Chapter – 5

Differential Calculus

Ex 5.1

Question 1.
Determine whether the following functions are odd or even?

(i) f(x) = $\left(\frac{a^x-1}{a^x+1}\right)$

(ii) f(x) = log(x^2 + $\sqrt{x^2+1}$)

(iii) f(x) = sin x + cos x

(iv) f(x) = x^2 – |x|

(v) f(x) = x + x^2

Solution:

(i) f(x) = $\left(\frac{a^x-1}{a^x+1}\right)$

$$f(-x)=\left(\frac{a^{-x}-1}{a^{-x}+1}\right)=\left(\frac{\frac{1}{a^x}-1}{\frac{1}{a^x}+1}\right)$$

$$=\frac{1-a^x}{1+a^x}=-\left(\frac{a^x-1}{a^x+1}\right)$$

Thus f(-x) = -f(x)

∴ f(x) is an odd function.

(ii) f(x) = log(x^2 + $\sqrt{x^2+1}$)

f(-x) = log($(-x)^2$ + $\sqrt{(-x)^2+1}$)

= log(x^2 + $\sqrt{x^2+1}$)

Thus f(-x) = f(x)

∴ f(x) is an even function.

(iii) $f(x) = \sin x + \cos x$

$f(-x) = \sin(-x) + \cos(-x)$

$= -\sin x + \cos x$

$= -[\sin x - \cos x]$

Since $f(-x) \neq -f(x)$ (or) $f(x) \neq -f(x)$

∴ f(x) is neither odd nor even function.

(iv) Given $f(x) = x^2 - |x|$
$f(-x) = (-x)^2 - |-x|$
$= x^2 - |x|$
$= f(x)$
∴ f(x) is an even function.

(v) $f(x) = x + x^2$
$f(-x) = (-x) + (-x)^2 = -x + x^2$
Since $f(-x) \neq f(x)$, $f(-x) \neq -f(x)$.
∴ f(x) is neither odd nor even function.

Question 2.
Let f be defined by $f(x) = x^3 - kx^2 + 2x$, $x \in R$. Find k, if 'f' is an odd function.

Solution:
For a polynomial function to be an odd function each term should have odd powers pf x. Therefore there should not be an even power of x term.
∴ $k = 0$.

Question 3.

If $f(x) = x^3 - \frac{1}{x^3}$, then show that $f(x) + f(\frac{1}{x}) = 0$

Solution:

f(x) = $x^3 - \frac{1}{x^3}$ .... (1)

$f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 - \frac{1}{\left(\frac{1}{x}\right)^3} = \frac{1}{x^3} - x^3$ .... (2)

(1) + (2) gives $f(x) + f\left(\frac{1}{x}\right) = x^3 - \frac{1}{x^3} + \frac{1}{x^3} - x^3 = 0$

Hence Proved.

Question 4.

If f(x) = $\frac{x+1}{x-1}$, then prove that f(f(x)) = x.

Solution:

$$f(x) = \frac{x+1}{x-1}$$

$$f(f(x)) = \frac{\frac{x+1}{x-1}+1}{\left(\frac{x+1}{x-1}\right)-1} = \frac{\frac{x+1+(x-1)!}{x-1}}{\frac{(x+1)-(x-1)}{x-1}}$$

$$= \frac{x+1+x-1}{x+1-x+1} = \frac{2x}{2} = x$$

Hence proved.

Question 5.

For f(x) = $\frac{x-1}{3x+1}$, write the expressions of f($\frac{1}{x}$) and $\frac{1}{f(x)}$.

Solution:

$$f(x) = \frac{x-1}{3x+1}$$

$$f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}-1}{3\left(\frac{1}{x}\right)+1} = \frac{\frac{(1-x)}{x}}{\left(\frac{3+x}{x}\right)}$$

$$= \frac{1-x}{3+x}$$

Also $\frac{1}{f(x)} = \frac{1}{\frac{x-1}{(3x+1)}} = \frac{3x+1}{x-1}$

Question 6.
If $f(x) = e^x$ and $g(x) = \log_e x$ then find
(i) (f + g) (1)
(ii) (fg) (1)
(iii) (3f) (1)
(iv) (5g) (1)

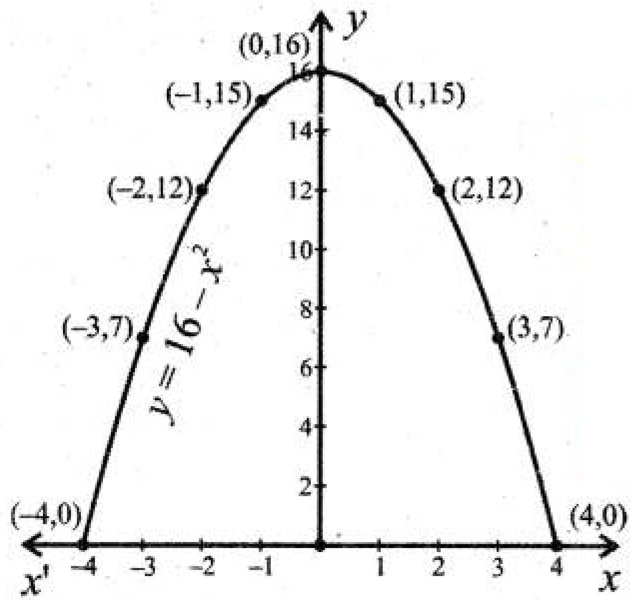
Solution:
(i) $(f+g)(1) = e^1 + \log_e 1 = e + 0 = e$
(ii) $(fg)(1) = f(1)\,g(1) = e^1 \log_e^1 = e \times 0 = 0$
(iii) $(3f)(1) = 3\,f(1) = 3\,e^1 = 3e$
(iv) $(5g)(1) = 5\,(g)(1) = 5 \log_e^1 = 5 \times 0 = 0$

Question 7.
Draw the graph of the following functions:
(i) $f(x) = 16 - x^2$
(ii) $f(x) = |x - 2|$
(iii) $f(x) = x|x|$
(iv) $f(x) = e^{2x}$
(v) $f(x) = e^{-2x}$
(vi) $f(x) = \frac{|x|}{x}$

Solution:
(i) $f(x) = 16 - x^2$
Let $y = f(x) = 16 - x^2$
Choose suitable values for x and determine y. Thus we get the following table.

x	–4	–3	–2	–1	0	1	2	3	4
y	0	7	12	15	16	15	12	7	0

Plot the points (-4, 0), (-3, 7), (-2, 12), (-1, 15), (0, 16), (1, 15), (2, 12), (3, 7), (4, 0).
The graph is as shown in the figure.



(ii) Let y = f(x) = |x – 2|

$$= \begin{cases} x-2 & \text{if} \quad x-2 \geq 0 \\ -(x-2) & \text{if} \quad x-2 < 0 \end{cases}$$

$$= \begin{cases} x-2 & \text{if} \quad x \geq 2 \\ -x+2 & \text{if} \quad x < 2 \end{cases}$$

$y = x - 2, x \geq 2$

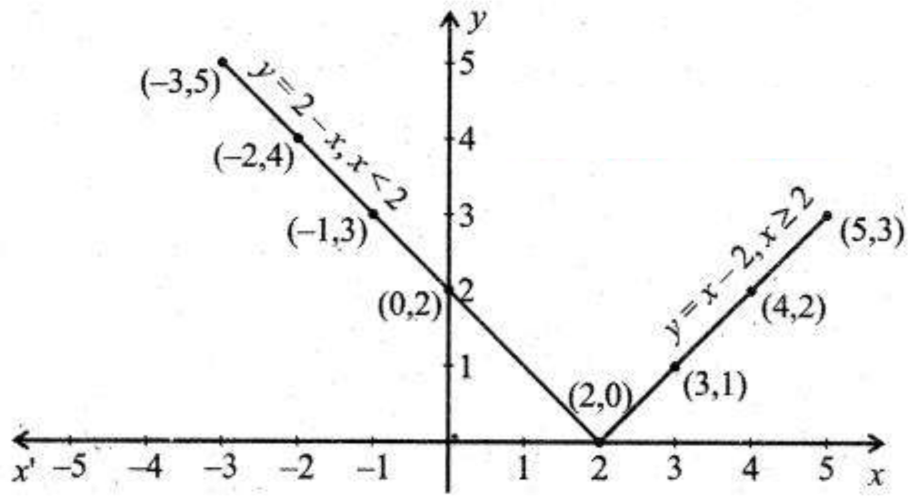
x	2	3	4	5
y	0	1	2	3

$y = -x + 2, x < 2$

x	0	–1	–2	–3
y	2	3	4	5

Plot the points (2, 0), (3, 1) (4, 2), (5, 3), (0, 2), (-1, 3), (-2, 4), (-3, 5) and draw a line.

The graph is as shown in the figure.



(iii) Let y = f(x) = x|x|

$$f(x) = \begin{cases} x(x) & \text{if} \quad x \geq 0 \\ x(-x) & \text{if} \quad x < 0 \end{cases}$$

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \geq 0 \\ -x^2 & \text{if} \quad x < 0 \end{cases}$$

$y = x^2, x \geq 0$

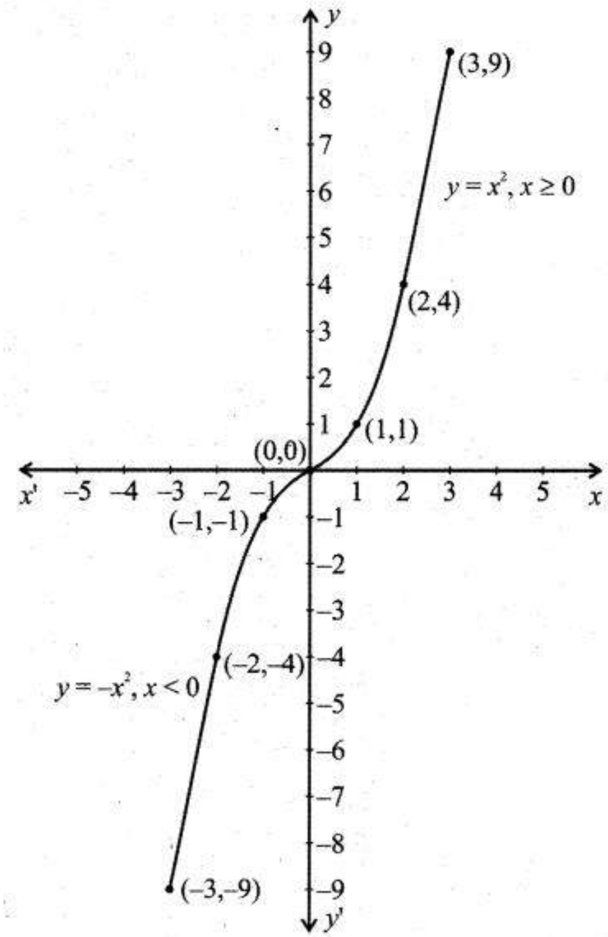
x	0	1	2	3
y	0	1	4	9

$y = -x^2, x < 0$

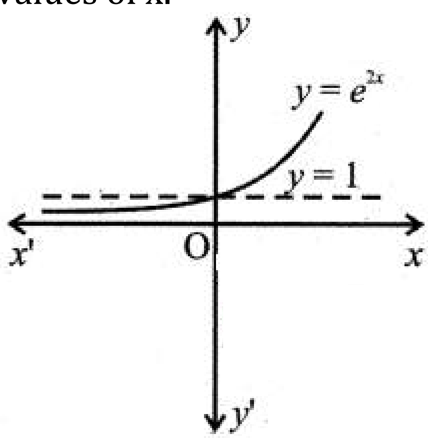
x	–1	–2	–3
y	–1	–4	–9

Plot the points (0, 0), (1, 1) (2, 4), (3, 9), (-1, -1), (-2, -4), (-3, -9) and draw a smooth curve.

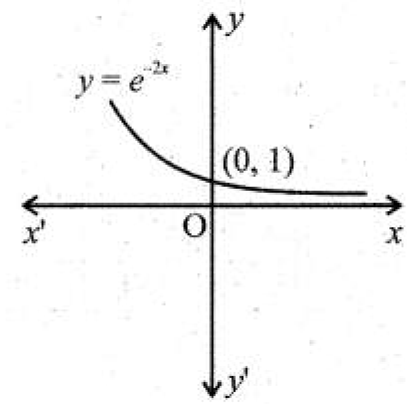
The graph is as shown in the figure.



(iv) For x = 0, f(x) becomes 1 i.e., the curve cuts the y-axis at y = 1.
For no real value of x, f(x) equals to 0. Thus it does not meet the x-axis for real values of x.



(v) For x = 0, f(x) becomes 1 i.e., the curve cuts the y-axis at y = 1.
For no real value of x, f(x) equal to 0. Thus it does not meet the x-axis for real values of x.

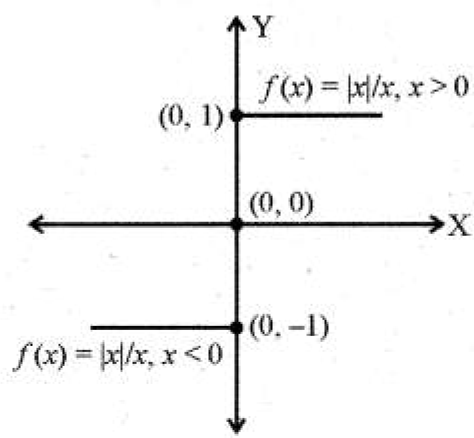


(vi) If f: R → R is defined by

$$f(x)=\begin{cases}\dfrac{|x|}{x} & \text{if} \quad x\neq 0\\ 0 & \text{if} \quad x=0\end{cases}$$

$$=\begin{cases}\dfrac{x}{x} & \text{if} \quad x>0\\ \dfrac{-x}{x} & \text{if} \quad x<0\end{cases}$$

$$=\begin{cases}1 & \text{if} \quad x>0\\ -1 & \text{if} \quad x<0\end{cases}$$



The domain of the function is R and the range is {-1, 0, 1}.

Ex 5.2

Question 1.
Evaluate the following:

(i) $\lim_{x \to 2} \frac{x^3+2}{x+1}$

(ii) $\lim_{x \to \infty} \frac{2x+5}{x^2+3x+9}$

(iii) $\lim_{x \to \infty} \frac{\sum n}{n^2}$

(iv) $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

(v) $\lim_{x \to a} \frac{x^{\frac{5}{8}} - a^{\frac{5}{8}}}{x^{\frac{2}{3}} - a^{\frac{2}{3}}}$

(vi) $\lim_{x \to 0} \frac{\sin^2 3x}{x^2}$

Solution:

(i) $\lim_{x \to 2} \frac{x^3+2}{x+1}$

$$\lim_{x \to 2} \frac{x^3+2}{x+1} = \frac{2^3+2}{2+1} = \frac{8+2}{3} = \frac{10}{3}$$

(ii) $\lim_{x \to \infty} \frac{2x+5}{x^2+3x+9}$

$$\lim_{x \to \infty} \frac{2x+5}{x^2+3x+9} = \lim_{x \to \infty} \frac{x\left(2+\frac{5}{x}\right)}{x^2\left(1+\frac{3}{x}+\frac{9}{x^2}\right)}$$

[Takeout x from numerator and take x^2 from the denominator]

$$= \lim_{x \to \infty} \frac{1}{x}\left(\frac{2+\frac{5}{x}}{1+\frac{3}{x}+\frac{9}{x^2}}\right)$$

$$= 0\left(\frac{2+0}{1+0+0}\right)$$

(iii) $\lim_{x\to\infty} \frac{\Sigma n}{n^2}$

$$= \lim_{x\to\infty} \frac{\frac{n(n+1)}{2}}{n^2}$$

$$= \lim_{x\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)}{2n^2} = \lim_{x\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right)$$

$$= \frac{1}{2}\left(1+\frac{1}{\infty}\right) = \frac{1}{2}(1+0) = \frac{1}{2}$$

(iv) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{5x}$

$$= \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{5x} \times \left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$= \lim_{x\to 0} \frac{(1+x)-(1-x)}{5x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x\to 0} \frac{1+x-1+x}{5x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x\to 0} \frac{2x}{5x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x\to 0} \frac{2}{5(\sqrt{1+0}+\sqrt{1-0})} = \frac{2}{5(1+1)} = \frac{1}{5}$$

(v) $\lim_{x\to a} \frac{x^{\frac{5}{8}}-a^{\frac{5}{8}}}{x^{\frac{2}{3}}-a^{\frac{2}{3}}}$

$$\lim_{x\to a} \frac{x^{\frac{5}{8}}-a^{\frac{5}{8}}}{x^{\frac{2}{3}}-a^{\frac{2}{3}}} = \lim_{x\to a} \frac{\left(\frac{x^{\frac{5}{8}}-a^{\frac{5}{8}}}{x-a}\right)}{\left(\frac{x^{\frac{2}{3}}-a^{\frac{2}{3}}}{x-a}\right)}$$

[Divide both numerator and denominator by x – a; $\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^n$]

$$= \frac{\lim_{x \to a} \frac{x^{\frac{5}{8}} - a^{\frac{5}{8}}}{x-a}}{\lim_{x \to a} \frac{x^{\frac{2}{3}} - a^{\frac{2}{3}}}{x-a}}$$

$$= \frac{\frac{5}{8} a^{\frac{5}{8}-1}}{\frac{2}{3} a^{\frac{2}{3}-1}} = \frac{5}{8} \times \frac{3}{2} \times \frac{a^{\frac{-3}{8}}}{a^{\frac{-1}{3}}} = \frac{15}{16} \times a^{-\frac{1}{24}}$$

(vi) $\lim_{x \to 0} \frac{\sin^2 3x}{x^2}$

$$= \lim_{x \to 0} \frac{\sin 3x}{x} \times \frac{\sin 3x}{x}$$

$$= \lim_{x \to 0} \frac{3 \sin 3x}{3x} \times \frac{3 \sin 3x}{3x}$$

$$= 3 \times 3 \lim_{x \to 0} \frac{\sin 3x}{3x} \times \lim_{x \to 0} \frac{\sin 3x}{3x}$$

$$= 9 \times 1 = 1 = 9$$

Question 2.

If $\lim_{x \to a} \frac{x^9 - a^9}{x-a} = \lim_{x \to 3}(x+6)$, find the value of a.

Solution:

$\lim_{x \to a} \frac{x^9 - a^9}{x-a} = \lim_{x \to 3}(x+6)$

$9 . a^{9-1} = 3 + 6$

$9 . a^8 = 9$

$a^8 = 1$

Taking squareroot on bothsides, we get

$\left(a^8\right)^{\frac{1}{2}} = \pm 1$

$a^4 = \pm 1$

But $a^4 = -1$ is impossssible.

$\therefore a^4 = 1$

Again taking squareroot, we get

$\left(a^4\right)^{\frac{1}{2}}$ = ±1

a^2 = ±1

a^2 = -1 is imposssible

∴ a^2 = 1

Again taking positive squareroot, a = ±1

Question 3.

If $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 448$, then find the least positive integer n.

Solution:

$$\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 448$$

$$\text{i.e., } n2^{n-1} = 7 \times 2^6$$

$$n \times 2^{n-1} = 7 \times 2^{7-1}$$

$$\therefore n = 7$$

$$\begin{array}{r|l} 2 & 448 \\ 2 & 224 \\ 2 & 112 \\ 2 & 56 \\ 2 & 28 \\ 2 & 14 \\ & 7 \end{array}$$

Question 4.

If f(x) = $\frac{x^7 - 128}{x^5 - 32}$, then find $\lim_{x \to 2} f(x)$

Solution:

$\lim_{x \to 2} f(x)$

$$= \lim_{x\to 2}\frac{x^7-128}{x^5-32} = \lim_{x\to 2}\frac{x^7-2^7}{x^5-2^5}$$

$$= \frac{\lim_{x\to 2}\frac{x^7-2^7}{x-2}}{\lim_{x\to 2}\frac{x^5-2^5}{x-2}}$$ [Divide both numerator and denominator by $x-2$]

$$= \frac{7\cdot 2^6}{5\cdot 2^4} \qquad \left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$= \frac{7}{5}\times 2^2 = \frac{28}{5}$$

Question 5.

Let f(x) = $\frac{ax+b}{x+1}$, if $\lim_{x\to 0} f(x) = 2$ and $\lim_{x\to\infty} f(x) = 1$, then show that f(-2) = 0

Solution:

Given that $\lim_{x\to 0} f(x) = 2$

$$\text{i.e., } \lim_{x\to 0}\frac{ax+b}{x+1} = 2$$

$$\frac{a(0)+b}{0+1} = 2$$

$$b = 2$$

Also given that $\lim_{x\to\infty} f(x) = 1$

$$\text{i.e., } \lim_{x\to\infty}\frac{ax+b}{x+1} = 1$$

$$\lim_{x\to\infty}\frac{x\left(a+\frac{b}{x}\right)}{x\left(1+\frac{1}{x}\right)} = 1$$

$$\lim_{x\to\infty}\frac{a+\frac{b}{x}}{1+\frac{1}{x}} = 1$$

$$\frac{a+0}{1+0} = 1$$

$$a = 1$$

$$\text{Now } f(x) = \frac{ax+b}{x+1}$$

$$f(x) = \frac{x+2}{x+1} \quad [\because a = 1, b = 2]$$

$$f(-2) = \frac{-2+2}{-2+1} = \frac{0}{1} = 0$$

Hence Proved.

Ex 5.3

Question 1.
Examine the following functions for continuity at indicated points.

$$\text{(a) } f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if} \quad x \neq 2 \\ 0 & \text{if} \quad x = 2 \end{cases} \text{ at } x = 2.$$

$$\text{(b) } f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if} \quad x \neq 3 \\ 6 & \text{if} \quad x = 3 \end{cases} \text{ at } x = 3.$$

Solution:

(a) $f(x) = \frac{x^2-4}{x-2}$, also given that f(2) = 0

$$L[f(x)]_{x=2} = \lim_{x \to 2^-} f(x)$$

[∵ x = 2 – h, where h → 0, x → 2]

$$= \lim_{h \to 0} f(2-h) \qquad [\because x = 2$$

$$= \lim_{h \to 0} \frac{(2-h)^2-4}{(2-h)-2} = \lim_{h \to 0} \frac{4+h^2-4h-4}{2-h-2}$$

$$= \lim_{h \to 0} \frac{h^2-4h}{-h}$$

$$= \lim_{h \to 0} \frac{h(h-4)}{-h} = \lim_{h \to 0} h-4$$

$$= \lim_{h\to 0} \frac{0-4}{-1} = 4$$

But $L[f(x)]_{x=2}\ f(2) = 0$

$$\therefore L[f(x)]_{x=2} \neq f(2)$$

∴ The given function is not continuous at x = 2.

(b) Given that f(x) = $\frac{x^2-9}{x-3}$ and f(3) = 6

$$L[f(x)]_{x=3} = \lim_{x\to 3^-} f(x)$$

[∵ x = 3 – h, where h → 0, x → 3]

$$= \lim_{h\to 0} f(3-h)$$

$$= \lim_{h\to 0} \frac{(3-h)^2-9}{3-h-3}$$

$$= \lim_{h\to 0} \frac{9+h^2-6h-9}{-h} = \lim_{h\to 0} \frac{h^2-6h}{-h}$$

$$= \lim_{h\to 0} \frac{h(h-6)}{-h} = \lim_{h\to 0} \frac{h-6}{-1} = \frac{0-6}{-1} = 6$$

$$R[f(x)]_{x=3^+} = \lim_{x\to 3^+} f(x)$$

[∵ x = 3 + h, where x → 3, h → 0]

$$= \lim_{h\to 0} f(3+h)$$

$$= \lim_{h\to 0} \frac{(3+h)^2-9}{3+h-3}$$

$$= \lim_{h\to 0} \frac{9+h^2+6h-9}{h}$$

$$= \lim_{h\to 0} \frac{h^2+6h}{h} = \lim_{h\to 0} \frac{h(h+6)}{h}$$

= 0 + 6

= 6

Also given that f(3) = 6

Thus $L[f(x)]_{x=3} = R[f(x)]_{x=3} = f(3)$

i.e., $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

∴ The given function f(x) is continuous at x = 3.

Question 2.
Show that f(x) = |x| is continuous at x = 0.

Solution:

Given that $f(x) = |x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$

$$L[f(x)]_{x=0} = \lim_{x\to 0^-} f(x)$$

[∵ x = 0 – h]

$= \lim_{h\to 0^-} f(0-h)$

$= \lim_{h\to 0^-} f(-h) = \lim_{h\to 0^-} |-h|$

$= \lim_{h\to 0^-} |h| = \lim_{h\to 0^-} h = 0$

$R[f(x)]_{x=0^+} = \lim_{x\to 0^+} f(x)$

$= \lim_{h\to 0^+} f(0+h) = \lim_{h\to 0^+} f(h) = \lim_{h\to 0^+} |h|$

$= \lim_{h\to 0} h$

$= 0$

[∵ |x| = x if x > 0]

Also $f(0) = |0| = 0$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

∴ f(x) is continuous at x = 0.

Ex 5.4

Question 1.
Find the derivative of the following functions from first principle.
(i) x^2
(ii) e^x
(iii) $\log(x + 1)$

Solution:
Let $f(x) = x^2$ then $f(x + h) = (x + h)^2$

Now $\frac{d}{dx}$ f(x)

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + h^2 + 2hx - x^2}{h} = \lim_{h \to 0} \frac{h^2 + 2hx}{h}$$

$$= \lim_{h \to 0} \frac{h(h+2x)}{h}$$

$$= \lim_{h \to 0} h + 2x$$

$$= 0 + 2x = 2x$$

Thus $\frac{d}{dx}(x^2) = 2x$

(ii) Let $f(x) = e^{-x}$ then $f(x + h) = e^{-(x+h)}$

Now $\frac{d}{dx}$ f(x)

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{-x-h} - e^{-x}}{h}$$

$$= \lim_{h\to 0} \frac{e^{-x} \cdot e^{-h} - e^{-x}}{h} = \lim_{h\to 0} e^{-x}\left(\frac{e^{-h}-1}{h}\right)$$

$$= e^{-x} \lim_{h\to 0}\left(\frac{\frac{1}{e^h}-1}{h}\right) = e^{-x} \lim_{h\to 0}\left(\frac{1-e^h}{e^h h}\right) = e^{-x} \lim_{h\to 0}\left(-\frac{e^h-1}{e^h h}\right)$$

$$= -e^{-x} \lim_{h\to 0}\left(\frac{1}{e^h} \times \frac{e^h-1}{h}\right)$$

$$= -e^{-x} \frac{1}{e^0} \times 1$$

$$\left[\because \lim_{x\to 0} \frac{e^x-1}{x} = 1\right]$$

$$\frac{d}{dx} f(x) = -e^{-x}$$

$$\therefore \frac{d}{dx}(e^{-x}) = -e^{-x}$$

(iii) Let $f(x) = \log(x + 1)$

Then $f(x + h) = \log(x + h + 1) = \log((x + 1) + h)$

Now $\frac{d}{dx}$ f(x)

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\log(x+1) + h) - \log(x+1)}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{(x+1)+h}{x+1}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(1 + \frac{h}{x+1}\right)}{h} = \lim_{h\to 0} \frac{\log\left(1+\frac{h}{x+1}\right)}{\left(\frac{h}{x+1}\right) \times (x+1)}$$

$$= \frac{1}{x+1} \lim_{h\to 0} \frac{\log\left(1+\frac{h}{x+1}\right)}{\frac{h}{x+1}}$$

$$\frac{d}{dx} f(x) = \frac{1}{x+1}$$

$$\therefore \frac{d}{dx} \log(x+1) = \frac{1}{x+1}$$

Ex 5.5

Question 1.

Differentiate the following with respect to x.

(i) $3x^4 - 2x^3 + x + 8$

(ii) $\frac{5}{x^4} - \frac{2}{x^3} + \frac{5}{x}$

(iii) $\sqrt{x} + \frac{1}{\sqrt[3]{x}} + e^x$

(iv) $\frac{3+2x-x^2}{x}$

(v) $x^3 e^x$

(vi) $(x^2 - 3x + 2)(x + 1)$

(vii) $x^4 - 3 \sin x + \cos x$

(viii) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

Solution:

(i) Let $y = 3x^4 - 2x^3 + x + 8$

$$\frac{dy}{dx} = \frac{d}{dx}(3x^4) - \frac{d}{dx}(2x^3) + \frac{d}{dx}(x) + \frac{d}{dx}(8)$$

$$= 3\frac{d}{dx}(x^4) - 2\frac{d}{dx}(x^3) + 1 + 0$$

$$= 3(4 \cdot x^{4-1}) - 2(3x^{3-1}) + 1$$

$$= 12x^3 - 6x^2 + 1$$

(ii) Let y = $\frac{5}{x^4} - \frac{2}{x^3} + \frac{5}{x}$

$$\frac{dy}{dx} = 5\frac{d}{dx}\left(\frac{1}{x^4}\right) - 2\frac{d}{dx}\left(\frac{1}{x^3}\right) + 5\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= 5\frac{d}{dx}(x^{-4}) - 2\frac{d}{dx}(x^{-3}) + 5x^{-1}$$

$$= 5(-4x^{-4-1}) - 2(-3x^{-3-1}) + 5(-1)\,x^{-1-1}$$

$$= -20x^{-5} + 6x^{-4} - 5x^{-2}$$

$$= \frac{-20}{x^5} + \frac{6}{x^4} - \frac{5}{x^2}$$

(iii) Let y = $\sqrt{x} + \frac{1}{\sqrt[3]{x}} + e^x$

$$y = x^{\frac{1}{2}} + x^{\frac{1}{3}} + e^x$$

$\left[\because \frac{1}{\sqrt[3]{x}} = \frac{1}{(x)^{\frac{1}{3}}} = x^{\frac{1}{3}}\right]$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{\frac{1}{2}}) + \frac{d}{dx}(x^{-\frac{1}{3}}) + \frac{d}{dx}(e^x)$$

$$= \frac{1}{2}x^{\frac{1}{2}-1} + \left(\frac{-1}{3}\right)x^{\frac{-1}{3}-1} + e^x$$

$$= \frac{1}{2}x^{\frac{-1}{2}} - \frac{1}{3}\frac{1}{x^{\frac{4}{3}}} + e^x$$

$$= \frac{1}{2}\frac{1}{x^{\frac{1}{2}}} - \frac{1}{3}\frac{1}{x^{\frac{4}{3}}} + e^x$$

$$= \frac{1}{2\sqrt{x}} - \frac{1}{3\sqrt[3]{x^4}} + e^x$$

(iv) Let $y = \frac{3+2x-x^2}{x}$

$$= \frac{3}{x} + \frac{2x}{x} - \frac{x^2}{x}$$

$$y = \frac{3}{x} + 2 - x$$

$$\frac{dy}{dx} = \frac{-3}{x^2} + 0 - 1$$

$$= \frac{-3}{x^2} - 1 = \frac{-3-x^2}{x^2} = -\frac{(3+x^2)}{x^2}$$

(v) Let $y = x^3 e^x$

[∵ We have product of two functions, so use product rule]

$= x^3 e^x + e^x (3x^2)$

$= e^x (x^3 + 3x^2)$

$= x^2 e^x (x + 3)$

(vi) Let $y = (x^2 - 3x + 2)(x + 1)$

$y = x^3 - 3x^2 + 2x + x^2 - 3x + 2$

$y = x^3 - 2x^2 - x + 2$

$\frac{dy}{dx} = 3x^2 - 4x - 1$

(vii) Let $y = x^4 - 3 \sin x + \cos x$

$\frac{dy}{dx} = \frac{d}{dx}(x4) - 3\frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos x)$

$= 4x^3 - 3 \cos x - \sin x$

(viii) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

$$\text{Let } y = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 = x + \frac{1}{x} + 2 \cdot \sqrt{x} \cdot \frac{1}{\sqrt{x}}$$

$$y = x + \frac{1}{x} + 2$$

$$\frac{dy}{dx} = 1 - \frac{1}{x^2} + 0 = 1 - \frac{1}{x^2}$$

Question 2.
Differentiate the following with respect to x.

(i) $\frac{e^x}{1+x}$

(ii) $\frac{x^2+x+1}{x^2-x+1}$

(iii) $\frac{e^x}{1+e^x}$

Solution:

(i) Let y = $\frac{e^x}{1+x}$

Let $y = \frac{e^x}{1+x}$

$$\frac{dy}{dx} = \frac{(1+x)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{(1+x)e^x - e^x \cdot 1}{(1+x)^2}$$

$$= \frac{e^x(1+x-1)}{(1+x)^2} = \frac{xe^x}{(1+x)^2}$$

(ii) Let y = $\frac{x^2+x+1}{x^2-x+1}$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x^2-x+1)\frac{d}{dx}(x^2+x+1) - (x^2+x+1)\frac{d}{dx}(x^2-x+1)}{(x^2-x+1)^2}$$

$$= \frac{(x^2-x+1)(2x+1) - (x^2+x+1)(2x-1)}{(x^2-x+1)^2}$$

$$= \frac{2x^3 - 2x^3 + x^2 - 2x^2 - 2x^2 + x^2 + 2x - x - 2x + x + 1 + 1}{(x^2-x+1)^2}$$

$$= \frac{-2x^2+2}{(x^2-x+1)^2} = \frac{-2(x^2-1)}{(x^2-x+1)^2} \text{ (or) } \frac{2(1-x^2)}{(x^2-x+1)^2}$$

(iii) Let y = $\frac{e^x}{1+e^x}$

$$\frac{dy}{dx} = \frac{(1+e^x)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+e^x)}{(1+e^x)^2}$$

$$= \frac{(1+e^x)e^x - e^x \cdot e^x}{(1+e^x)^2} = \frac{e^x(1+e^x-e^x)}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

Question 3.
Differentiate the following with respect to x.
(i) x sin x
(ii) e^x sin x
(iii) e^x (x + log x)
(iv) sin x cos x
(v) x^3 e^x

Solution:
(i) Let y = x sin x

$$\frac{dy}{dx} = x\,\frac{d}{dx}(\sin x) + \sin x\,\frac{d}{dx}(x)$$

= x cos x + sin x . 1

= x cos x + sin x

(ii) Let y = e^x sin x

$$\frac{dy}{dx} = e^x\,\frac{d}{dx}(\sin x) + \sin x\,\frac{d}{dx}(e^x)$$

= e^x cos x + sin x e^x

= e^x (cos x + sin x)

(iii) Let y = e^x (x + log x)

$$\frac{dy}{dx} = e^x\frac{d}{dx}(x+\log x) + (x+\log x)\frac{d}{dx}(e^x)$$

$$= e^x \left(1 + \frac{1}{x}\right) + (x + \log x)\, e^x$$

$$= e^x \left[1 + \frac{1}{x} + x + \log x\right]$$

(iv) Let y = sin x cos x

$\frac{dy}{dx}$ = sin x $\frac{d}{dx}$ (cos x) + cos x $\frac{d}{dx}$ (sin x)

= sin x (-sin x) + cos x cos x

= $-\sin^2 x + \cos^2 x$

= $\cos^2 x - \sin^2 x$

= cos 2x [∵ cos 2x = $\cos^2 x - \sin^2 x$]

(or) y = sin x cos x

y = $\frac{1}{2}$ (2 sin x cos x)

y = $\frac{1}{2}$ sin 2x

$\frac{dy}{dx}$ = $\frac{1}{2}$ cos 2x . 2 = cos 2x

(v) Let y = $x^3 e^x$

$\frac{dy}{dx}$ = $x^3 \frac{d}{dx} (e^x) + e^x \frac{d}{dx} (x^3)$

= $x^3 (e^x) + e^x (3x^2)$

= $x^2 e^x (x + 3)$

Question 4.
Differentiate the following with respect to x.

(i) $\sin^2 x$

(ii) $\cos^2 x$

(iii) $\cos^3 x$

(iv) $\sqrt{1+x^2}$

(v) $(ax^2 + bx + c)^n$

(vi) $\sin(x^2)$

(vii) $\frac{1}{\sqrt{1+x^2}}$

Solution:

For the following problems chain rule to be used:

$\frac{d}{dx}$ f(g(x)) = f'(g(x)) . g'(x)

$\frac{d}{dx}$ $[f(x)]^n = n[f(x)]^{n-1}$ × $\frac{d}{dx}$ f(x)

(i) Let y = $\sin^2$ x = $(\sin x)^2$

$\frac{dy}{dx}$ = $2(\sin x)^{2-1}$ $\frac{d}{dx}$ (sin x)

= 2 sin x (cos x)

= sin 2x

(ii) y = $\cos^2$ x = $(\cos x)^2$

$\frac{dy}{dx}$ = $2(\cos x)^{2-1}$ $\frac{d}{dx}$ (cos x)

= 2 cos x (-sin x)

= -2 sin x cos x

= -sin 2x

(iii) y = $\cos^3$ x

y = $(\cos x)^3$

$\frac{dy}{dx}$ = $3(\cos x)^{3-1}$ $\frac{d}{dx}$ (cos x)

= 3 $\cos^2$ x (-sin x)

= -3 $\cos^2$ x sin x

= -3 cos x (sin x cos x) [Multiply and divide by 2]

$= \frac{-3}{2}$ cos x (2 sin x cos x)

$= \frac{-3}{2}$ cos x sin 2x

(iv) Let y = $\sqrt{1+x^2}$

y = $\left(1+x^2\right)^{\frac{1}{2}}$

Here f(x) = 1 + x^2; n = $\frac{1}{2}$

$$= \frac{1}{2}(1+x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1+x^2)$$

$$= \frac{1}{2}(1+x^2)^{-\frac{1}{2}}(0+2x)$$

$$= \frac{1}{2}\frac{1}{(1+x^2)^{\frac{1}{2}}}(2x)$$

$$= \frac{1}{2}\frac{1}{\sqrt{1+x^2}}(2x) = \frac{x}{\sqrt{1+x^2}}$$

(v) Let y = $(ax^2 + bx + c)^n$

$\frac{dy}{dx}$ = $n(ax^2 + bx + c)^{n-1}$ $\frac{d}{dx}$ $(ax^2 + bx + c)$

= $n(ax^2 + bx + c)^{n-1}$ (2ax + b)

(vi) Let y = $\sin(x^2)$

$\frac{d}{dx}$ f(g(x)) = f'(g(x)) . g'(x)

Here f = sin x, g = x^2

$\frac{dy}{dx}$ = $\cos(x^2)$ $\frac{d}{dx}$ (x^2)

= $\cos(x^2)$ (2x)

= 2x $\cos(x^2)$

(vii) Let y = $\frac{1}{\sqrt{1+x^2}}$

y = $\left(1+x^2\right)^{-\frac{1}{2}}$

Here n = $-\frac{1}{2}$; f(x) = 1 + x^2

$$\frac{dy}{dx} = -\frac{1}{2}(1+x^2)^{-\frac{1}{2}-1}\frac{d}{dx}(1+x^2)$$

$$= -\frac{1}{2}(1+x^2)^{-\frac{3}{2}}(0+2x)$$

$$= -\frac{1}{2}\frac{1}{(1+x^2)^{\frac{3}{2}}}(2x)$$

$$= -\frac{x}{\sqrt{(1+x^2)^3}}$$

$$= \frac{-x}{\sqrt{(1+x^2)^2}\sqrt{1+x^2}} = \frac{-x}{(1+x^2)\sqrt{1+x^2}}$$

Ex 5.6

Question 1.
Find dy/dx for the following functions:
(i) xy – tan(xy)
(ii) $x^2 - xy + y^2 = 1$
(iii) $x^3 + y^3 + 3axy = 1$

Solution:
(i) Given xy = tan(xy)
Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(xy) = \frac{d}{dx}\tan(xy)$$

$$x\frac{d}{dx}(y) + y\frac{d}{dx}(x) = \sec^2(xy)\frac{d}{dx}(xy)$$

$$x\frac{dy}{dx} + y(1) = \sec^2(xy)\left[x\frac{dy}{dx} + y\frac{d}{dx}(x)\right]$$

$$x\frac{dy}{dx} + y = \sec^2(xy)\left[x\frac{dy}{dx} + y\right]$$

$$x\frac{dy}{dx} + y = x\sec^2(xy)\frac{dy}{dx} + y\sec^2(xy)$$

$$x\frac{dy}{dx} - x\sec^2(xy)\frac{dy}{dx} = y\sec^2(xy) - y$$

$$\frac{dy}{dx}[x - x\sec^2(xy)] = y[\sec^2(xy) - 1]$$

$$\frac{dy}{dx} = \frac{y[\sec^2(xy) - 1]}{x[1 - \sec^2(xy)]}$$

$$= \frac{y}{x}(-1) = \frac{-y}{x}$$

(ii) $x^2 - xy + y^2 = 7$

Differentiating both side with respect to x,

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

$$2x - [x\frac{d}{dx}(y) + y\frac{d}{dx}(x)] + 2y\frac{dy}{dx} = 0$$

$$2x - [x\frac{dy}{dx} + y \cdot 1] + 2y\frac{dy}{dx} = 0$$

$$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}[2y - x] = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

(iii) $x^3 + y^3 + 3axy = 1$

Differentiating both sides with respect to x,

$$3x^2 + 3y^2 \frac{dy}{dx} + 3a\left(x\frac{dy}{dx} + y \cdot 1\right) = 0$$

$$3[x^2 + y^2\frac{dy}{dx} + ax\frac{dy}{dx} + ay] = 0$$

$$x^2 + y^2\frac{dy}{dx} + ax\frac{dy}{dx} + ay = 0$$

$$\frac{dy}{dx}[y^2 + ax] = -x^2 - ay$$

$$\frac{dy}{dx} = \frac{-x^2 - ay}{y^2 + ax} = -\frac{(x^2 + ay)}{y^2 + ax}$$

$$= -\left[\frac{x^2 + ay}{y^2 + ax}\right]$$

Question 2.

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ and x ≠ y, then prove that $\frac{dy}{dx} = -\frac{1}{(x+1)^2}$.

Solution:

Given $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides we get

⇒ $x^2 (1 + y) = y^2 (1 + x)$

⇒ $x^2 + x^2y = y^2 + y^2x$

⇒ $x^2 - y^2 + x^2y - y^2x = 0$

⇒ $(x + y) (x - y) + xy(x - y) = 0$

⇒ $(x - y) [(x + y) + xy] = 0$

∴ $x - y = 0$ (or) $x + y + xy = 0$

$x = y$ (or) $x + y + xy = 0$

Given that $x \neq y$

$x + y + xy = 0$

⇒ $y + xy = -x$

⇒ $y(1 + x) = -x$

$$y = \frac{-x}{1+x} = -\left(\frac{x}{1+x}\right)$$

$$\frac{dy}{dx} = -\left[\frac{(1+x)1 - x(1+0)}{(1+x)^2}\right]$$

$$= -\left[\frac{1+x-x}{(1+x)^2}\right] = -\left[\frac{1}{(1+x)^2}\right] = -\frac{1}{(1+x)^2}$$

Hence proved.

Question 3.
If 4x + 3y = log(4x – 3y), then find dy/dx

Solution:
Given 4x + 3y = log(4x – 3y)
Differentiating both sides with respect to x,

$$4(1) + 3\frac{dy}{dx} = \frac{1}{(4x-3y)}\frac{d}{dx}(4x-3y)$$

$$4 + 3\frac{dy}{dx} = \frac{1}{4x-3y}\left(4(1) - 3\frac{dy}{dx}\right)$$

$$(4x-3y)\left(4 + 3\frac{dy}{dx}\right) = 4 - 3\frac{dy}{dx}$$

$$16x + 12x\frac{dy}{dx} - 12y - 9y\frac{dy}{dx} = 4 - 3\frac{dy}{dx}$$

$$12x\frac{dy}{dx} + 3\frac{dy}{dx} - 9y\frac{dy}{dx} = 4 - 16x + 12y$$

$$\frac{dy}{dx}[12x + 3 - 9y] = 4[1 - 4x + 3y]$$

$$\frac{dy}{dx} = \frac{4[1-4x+3y]}{3[4x+1-3y]}$$

$$= \frac{4}{3}\frac{[1-4x+3y]}{[1+4x-3y]}$$

Ex 5.7

Question 1.
Differentiate the following with respect to x.

(i) $x^{\sin x}$

(ii) $(\sin x)^x$

(iii) $(\sin x)^{\tan x}$

(iv) $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x^2+x+1)}}$

Solution:
(i) Let $y = x^{\sin x}$
Taking logarithm on both sides we get,
$\log y = \log(x^{\sin x})$
$\log y = \sin x \log x$

Differentiating with respect to x,

$$\frac{1}{y}\cdot\frac{dy}{dx} = \sin x\,\frac{d}{dx}(\log x) + \log x\,\frac{d}{dx}(\sin x)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \sin x\left(\frac{1}{x}\right) + \log x\,(\cos x)$$

$$\therefore\ \frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x \log x\right]$$

$$\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$$

(ii) Let $y = (\sin x)^x$
Taking logarithm on both sides we get,

$\log y = x \log(\sin x)$
Differentiating with respect to x,

$\frac{1}{y} \cdot \frac{dy}{dx}$ = x $\frac{d}{dx}$ log(sin x) + log(sin x) $\frac{d}{dx}$ (x)

$\frac{1}{y} \cdot \frac{dy}{dx}$ = x $\frac{1}{\sin x}$ (cos x) + log(sin x) (1)

$\frac{1}{y} \cdot \frac{dy}{dx}$ = x cot x + log(sin x)

$\frac{dy}{dx}$ = y[x cot x + log(sin x)]

$\frac{dy}{dx}$ = $(\sin x)^x$ [x cot x + log(sin x)]

(iii) Let y = $(\sin x)^{\tan x}$

Taking logarithm on both sides we get,

log y = tan x log(sin x)

Differentiating with respect to x,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \tan x \frac{d}{dx}(\log(\sin x)) + \log(\sin x)\frac{d}{dx}(\tan x)$$

$$= \tan x \frac{1}{\sin x}(\cos x) + \log(\sin x)\sec^2 x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{\sin x}{\cos x} \times \frac{\cos x}{\sin x} + \log(\sin x)(\sec^2 x)$$

$$= 1 + \log(\sin x)\sec^2 x$$

$$\frac{dy}{dx} = y[1 + \sec^2 x \log(\sin x)]$$

$$= (\sin x)^{\tan x}[1 + \sec^2 x \log(\sin x)]$$

(iv) Let y = $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x^2+x+1)}}$

y = $\left(\frac{(x-1)(x-2)}{(x-3)(x^2+x+1)}\right)^{\frac{1}{2}}$

Taking logarithm on both sides we get,

log y = $\frac{1}{2}$ {[log(x – 1) + log(x – 2)] – [(log(x – 3) + log(x^2 + x + 1)]}

log y = $\frac{1}{2}$ [log(x – 1) + log(x – 2) – log(x – 3) – log(x^2 + x + 1)]

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x-1}(1-0)+\frac{1}{x-2}(1-0)-\frac{1}{x-3}(1-0)-\frac{1}{x^2+x+1}(2x+1)\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{2x+1}{x^2+x+1}\right]$$

$$\therefore\frac{dy}{dx}=\frac{1}{2}y\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{2x+1}{x^2+x+1}\right]$$

$$=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x^2+x+1)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{2x+1}{x^2+x+1}\right]$$

Question 2.

If $x^m . y^n = (x + y)^{m+n}$, then show that $\frac{dy}{dx}=\frac{y}{x}$

Solution:

$x^m . y^n = (x + y)^{m+n}$
Taking logarithm on both sides we get,
$m \log x + n \log y = (m + n) \log(x + y)$
Differentiating with respect to x,

$$m\frac{1}{x}+n\frac{1}{y}\frac{dy}{dx}=(m+n)\frac{1}{(x+y)}\left(1+\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}\left(\frac{n}{y}-\frac{m+n}{x+y}\right)=\frac{m+n}{x+y}-\frac{m}{x}$$

$$\frac{dy}{dx}\left(\frac{nx+ny-my-ny}{y(x+y)}\right)=\left(\frac{mx+nx-mx-my}{x(x+y)}\right)$$

$$\frac{dy}{dx}\left(\frac{nx-my}{y(x+y)}\right)=\frac{(nx-my)}{x(x+y)}$$

$$\therefore\frac{dy}{dx}=\frac{y}{x}$$

Hence proved.

Ex 5.8

Question 1.

Find dy/dx of the following functions:

(i) $x = ct, y = c/t$

(ii) $x = \log t, y = \sin t$

(iii) $x = a \cos^3\theta, y = a \sin^3\theta$

(iv) $x = a(\theta - \sin \theta), y = a(1 - \cos \theta)$

Solution:

(i) $x = ct, y = \frac{c}{t}$

$$x = ct \quad y = \frac{c}{t}$$

$$\frac{dx}{dt} = c \quad \frac{dy}{dt} = c\left(\frac{-1}{t^2}\right)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{c\left(\frac{-1}{t^2}\right)}{c} = \frac{-1}{t^2}$$

(ii) $x = \log t, y = \sin t$

$$x = \log t \quad \Big| \quad y = \sin t$$

$$\frac{dx}{dt} = \frac{1}{t} \quad \Big| \quad \frac{dy}{dt} = \cos t$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{\left(\frac{1}{t}\right)} = t \cos t$$

(iii) $x = a \cos^3\theta, y = a \sin^3\theta$

we have $x = a \cos^3\theta$; $y = a \sin^3\theta$

Now, $\therefore \frac{dx}{d\theta} = -3a \cos^2\theta \sin\theta$ and $\frac{dy}{d\theta} = 3a \sin^2\theta \cos\theta$

Therefore $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$

(iv) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$

$$x = a(\theta - \sin\theta) \quad \Big| \quad y = a(1 - \cos\theta)$$

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \quad \Big| \quad \frac{dy}{dx} = a(0 - (-\sin\theta) = a\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1-\cos\theta)}$$

$$= \frac{a(2\sin\frac{\theta}{2}\cos\frac{\theta}{2})}{2\sin^2\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

$$\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

Question 2.
Differentiate $\sin^3 x$ with respect to $\cos^3 x$.

Solution:
Let $u = \sin^3 x = (\sin x)^3$; $v = \cos^3 x = (\cos x)^3$
$du/dx = 3(\sin x) \cos x$; $dv/dx = 3(\cos x)^2 (-\sin x)$

$$\frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)} = \frac{3\sin^2 x\cos x}{3\cos^2 x(-\sin x)} = \frac{-\sin x}{\cos x} = -\tan x$$

Question 3.
Differentiate $\sin^2 x$ with respect to x^2.

Solution:
Let $u = (\sin x)^2$; $v = x^2$
$du/dx = (2\sin x)(\cos x) = \sin 2x$; $dv/dx = 2x$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\sin 2x}{2x}$$

Ex 5.9

Question 1.
Find y_2 for the following functions:
(i) $y = e^{3x+2}$
(ii) $y = \log x + a^x$
(iii) $x = a \cos\theta, y = a \sin\theta$

Solution:
(i) $y = e^{3x+2}$

$$y_1 = \frac{dy}{dx} = e^{3x+2} \frac{d}{dx}(3x+2)$$
$$= e^{3x+2}(3(1)+0)$$
$$= 3e^{3x+2}$$
$$y_2 = \frac{d^2y}{dx^2} = 3\left[\frac{d}{dx}(e^{3x+2})\right]$$
$$= 3[3e^{3x+2}]$$
$$= 9e^{3x+2}$$
$$= 9y$$

(ii) $y = \log x + a^x$

$$y_1 = \frac{dy}{dx} = \frac{1}{x} + a^x \log a \qquad \left[\because \frac{d}{dx}(a^x) = a^x \log a\right]$$
$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) + \log a \frac{d}{dx}(a^x)$$
$$= \frac{-1}{x^2} + (\log a)(a^x \log a)$$
$$= \frac{-1}{x^2} + a^x (\log a)^2$$

(iii) $x = a\cos\theta$, $y = a\sin\theta$

$$\frac{dx}{d\theta} = a(-\sin\theta) = -a\sin\theta \text{ .... (i)}$$

$$\frac{dy}{d\theta} = a(\cos\theta)$$

$$\therefore y_1 = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\cos\theta}{-a\sin\theta}$$

$$y_1 = \frac{dy}{dx} = -\cot\theta$$

$$y_2 = \frac{d^2y}{dx^2} = -(-\operatorname{cosec}^2\theta)\frac{d\theta}{dx}$$

$$= \operatorname{cosec}^2\theta \frac{d\theta}{dx}$$

$$= \operatorname{cosec}^2\theta \frac{1}{\left(\frac{dx}{d\theta}\right)} \Rightarrow \operatorname{cosec}^2\theta \frac{1}{-a\sin\theta} \text{ using } (i)$$

$$= \operatorname{cosec}^2\theta \times \frac{\operatorname{cosec}\theta}{-a}$$

$$= \frac{-1}{a}\operatorname{cosec}^3\theta$$

Question 2.

If $y = 500e^{7x} + 600e^{-7x}$, then show that $y_2 - 49y = 0$.

Solution:

$y = 500e^{7x} + 600e^{-7x}$

$$y_1 = \frac{dy}{dx} = 500\frac{d}{dx}(e^{7x}) + 600\frac{d}{dx}(e^{-7x})$$

$$= 500(7e^{7x}) + 600(-7e^{-7x})$$

$$y_2 = \frac{d^2y}{dx^2} = 500 \times 7\frac{d}{dx}(e^{7x}) + 600(-7)\frac{d}{dx}(e^{-7x})$$

$$= 500 \times 7(7e^{7x}) + 600 \times (-7)(-7)e^{-7x}$$

$$= 500 \times 49e^{7x} + 600 \times 49e^{-7x}$$

$$y_2 = 49[500e^{7x} + 600e^{-7x}] = 49y$$

(or) $y_2 - 49y = 0$

Question 3.
If $y = 2 + \log x$, then show that $xy_2 + y_1 = 0$.

Solution:
$y = 2 + \log x$

$$y_1 = 0 + \frac{1}{x}$$

$$\text{i.e., } y_1 = \frac{1}{x}$$

$$\therefore y_2 = -\frac{1}{x^2}$$

$$\text{Now } xy_2 + y_1 = x\left(-\frac{1}{x^2}\right) + \frac{1}{x} = \frac{-1}{x} + \frac{1}{x} = 0$$

Question 4.
If = a cos mx + b sin mx, then show that $y_2 + m^2y = 0$.

Solution:
$y = a \cos mx + b \sin mx$
$y_1 = a\, d/dx (\cos mx) + b\, d/dx (\sin mx)$
$[\because d/dx (\sin mx) = \cos mx\, d/dx (mx) = (\cos mx) . m]$
$= a(-\sin mx) . m + b(\cos mx) . m$
$= -am \sin mx + bm \cos mx$
$y_2 = -am(\cos mx) . m + bm(-\sin mx) . m$
$= -am^2 \cos mx - bm^2 \sin mx$
$= -m^2 [a \cos mx + b \sin mx]$
$= -m^2y$
$\therefore y_2 + m^2y = 0$

Question 5.
If $y = \left(x + \sqrt{1 + x^2}\right)^m$, then show that $(1 + x^2) y_2 + xy_1 - m^2y = 0$

Solution:

$$y = (x + \sqrt{1+x^2})^m$$

$$y_1 = m(x + \sqrt{1+x^2})^{m-1}\left\{1 + \frac{2x}{2\sqrt{1+x^2}}\right\}$$

$$= m(x + \sqrt{1+x^2})^{m-1}\left\{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right\} = \frac{m(x+\sqrt{1+x^2})^m}{\sqrt{1+x^2}}$$

$$y_1 = \frac{my}{\sqrt{1+x^2}}$$

Squaring both sides we get,

$$y_1^2 = \frac{m^2y^2}{(1+x^2)}$$

$(1 + x^2)(y_1^2) = m^2y^2$

Differentiating with respect to x, we get

$(1 + x^2) . 2(y_1)(y_2) + (y_1)^2 (2x) = 2m^2yy_1$

Dividing both sides by $2y_1$ we get,

$(1 + x^2) y_2 + xy_1 = m^2y$

$\Rightarrow (1 + x^2) y_2 + xy_1 - m^2y = 0$

Question 6.

If $y = \sin(\log x)$, then show that $x^2y_2 + xy_1 + y = 0$.

Solution:

$y = \sin(\log x)$

$y_1 = \cos(\log x)\ d/dx\ (\log x)$

$y_1 = \cos(\log x) . 1x$

$\therefore xy_1 = \cos(\log x)$

Differentiating both sides with respect to x, we get

$xy_2 + y_1(1) = -\sin(\log x) . 1/x$

$\Rightarrow x[xy_2 + y_1] = -\sin(\log x)$

$\Rightarrow x^2y_2 + xy_1 = -y$

$\Rightarrow x^2y_2 + xy_1 + y = 0$

Ex 5.10

Question 1.
If $f(x) = x^2 - x + 1$ then $f(x + 1)$ is:
(a) x^2
(b) x
(c) 1
(d) $x^2 + x + 1$

Answer:
(d) $x^2 + x + 1$
Hint:
$f(x) = x^2 - x + 1$
$f(x + 1) = (x + 1)^2 - (x + 1) + 1$
$= x^2 + 2x + 1 - x - 1 + 1$
$= x^2 + x + 1$

Question 2.
If $f(x) = \begin{cases} x^2 - 4x & \text{if } x \geq 2 \\ x + 2 & \text{if } x < 2 \end{cases}$, then $f(5)$ is
(a) -1
(b) 2
(c) 5
(d) 7

Answer:
(c) 5
Hint:
$$f(x) = \begin{cases} x^2 - 4x & \text{if} \quad x \geq 2 \\ x + 2 & \text{if} \quad x < 2 \end{cases}$$
$f(5) = 5^2 - 4(5) = 25 - 20 = 5$
[For x = 5 take $f(x) = x^2 - 4x$]

Question 3.
If $f(x) = \begin{cases} x^2 - 4x & \text{if } x \geq 2 \\ x + 2 & \text{if } x < 2 \end{cases}$, then $f(0)$ is
(a) 2

(b) 5
(c) -1
(d) 0

Answer:
(a) 2
Hint:

$$f(x) = \begin{cases} x^2 - 4x & \text{if} \quad x \geq 0 \\ x + 2 & \text{if} \quad x < 2 \end{cases}$$

f(0) = 0 + 2 = 2
[For x = 0 take f(x) = x + 2]

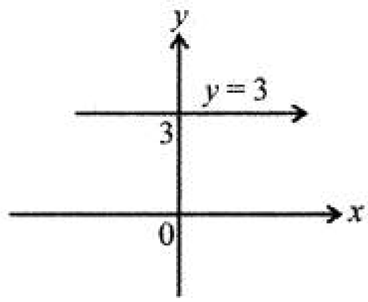
Question 4.
If f(x) = $\frac{1-x}{1+x}$ then f(-x) is equal to:
(a) -f(x)
(b) $\frac{1}{f(x)}$
(c) $-\frac{1}{f(x)}$
(d) f(x)

Answer:
(b) $\frac{1}{f(x)}$
Hint:
f(x) = $\frac{1-x}{1+x}$

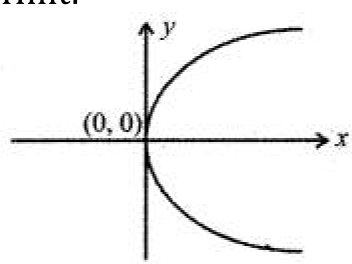
$$f(-x) = \frac{1-(-x)}{1+(-x)} = \frac{1+x}{1-x} = \frac{1}{\left(\frac{1-x}{1+x}\right)} = \frac{1}{f(x)}$$

Question 5.
The graph of the line y = 3 is:
(a) Parallel to x-axis
(b) Parallel to the y-axis
(c) Passing through the origin
(d) Perpendicular to the x-axis

Answer:
(a) Parallel to x-axis

Hint:


Question 6.
The graph of $y = 2x^2$ is passing through:
(a) (0, 0)
(b) (2, 1)
(c) (2, 0)
(d) (0, 2)

Answer:
(a) (0, 0)
Hint:


$y = 2x^2$
Put x = 0, y = 0 the equation is satisfied.

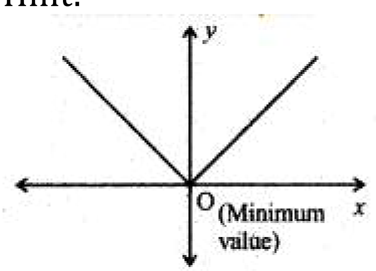
Question 7.
The graph of $y = e^x$ intersect the y-axis at:
(a) (0, 0)
(b) (1, 0)
(c) (0, 1)
(d) (1, 1)

Answer:
(c) (0, 1)

Hint:
$y = e^x$
Put x = 0, we get $y = e^0 = 1$.
∴ The graph intersects the y-axis at (0, 1)

Question 8.
The minimum value of the function f(x) = |x| is:
(a) 0
(b) -1
(c) +1
(d) ∞

Answer:
(a) 0
Hint:



f(x) = |x|
f(0) = |0| = 0

Question 9.
Which one of the following functions has the property f(x) = f(1/x)?
(a) $f(x) = \frac{x^2-1}{x}$
(b) $f(x) = \frac{1-x^2}{x}$
(c) f(x) = x
(d) $f(x) = \frac{x^2+1}{x}$

Answer:

(d) $f(x) = \frac{x^2+1}{x}$
Hint:

$f(x) = f(\frac{1}{x})$

take $f(x) = \frac{x^2+1}{x}$

$$f\left(\frac{1}{x}\right) = \frac{\left(\frac{1}{x}\right)^2+1}{\frac{1}{x}} = \left(\frac{1}{x^2}+1\right)x = \frac{1+x^2}{x^2} \times x = \frac{x^2+1}{x} = f(x)$$

Question 10.

If $f(x) = 2^x$ and $g(x) = \frac{1}{2^x}$ then (fg)(x) is:

(a) 1

(b) 0

(c) 4^x

(d) $\frac{1}{4^x}$

Answer:

(a) 1

Hint:

$(fg)\ x = f(x)\ g(x) = 2^x \times \frac{1}{2^x} = 1$

Question 11.

Which of the following function is neither even nor odd?

(a) $f(x) = x^3 + 5$

(b) $f(x) = x^5$

(c) $f(x) = x^{10}$

(d) $f(x) = x^2$

Answer:

(a) $f(x) = x^3 + 5$

Hint:

Since it has a constant term 5.

$f(x) = x^3 + 5$

$f(-x) = (-x)^3 + 5 = -x^3 + 5.$

It is not either f(x) or -f(x).

Question 12.
$f(x) = -5$, for all $x \in R$ is a:
(a) an identity function
(b) modulus function
(c) exponential function
(d) constant function

Answer:
(d) constant function

Question 13.
The range of $f(x) = |x|$, for all $x \in R$ is:
(a) $(0, \infty)$
(b) $[0, \infty)$
(c) $(-\infty, \infty)$
(d) $[1, \infty)$

Answer:
(b) $[0, \infty)$
Hint:
$[0, \infty)$ since in this interval 0 is included and $f(0) = 0$.

Question 14.
The graph of $f(x) = e^x$ is identical to that of:
(a) $f(x) = a^x, a > 1$
(b) $f(x) = a^x, a < 1$
(c) $f(x) = a^x, 0 < a < 1$
(d) $y = ax + b, a \neq 0$

Answer:
(a) $f(x) = a^x, a > 1$

Question 15.
If $f(x) = x^2$ and $g(x) = 2x + 1$ then $(fg)(0)$ is:
(a) 0
(b) 2
(c) 1
(d) 4

Answer:
(a) 0
Hint:
(fg)(0) = f(o) g(o)
= 0^2 (2(0) + 1)
= 0(1)
= 0

Question 16.
$\lim_{\theta \to 0} \frac{\tan\theta}{\theta} =$
(a) 1
(b) ∞
(c) -∞
(d) θ

Answer:
(a) 1 (By formula)

Question 17.
$\lim_{x \to 0} \frac{e^x - 1}{x} =$
(a) e
(b) nx^{n-1}
(c) 1
(d) 0

Answer:
(c) 1 (By formula)

Question 18.

For what value of x, f(x) = $\frac{x+2}{x-1}$ is not continuous?
(a) -2
(b) 1
(c) 2

(d) -1

Answer:
(b) 1

Hint:

When x = 1, $\frac{x+2}{x-1}$ is not defined.

Question 19.
A function f(x) is continuous at x = a $\lim_{x \to a} f(x)$ is equal to:

(a) f(-a)

(b) f($\frac{1}{a}$)

(c) 2f(a)

(d) f(a)

Question 20.
$\frac{d}{dx}\left(\frac{1}{x}\right)$ is equal to:
(a) $-\frac{1}{x^2}$
(b) $-\frac{1}{x}$
(c) log x
(d) $\frac{1}{x^2}$

Answer:
(a) $-\frac{1}{x^2}$

Question 21.
$\frac{d}{dx}$ ($5e^x$ – 2 log x) is equal to:
(a) $5e^x - \frac{2}{x}$
(b) $5e^x$ – 2x
(c) $5e^x - \frac{1}{x}$
(d) 2 log x

Answer:
(a) $5e^x - \frac{2}{x}$
Hint:

$$\frac{d}{dx}(5e^x - 2\log x) = \frac{d}{dx}(5e^x) - 2\frac{d}{dx}(\log x)$$

$$= 5e^x - 2 \times \frac{1}{x} \Rightarrow 5e^x - \frac{2}{x}$$

Question 22.
If y = x and z = $\frac{1}{x}$ then $\frac{dy}{dz}$ =
(a) x^2
(b) 1
(c) $-x^2$
(d) $\frac{-1}{x^2}$

Answer:
(c) $-x^2$
Hint:

$$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\frac{d}{dx}(x)}{\frac{d}{dx}\left(\frac{1}{x}\right)} = \frac{1}{-\frac{1}{x^2}} = -x^2$$

Question 23.
If y = e^{2x} then $\frac{d^2y}{dx^2}$ at x = 0 is:
(a) 4
(b) 9
(c) 2
(d) 0

Answer:
(a) 4

Hint:

$$y = e^{2x}$$

$$\frac{dy}{dx} = 2e^{2x}$$

$$\frac{d^2y}{dx^2} = 2e^{2x} \therefore \left(\frac{d^2y}{dx^2}\right)_{x=0}$$

$$= 4 \times e^0 = 4 \times 1 = 4$$

Question 24.
If y = log x then y_2 =

(a) $\frac{1}{x}$

(b) $-\frac{1}{x^2}$

(c) $-\frac{2}{x^2}$

(d) e2

Answer:
(b) $-\frac{1}{x^2}$

Hint:

y = log x

$\therefore y_1 = \frac{1}{x}$

$\therefore y_2 = \frac{-1}{x^2}$

Question 25.
$\frac{d}{dx}(a^x) =$

(a) $\frac{1}{x \log_e a}$

(b) a^a

(c) x $\log_e a$

(d) $a^x \log_e a$

Answer:
(d) $a^x \log_e a$ (by formula)