And the maximum value of P is

$$P = 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$$

$$= \frac{2}{9} + \frac{2}{9} + \frac{2}{9}$$

$$= \frac{2}{3}$$

Hence P is at most $\frac{2}{3}$

Answer 55E.

The vertical deviation of the point (x_i, y_i) from the line is

$$d_i = y_i - (mx_i + b)$$

Now, in order to make the sum of squares minimum, we are to minimize

$$S = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (y_i - mx_i - b)^2$$

S will have its extreme values when

$$\frac{\partial S}{\partial m} = 0 \quad \text{and} \quad \frac{\partial S}{db} = 0$$

Differentiating S partially with respect to b,
We get

$$\frac{\partial S}{\partial b} = \frac{\partial}{\partial b} \sum_{i=1}^{n} (y_i - mx_i - b)^2$$

$$= \sum_{i=1}^{n} 2(y_i - mx_i - b) \cdot (-1)$$

$$= -2\sum_{i=1}^{n} (y_i - mx_i - b)$$

Differentiating S partially with respect to m,
We get,

$$\frac{\partial S}{\partial m} = \frac{\partial}{\partial m} \sum_{i=1}^{n} (y_i - mx_i - b)^2$$

$$= \sum_{i=1}^{n} 2(y_i - mx_i - b) \cdot (-x_i)$$

$$= -2\sum_{i=1}^{n} x_i (y_i - mx_i - b)$$

S will have extreme values when $\frac{\partial S}{\partial b} = 0$, $\frac{\partial S}{\partial m} = 0$

Now, $\frac{\partial S}{\partial b} = 0$ gives,

$$\sum_{i=1}^{n} (y_i - mx_i - b) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} 1 = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} y_i = m\sum_{i=1}^{n} x_i + bn \qquad \text{------ (1)} \qquad \left[\text{since } \sum_{i=1}^{n} 1 = n\right]$$

Also, from $\frac{\partial S}{\partial b} = 0$, we have,

$$-2\sum_{i=1}^{n} x_i \left(y_i - mx_i - b\right) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n}\left(x_i y_i - mx_i^2 - bx_i\right) = 0$$

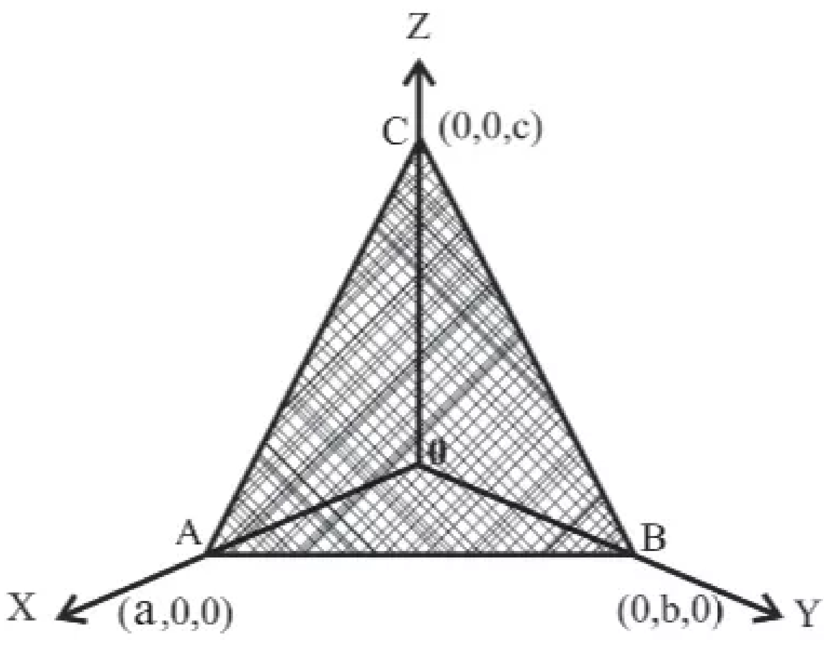
$$\Rightarrow \quad \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} mx_i^2 + \sum_{i=1}^{n} bx_i$$

$$\Rightarrow \quad \sum_{i=1}^{n} x_i y_i = m\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i \qquad \text{------ (2)}$$

Solving equation (1) and (2) we can find the values of m and b.
Hence, the line of best fit is obtained when

$$\boxed{\begin{aligned} &m\sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i \\ &\sum_{i=1}^{n} x_i y_i = m\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i \end{aligned}}$$

Answer 56E.



Let the plane passing through the point (1,2,3) and cuts off the smallest volume in the first octant cuts off intercepts a, b and c from the co-ordinate axes. Let the required plane meets the co-ordinate axes at points A, B, C respective.
Therefore the equation of plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Since it passes through (1, 2, 3).

Therefore, $$\frac{1}{a}+\frac{2}{b}+\frac{3}{c}=1$$

$$\Rightarrow \quad \frac{3}{c}=1-\frac{1}{a}-\frac{2}{b}$$

$$\frac{3}{c}=\frac{ab-b-2a}{ab}$$

Or, $$c=\frac{3ab}{ab-b-2a}$$

Now, the plane forms tetrahedron in the first octant OABC and the volume of tetrahedron

$$V=\frac{1}{6}abc$$

$$=\frac{1}{6}ab\frac{3ab}{ab-b-2a}$$

$$=\frac{a^2b^2}{ab-b-2a}$$

Differentiating v partially with respect to a,

$$\frac{\partial v}{\partial a}=\frac{\partial}{\partial a}\left(\frac{a^2b^2}{ab-b-2a}\right)$$

$$=b^2\frac{\partial}{\partial a}\left(\frac{a^2}{(ab-b-2a)}\right)$$

$$=\frac{b^2\left[(ab-b-2a)\frac{\partial}{\partial a}a^2-a^2\frac{\partial}{\partial a}(ab-b-2a)\right]}{(ab-b-2a)^2}$$

$$=\frac{b^2\left[(ab-b-2a)2a-a^2(b-2)\right]}{(ab-b-2a)^2}$$

$$=\frac{ab^2\left[2ab-2b-4a-ab+2a\right]}{(ab-b-2a)^2}$$

$$=\frac{ab^2\left[ab-2a-2b\right]}{(ab-b-2a)^2}$$

Differentiating v partially with respect to b,

$$\frac{\partial v}{\partial b}=\frac{\partial}{\partial b}\left(\frac{a^2b^2}{(ab-b-2a)}\right)$$

$$=a^2\frac{\partial}{\partial b}\left(\frac{b^2}{ab-b-2a}\right)$$

$$=\frac{a^2\left[(ab-b-2a)\frac{\partial}{\partial b}b^2-\frac{\partial}{\partial b}(ab-b-2a)\right]}{(ab-b-2a)^2}$$

$$=\frac{a^2\left[(ab-b-2a)2b-b^2(a-1)\right]}{(ab-b-2a)^2}$$

$$=\frac{a^2b\left[2ab-2b-4a-ab+b\right]}{(ab-b-2a)^2}$$

$$=\frac{a^2b\left[ab-4a-b\right]}{(ab-b-2a)^2}$$

For critical points, $\frac{\partial v}{\partial a}=0$ and $\frac{\partial v}{\partial b}=0$

Now, $\frac{\partial v}{\partial a}=0$ gives,

$$\frac{ab^2\left[ab-2a-2b\right]}{(ab-b-2a)^2}=0$$

$\Rightarrow$ $a=0$ or $b=0$ or $ab-2a-2b=0$

Since a and b cannot be zero [if a = 0, b = 0, tetrahedron will not formed]

Therefore,

$$ab-2a-2b=0$$

$\Rightarrow$ $ab=2a+2b$

$\Rightarrow$ $\frac{1}{a}+\frac{1}{b}=\frac{1}{2}$ ----- (1)

Also, from $\frac{\partial v}{\partial b}=0$

$$\frac{a^2b\left[ab-4a-b\right]}{ab-b-2a}=0$$

$\Rightarrow$ $a=0$ or $b=0$ or $ab-4a-b=0$

Since $a\neq 0, b\neq 0$ Therefore,

$$ab-4a-b=0$$

$\Rightarrow$ $ab=4a+b$

$\Rightarrow$ $\frac{1}{a}+\frac{4}{b}=1$ ------ (2)

Subtracting equation (1) from equation (2)

$$\frac{3}{b}=\frac{1}{2}$$

$$\Rightarrow \quad b=6$$

Putting value of b = 6, in equation (1) we get.

$$\frac{1}{a}+\frac{1}{6}=\frac{1}{2}$$

$$\frac{1}{a}=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$$

$$\Rightarrow \quad a=3$$

Putting values of a and b in the value of c,

We get, $$c=\frac{3ab}{ab-b-2a}=\frac{3\times3\times6}{3\times6-6-2\times3}$$

$$=\frac{54}{6}$$

$$=9.$$

Now, differentiating $\frac{\partial v}{\partial a}$ partially with respect to a,

$$\frac{\partial^2 v}{\partial a^2}=\frac{\partial}{\partial a}\frac{ab^2(ab-2a-2b)}{(ab-b-2a)^2}$$

$$=b^2\frac{\partial}{\partial a}\frac{(a^2b-2a^2-2ab)}{(ab-b-2a)^2}$$

$$=\frac{b^2\left[(ab-b-2a)^2\frac{\partial}{\partial a}(a^2b-2a^2-2ab)-(a^2b-2a^2-2ab)\frac{\partial}{\partial a}(ab-b-2a)^2\right]}{(ab-b-2a)^4}$$

$$=\frac{b^2\left[(ab-b-2a)^2\cdot(2ab-4a-2b)-(a^2b-2a^2-2ab)\cdot 2(ab-b-2a)\cdot(b-2)\right]}{(ab-b-2a)^4}$$

$$=\frac{b^2\left[(ab-b-2a)(2ab-4a-2b)-(a^2b-2a^2-2ab)2(b-2)\right]}{(ab-b-2a)^3}$$

Therefore,

$\frac{\partial^2 v}{\partial a^2}$ at a = 3 and b = 6.

$$=12$$

Differentiating $\frac{\partial v}{\partial a}$ partially with respect to b,

$$\frac{\partial^2 v}{\partial b \partial a} = \frac{\partial}{\partial b}\left(\frac{\partial v}{\partial a}\right)$$

$$= \frac{\partial}{\partial b} \quad \frac{ab^2(ab-2a-2b)}{(ab-b-2a)^2}$$

$$= a\frac{\partial}{\partial b} \quad \frac{(ab^3-2ab^2-2b^3)}{(ab-b-2a)^2}$$

$$= \frac{a\left[(ab-b-2a)^2\frac{\partial}{\partial b}(ab^3-2ab^2-2b^3)-(ab^3-2ab^2-2b^3)\frac{\partial}{\partial b}(ab-b-2a)^2\right.}{(ab-b-2a)^4}$$

$$= \frac{a\left[(ab-b-2a)^2(3ab^2-4ab-6b^2)-(ab^3-2ab^2-2b^3)2(ab-b-2a)(a-1)\right.}{(ab-b-2a)^4}$$

$$= \frac{a\left[(ab-b-2a)(3ab^2-4ab-6b^2)-2(a-1)(ab^3-3ab^2-2b^3)\right]}{(ab-b-2a)^3}$$

Therefore at a =3 and b=6 , $\frac{\partial^2 v}{\partial b \partial a}$ =3

Differentiating $\frac{\partial v}{\partial b}$ partially with respect to b,

$$\frac{\partial^2 v}{\partial b^2} = \frac{\partial}{\partial b}\left(\frac{\partial v}{\partial b}\right)$$

$$= \frac{\partial}{\partial b} \quad \frac{a^2b(ab-4a-b)}{(ab-b-2a)^2}$$

$$= a^2 \frac{\partial}{\partial b} \frac{(ab^2 - 4ab - b^2)}{(ab - b - 2a)^2}$$

$$= \frac{a^2 \left[(ab - b - 2a) \frac{\partial}{\partial b}(ab^2 - 4ab - b^2) - (ab^2 - 4ab - b^2) \frac{\partial}{\partial b}(ab - b - 2a)^2 \right]}{(ab - b - 2a)^4}$$

$$= \frac{a^2 \left[(ab - b - 2a)^2 (2ab - 4a - 2b) - (ab^2 - 4ab - b^2) 2(ab - b - 2a)(a - 1) \right]}{(ab - b - 2a)^4}$$

$$= \frac{a^2 \left[(2ab - 4a - 2b)(ab - b - 2a) - 2(ab^2 - 4ab - b^2)(a - 1) \right]}{(ab - b - 2a)^3}$$

Therefore,

At a=3 and b=6 , $\frac{\partial^2 v}{\partial b^2}$ =3

Now, D at a = 3 and b = 6 is

$$D = \left(\frac{\partial^2 v}{\partial a^2}\right)\left(\frac{\partial^2 v}{\partial b^2}\right) - \left[\frac{\partial^2 v}{\partial b \partial a}\right]^2$$

$$= 12 \times 3 - (3)^2$$

$$= 36 - 9$$

$$= 27$$

$$> 0$$

Therefore, volume v is minimum at a = 3, b = 6 and c = 9 by second derivative test and equation of plane is,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\Rightarrow \quad \frac{x}{3} + \frac{y}{6} + \frac{z}{9} = 1$$

$$\Rightarrow \quad \frac{6x + 3y + 2z}{18} = 1$$

$$\Rightarrow \quad 6x + 3y + 2z = 18$$

Hence,

The equation of plane passing through(1,2,3) and cuts off the smallest volume in the first octant is $6x + 3y + 2z = 18$