By definition $e = \frac{SP}{PM}$ **(i)**

Let Z be the foot of the perpendicular from S to l. Let A and A' be two points that divide $\overline{SZ}$ from S in the ratio $e : 1$ and $-e : 1$ respectively.

Thus, $\frac{SA}{AZ} = e$. Also, $\frac{SA'}{A'Z} = e$

SA = distance of focus S from A

AZ = perpendicular distance of A from l. This holds for A' also.

Thus, $\frac{SA}{AZ} = \frac{SA'}{A'Z} = e$ and hence A and A' are both on the ellipse. Suppose C is the mid-point of $\overline{AA'}$. Let C be the origin and direction of $\overrightarrow{CA}$ as the positive direction of the X-axis. Let CA = a. Hence coordinates of A and A' are $(a, 0)$ and $(-a, 0)$ respectively. Let the coordinates of S be $(p, 0)$ and coordinates of Z be $(q, 0)$. As A$(a, 0)$ divides $\overline{SZ}$ from S in ratio $e : 1$, we get

$$a = \frac{eq + p}{e + 1} \quad \textbf{(ii)}$$

Similarly for A' the ratio of division is $-e : 1$.

$$-a = \frac{-eq + p}{-e + 1} \quad \textbf{(iii)}$$

From (ii) and (iii) we have,

$eq + p = ae + a$ and $-eq + p = ae - a$. Solving these equations for p and q,

$p = ae$ and $q = \frac{a}{e}$

Thus focus is S$(ae, 0)$ and coordinates of Z are $\left(\frac{a}{e}, 0\right)$. The directrix passes through Z and it is a vertical line. Its equation is $x = \frac{a}{e}$.

Let P(x, y) be any point on the ellipse. Then from (i)

$\frac{SP}{PM} = e \Leftrightarrow SP = e(PM)$

$\Leftrightarrow SP^2 = e^2(PM^2)$ **(iv)**

Here PM = distance of P(x, y) from the line l,

= distance of P(x, y) from the line $x - \frac{a}{e} = 0$

$$= \frac{\left|x - \frac{a}{e}\right|}{\sqrt{1 + 0}} \qquad \left(\textbf{by the formula } \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}\right)$$

$$= \left|x - \frac{a}{e}\right|$$

$$\therefore \quad PM^2 = \left(x - \frac{a}{e}\right)^2 \quad \textbf{(v)}$$

Also, $SP^2 = (x - ae)^2 + y^2$ **(vi)**

Using (v) and (vi) in (iv), we get

$$\frac{SP}{PM} = e \Leftrightarrow (x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2$$

$$\Leftrightarrow (x - ae)^2 + y^2 = e^2\left(x^2 - \frac{2ax}{e} + \frac{a^2}{e^2}\right)$$

$$\Leftrightarrow x^2 - 2aex + y^2 + a^2e^2 = e^2x^2 - 2aex + a^2$$

$$\Leftrightarrow x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

$$\Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \qquad \text{(vii)}$$

Now, as $a > 0$ and $e < 1$, $a^2(1 - e^2) > 0$

Thus, we can choose $b > 0$ such that $a^2(1 - e^2) = b^2$. So (vii) takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is called the standard equation of the ellipse.

Conclusion :

(1) If the equation of an ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the relation $b^2 = a^2(1 - e^2)$ $(a > b)$ can be used to determine eccentricity of the ellipse.

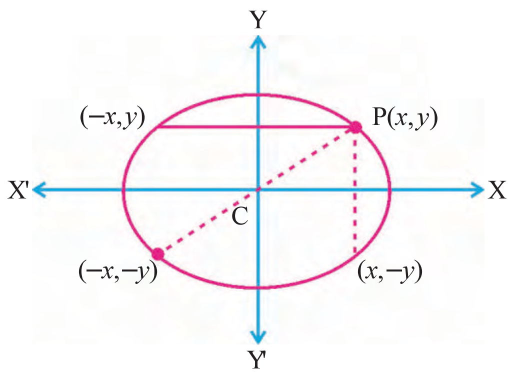

Figure 8.16

(2) Symmetry :

From the standard equation of an ellipse we observe that for any point P(x, y) on the ellipse

(i) the point $(x, -y)$ is also on the ellipse, that is, the ellipse is symmteric about X-axis.

(ii) the point $(-x, y)$ is on the ellipse, that is, the ellipse is symmetric about Y-axis.

(iii) the point $(-x, -y)$ is on the ellipse, that is the ellipse is symmetric about the origin C(0, 0). This point C is called centre of the ellipse. And hence ellipse is also called a **central conic.**

(3) Intersection with coordinate axes :

In the derivation of the equation of an ellipse we have taken A$(a, 0)$ and A'$(-a, 0)$ on the ellipse. Thus the ellipse intersects X-axis at $x = \pm a$. To find the intersection of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with Y-axis, we put $x = 0$ and hence we get $y = \pm b$. Thus the ellipse intersects Y-axis in point B$(0, b)$ and B'$(0, -b)$ as shown in the figure 8.16. Similarly it can be observed that the ellipse intersects X-axis in A and A' by taking $y = 0$ in the equation of the ellipse. These points A, A', B and B' are called vertices of the ellipse.

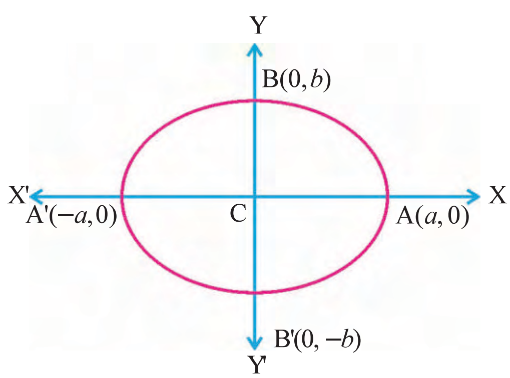

Figure 8.17

(4) Two pairs of focus and directrix :

The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b)$ **(i)**

We know that, $b^2 = a^2(1 - e^2)$

$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$

$\therefore \quad x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

$\therefore \quad x^2 - x^2e^2 + y^2 = a^2 - a^2e^2$

$\therefore \quad x^2 + 2aex + a^2e^2 + y^2 = x^2e^2 + a^2 + 2aex$

$\therefore \quad (x + ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2$ **(ii)**

To interprete (ii) we take S' = $(-ae, 0)$ and l' the line $x + \frac{a}{e} = 0$.

Now, the perpendicular distance of P(x, y) from l' (say PM') is given by

$$PM' = \frac{\left|x + \frac{a}{e}\right|}{\sqrt{1+0}} = \left|x + \frac{a}{e}\right|$$

$\therefore \quad PM'^2 = \left(x + \frac{a}{e}\right)^2$ **(iii)**

Also, $S'P^2 = (x + ae)^2 + y^2$ **(iv)**

From (iii) and (iv), (ii) gives,

$(S'P)^2 = e^2(PM')^2$

$\therefore \quad \frac{S'P}{PM'} = e$

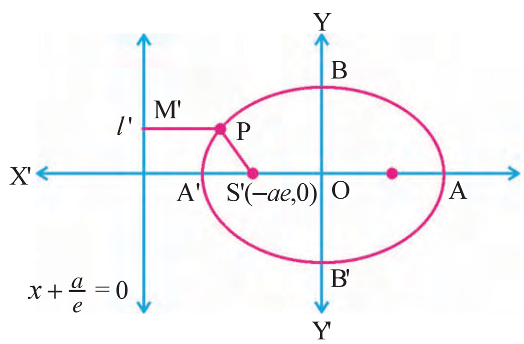

Figure 8.18

By the definition of eccentricity, S' can be taken as focus and l' as directrix. Thus an ellipse has two foci $(\pm ae, 0)$ and two corresponding directrices $x \mp \frac{a}{e} = 0$.

(5) It was seen that an ellipse is symmetric about $\overline{AA'}$ and $\overline{BB'}$. These line segments are called **axes of the ellipse**. Also AA' = $2a$ and BB' = $2b$ and $b < a$. Thus **$\overline{AA'}$ is called major axis** and **$\overline{BB'}$ is called minor axis** and b is called the length of **semi-minor axis, a is called the length of semi-major axis.**

Here major axis is along X-axis. If major axis is along Y-axis. Then foci of the ellipse are on Y-axis and directrices are parallel to X-axis. The equation of such an ellipse is,

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $b > a$ and also $a^2 = b^2(1 - e^2)$.

Also, the coordinates of foci are $(0, \pm be)$, the equations of corresponding directrices are $y \mp \frac{b}{e} = 0$.

(6) In analogy with the case of a parabola, chord and focal chord of an ellipse are defined. But, as an ellipse has two foci, it has two latera-recta (figure 8.19). As shown in the figure end-points of latera-recta in different quadrants are denoted by L_1, L_2, L_3 and L_4. $\overline{L_1L_4}$ and $\overline{L_2L_3}$ are two latera-recta.

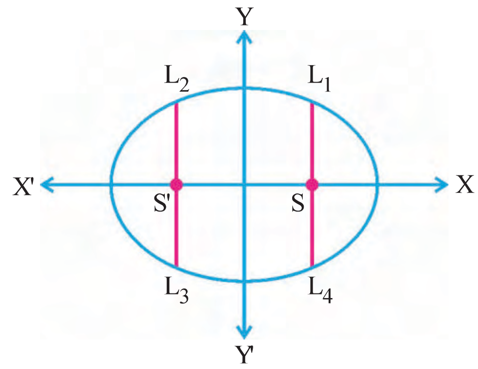

Figure 8.19

(7) Length of latera-recta :

Consider a latus-rectum $\overline{L_1L_4}$ passing through the focus S(ae, 0). Since $\overline{L_1L_4}$ is parallel to Y-axis, its length is the difference of y-coordinates of L_1 and L_4. To determine y-coordinates of L_1 and L_4, we put $x = ae$ in the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Thus, we get

$$e^2 + \frac{y^2}{b^2} = 1$$

$\therefore \quad y^2 = b^2(1 - e^2)$

But $1 - e^2 = \frac{b^2}{a^2}$

$\therefore \quad y^2 = \frac{b^4}{a^2}$

$\therefore \quad y = \pm \frac{b^2}{a}$

$\therefore$ y-coordinates of L_1 and L_4 are $\frac{b^2}{a}$ and $-\frac{b^2}{a}$ respectively. Hence

$$L_1L_4 = \frac{b^2}{a} - \left(-\frac{b^2}{a}\right) = \frac{2b^2}{a}$$

$$L_1\left(ae, \frac{b^2}{a}\right) \text{ and } L_4\left(ae, -\frac{b^2}{a}\right).$$

$$\text{Similarly } L_2 = \left(-ae, \frac{b^2}{a}\right) \text{ and } L_3 = \left(-ae, -\frac{b^2}{a}\right).$$

$\therefore$ **The length of a latus-rectum = $\frac{2b^2}{a}$**

Example 18 : Obtain the equation of the ellipse whose focus has coordinates (2, 0), the equation of corresponding directrix is $x - 5 = 0$ and eccentricity is $\frac{1}{\sqrt{2}}$.

Solution : Let P(x, y) be any point on the ellipse, S be the focus and PM the perpendicular distance of P from directrix.

$\therefore$ $SP^2 = e^2PM^2$

$\therefore$ $(x-2)^2 + y^2 = \left(\frac{1}{\sqrt{2}}\right)^2 (x-5)^2$

$\therefore$ $2(x^2 - 4x + 4 + y^2) = x^2 - 10x + 25$

$\therefore$ $x^2 + 2y^2 + 2x - 17 = 0$ is the equation of required ellipse.

Example 19 : By shifting the origin to (1, 2), prove that $\frac{(x-1)^2}{16} + \frac{(y-2)^2}{9} = 1$ is the equation of an ellipse. Also find the coordinates of foci and the equation of directrices.

Solution : In standard notations taking $x = x' + 1$, $y = y' + 2$,

the transformed equation takes the form $\frac{(x')^2}{16} + \frac{(y')^2}{9} = 1$, which represents an ellipse.

$a^2 = 16$, $b^2 = 9$

As $b^2 = a^2(1 - e^2)$, we get $9 = 16(1 - e^2)$

$\therefore$ $e^2 = 1 - \frac{9}{16} = \frac{7}{16}$

$\therefore$ $e = \frac{\sqrt{7}}{4}$ **($e > 0$)**

The coordinates of foci $(\pm ae, 0) = (\pm\sqrt{7}, 0)$ and the equations of corresponding directrices are

$x' \mp \frac{16}{\sqrt{7}} = 0$ **(in $x' - y'$ coordinate system)**

$\therefore$ In the original coordinate system the coordinates of foci are $\left(1 \pm \frac{\sqrt{7}}{4}, 2\right)$ and the equations of corresponding directrices are $x - 1 \mp \frac{16}{\sqrt{7}} = 0$.

Example 20 : Find the coordinates of foci, the equations of directrices, eccentricity and length of the latus-rectum for each of the following ellipses :

(1) $\frac{x^2}{9} + y^2 = 1$ (2) $4x^2 + y^2 = 25$

Solution : (1) $\frac{x^2}{9} + y^2 = 1$ gives $a^2 = 9$, $b^2 = 1$. So $a = 3$, $b = 1$.

As $a > b$, the major axis is along X-axis.

(i) Eccentricity : We have $b^2 = a^2(1 - e^2)$

$\therefore$ $1 = 9(1 - e^2)$

$\therefore$ $\frac{1}{9} = 1 - e^2$

$\therefore$ $e^2 = \frac{8}{9}$

$\therefore$ $e = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$

(ii) Foci : $(\pm ae, 0) = \left(\pm 3\left(\frac{\sqrt{8}}{3}\right), 0\right) = (\pm 2\sqrt{2}, 0)$

(iii) Directrices : $x = \pm\frac{a}{e}$

$\therefore$ $x = \pm 3\left(\frac{3}{\sqrt{8}}\right) = \pm\frac{9}{\sqrt{8}} = \pm\frac{9}{2\sqrt{2}}$

The equations of directrices are $x \pm \frac{9}{2\sqrt{2}} = 0$.

(iv) Length of latus-rectum : $\frac{2b^2}{a} = \frac{2}{3}$

(2) From the given equation, we get $\frac{4x^2}{25} + \frac{y^2}{25} = 1$ i.e. $\frac{x^2}{\left(\frac{25}{4}\right)} + \frac{y^2}{25} = 1$

Thus, $a^2 = \frac{25}{4}$, $b^2 = 25$

$\therefore$ $a = \frac{5}{2}$, $b = 5$. Hence $b > a$

$\therefore$ The major axis is along Y-axis.

(i) Eccentricity : $a^2 = b^2(1 - e^2)$

$\therefore$ $\frac{25}{4} = 25(1 - e)^2$

$\therefore$ $1 - e^2 = \frac{1}{4}$

$\therefore$ $e^2 = \frac{3}{4}$

$\therefore$ $e = \frac{\sqrt{3}}{2}$

(ii) Foci : $(0, \pm be) = \left(0, \pm 5\left(\frac{\sqrt{3}}{2}\right)\right) = \left(0, \pm \frac{5\sqrt{3}}{2}\right)$

(iii) Directrices : $y = \pm\frac{b}{e}$

So, $y = \pm 5\left(\frac{2}{\sqrt{3}}\right)$

$\therefore$ $y \pm \frac{10}{\sqrt{3}} = 0$ are the equations of directrices.

(iv) Length of latus-rectum : $\frac{2a^2}{b} = 2\left(\frac{25}{4}\right)\left(\frac{1}{5}\right) = \frac{5}{2}$

Example 21 : In each of the following cases, find the standard equation of the ellipse :

(1) Length of the major axis 6, eccentricity $\frac{1}{3}$ and major axis along X-axis.

(2) Length of the latus-rectum 8, eccentricity $\frac{1}{\sqrt{2}}$, major axis along Y-axis.

Solution : (1) Here major axis is along X-axis and length of the major axis is 6.

$\therefore$ $2a = 6$. So, $a = 3$

Hence $a^2 = 9$. Further $e = \frac{1}{3}$

Now, $b^2 = a^2(1 - e^2)$

$\therefore$ $b^2 = 9(1 - e^2) = 9\left(1 - \frac{1}{9}\right) = 9\left(\frac{8}{9}\right) = 8$

$\therefore$ The equation of the ellipse is $\frac{x^2}{9} + \frac{y^2}{8} = 1$.

(2) Here the major axis is along Y-axis.

$\therefore$ Length of the latus-rectum $\frac{2a^2}{b} = 8$. Hence $a^2 = 4b$ **(i)**

Also, eccentricity $e = \frac{1}{\sqrt{2}}$ and $a^2 = b^2(1 - e^2) = b^2\left(1 - \frac{1}{2}\right)$

$\therefore$ $a^2 = \frac{1}{2}b^2$ **(ii)**

From (i) and (ii), we get

$$\frac{1}{2}b^2 = 4b$$

$\therefore \quad b^2 - 8b = 0$

$\therefore \quad b = 8$ as $b \neq 0$.

$\therefore \quad b^2 = 64$

$\therefore \quad a^2 = \frac{b^2}{2} = \frac{64}{2} = 32$

Thus, the equation of the ellipse is $\frac{x^2}{32} + \frac{y^2}{64} = 1$.

Example 22 : Find the equation of ellipse whose major axis is along X-axis, length of semi-minor axis is 4 and distance between two foci is 5.

Solution : Here, length of the semi-minor axis $b = 4$. Major axis is along X-axis

Let S$(ae, 0)$, S'$(-ae, 0)$ be foci. Then the distance between them is SS' $= 2ae = 5$.

$\therefore \quad ae = \frac{5}{2}$ **(i)**

Also, $b^2 = a^2(1 - e^2) = a^2 - a^2e^2$

$16 = a^2 - \left(\frac{5}{2}\right)^2 = a^2 - \frac{25}{4}$ **(from (i))**

$\therefore \quad a^2 = 16 + \frac{25}{4} = \frac{89}{4}$

Thus, the equation of the required ellipse is $\frac{x^2}{\frac{89}{4}} + \frac{y^2}{16} = 1$.

$\therefore \quad \frac{4x^2}{89} + \frac{y^2}{16} = 1$

Exercise 8.4

1. Find the standard equation of the ellipse in each of the following :

(1) Foci $(\pm 2, 0)$, eccentricity $= \frac{1}{2}$

(2) Foci $(\pm 4, 0)$, vertices $(\pm 5, 0)$

(3) Length of the semi-minor axis 6, eccentricity $\frac{4}{5}$, major axis along X-axis.

(4) A focus $(0, 4)$, eccentricity $\frac{4}{5}$

(5) Eccentricity $\frac{2}{3}$, length of a latus-rectum 5, major axis along X-axis.

(6) Length of semi-major axis 4, eccentricity $\frac{1}{2}$, major axis along X-axis.

(7) Length of semi-minor axis 8, a focus $(0, 6)$.

2. If possible, find the equation of the ellipse whose foci are $(\pm 3, 0)$ and which passes through the point $(4, 1)$.

3. Find the coordinates foci, eccentricity, the equations of directrices and length of the latus-rectum for the following ellipses :

(1) $\frac{x^2}{4} + \frac{y^2}{9} = 1$ (2) $\frac{x^2}{36} + \frac{y^2}{20} = 1$ (3) $x^2 + 2y^2 = 100$

(4) $\frac{x^2}{43} + \frac{7y^2}{688} = 1$ (5) $5x^2 + 9y^2 = 81$

4. Find the eccentricity of the ellipse in which the distance between the two directrices is three times the distance between the foci.

5. Find the equations of directrices of the ellipse $16x^2 + 25y^2 = 1600$. Show that the point $(5\sqrt{3}, 4)$ lies on the ellipse. Find the ratio of distance of this point from a directrix to its distance from the corresponding focus.

6. Show that the line $x + y = 3$ contains to a focal chord of the ellipse $20x^2 + 36y^2 = 405$.

7. Find the equation of the ellipse passing through the points (4, 3) and (−1, 4).

8. Find the equation of the ellipse having eccentricity $\frac{1}{2}$, a focus (3, 2) and corresponding directrix $y = 5$.

9. Shift the origin (2, 1) and prove that $\frac{(x-2)^2}{4} + \frac{(y-1)^2}{9} = 1$ represents an ellipse. Find the coordinates of foci and the equations of directrices.

*

8.12 Parametric Equations of an Ellipse

The equation of an ellipse is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Hence $\left(\frac{x}{a}, \frac{y}{b}\right)$ is on the unit circle.

Sum of two squares is 1.

$\therefore$ $\exists\, \theta \in (-\pi, \pi]$ such that $\frac{x}{a} = cos\theta$, $\frac{y}{b} = sin\theta$

$\therefore$ $x = acos\theta$, $y = bsin\theta$

Further elimination of θ from $x = acos\theta$, $y = bsin\theta$ gives $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Thus we see that $x = acos\theta$, $y = bsin\theta$, $\theta \in (-\pi, \pi]$ are parametric equations of the ellipse. The point $(a\,cos\theta, b\,sin\theta)$ on the ellipse is called the θ-point.

Properties of an Ellipse :

Property 1 : The distance of a focus of an ellipse from an end-point of the minor axis is equal to the length of the semi-major axis.

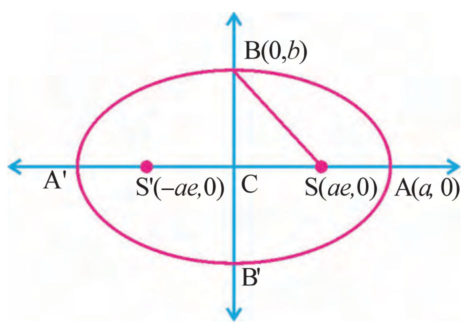

Figure 8.20

Proof : An end-point of the minor axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is B(0, b). The coordinates of one of the focus S are (ae, 0).

$\therefore$ $SB^2 = a^2e^2 + b^2 = a^2e^2 + a^2(1 - e^2) = a^2$

$\therefore$ $SB = a$

Similarly, for S'($-ae$, 0); $S'B^2 = a^2e^2 + b^2 = a^2$

$\therefore$ $S'B = a$

Also, the other end-point of the minor axis is B'(0, $-b$). For this point we also can show that, $SB' = a = S'B'$.

Property 2 : If S is a focus and A and A' are extremities of the major axis, then AS $\cdot$ A'S = b^2.

Proof : Here focus is S(ae, 0), A(a, 0) and A'($-a$, 0).

$$\therefore \quad AS \cdot A'S = \sqrt{(a-ae)^2}\sqrt{(a+ae)^2}$$

$$= a(1-e)\,a(1+e) \qquad \mathbf{(0 < e < 1)}$$

$$= a^2(1-e^2) = b^2$$

Property 3 : For every point P(x, y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, SP + S'P = $2a$, where S and S' are foci and $b < a$.

Proof : The directricies of the ellipse are $x \pm \frac{a}{e} = 0$. Thus the distance of the point P(x, y) from respective directrices is $\left|\frac{a}{e} \mp x\right|$. By definition of the ellipse we have,

$$SP = e\left|\frac{a}{e} - x\right| = |a - ex|$$

$$S'P = e\left|\frac{a}{e} + x\right| = |a + ex|$$

Also as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so $\frac{x^2}{a^2} \le 1$

$\therefore \quad |x| \le a$. Also $e < 1$

$\therefore \quad |ex| < a$ or $-a < ex < a$

$\therefore \quad a - ex > 0$ and also $a + ex > 0$

$\therefore \quad$ SP = $a - ex$, S'P = $a + ex$

$\therefore \quad$ SP + S'P = $2a$

The converse of above property is also true. That is, the set of all points in the plane, the sum of whose distances from two fixed points in the plane is a constant is an ellipse whose major axis has the same length as the constant.

To prove this result we proceed as follows :

Suppose S(c, 0) and S'($-c$, 0) are two fixed points in the plane. These points are selected so that the origin C the is mid-point of $\overline{SS'}$ and the direction of $\overrightarrow{CS}$ is the positive direction of the X-axis. Suppose P is a point in the plane such that SP + S'P = $2a$, where a is a constant. ($a \ne c$)

P $\notin \overline{SS'}$ **(If P $\in \overline{SS'}$, SP + S'P = SS' i.e. $2a = 2c$)**

$\therefore \quad$ SP + S'P > SS'

$\therefore \quad 2a > 2c$ **(i)**

Now, SP + S'P = $2a$

$$\therefore \quad \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\therefore \quad \sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

$$\therefore \quad (x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

$\therefore \quad a\sqrt{(x-c)^2 + y^2} = a^2 - cx$

$\therefore \quad \sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x$

$\therefore$ Taking $\frac{c}{a} = e$, $\sqrt{(x-c)^2 + y^2} = a - ex$

$\therefore \quad \sqrt{(x-ae)^2 + y^2} = a - ex \qquad$ **$(c = ae)$**

$\therefore \quad (x - ae)^2 + y^2 = (a - ex)^2$

$\therefore \quad x^2 - 2aex + a^2e^2 + y^2 = a^2 - 2aex + e^2x^2$

$\therefore \quad x^2 (1 - e^2) + y^2 = a^2 (1 - e^2)$

$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$

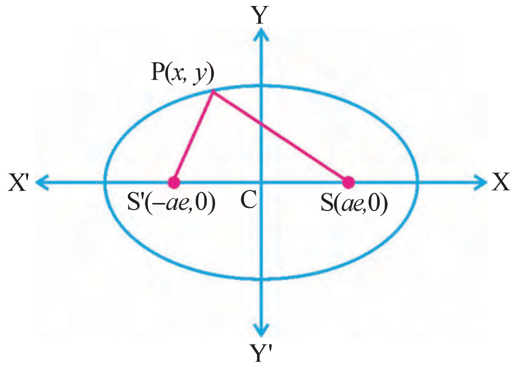

Figure 8.21

Since by (i) $a > c$, $e = \frac{c}{a} < 1$. Hence $a^2 (1 - e^2) > 0$.

Thus there exists a positive real number b such that $b^2 = a^2(1 - e^2)$.

Thus, we get $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

This is an ellipse with length of major axis equal to $2a$.

This property is often used as a definition of an ellipse.

An important application of ellipse :

If a source of light (or sound or in general any wave) is placed at one focus S of an elliptic mirror, then after reflection from the mirror, light will reach the other focus S'.

This property of ellipses was used by ancient Indian architects in construction of whispering gallaries. Some whispering galaries are found at Bijapur in Karnataka and Golkonda Fort in Hydrabad. In the design of telescopes this property of an ellipse is also used.

Further, in medical science, this property of ellipses is used in lithotripper which is used to break stones in kidney or bladder. Here, the lithotripper is placed at one focus of an ellipse and ultra-high frequency, shock-waves are produced at the other focus. The reflected waves break the kidney stone.

Example 23 : Find parametric equations of the ellipse $3x^2 + 5y^2 = 15$.

Solution : Dividing given equation by 15, we get

$$\frac{x^2}{5} + \frac{y^2}{3} = 1$$

Thus we get $a = \sqrt{5}$, $b = \sqrt{3}$ and hence parametric equations of the ellipse are $x = \sqrt{5}\,cos\theta$, $y = \sqrt{3}\,sin\theta$. $\theta \in (-\pi, \pi]$

Example 24 : Find the coordinates of foci, the equations of directrices and eccentricity of the ellipse, $x = 2cos\theta$, $y = 5sin\theta$.

Solution : Here $a = 2$, $b = 5$. Since $b > a$ major axis of the ellipse is along Y-axis.

(1) Eccentricity : We have $a^2 = b^2(1 - e^2)$

$\therefore \quad 4 = 25(1 - e^2)$

$\therefore \quad \frac{4}{25} = 1 - e^2$

$\therefore \quad e^2 = 1 - \frac{4}{25} = \frac{21}{25}$. Thus $e = \frac{\sqrt{21}}{5}$

(2) The coordinates of Foci : $(0, \pm be) = \left(0, \pm 5\frac{\sqrt{21}}{5}\right) = (0, \pm\sqrt{21})$

(3) The equations of Directrices : $y = \pm\frac{b}{e} = \pm 5 \times \frac{5}{\sqrt{21}} = \pm\frac{25}{\sqrt{21}}$

Exercise 8.5

1. Obtain parametric equations of the following ellipses :

(1) $\frac{x^2}{16} + \frac{y^2}{9} = 1$ (2) $\frac{x^2}{16} + \frac{y^2}{12} = 1$

(3) $3x^2 + 4y^2 - 12 = 0$ (4) $\frac{x^2}{16} + \frac{y^2}{7} = 1$

(5) $x^2 + 2y^2 - 18 = 0$

2. Find eccentricity and foci of the following ellipses :

(1) $x = 2cos\theta$, $y = 3sin\theta$, $\theta \in (-\pi, \pi]$

(2) $3x = 5cos\theta$, $5y = 7sin\theta$, $\theta \in (-\pi, \pi]$

(3) $x = 4cos\theta$, $y = 3sin\theta$, $\theta \in (-\pi, \pi]$

3. If the sum of distances of a variable point P from points S(1, 0) and S'(−1, 0) is constant and equal to 8, then find the set of points.

*

Hyperbola : Hyperbola is an important curve used in military sciences. For example, source of a fired bullet can be determined by properties of a hyperbola and intensity of sound.

A conic with eccentricity $e > 1$ is called a hyperbola.

Standard Equation of a Hyperbola :

Suppose S is the focus, line l is the directrix and e is the eccentricity of a hyperbola. Let Z be the foot of the perpendicular on l drawn from S. Now let A and A' divide $\overline{SZ}$ from S in the ratio $e : 1$ and $-e : 1$ respectively. Since $\frac{SA}{AZ} = e$ and $\frac{SA'}{A'Z} = e$, A and A' are on the hyperbola.

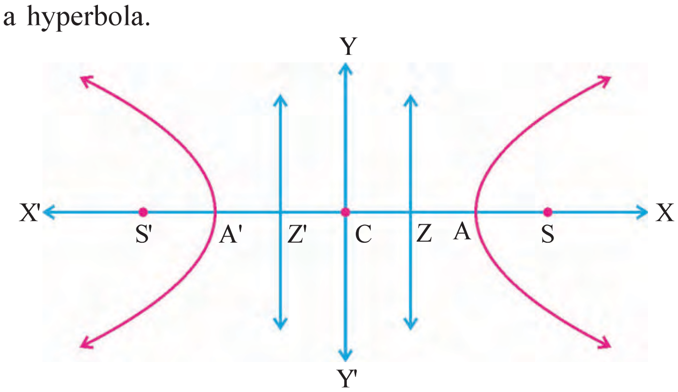

Figure 8.22

Let AA' = $2a$ and C is the mid-point of $\overline{AA'}$. Also CA = CA' = a.

Let C be the origin and take $\overrightarrow{CA}$ as the positive direction of X-axis. Then A = $(a, 0)$ and A' = $(-a, 0)$. Let the coordinates of S and Z be $(p, 0)$ and $(q, 0)$ respectively. Since A and A' divide $\overline{SZ}$ in the ratio e and $-e$,

$$\frac{eq + p}{e + 1} = a \text{ and } \frac{-eq + p}{-e + 1} = -a$$

$\therefore \quad eq + p = ae + a$ and $-eq + p = ae - a$

$\therefore\quad p = ae$ and $q = \dfrac{a}{e}$

$\therefore$ The coordinates of the focus S are $(ae, 0)$ and the equation of the directrix l is $x = \dfrac{a}{e}$.

Suppose $P(x, y)$ is a point on the hyperbola and M is the foot of the perpendicular on directrix l drawn from P. Thus coordinates of M are $\left(\dfrac{a}{e}, y\right)$.

Now, $\dfrac{SP}{PM} = e \Leftrightarrow SP^2 = e^2 PM^2$

$$\Leftrightarrow (x - ae)^2 + y^2 = (ex - a)^2$$

$$\Leftrightarrow x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2$$

$$\Leftrightarrow (e^2 - 1)x^2 - y^2 = a^2(e^2 - 1)$$

$$\Leftrightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

Here $a^2 > 0$ and $e > 1$. Hence $e^2 - 1 > 0$

$\therefore\quad a^2(e^2 - 1) > 0$. Thus there exists a real number b such that $a^2(e^2 - 1) = b^2$.

$\therefore\quad$ **$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is the standard equation of a hyperbola.**

Some conclusions can be drawn from the standard equation, they are discussed below :

(1) Symmetry :

Hyperbola is symmetric about both the axes and also symmetric about the origin. Also, origin is centre and hence hyperbola is also a central conic.

(2) Intersection with axes :

To obtain intersection of a hyperbola with axes, we put $y = 0$ in the equation of the hyperbola.

We get, $\dfrac{x^2}{a^2} = 1 \Rightarrow x = \pm a$

So the hyperbola intersects X-axis in the points $A(a, 0)$ and $A'(-a, 0)$. A and A' are called the vertices of the hyperbola.

Putting $x = 0$ in the equation of hyperbola we get $y^2 = -b^2$. As $b \neq 0$, for no real value of y, $y^2 = -b^2$. Thus hyperbola does not intersect Y-axis. In analogy with ellipse the points $B(0, b)$ and $B'(0, -b)$ are also called vertices of the hyperbola, here we note that these points are not on the hyperbola. In the case of a hyperbola $\overline{AA'}$ and $\overline{BB'}$ are called **Transverse axis** and **Conjugate axis** respectively.

The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, does not intersect Y-axis but it lies on both sides of the Y-axis. Two parts of the hyperbola have no point in common and they are called **branches** of the hyperbola.

(3) A second pair of focus and directrix :

The equation of hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

$$\therefore\quad \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

$\therefore \quad (e^2 - 1)x^2 - y^2 = a^2(e^2 - 1)$

$\therefore \quad x^2 + 2aex + a^2e^2 + y^2 = a^2 + 2aex + e^2x^2$

$\therefore \quad (x + ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2$

Let S'$(-ae, 0)$ and line $l' : x + \frac{a}{e} = 0$

Let M' be the foot of perpendicular drawn from P(x, y) to l'.

$\therefore \quad (\text{S'P})^2 = e^2(\text{P'M})^2$

$\therefore$ The second directrix of the hyperbola is $x + \frac{a}{e} = 0$.

Thus, for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, there are two foci $(\pm ae, 0)$ and corresponding directrices are $x \mp \frac{a}{e} = 0$.

(4) Chords, Focal chords and Latera-recta :

A line segment joining two points of a hyperbola is called a **chord** of the hyperbola. If a chord passes through a focus, then it is called a **focal chord** of the hyperbola. A focal chord perpendicular to the transverse axis of the hyperbola is called a **latus-rectum** of the hyperbola.

(5) Length of a latus-rectum :

Consider a latus-rectum $\overline{L_1L_4}$ passing through a focus S$(ae, 0)$, as shown in the figure 8.23. The equation of the latus-rectum $\overleftrightarrow{L_1L_4}$ is $x = ae$. Thus x-coordinates of L_1 and L_4 both are ae. Using $x = ae$ in the equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

$$\frac{(ae)^2}{a^2} - \frac{y^2}{b^2} = 1$$

$\therefore \quad \frac{y^2}{b^2} = e^2 - 1$

$\therefore \quad y^2 = b^2(e^2 - 1)$

$\qquad = b^2 \cdot \frac{b^2}{a^2}$

$\qquad = \frac{b^4}{a^2}$

$\therefore \quad y = \pm \frac{b^2}{a}$

$\therefore \quad L_1\left(ae, \frac{b^2}{a}\right)$ and $L_4\left(ae, -\frac{b^2}{a}\right)$

$\therefore \quad L_1L_4 = \frac{2b^2}{a}$

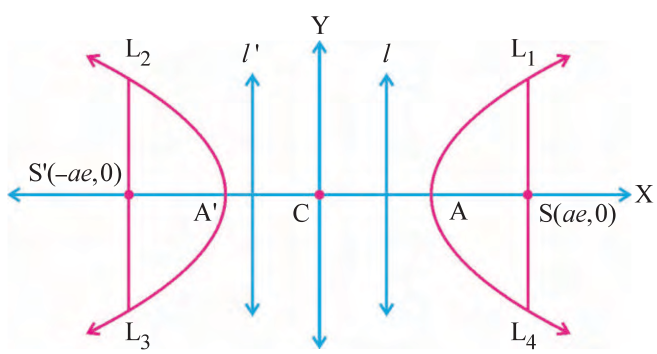

Figure 8.23

(6) Another form of the equation of a hyperbola :

In analogy with ellipse, we can consider hyperbola with transverse axis along Y-axis. The equation would be

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

This hyperbola is said to be **conjugate hyperbola** of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Parametric equations of hyperbola :

Comparing the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with the trigonometric identity,

$sec^2\theta - tan^2\theta = 1.$

Now for a given point (x, y) on the hyperbola, we choose θ such that $-\pi < \theta \leq \pi$; $\theta \neq \frac{\pi}{2}, -\frac{\pi}{2}$ such that $x = a\,sec\theta$, $y = b\,tan\theta$.

Conversely, for any $\theta \in (-\pi, \pi] - \left\{\frac{\pi}{2}, -\frac{\pi}{2}\right\}$, if we take $x = a\,sec\theta$, $y = b\,tan\theta$, then the point (x, y) is on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Here θ is a parameter. In analogy with earlier situations the point $(a\,sec\theta, b\,tan\theta)$ is referred to as θ-point of the hyperbola. Similarly parametric equations of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ are $x = a\,tan\theta$, $y = b\,sec\theta$, $\theta \in (-\pi, \pi] - \left\{\frac{\pi}{2}, -\frac{\pi}{2}\right\}$.

Rectangular Hyperbola :

If $a^2 = b^2$ for hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then it is called a **rectangular hyperbola**. Thus the standard equation of a rectangular hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1 \quad \text{or} \quad x^2 - y^2 = a^2$$

Eccentricity : For a hyperbola, eccentricity is given by $b^2 = a^2(e^2 - 1)$.

For a rectangular hyperbola, we have $a^2 = b^2$.

$\therefore \quad a^2 = a^2(e^2 - 1)$

$\therefore \quad e^2 = 2$

$\therefore \quad e = \sqrt{2}$ **(as $e > 1$)**

θ-point : A θ-point on a rectangular hyperbola is $(a\,sec\theta, a\,tan\theta)$.

Length of a latus-rectum : Length of the latus-rectum of a hyperbola is given by $\frac{2b^2}{a}$. Here $b^2 = a^2$. Hence length of the latus-rectum of a rectangular hyperbola is $2a$.

Properties of a hyperbola :

If S and S' are foci of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and P is any point on the hyperbola then $|\,SP - S'P\,|$ is constant.

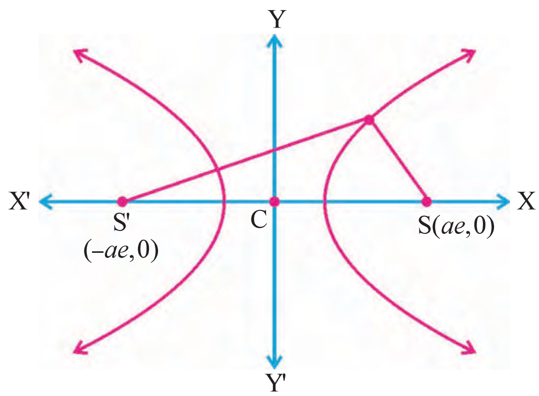

Figure 8.24

Proof : The foci are $S(ae, 0)$ and $S(-ae, 0)$.

Now, $SP = ePM$

$= e\left|x - \frac{a}{e}\right|$

Here $\overline{PM}$ is perpendicular to the directrix $x = \frac{a}{e}$ from $P(x, y)$.

$\therefore \quad SP = ePM = e\left|x - \frac{a}{e}\right| = |\,ex - a\,|$

$\therefore \quad SP = |\,ex - a\,|$. Similarly $S'P = |\,ex + a\,|$

$\therefore\quad (SP - S'P)^2 = SP^2 + S'P^2 - 2SP \cdot S'P$

$= (ex - a)^2 + (ex + a)^2 - 2\,|\,e^2x^2 - a^2\,|$

$= (ex - a)^2 + (ex + a)^2 - 2(e^2x^2 - a^2)$ **$(e^2 > 1, x^2 \geq a^2 \Rightarrow e^2x^2 > a^2)$**

$= 4a^2$

$\therefore\quad |\,SP - S'P\,| = 2a$

Note : The converse of above is also true. Thus we have an equivalent definition, "hyperbola is the set of points (in a plane), the difference of whose distance from two fixed points in the plane is constant."

Using this definition also the equation of a hyperbola can be derived.

Suppose S and S' are two fixed points and let P be a point in the plane so that $|\,SP - S'P\,| = 2a$.

Let $(c, 0)$ and $(-c, 0)$ be the coordinates of S and S' respectively and mid-point C of $\overline{SS'}$ be the origin.

$\therefore\quad \left|\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}\right| = 2a$

$\therefore\quad \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a$

$\therefore\quad \sqrt{(x+c)^2 + y^2} = \sqrt{(x-c)^2 + y^2} \pm 2a$

$\therefore\quad (x + c)^2 + y^2 = (x - c)^2 + y^2 \pm 4a\sqrt{(x-c)^2 + y^2} + 4a^2$

$\therefore\quad cx - a^2 = \pm\, a\sqrt{(x-c)^2 + y^2}$

$\therefore\quad \left(\frac{c}{a}x - a\right)^2 = (x - c)^2 + y^2$

Taking $\frac{c}{a} = e$, $c = ae$

$\therefore\quad (ex - a)^2 = (x - ae)^2 + y^2$

$\therefore\quad (x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2$ **(i)**

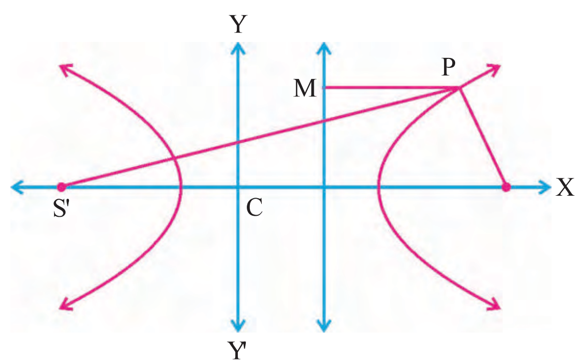

Figure 8.25

Further, $S = (c, 0) = (ae, 0)$

Suppose $l : x - \frac{a}{e} = 0$ is a line, then from (i)

$\therefore\quad (SP)^2 = e^2(PM)^2$

$\therefore\quad \frac{SP}{PM} = e$

Also $|\,SP - S'P\,| = 2a < SS' = 2c$ $\left(P \notin \overleftrightarrow{SS'} - \overline{SS'}\right)$

$\therefore\quad \frac{c}{a} > 1$

$\therefore\quad e > 1$

$\therefore\quad$ The point set of P is a hyperbola with eccentricity $e > 1$.

Example 25 : Obtain the equation of the hyperbola whose focus is $(0, 1)$, the equation of the directrix is $x + 3 = 0$ and eccentricity is $\sqrt{2}$.

Solution : $SP^2 = e^2PM^2$

$\therefore \quad x^2 + (y-1)^2 = 2(x+3)^2$

$\therefore \quad x^2 + y^2 - 2y + 1 = 2(x^2 + 6x + 9)$

$\therefore \quad x^2 - y^2 + 12x + 2y + 17 = 0$ is the equation of the required hyperbola.

Example 26 : By shifting origin to $(-1, -2)$, show that $(x+1)^2 - (y+2)^2 = 16$ represents a hyperbola. Find its eccentricity, coordinates of foci and equation of directrices.

Solution : In the standard notations taking $x = x' - 1$, $y = y' - 2$,

$(x')^2 - (y')^2 = 16$

This equation represents a rectangular hyperbola with $a = b = 4$ and $e = \sqrt{2}$.

$\therefore$ The coordinates of foci are $(\pm 4\sqrt{2}, 0)$ and the corresponding equations of directrices are $x' \mp 2\sqrt{2} = 0$. **(in $x' - y'$ system)**

$\therefore$ In original coordinates system, coordinates of foci are $(\pm 4\sqrt{2} - 1, -2)$ and

The equations of directrices are $x + 1 \pm 2\sqrt{2} = 0$.

Example 27 : Point P is a variable point such that difference of its distances from fixed points S and S', which are 12 units apart, is constant 8. Find the point set of P.

Solution : $| SP - S'P | = 2a = 8$

$\therefore \quad a = 4$, $SS' = 2c = 12$. Hence $c = 6$

$e = \frac{c}{a} = \frac{6}{4} = \frac{3}{2}$

Now, $b^2 = a^2(e^2 - 1) = 16\left(\frac{9}{4} - 1\right) = 36 - 16 = 20$

$\therefore$ The equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{20} = 1$.

Example 28 : For the following hyperbola, find the coordinates of foci, the equations of directrices, eccentricity, length of the latus-rectum and length of transverse and conjugate axes :

(1) $x^2 - 16y^2 = 16$ (2) $\frac{x^2}{25} - \frac{y^2}{24} = 1$

(3) $\frac{y^2}{25} - \frac{x^2}{9} = 1$ (4) $x^2 - y^2 = 4$

Solution :

(1) This equation can be written as $\frac{x^2}{16} - \frac{y^2}{1} = 1$.

$\therefore \quad a = 4$, $b = 1$

as $b^2 = a^2(e^2 - 1)$, $\quad 1 = 16(e^2 - 1)$

$\therefore \quad e^2 - 1 = \frac{1}{16}$ or $e^2 = \frac{17}{16}$

$\therefore \quad e = \frac{\sqrt{17}}{4}$

Foci are $(\pm ae, 0) = \left(\pm 4\left(\frac{\sqrt{17}}{4}\right), 0\right) = (\pm\sqrt{17}, 0)$.

Directrices are $x = \pm \frac{a}{e}$ i.e. $x = \pm 4\left(\frac{4}{\sqrt{17}}\right)$.

$\therefore \quad x = \pm \frac{16}{\sqrt{17}}$ are equations of directrices.

Length of the latus-rectum $= \frac{2b^2}{a} = \frac{2}{4} = \frac{1}{2}$

Length of the transverse axis $= 2a = 8$

Length of the conjugate axis $= 2b = 2$

(2) Here $a^2 = 25,\ b^2 = 24$

$\therefore\ b^2 = a^2(e^2 - 1)$

$\therefore\ 24 = 25(e^2 - 1)$

$\therefore\ e^2 - 1 = \frac{24}{25}$

$\therefore\ e^2 = \frac{49}{25}$

$\therefore\ e = \frac{7}{5}$

Foci : $(\pm ae,\ 0) = \left(\pm 5\left(\frac{7}{5}\right), 0\right) = (\pm 7,\ 0)$

Directrices : $x = \pm \frac{a}{e} = \pm \frac{5}{\left(\frac{7}{5}\right)} = \pm \frac{25}{7}$

$\therefore$ The equations of directrices are $x = \pm \frac{25}{7}$.

Length of the latus-rectum $= \frac{2b^2}{a} = \frac{2(24)}{5} = \frac{48}{5}$

Length of the transverse axis $= 2a = 10$

Length of the conjugate axis $= 2b = 2\sqrt{24} = 4\sqrt{6}$

(3) In this hyperbola, directrices are parallel to X-axis. Here $a^2 = 9,\ b^2 = 25$

For eccentricity, we have

$\therefore\ a^2 = b^2(e^2 - 1)$

$\therefore\ 9 = 25(e^2 - 1)$

$\therefore\ e^2 = 1 + \frac{9}{25} = \frac{34}{25}$

$\therefore\ e = \frac{\sqrt{34}}{5}$

Foci : $(0,\ \pm be) = \left(0, \pm 5\left(\frac{\sqrt{34}}{5}\right)\right) = (0,\ \pm\sqrt{34})$

Directrices : $y = \pm \frac{b}{e} = \pm 5\left(\frac{5}{\sqrt{34}}\right) = \pm \frac{25}{\sqrt{34}}$

$\therefore\ y = \pm \frac{25}{\sqrt{34}}$ are equations of directrices of the ellipse.

Length of the latus-rectum $= \frac{2a^2}{b} = \frac{2 \cdot 9}{5} = \frac{18}{5}$

Length of the transverse axis $= 2b = 10$

Length of the conjugate axis $= 2a = 6$

(4) This equation can be written $\frac{x^2}{4} - \frac{y^2}{4} = 1$. This is a rectangular hyperbola. $a^2 = b^2 = 4$

Eccentricity : $e = \sqrt{2}$, the coordinates of foci $(\pm 2\sqrt{2},\ 0)$, the equations of directrices : $x = \pm\sqrt{2}$

Length of the latus-rectum $= 2a = 4$

Length of the transverse axis $= 2a = 4$

Length of the conjugate axis $= 2b = 4$

Example 29 : Find the equation of the hyperbola from the following conditions :

(1) Foci $(\pm 7, 0)$, vertices $(\pm 5, 0)$

(2) Foci $(0, \pm 3)$, eccentricity = 2

(3) Distance between foci 16 (foci on X-axis), eccentricity = $\sqrt{2}$

Solution : (1) Here, foci are $(\pm ae, 0) = (\pm 7, 0)$

$\therefore \quad ae = 7$ **(i)**

Now vertices are $(\pm 5, 0)$.

$\therefore \quad a = 5$ **(ii)**

$\therefore \quad ae = 5e = 7$

$\therefore \quad e = \frac{7}{5}$

Now $b^2 = a^2(e^2 - 1) = 25\left(\frac{49}{25} - 1\right) = 24$

$\therefore$ The equation of the hyperbola is $\frac{x^2}{25} - \frac{y^2}{24} = 1$.

(2) Foci $(0, \pm 3)$. Foci are on Y-axis. Thus directrices are parallel to X-axis.

Given that $e = 2$

$\therefore \quad be = 3$

$\therefore \quad 2b = 3$

$\therefore \quad b = \frac{3}{2}$

Now $a^2 = b^2(e^2 - 1)$

$\therefore \quad a^2 = \frac{9}{4}(4 - 1) = \frac{9}{4}(3) = \frac{27}{4}$

$\therefore$ The equation of the hyperbola is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

$\therefore$ The equation of the hyperbola is $\frac{y^2}{\frac{9}{4}} - \frac{x^2}{\frac{27}{4}} = 1$

$\therefore \quad \frac{4y^2}{9} - \frac{4x^2}{27} = 1$

(3) Distance between foci $= 2ae = 16$. Thus $ae = 8$ **(i)**

$e = \sqrt{2}$

$\therefore \quad a\sqrt{2} = 8$

$a = \frac{8}{\sqrt{2}} = 4\sqrt{2}$

Now $b^2 = a^2(e^2 - 1) = (4\sqrt{2})^2(2 - 1) = 32$

$\therefore$ The equation of the hyperbola is $\frac{x^2}{(4\sqrt{2})^2} - \frac{y^2}{32} = 1$ or $x^2 - y^2 = 32$

Exercise 8.6

1. Find the coordinates of foci, the equations of directrices, length of the latus-rectum, lengths of transverse and conjugate axes of the following hyperbolas :

(1) $\frac{x^2}{100} - \frac{y^2}{25} = 1$ (2) $x^2 - y^2 = 64$ (3) $2x^2 - 3y^2 = 5$

(4) $9y^2 - 16x^2 = 144$ (5) $\frac{y^2}{25} - \frac{x^2}{39} = 1$

2. Find the equation of the hyperbola for the following situations. Also write their parametric equations :

(1) Eccentricity $e = \frac{4}{3}$, Vertices $(0, \pm 7)$

(2) Foci $(\pm \sqrt{13}, 0)$, Eccentricity $\frac{\sqrt{13}}{3}$

(3) Foci $(\pm 3\sqrt{5}, 0)$, Length of the latus-rectum = 8

(4) Foci $(0, \pm 8)$, Eccentricity $\sqrt{2}$

(5) Distance between foci (on Y-axis) = 10, Eccentricity $\frac{5}{4}$

3. If the eccentricities of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ are e_1 and e_2 respectively, then prove that $e_1^2 + e_2^2 = e_1^2 e_2^2$.

4. Find the equation of the hyperbola for which distance from one vertex to two foci are 9 and 1.

5. Write parametric equations of the hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$.

*

Miscellaneous Problems :

Example 30 : The two supporting pillars of a suspension bridge in the shape of a parabola are 30 *m* high and 200 *m* apart. The height of the bridge above its centre is 5 *m*. There is a pillar of height 11.25 *m*. Find its distance from the centre.

Solution : As shown in the figure 8.26 CAB is the suspension bridge in the shape of a parabola. The centre of parabola is vertex, which is at height 5 *m*. Taking A as origin, $\overleftrightarrow{OA}$ as Y-axis, the equation of the parabola is $x^2 = 4ay$. Now the coordinates of O are $(0, -5)$, thus by shifting the origin at O, the equation of the parabola is,

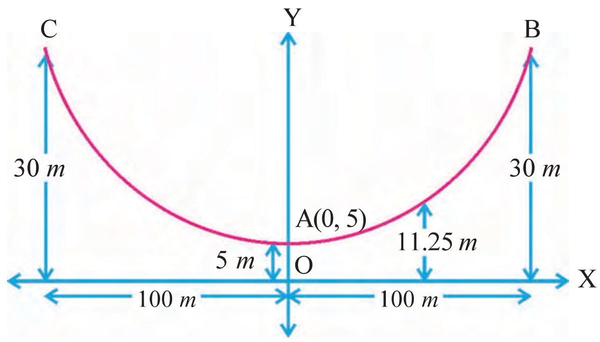

Figure 8.26

$$(x')^2 = 4a(y' - 5) \quad \text{(i)}$$

For the supports C and B, we are given that coordinates are $(-100, 30)$ and $(100, 30)$ respectively. Using these in (i), we get

$$(100)^2 = 4a(30 - 5)$$

$\therefore$ $10000 = 100a$

$\therefore$ $a = 100$

Thus, (i) gives $x^2 = 400(y - 5)$ **(ii)**

Further to find the distance of supports at height 11.25, we substitute $y = 11.25$ in (ii).

$$x^2 = 400(11.25 - 5) = 400(6.25) = 2500$$

$\therefore$ $x = \pm 50$

Hence there are two supports on each side of the centre at distance 50 *m* from the centre having heights 11.25 *m*.

Example 31 : A 12 *m* long rod slides in such a way that its ends stay on the two axes. Find the point-set of the point on the rod 3 *m* away from its end-point on the X-axis.

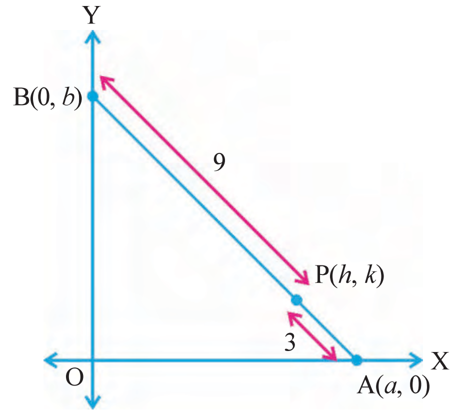

Figure 8.27

Solution : The end-points of the rod are A(a, 0) and B(0, b) and the point on the rod 3 *m* away from A is P(h, k).

Thus, AP = 3 *m*, PB = 9 *m*

$\therefore$ P divides $\overline{AB}$ from A's side in the ratio 1 : 3.

$\therefore \quad h = \frac{3a}{4}$ and $k = \frac{b}{4}$

$\therefore \quad a = \frac{4h}{3}$ and $b = 4k$

Now, in the right ΔAOB, $OA^2 + OB^2 = AB^2$. So $a^2 + b^2 = 144$

$\therefore \quad \frac{16h^2}{9} + \frac{16k^2}{1} = 144$

$\therefore \quad \frac{h^2}{81} + \frac{k^2}{9} = 1$

$\therefore$ Point-set of P is $\frac{x^2}{81} + \frac{y^2}{9} = 1$. It is an ellipse.

Example 32 : The orbit of the earth around the sun is an ellipse. The sun is at one of the foci of this ellipse. If the length of the major axis of this ellipse is 300 million *km* and the eccentricity is 0.0167, find the minimum and maximum distance of the earth from the sun.

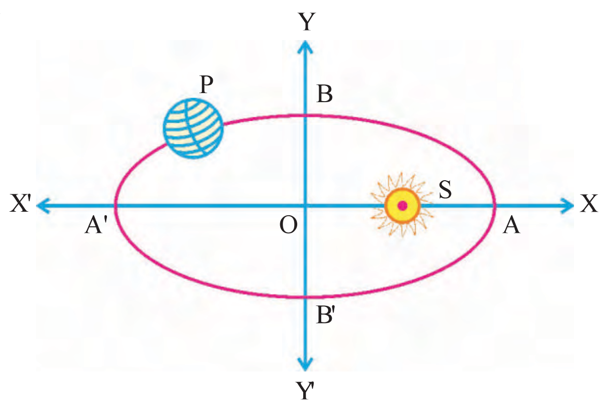

Figure 8.28

Solution : Take the focus of the orbit at S (where the sun is) and take a point P on elliptical orbit. Then the focal distance of P is

$SP = a(1 - e\cos\theta)$.

Now, $2a = 3 \times 10^8$ *km*

$\therefore \quad a = 1.5 \times 10^8$ *km*

$\therefore \quad SP = 1.5 \times 10^8$ *km* $(1 - 0.0167 \cos\theta)$

When the earth-sun distance is minimum, the earth is on the major axis at its end. So $\theta = 0$ and $\cos\theta = 1$. Hence minimum distance of sun from earth is

1.5×10^8 *km* $(1 - 0.0167 \cos\theta)$

$= 1.5 \times 10^8 (1 - 0.0167)$

$=$ 147,495,000 *km*

Earth is at its maximum distance when it is at the other end of the major axis and away from S. The maximum distance is

$1.5 \times 10^8 (1 - 0.0167(-1))$ *km* $= 1.5 \times 10^8 (1 + 0.0167)$ *km*

$=$ 152,505,000 *km*

Exercise 8

1. Find the equation of the circle having (1, 2), (2, −3) as extremities of a diameter.

2. Find the equation of the circle which passes through the points (4, 0), (−4, 0) and (0, 8).

3. Find the equation of the circle concentric with $x^2 + y^2 - 4x - 6y - 5 = 0$ and touching X-axis.

4. Find the focus and the length of the latus-rectum of the parabola $y^2 = x$.

5. Find the standard equation of the ellipse whose foci are on X-axis and 8 units apart from each other and eccentricity is $\frac{1}{3}$.

6. Obtain the standard equation of hyperbola having directrix parallel to X-axis.

7. Using definition, find the equation of parabola having focus at (−4, 0) and directrix $x = 2$.

8. A cross-section of a parabolic reflector is shown. The diameter of opening at the focus is 10 *cm*. Find the equation of the porabola. Find diameter of the opening $\overline{PQ}$ at 11 *cm* from the vertex. (See figure 8.29)

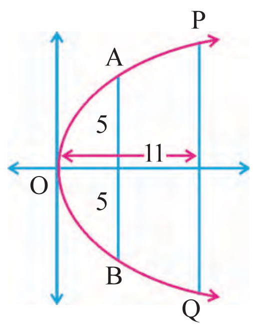

Figure 8.29

9. A parabolic reflector is 24 *cm* in diameter and 6 *cm* deep. Find coordinates of the focus.

10. An arch is in the form a semi-ellipse. It is 10 *m* wide and 4 *m* high at the centre. Find the height of the arc at a point 2 *m* from one end.

11. A toy train moves such that sum of its distances from two signals is always constant and equal to 10 *m* and the distance between the signals is 8 *m*. Find the path traced by the train.

12. Select the proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The equation of the circle whose extremities of a diameter are centres of the circles, $x^2 + y^2 + 6x - 14y = 1$ and $x^2 + y^2 - 4x + 10y = 2$ is ... ☐

(a) $x^2 + y^2 + x - 2y - 41 = 0$ (b) $x^2 + y^2 + x + 2y - 41 = 0$

(c) $x^2 + y^2 + x + 2y + 41 = 0$ (d) $x^2 + y^2 - x - 2y - 41 = 0$

(2) If one end of a diameter of the circle $x^2 + y^2 - 8x - 4y + 5 = 0$ has coordinates (−3, 2), then the coordinates of the other end are ... ☐

(a) (5, 3) (b) (6, 2) (c) (1, −8) (d) (11, 2)

(3) If a circle has centre on X-axis, radius 5 and it passes through the point (2, 3), then the equation of the circle is ... ☐

(a) $x^2 + y^2 - 12x + 11 = 0$ (b) $x^2 + y^2 - 12y + 11 = 0$

(c) $x^2 + y^2 - 12x - 11 = 0$ (d) $x^2 + y^2 - 4x + 12y = 0$

(4) The equation of circle, with centre at (4, 5) and passing through the centre of the circle $x^2 + y^2 + 4x - 6y = 12$ is ... ☐

(a) $x^2 + y^2 + 8x - 10y + 1 = 0$ (b) $x^2 + y^2 - 8x - 10y + 1 = 0$

(c) $x^2 + y^2 - 8x + 10y - 1 = 0$ (d) $x^2 + y^2 - 8x - 10y - 1 = 0$

(5) Area of the circle centred at (1, 2) and passing through the point (4, 6) is ...

(a) 30π sq units (b) 5π sq units (c) 15π sq units (d) 25π sq units

(6) Coordinates of the centre of the circle passing through the points (0, 0), (a, 0), (0, b) are ...

(a) $\left(\frac{b}{2}, \frac{a}{2}\right)$ (b) $\left(\frac{a}{2}, \frac{b}{2}\right)$ (c) (b, a) (d) (a, b)

(7) The parametric equations of the parabola $x^2 = 4ay$ are ....

(a) $x = at^2, y = at^2$ (b) $x = 2at, y = 2at$ (c) $x = 2at, y = at^2$ (d) $x = 2at^2, y = at$

(8) The line $2x - 3y + 8 = 0$ intersects the parabola $y^2 = 8x$ in P and Q. The mid-point of $\overline{PQ}$ is ...

(a) (2, 4) (b) (8, 8) (c) (5, 6) (d) (6, 5)

(9) The eccentricity of the ellipse whose latus-rectum is half of the minor axis is ...

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\sqrt{2}$

(10) The eccentricity of the ellipse whose minor axis is equal to the distance between foci is ...

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{2}}{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{2}{\sqrt{3}}$

(11) The eccentricity of the ellipse $9x^2 + 25y^2 = 225$ is ...

(a) $\frac{2}{5}$ (b) $\frac{4}{5}$ (c) $\frac{3}{5}$ (d) 0

(12) Length of the latus-rectum of the ellipse $4x^2 + 9y^2 = 1$ is ...

(a) $\frac{4}{9}$ (b) $\frac{9}{4}$ (c) $\frac{2}{9}$ (d) $\frac{2}{3}$

(13) ...... is a focus of the ellipse $9x^2 + 4y^2 = 36$.

(a) $(\sqrt{5}, 0)$ (b) $(0, \sqrt{5})$ (c) $(3\sqrt{5}, 0)$ (d) $(0, 3\sqrt{5})$

(14) Length of the major axis of the ellipse $25x^2 + 9y^2 = 1$ is ...

(a) $\frac{2}{5}$ (b) $\frac{2}{3}$ (c) $\frac{1}{5}$ (d) $\frac{1}{9}$

(15) The foci of the hyperbola $9x^2 - 16y^2 = 144$ are ...

(a) $(\pm 4, 0)$ (b) $(0, \pm 4)$ (c) $(\pm 5, 0)$ (d) $(0, \pm 5)$

(16) The length of the latus-rectum of the hyperbola $16x^2 - 9y^2 = 144$ is ...

(a) $\frac{32}{3}$ (b) $\frac{16}{3}$ (c) $\frac{8}{3}$ (d) $\frac{4}{3}$

(17) The eccentricity of the hyperbola $16y^2 - 9x^2 = 144$ is ...

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $\frac{5}{4}$ (d) $\frac{4}{5}$

(18) The eccentricity of the hyperbola $x^2 - 4y^2 = 1$ is ... $\square$

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{5}}{2}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{2}{\sqrt{5}}$

(19) If the parabola $y^2 = 4ax$ passes through the point $(2, -6)$, then the length of the latus-rectum is ... $\square$

(a) 9 (b) 16 (c) 18 (d) 8

(20) The length of the latus-rectum of the ellipse $5x^2 + 9y^2 = 45$ is ... $\square$

(a) $\frac{5\sqrt{5}}{3}$ (b) $\frac{5}{3}$ (c) $\frac{2\sqrt{5}}{3}$ (d) $\frac{10}{3}$

*

Summary

We have studied following points in this chapter :

1. Standard equation of a circle : $x^2 + y^2 = r^2$

General equation of a circle : $(x - h)^2 + (y - k)^2 = r^2$

2. Centre of the circle : $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$ if $g^2 + f^2 - c > 0$ and does not represent a circle if $g^2 + f^2 - c \le 0$.

3. The equation of a parabola $y^2 = 4ax$, Parametric equations $x = at^2$, $y = 2at$, $t \in R$, Latus-rectum $4|a|$.

4. A property of a parabola : for a focal chord $t_1t_2 = -1$

5. Standard equation of the ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **$(a > b)$**

Foci $(\pm ae, 0)$, the equations of the directrices $x \pm \frac{a}{e} = 0$

Parametric equations $x = a\cos\theta$, $y = b\sin\theta$, $0 \in [0, 2\pi)$, length of the latus-rectum $\frac{2b^2}{a}$, major axis $2a$, minor axis $2b$.

6. A property of an ellipse : SP + S'P = $2a$

7. Standard equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Foci $(\pm ae, 0)$, Equations of directrices $x \mp \frac{a}{e} = 0$

Parametric equations $x = a\sec\theta$, $y = b\tan\theta$, $\theta \in R - \left\{(2k - 1)\frac{\pi}{2} \,\middle|\, k \in Z\right\}$, length of latus-rectum $\frac{2b^2}{a}$.

8. A property of hyperbola : $|SP - S'P| = 2a$

APPENDIX

Intersection of a Double Cone and a Plane

Let l be a fixed vertical line and m be another line intersecting it at a fixed point V and let the measure of the angle made by m with l be α $(0 < \alpha < \frac{\pi}{2})$, as shown in figure A.1. Suppose the line m is rotated around the line l in such a way that the angle α remains constant. Then the surface generated is called a **right circular cone**. The point of intersection V separates the cone in two parts. Hence it is called a double napped cone or a double cone. For simplicity we will refer this as a cone. Since the lines l and m are of infinite extent, the cone is extending indefinitely in both directions (figure A.2). The point V is called the **vertex**. The line l is the **axis** of the cone and the rotating line m is called a **generator** of the cone, and two parts of the cone are called **napes**. We note that looking at a given cone we cannot observe the line m actually. Any of the line on the surface of the cone can be taken as the generator.

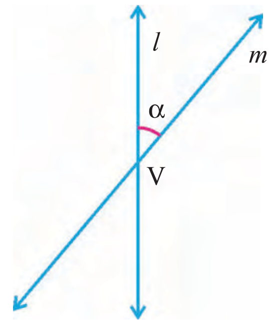

Figure A.1

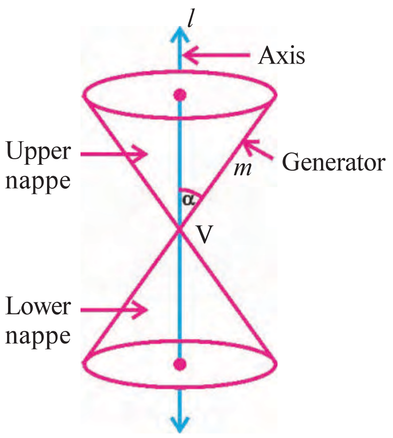

Figure A.2

Now we consider the intersection of a plane with a cone, the section so obtained is called a **conic section**. Thus, conic sections are the curves obtained by intersecting a **right circular cone by a plane and hence the name conics.**

There are many possibilities when we consider intersection of a cone with a plane depending on the position of the intersecting plane with respect to the cone and by the angle made by it with the vertical axis of the cone. Let β $(0 < \beta < \frac{\pi}{2})$ be the angle made by the plane with the vertical axis of the cone (figure A.3). There are two possibilities : (1) the plane passes through the vertex; or, (2) the plane does not pass through the vertex. Accordingly the intersection takes place at vertex or at any other part of the napes above or below the vertex.

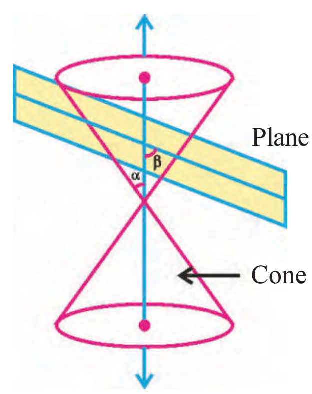

Figure A.3

Various situations of intersection are discussed below; in each case above two possibilities are discussed separately.

Let the angle made by the plane with the axis of the cone be right angle, i.e. $\beta = \frac{\pi}{2}$. If the plane passes through the vertex, then the intersection is the vertex itself (figure A.4 (a)); and if the

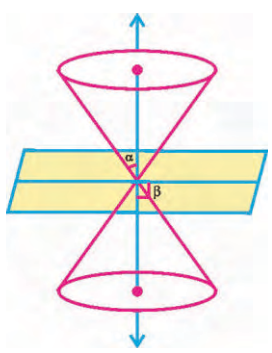

Figure A.4(a)

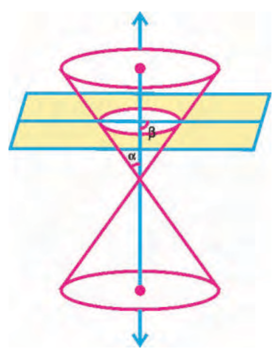

Figure A.4(b)

plane does not pass through the vertex, then the intersection is a circle, either in the upper nape of the cone or the lower nape of the cone depending on the position of the plane as shown in the figure A.4(b). In the first case we got the intersection as a point. Thus it is a degenerate case of the circle.

Suppose $\alpha < \beta < \frac{\pi}{2}$ again. If the plane is passing through the vertex, then the intersection is the vertex itself. If it is not the case, then the intersection is an ellipse (figure A.5). Here also, the first case is degenerate ellipse – a point. (Try to visualize this!).

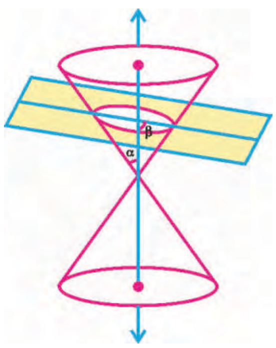

Figure A.5(a)

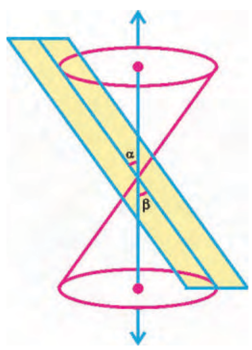

Figure A.5(b)

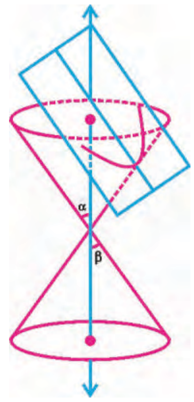

Figure A.5(c)

Now, consider the case, when $\alpha = \beta$. In this case the intersecting plane is parallel to a generator. If the plane passes through the vertex, then the intersection is a straight line. It can be seen that the line of intersection is a generator of the cone. If the vertex is not on the plane, then the intersection is a parabola as shown in figure A.5(c). The intersection being a straight line is actually degenerate parabola, i.e. as if the parabola is opened up straight to get the line.

Finally, consider the case $\beta < \alpha$. In this case the plane intersects both the napes. This did not happen in earlier cases. The intersection is a hyperbola and it has two branches as shown in the Figure A.6. Here the degeneracy occurs in a particular case. In this case the plane passes through the vertex and the intersection is a pair of lines.

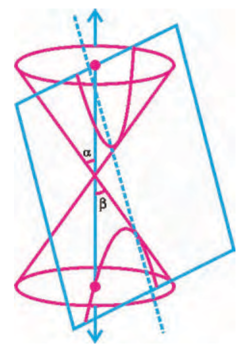

Figure A.6

In this section we have seen that, circle, ellipse, parabola and hyperbola are various conics, with point, line or a pair of lines as degenerate cases. This discussion about conics is useful for the practical consideration.

—❖—

Some of Bhaskara's contributions to mathematics include the following :

- A proof of the Pythagorean theorem by calculating the same area in two different ways and then cancelling out terms to get $a^2 + b^2 = c^2$.
- In Lilavati, solutions of quadratic, cubic and quartic indeterminate equations are explained.
- Solutions of indeterminate quadratic equations (of the type $ax^2 + b = y^2$).
- A cyclic Chakravala method for solving indeterminate equations of the form $ax^2 + bx + c = y$. The solution to this equation was traditionally attributed to William Brouncker in 1657, though his method was more difficult than the Chakravala method.
- The first general method for finding the solutions of the problem $x^2 - ny^2 = 1$ (so-called "Pell's equation") was given by Bhaskara II.

Chapter 9

THREE DIMENSIONAL GEOMETRY

As far as the laws of mathematics refer to reality they are not certain and as far as they are certain they do not refer to reality.

– Albert Einstein

9.1 Introduction

Earlier the concepts of plane coordinate geometry were initiated by French mathematician **René Descartes** and simultaneously also by **Fermat** in the beginning of 17th century. It was later systematized by **Bernoulli** and **Euler** in the 18th century. In the 19th century, it was further extended to highter dimensions and found interesting applications in the last century only.

In this chapter, we will discuss some basic concepts of quantities called vectors useful in mathematics and sciences. Also the study of coordinate geometry in plane will be extended to three dimensions, i.e. we will discuss coordinate geometry in the space. This is useful in studying solid objects and things in the space around us. We will use vectors as a tool to discuss three dimensional geometry.

9.2 Vectors

Some physical quantities require magnitude and direction both to completely specify position and application. Such quantities are called **vectors**. Velocity is a vector, as its complete description requires both magnitude as well as direction. Otherwise the meaning is incomplete. We already know about the representation of complex numbers in the Argand plane. In a polar representation of a complex number $z = r(cos\theta + i\, sin\theta)$, there are two important parameters r and θ. Here r is its magnitude and by θ, we can decide its direction. Thus, every complex number is a vector as it has both magnitude and direction. Suppose Dev walks 300 *m* towards East and then he walks 400 *m* towards North. Hence to know his final position from original position, we should know direction and magnitude both. This is also a primary illustration of a vector.

In mathematics also we can think of quantities that have both magnitude and direction. For instance, we are familiar with the set R^2 of ordered points of real numbers. Also it is known that there is a one-one correspondence between R^2 and the points in a plane. Taking O(0, 0) as the origin, we can associate magnitude and direction with any element other than O, say (1, −2) of R^2. Suppose the point P represents (1, −2) in the plane. Then with (1, −2), we can associate the magnitude of $\overline{OP}$ (that is length $OP = \sqrt{(1)^2 + (-2)^2}$ and the direction of $\overrightarrow{OP}$. Thus (1, −2) can be regarded as a vector. Similarly, it is possible to regard elements of set of ordered real triplets of R^3.

Having considered elements of R^2 or R^3 as vectors, we can think of the collection R^2 or R^3 of vectors as vector spaces.

9.3 Vectors in R^2 and R^3

Taking R^2 and R^3 as sets of ordered pairs and triplets of real numbers respectively, an element in R^2 or R^3 is denoted by a letter with an overhead bar, say $\overline{x}$. Thus, $\overline{x} = (x_1, x_2, x_3)$ in R^3 and $\overline{x} = (x_1, x_2)$ in R^2.

We first define the notion of equality in R^2 and R^3 as follows :

In R^2, $(x_1, x_2) = (y_1, y_2)$ if $x_1 = y_1$ and $x_2 = y_2$.

In R^3, $(x_1, x_2, x_3) = (y_1, y_2, y_3)$ if $x_1 = y_1$, $x_2 = y_2$ and $x_3 = y_3$.

Thus (1, 2) and (2, 1) are distinct elements in R^2.

In the further discussion, we shall study R^3 in detail. All these results would be essentially true for R^2 also.

Definition : Let $\overline{x} = (x_1, x_2, x_3)$ and $\overline{y} = (y_1, y_2, y_3)$ be two elements of R^3. Their addition is defined by $\overline{x} + \overline{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$. Thus if $\overline{z} = (z_1, z_2, z_3) = \overline{x} + \overline{y}$, then $z_1 = x_1 + y_1, z_2 = x_2 + y_2, z_3 = x_3 + y_3$.

Clearly, for $\overline{x} \in R^3$, $\overline{y} \in R^3$ we have $\overline{x} + \overline{y} \in R^3$ i.e. the addition defined above has closure property. $\overline{x} + \overline{y}$ is called the sum of $\overline{x}$ and $\overline{y}$.

Definition : Let $\overline{x} = (x_1, x_2, x_3)$ and $k \in R$. We define multiplication of $\overline{x}$ by k as $k\overline{x} = (kx_1, kx_2, kx_3)$.

Obviously, for $k \in R$ and $\overline{x} \in R^3$, $k\overline{x} \in R^3$.

Some obvious results :

For any $\overline{x}, \overline{y}, \overline{z} \in R^3$ and $k, l \in R$

(i) $\overline{x} + \overline{y} = \overline{y} + \overline{x}$ (Commutative law)

(ii) $\overline{x} + (\overline{y} + \overline{z}) = (\overline{x} + \overline{y}) + \overline{z}$ (Associative law)

(iii) If $\overline{0} = (0, 0, 0)$, then $\overline{x} + \overline{0} = \overline{x}$ (Existence of identity)

Identity element is unique.

(iv) For each $\overline{x} \in R^3$, $\exists\, \overline{y} \in R^3$ such that $\overline{x} + \overline{y} = \overline{0}$ (Existence of inverse)

(It can be proved that if $\overline{x} = (x_1, x_2, x_3)$, then $\overline{y} = (-x_1, -x_2, -x_3)$ so that $\overline{x} + \overline{y} = \overline{0}$. $\overline{y}$ is called an additive inverse of $\overline{x}$ and for every $\overline{x}$ there correspond a unique $\overline{y}$. Additive inverse of $\overline{x}$ is denoted by $-\overline{x}$.

$\therefore \quad -\overline{x} = (-x_1, -x_2, -x_3)$

(v) $k(\overline{x} + \overline{y}) = k\overline{x} + k\overline{y}$

(vi) $(k + l)\overline{x} = k\overline{x} + l\overline{x}$

(vii) $(kl)\overline{x} = k(l\overline{x})$

(viii) $1\overline{x} = \overline{x}$

The set R^3 with all above properties is called a vector space over R. There are other sets also which are vector spaces. **Mathematically, elements of a vector space are called vectors. Thus any element of R^3 is called a vector. R^2 is also a vector space over R.**

The sum defined above in R^3 (or R^2) is called a vector sum. When R^3 (or R^2) is considered as a vector space over R, the elements of R are called scalars. Thus a real number is a scalar in this context. Accordingly for $k \in R$, $\bar{x} \in R^3$, $k\bar{x}$ is called the multiplication of vector $\bar{x}$ by a scalar k. The product $k\bar{x}$ is a vector. $\bar{0} = (0, 0, 0)$ is called the zero vector.

9.4 Magnitude of a Vector

If $\bar{x} = (x_1, x_2, x_3)$, then the **magnitude of $\bar{x}$**, is defined as $\sqrt{x_1^2 + x_2^2 + x_3^2}$ and it is denoted by $|\bar{x}|$. Thus, $|\bar{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

In a similar manner for a vector $\bar{x}$ in R^2, magnitude is defined. If $\bar{x} = (x_1, x_2)$, then $|\bar{x}| = \sqrt{x_1^2 + x_2^2}$.

The following are obvious results :

(1) $|\bar{x}| \geq 0$ because $|\bar{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} \geq 0$

(2) $|\bar{x}| = 0 \Leftrightarrow \bar{x} = \bar{0}$

(3) $|k\bar{x}| = |(kx_1, kx_2, kx_3)|$

$$= \sqrt{k^2x_1^2 + k^2x_2^2 + k^2x_3^2}$$

$$= \sqrt{k^2\left(x_1^2 + x_2^2 + x_3^2\right)}$$

$$= \sqrt{k^2}\sqrt{x_1^2 + x_2^2 + x_3^2}$$

$= |k||\bar{x}|$; Here $\sqrt{k^2} = |k|$ is the magnitude of the real number k and

$|\bar{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the magnitude of vector $\bar{x}$.

$\therefore$ $|k\bar{x}| = |k||\bar{x}|$

Definition : A vector $\bar{x}$ is said to be unit vector, if $|\bar{x}| = 1$.

Some examples of unit vectors in R^2 are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, (1, 0), (0, −1), $(\sin\alpha, \cos\alpha)$, $\alpha \in R$. In R^3, some such examples are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, (1, 0, 0), $\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. For $\theta, \alpha \in R$, $(\cos\theta \sin\alpha, \cos\theta \cos\alpha, \sin\theta)$ is also a unit vector.

Example 1 : If $\bar{u} = (3, -1, 4)$, $\bar{v} = (1, -2, -3)$, find $3\bar{u} + \bar{v}$.

Solution : $3\bar{u} + \bar{v} = 3(3, -1, 4) + (1, -2, -3)$

$= (9, -3, 12) + (1, -2, -3)$

$= (9 + 1, -3 - 2, 12 - 3)$

$= (10, -5, 9)$

Example 2 : Find $\bar{x} - 2\bar{y}$, where $\bar{x} = (1, -1, 3)$, $\bar{y} = (1, 1, 1)$.

Solution : $\bar{x} - 2\bar{y} = \bar{x} + (-2)\bar{y}$

$= (1, -1, 3) + (-2)(1, 1, 1)$

$= (1, -1, 3) + (-2, -2, -2)$

$= (1 - 2, -1 - 2, 3 - 2)$

$= (-1, -3, 1)$

Example 3 : For vectors $\bar{x}, \bar{y}, \bar{z}$ in R^3, show that, $\bar{x} + \bar{y} = \bar{x} + \bar{z} \Rightarrow \bar{y} = \bar{z}$.

Solution : Let $\bar{x} = (x_1, x_2, x_3)$, $\bar{y} = (y_1, y_2, y_3)$ and $\bar{z} = (z_1, z_2, z_3)$.

$\bar{x} + \bar{y} = \bar{x} + \bar{z}$

$\therefore$ $(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1, x_2, x_3) + (z_1, z_2, z_3)$

$\therefore$ $(x_1 + y_1, x_2 + y_2, x_3 + y_3) = (x_1 + z_1, x_2 + z_2, x_3 + z_3)$

$\therefore$ $x_1 + y_1 = x_1 + z_1, x_2 + y_2 = x_2 + z_2, x_3 + y_3 = x_3 + z_3$

$\therefore$ $y_1 = z_1, y_2 = z_2, y_3 = z_3$

$\therefore$ $(y_1, y_2, y_3) = (z_1, z_2, z_3)$

$\therefore$ $\bar{y} = \bar{z}$

Another method :

$\bar{x} + \bar{y} = \bar{x} + \bar{z}$

$\therefore$ $(-\bar{x}) + (\bar{x} + \bar{y}) = (-\bar{x}) + \bar{x} + \bar{z}$ **($-\bar{x}$ exists uniquely)**

$\therefore$ $(-\bar{x} + \bar{x}) + \bar{y} = (-\bar{x} + \bar{x}) + \bar{z}$

$\therefore$ $\bar{0} + \bar{y} = \bar{0} + \bar{z}$

$\therefore$ $\bar{y} = \bar{z}$

Example 4 : Solve : $x(3, 1) + y(4, 2) = (1, 0)$

Solution : $x(3, 1) + y(4, 2) = (1, 0)$

$\therefore$ $(3x, x) + (4y, 2y) = (1, 0)$

$\therefore$ $(3x + 4y, x + 2y) = (1, 0)$

$\therefore$ $3x + 4y = 1, x + 2y = 0$

$\therefore$ $x = 1, \quad y = -\frac{1}{2}$

Exercise 9.1

1. Find :

(1) $x_1(1, 0) + x_2(0, 1)$; $(x_1, x_2 \in R)$

(2) $x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$; $(x, y, z \in R)$

(3) $2(1, 2, 1) + 3(1, -2, 0)$

(4) $2(1, -1, -1) - 2(-1, 1, 1)$

(5) $-2(1, 2, 3) + (1, 0, -1)$

(6) $3(1, -1, 0) - (2, 2, 2)$

2. Solve the following equations to find x and y :

(1) $x(3, 2) + y(1, -1) = (2, 3)$

(2) $x(1, 1) + y(1, -1) = (0, 0)$

(3) $y(1, 2) = x(3, 1) + (1, 3)$

(4) $x(1, 0) + y(0, 1) = \bar{0}$

3. Find magnitude of the following vectors :

(1) $(1, 1, 1)$

(2) $(1, -1, -1)$

(3) $(3, -4, 0)$

(4) $(-1, -2, -3)$

(5) $(2, 3, -5)$

(6) $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

4. Verify $|\overline{x} + \overline{y}| \le |\overline{x}| + |\overline{y}|$ for the following vector $\overline{x}$ and $\overline{y}$.

(1) $\overline{x} = (1, -1, 2)$, $\overline{y} = (1, 2, 4)$ (2) $\overline{x} = \left(\frac{-3}{2}, 9, -9\right)$, $\overline{y} = (-1, 6, -6)$

5. If $\overline{u} = (2, 3)$ and $\overline{v} = (2k, k + 2)$ are equal then, find k.

6. If $\overline{u} = \left(\frac{-1}{2}, \frac{3}{5}, 0\right)$ and $\overline{v} = \left(\frac{1}{6}, \frac{-2}{3}, 0\right)$, find $3\overline{u} - 2\overline{v}$.

*

9.5 Direction of a Vector

As discussed earlier vectors in physics are specified with magnitude and direction both. Now we shall associate a **direction** with every non-zero vector. We will restrict our discussion about direction to define equality of directions of two non-zero vectors, two non-zero vectors with opposite directions and two non-zero vectors with different directions. This discussion will help in giving geometric meaning to the vectors in R^2 and R^3.

Suppose $\overline{x}$ and $\overline{y}$ are two non-zero vectors in R^2 or R^3. $\overline{x}$ and $\overline{y}$ are said to have the same direction, if $\overline{y} = k\overline{x}$ for some real number $k > 0$. If $k < 0$ and $\overline{y} = k\overline{x}$, then $\overline{x}$ and $\overline{y}$ are said to have opposite directions. Further, if $\overline{x}$ and $\overline{y}$ have neither same nor opposite directions, then they have different directions. If directions of $\overline{x}$ and $\overline{y}$ are equal, then they are called equi-directed vectors. If $\overline{x}$ and $\overline{y}$ have opposite directions then they are called vectors of opposite directions.

Thus, $(1, -1, 1)$ and $(2, -2, 2)$ have same direction, because

$(2, -2, 2) = 2(1, -1, 1)$ and $2 > 0$

Also $(-1, 1, -1) = (-1)(1, -1, 1)$. So $(1, -1, 1)$ and $(-1, 1, -1)$ have opposite directions.

The vectors $(1, -1, 1)$ and $(2, 0, 2)$ have different directions, because there is no $k \in R$ such that $(1, -1, 1) = k(2, 0, 2)$.

The direction determined by a non-zero vector (x_1, x_2, x_3) is denoted by $<x_1, x_2, x_3>$. The direction opposite to $<x_1, x_2, x_3>$ is denoted by $-<x_1, x_2, x_3>$.

If $k > 0$ then $<kx_1, kx_2, kx_3> = <x_1, x_2, x_3>$ and if $k < 0$ then $<kx_1, kx_2, kx_3> = <-x_1, -x_2, -x_3>$. We note that, we can not write $(kx_1, kx_2, kx_3) = (x_1, x_2, x_3)$ unless $k = 1$.

9.6 Magnitude and Direction of a Vector and Unit Vector

Theorem 1 : Non-zero vectors $\overline{x}$ and $\overline{y}$ are equal if and only if $|\overline{x}| = |\overline{y}|$ and $\overline{x}$ and $\overline{y}$ have the same direction.

Proof : Suppose $\overline{x} = \overline{y}$

$\therefore$ $(x_1, x_2, x_3) = (y_1, y_2, y_3)$

$\therefore$ $x_1 = y_1, x_2 = y_2, x_3 = y_3$

$\therefore$ $|\overline{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{y_1^2 + y_2^2 + y_3^2} = |\overline{y}|$

Also since $\overline{x} = \overline{y}$, $\overline{x} = k\overline{y}$ with $k = 1 > 0$

$\therefore$ $\overline{x}$ and $\overline{y}$ have the same direction,

i.e. $<x_1, x_2, x_3> = <y_1, y_2, y_3>$

Thus, $\overline{x} = \overline{y} \Rightarrow |\overline{x}| = |\overline{y}|$ and $\overline{x}$ and $\overline{y}$ have the same direction.

Conversely, suppose $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$, $|\bar{x}| = |\bar{y}|$ and $\bar{x}$ and $\bar{y}$ have the same direction.

As $\bar{x}$ and $\bar{y}$ have the same direction, so $\bar{y} = k\bar{x}$ for some $k > 0$.

Now, $|\bar{y}| = |k\bar{x}| = |k|\,|\bar{x}|$

But we are given that $|\bar{x}| = |\bar{y}|$ So, $|\bar{x}| = |k|\,|\bar{x}|$

As $\bar{x} \neq \bar{0}$, $|k| = 1$

$\therefore$ $k = \pm 1$ But $k > 0$

$\therefore$ $k = 1$

$\therefore$ $\bar{y} = k\bar{x} = 1\bar{x} = \bar{x}$

$\therefore$ $|\bar{x}| = |\bar{y}|$ and $\bar{x}$, $\bar{y}$ have the same direction $\Rightarrow \bar{x} = \bar{y}$

This theorem is in confirmity with the definition of a vector generally given in physics.

Theorem 2 : If $\bar{x} \neq \bar{0}$, then there is a unique unit vector in the direction of $\bar{x}$.

Proof : As $\bar{x} \neq \bar{0}$, so $|\bar{x}| \neq 0$.

Let $\bar{y} = \frac{\bar{x}}{|\bar{x}|} = k\bar{x}$; where $k = \frac{1}{|\bar{x}|} > 0$

$\therefore$ $|\bar{y}| = |k\bar{x}| = |k|\,|\bar{x}| = \left|\frac{1}{|\bar{x}|}\right| |\bar{x}| = \frac{1}{|\bar{x}|}\,|\bar{x}| = 1$ $\quad (\left|\,|\bar{x}|\,\right| = |\bar{x}|)$

$\therefore$ $\bar{y}$ is a unit vector and as $\bar{y} = k\bar{x}$ with $k > 0$. $\bar{y}$ is in the same direction as $\bar{x}$ has.

To prove uniqueness of unit vector $\bar{y}$, suppose $\bar{z}$ is also a unit vector in the same direction as $\bar{x}$ has. Then, $|\bar{y}| = |\bar{z}| = 1$ and $\bar{y}$ and $\bar{z}$ are in the same direction (the direction of $\bar{x}$).

$\therefore$ By theorem 1, $\bar{y} = \bar{z}$

Thus, there is a unique unit vector in the direction of every non-zero vector.

To find the unit vector in the direction of $\bar{x} = (2, 1, 2)$, we note that

$$|\bar{x}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4 + 1 + 4} = 3$$

So, $\bar{y} = \frac{\bar{x}}{|\bar{x}|} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$, is the required vector.

9.7 Three Dimensional Coordinate Geometry

Our study of geometry so far was confined to a plane. Many times we need to study objects which are not in a plane. In fact in actual life, the concept of plane is inadequate. For example, consider the position of a ball thrown in space at different points of time or when a kite is flying in the sky. Its position from time to time changes in the space. Recall that to locate the position of a point in a plane; we need two intersecting mutually perpendicular lines in the plane. These lines are called the **coordinate axes labelled as X-axis and Y-axis**; and the absolute values of coordinates of the point are distances measured perpendicular to the axes. These are called the coordinates of the point with respect to the axes. Thus using these lines, we can associate a unique ordered pair of two real numbers to every point in the plane. Also for each given ordered pair of real numbers, a unique point in the plane can be found of which the given pair are the coordinates. Thus there is a one-to-one correspondence between points in a plane and the set R^2.

If we were to locate the position of a point in the spaces, then two real numbers are not sufficient. For example, to locate the central tip of a ceiling fan in a room, we will require the perpendicular distances of the point to be located from two perpendicular walls of the room and the height of the point from the floor of the room. Therefore, we need three numbers representing the perpendicular distances of the point from three mutually perpendicular planes, namely the floor of the room and two adjacent walls of the room. In general, a point in the space can be located by describing its **perpendicular distances** from three mutually perpendicular planes. Its position can be determined using these distances. These mutually perpendicular planes are called **coordinate planes**. In analogy with coordinates of a point in XY-plane, here also a coordinate of a point in space can be positive or negative. So, a point in space has three coordinates. Also, for a given triplet of real numbers, we can find a point in the space for which the given triplet represents coordinates. Here we note that there is one-one correspondence between R^3 and points in the space. In this Chapter, we shall study the basic concepts of geometry in **three dimensional space.**

9.8 Coordinate Axes and Coordinate Planes in Three Dimensional Space

In the case of plane, two mutually perpendicular lines are taken as reference lines. While assigning coordinates to a point in the space three mutually perpendicular planes are taken as reference. Consider three planes intersecting at a point O such that these three planes are mutually perpendicular (figure 9.1). Among these three planes any two planes intersect along the lines X'OX, Y'OY and Z'OZ, called the X-axis, Y-axis and Z-axis, respectively. We may note that these lines are mutually perpendicular to each other. Since these lines are mutually perpendicular, they constitute the **rectangular coordinate system**. We will refer to these three mutually perpendicular lines drawn passing through the point O as **coordinate axes** or simply **axes** (figure 9.2).

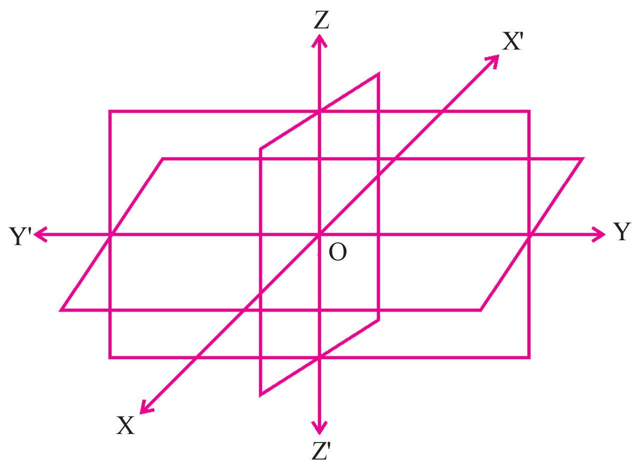

Figure 9.1

The point O is called the origin of the coordinate system. The planes XOY, YOZ and ZOX, called, respectively the **XY-plane, YZ-plane** and the **ZX-plane**, are known as the three **coordinate planes**. We will take the XOY plane as the plane of the paper and the line passing through O perpendicular to the plane as the line ZOZ'. If the plane of the paper is considered as horizontal, then the line Z'OZ will be vertical. In the case of plane we have seen that the coordinate axes divide the plane into four parts called quadrants, in the same manner the three coordinate planes divide the space into eight parts known as octants. These octants could be named as XOYZ, X'OYZ, X'OY'Z, XOY'Z, XOYZ', X'OYZ', X'OY'Z' and XOY'Z' and denoted by octant I, II, III, ..., VIII, respectively.

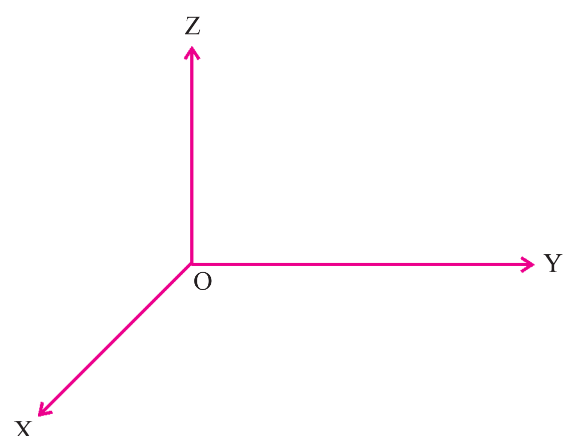

Figure 9.2

Note : The coordinate system discussed here is one of the methods for assigning coordinates to a point in the space. This is called Cartesian coordinate system, named after French mathematician **René Des Cartes**. There are other popular coordinate systems also.

Coordinates of a Point in the Space

Following the method of assigning coordinates to a point in the plane with the help of coordinate axes and the origin, we will now discuss how to associate three coordinates to a given point in the space. Also we will see how a given triplet of real numbers can be associated with a point in the space.

Through the point P in the space, we draw three planes parallel to the coordinate planes, meeting the X-axis, Y-axis and Z-axis in the points A, B and C, respectively as shown in the figure 9.3. Let A(x, 0, 0), B(0, y, 0) and C(0, 0, z). Then, the point P will have the coordinates x, y and z and we write P(x, y, z). Conversely, given real numbers x, y and z, we locate the three points A(x, 0, 0), B(0, y, 0) and C(0, 0, z) on X-axis, Y-axis and Z-axis respectively. Through the points A, B and C we draw planes parallel to the YZ-plane, ZX-plane and XY-plane, respectively. The point of intersection of these three planes, namely ADPF, BDPE and CEPF is obviously the point P, which corresponds to the ordered triplet (x, y, z). We observe that if P(x, y, z) is any point in the space, then $|x|$, $|y|$ and $|z|$ are perpendicular distances from YZ, ZX and XY planes, respectively. Thus, there is a one to one correspondence between the points in the space and ordered triplets (x, y, z) of real numbers. Thus, the space is identified with the set R^3 of ordered triplets.

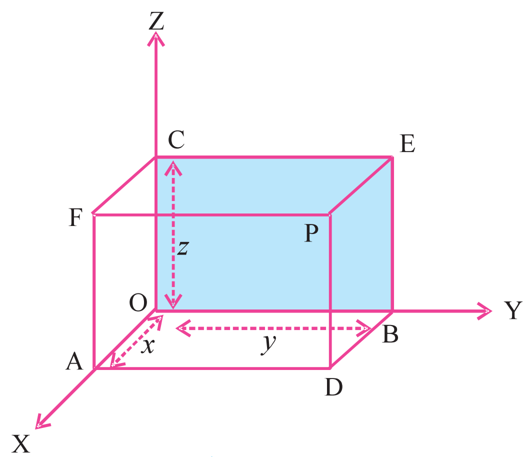

Figure 9.3

Note : The coordinates of the origin O are (0, 0, 0). The coordinates of any point on the X-axis will be (x, 0, 0) and the coordinates of any point in the YZ-plane will be as (0, y, z). Similar remarks apply to the other coordinate axes and other coordinate planes.

Remark : The combination of positive and negative coordinates of a point determines the octant in which the point lies. The following table shows this fact :

Table 9.1

Octants → Coordinates ↓	I OXYZ	II OX'YZ	III OX'Y'Z	IV OXY'Z	V OXYZ'	VI OX'YZ'	VII OX'Y'Z'	VIII OXY'Z'
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

Example 5 : Let coordinates of the vertex A of a cuboid be (1, 3, 2) as shown in the figure 9.4. $\overline{AB}$ is perpendicular to Z-axis. Find z-coordinate of the vertex B. If the side $\overline{AB}$ measures 3, then find y-coordinate of B.

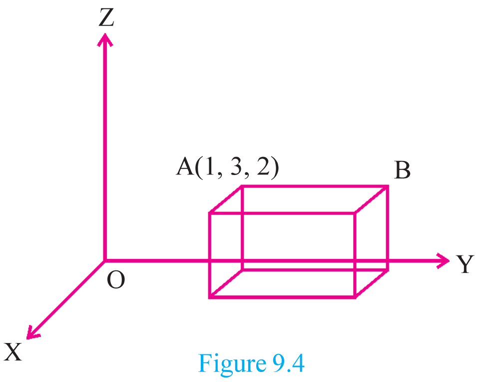

Figure 9.4

Solution : Vertices A and B are on the same heights and hence their z-coordinates are equal and hence z-coordinate of B is 2.

Now, side $\overline{AB}$ is parallel to Y-axis.

Thus y-coordinate of B = y-coordinate of A + 3 = 3 + 3 = 6

Exercise 9.2

1. Fill in the blank in the column, in the following table, by writing the name of the octant of the point in first column :

Point	Octant
(1, 2, 3)	
(1, −2, −4)	
($\sqrt{2}$, 2, −1)	
(−1, −2, 0)	
(−1, −1, −1)	

2. Ram starts walking from a point (−1, 2, 0). He walks 1 unit along $\overrightarrow{OX}$ and then moves in the $\overrightarrow{OY'}$ direction and walks further 2 units. What will be Ram's final position ?

*

9.9 Geometric Representation of Vector

Suppose P is a point in the coordinate plane other than the origin. The line segment $\overline{OP}$ with the direction from O to P, i.e. the direction of the $\overrightarrow{OP}$ will be denoted by $\overrightarrow{\mathbf{OP}}$. Thus, $\overrightarrow{\mathbf{OP}}$ is a directed line segment with the same direction as the ray OP.

We know that any point P in the coordinate plane can be identified with an ordered pair of real numbers, say (x_1, x_2) and conversely, corresponding to any ordered pair of real numbers (x_1, x_2), there exists a point in the plane. We say that the coordinates of the point are (x_1, x_2). In this manner the plane is identified with the set R^2 of ordered pairs of real numbers. Thus we will use R^2 and plane interchangeably.

Position Vector : Let P be a point other than origin in the coordinate plane having coordinates (x_1, x_2). The directed line segment $\overrightarrow{\mathbf{OP}}$ is called the position vector of the point P with respect to the origin O. The coordinates x_1 and x_2 of the point P are taken as components of the position vector $\overrightarrow{\mathbf{OP}}$. For simplicity (x_1, x_2) will be called the position vector of the point P.

The position vector of the origin has components 0 and 0. Using the definitions of addition of two vectors and multiplication by a scalar it is easy to define addition of two position vectors and multiplication of a position vector by a scalar.

Now we consider a line segment $\overline{AB}$. It is possible to associate direction with this line segment in analogy with the concept of a vector. The direction of the line segment $\overline{AB}$ is same as the direction of the ray from the point A towards the point B. Thus we define directed line segment $\overrightarrow{AB}$ whose length is AB and direction is the same as the direction of the ray $\overrightarrow{AB}$. Using this we define the position vector of point B with respect to point A as the directed line segment $\overrightarrow{AB}$. Here position vector of a point with respect to itself is zero vector.

Look at the following diagram :

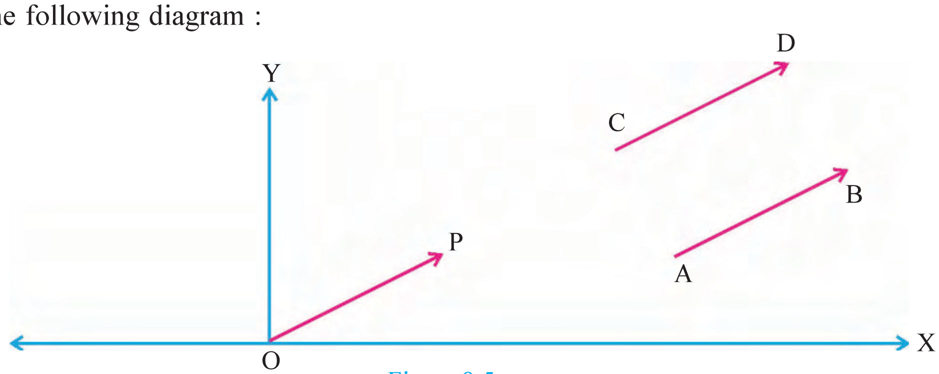

Figure 9.5

We define equality of two directed line segments in analogy with equality of two vectors. Thus, $\overrightarrow{AB} = \overrightarrow{CD}$, if AB = CD and $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same direction. For every $\overrightarrow{AB}$ there is a directed line segment $\overrightarrow{OP}$, such that $\overrightarrow{AB} = \overrightarrow{OP}$. In the figure, it can be observed that $\overrightarrow{AB} = \overrightarrow{OP}$ and also $\overrightarrow{CD} = \overrightarrow{OP}$. In fact, in the plane there are infinitely many directed line segment that are equal (as directed line segments) but distinct as line segments. For every directed line segment $\overrightarrow{AB}$ there is a position vector $\overrightarrow{OP}$ such that $\overrightarrow{AB} = \overrightarrow{OP}$. Thus, $\overrightarrow{OP}$ represents the class of all directed line segments that are equal to $\overrightarrow{AB}$. The position vectors like $\overrightarrow{OP}$ are called **bound vectors** because one of their end-points namely, O is fixed, whereas the other directed line segments equivalent to $\overrightarrow{OP}$ (like $\overrightarrow{AB}$) are called **free vectors** as both their end-points can be chosen arbitrarily, without changing the vector.

Now look at the figure 9.6.

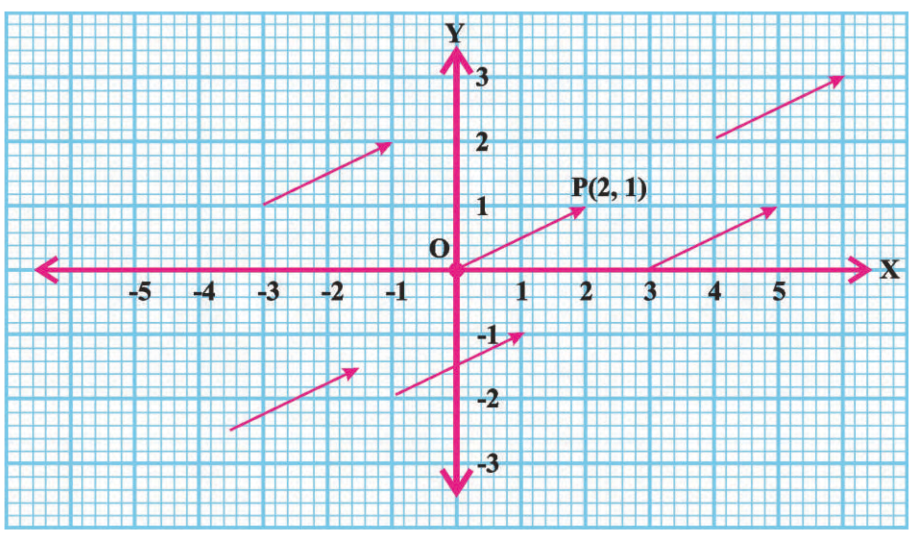

Figure 9.6

Here all the segments are directed in the same way and the end-point of each is obtained by moving horizontally 2 unit towards right and then 1 units vertically upwards (like moving a knight on the chess board) from the initial point. This means each is equal to the position vector (2, 1). In other words the vector (2, 1) represents all the vectors in the figure 9.6. Thus, for every free vector, there exists a bound vector equal to the given vector.

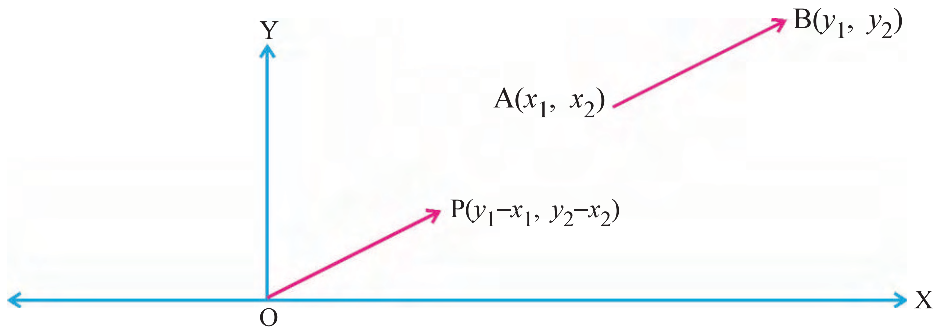

Figure 9.7

Suppose A(x_1, x_2), B(y_1, y_2) and P($y_1 - x_1$, $y_2 - x_2$) are points as shown in the figure 9.7. We have direction of $\overrightarrow{AB}$ = direction of $\overrightarrow{OP}$ and AB = OP = $\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$. Thus free vector $\overrightarrow{AB}$ is equal to the bound vector $\overrightarrow{OP}$. Also,

$\overrightarrow{AB} = \overrightarrow{OP}$ **(they have the same direction and the same magnitude)**

$= (y_1 - x_1, y_2 - x_2)$

$= (y_1, y_2) - (x_1, x_2)$

= **Position vector of B − Position vector of A**

In a similar manner, we can define position vector of point in the space. Also we define free vectors and bound vectors in the space analogously. Suppose A(x_1, x_2, x_3), B(y_1, y_2, y_3) and P($y_1 - x_1$, $y_2 - x_2$, $y_3 - x_3$) are points in the space. Then we, write the free vector $\overrightarrow{AB}$ as,

$\overrightarrow{AB} = \overrightarrow{OP} = (y_1 - x_1, y_2 - x_2, y_3 - x_3)$

$= (y_1, y_2, y_3) - (x_1, x_2, x_3)$

= **Position vector of B − Position vector of A**

Also, corresponding to this free vector $\overrightarrow{AB}$ there is a bound vector $\overrightarrow{OP}$ such that

$\overrightarrow{AB} = \overrightarrow{OP}$

This is how, we represent a vector in space geometrically.

Example 5 : In each of the following pairs of vectors, determine whether the two vectors have the same or opposite directions or different directions :

(1) (1, 1, 1), (2, 2, 2)

(2) (1, −1, 2), (0.5, −0.5, 1)

(3) (1, −1, 0), (0, 1, −1)

(4) (3, 6, −9), (−1, −2, 3)

(5) (1, 0, 0), (0, 1, 0)

(6) (2, 5, 7), (−2, 5, −7)

Solution : (1) (2, 2, 2) = 2(1, 1, 1). Here $k = 2 > 0$

∴ The vectors have the same direction.