LU Decomposition :

→ Using LU - Decomposition to solve System of equ.

(i) $\boxed{AX = B}$

(i) Given A, Find L and U such that $A = LU$
Hence
$\cancel{A=LU}$ $LUX = B$

(ii) Let, $Y = UX$, such that $LY = BY$. Solve this triangular system for Y.

(iii) Finally solve the triangular system, $UX = Y$ for X.

- LU Decomposition of matrix -

$A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Where, $l_{11} = l_{22} = l_{33} = 1$

** → Some times, matrices does not have LU Decomposition.

ex

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

$A_1 = 1$ $\quad A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$ $\quad A_3 = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$.

$|A_1| = 1 \neq 0$ $\quad |A_2| = 2 \neq 0$ $\quad |A_3| = 10 \neq 0$

** → An invertible matrix A has an LU Decompo- provided that all its leading submatrices have non-zero determinent.

→ Matrix 'A' LU-decomposition possible.

ex

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix}$$

$A_1 = 1$ $\quad A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\quad A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix}$

$|A_1| = 1 \neq 0$ $\quad |A_2| = 0 \cancel{\neq} 0$ $\quad |A_3| = ?$

→ LU Decomposition not possible. $|A_2| = 0$

ex. find an LU-Decomposition of $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$

$\Rightarrow \quad A = LU$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{31} & 1 \end{bmatrix} \times \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21} u_{11} & l_{21}u_{12} + u_{22} & l_{31}u_{13} + \cancel{l_{31}} u_{23} \\ l_{31} u_{11} & l_{31}u_{12} + l_{31}u_{22} & l_{31}u_{13} + l_{31}u_{23} + u_{33} \end{bmatrix}$$

by comparing and equating elemenwise we get-

1st row, $u_{11} = 1$, $u_{12} = 2$; $u_{13} = 4$

2nd row, $l_{21} u_{11} = 3 \Rightarrow l_{21} = 3$

$u_{22} = 2$

$u_{23} = 2$

3rd row,

$l_{31} u_{11} = 2 \Rightarrow l_{31} = 2$

$l_{32} = 1$

$l_{33} = 3$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

→ in exam they give one matrix and say find other.

ex) solve the system of equations -

$$x_1 + 2x_2 + 4x_3 = 3$$
$$3x_1 + 8x_2 + 14x_3 = 13$$
$$2x_1 + 6x_2 + 13x_3 = 4$$

$\Rightarrow \quad AX = B$

$\Rightarrow LUX = B$ —— (1)

From prev example, $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

let, $Y = UX$, put in eqn(i)

$LY = B$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

$\Rightarrow \quad y_1 = 3$

$3y_1 + y_2 = 13, \Rightarrow y_2 = 4$

$2y_1 + y_2 + y_3 = 4 \Rightarrow y_3 = -6$

6 4

since, $U.X = Y$, put y in this eqn.

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} \qquad X = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}.$$

$\Rightarrow \quad 3x_3 = -6 \Rightarrow x_3 = -2$ $\qquad -8 + 3 + 8$

$2x_2 + 2x_3 = 4 \Rightarrow x_2 = 4$

$x_1 + 2x_2 + 4x_3 = 3 \Rightarrow x_1 = 3$