$\therefore \quad cos\left(\frac{3\pi}{2} - \beta\right) = -sin\beta$

Now have a look at the figure 4.3,

$\pi - \beta$	β
$\pi + \beta$	$2\pi - \beta$

Figure 4.3

For any such transformations, the trigonometric functions remain same. *sin* → *sin*, *cos* → *cos* etc.

Choice of sign is according to the quadrant of the original function. The trigonometric point $P(\pi + \beta)$ is in the third quadrant and $sin(\pi + \beta)$ is $-ve$ in the third quadrant.

Hence, $sin(\pi + \beta) = -sin\beta$,

$tan(\pi + \beta) = tan\beta$ **($tan(\pi + \beta)$ is $+ve$ in third quadrant.)**

Now $P(2\pi - \beta)$ is in the fourth quadrant.

Hence, $sec(2\pi - \beta) = sec\beta$, $cosec(2\pi - \beta) = -cosec\beta$

(as *sec* takes $+ve$ and *cosec* takes $-ve$ values in the fourth quadrant.)

Now, let us find $sin\left(\frac{38\pi}{3}\right)$ and $cos\left(\frac{61\pi}{4}\right)$ using there rules.

$$sin\left(\frac{38\pi}{3}\right) = sin\left(\frac{36\pi + 2\pi}{3}\right)$$

$$= sin\left(12\pi + \frac{2\pi}{3}\right)$$

$$= sin\frac{2\pi}{3} \quad \textbf{(12}\boldsymbol{\pi}\textbf{ is a period of } \textit{sine} \textbf{ function.)}$$

$$= sin\left(\frac{3\pi - \pi}{3}\right)$$

$$= sin\left(\pi - \frac{\pi}{3}\right)$$

$$= sin\frac{\pi}{3} \quad \textbf{(}\textit{sine}\textbf{ takes } +ve \textbf{ values in the second quadrant.)}$$

$$= \frac{\sqrt{3}}{2}$$

$$cos\left(\frac{61\pi}{4}\right) = cos\left(\frac{60\pi + \pi}{4}\right)$$

$$= cos\left(15\pi + \frac{\pi}{4}\right)$$

$$= cos\left(14\pi + \pi + \frac{\pi}{4}\right)$$

$$= cos\left(\pi + \frac{\pi}{4}\right) \quad \textbf{(14}\boldsymbol{\pi}\textbf{ is a period of } \textit{cosine} \textbf{ function.)}$$

$$= -cos\frac{\pi}{4} \quad \textbf{(}\textit{cosine}\textbf{ takes } -ve \textbf{ values in the third quadrant.)}$$

$$= -\frac{1}{\sqrt{2}}$$

The Principle Period of *tan* :

We know that $sin(\pi + \theta) = -sin\theta$, $cos(\pi + \theta) = -cos\theta$. So $tan(\pi + \theta) = tan\theta$

Thus, π is a period of *tan*. Now we will prove that π is the principal period of *tan.*

Suppose the principal period of *tan* is p.

Now, $tan(\theta + p) = tan\theta,\ \forall\theta,\ \theta + p \in R - \left\{(2k+1)\frac{\pi}{2} \mid k \in Z\right\}$

In particular, taking $\theta = 0$, we get

$tanp = 0$

$\therefore\ p = k\pi$

$\therefore$ The least positive value of p is π.

Thus, π is the principal period of *tan*.

Example 1 : Evaluate : (1) $cos120^{\circ}$ (2) $sin\left(\frac{-17\pi}{4}\right)$ (3) $tan\left(\frac{13\pi}{4}\right)$ (4) $3sec\left(\frac{-7\pi}{4}\right)$

Solution : (1) $cos120^{\circ} = cos(90^{\circ} + 30^{\circ}) = -sin30^{\circ} = \frac{-1}{2}$ $\left(cos\left(\frac{\pi}{2}+\theta\right) = -sin\theta\right)$

$\therefore\ cos120^{\circ} = \frac{-1}{2}$

(2) $sin\left(\frac{-17\pi}{4}\right) = -sin\left(\frac{17\pi}{4}\right)$

$= -sin\left(\frac{16\pi+\pi}{4}\right)$

$= -sin\left(4\pi+\frac{\pi}{4}\right) = -sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$ (4π is a period of *sine*.)

$\therefore\ sin\left(\frac{-17\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

(3) $tan\left(\frac{13\pi}{4}\right) = tan\left(\frac{12\pi+\pi}{4}\right) = tan\left(3\pi+\frac{\pi}{4}\right)$

$= tan\frac{\pi}{4} = 1$ (3π is a period of *tan*.)

$\therefore\ tan\left(\frac{13\pi}{4}\right) = 1$

(4) $3sec\left(\frac{-7\pi}{4}\right) = 3sec\left(\frac{7\pi}{4}\right)$ $(sec(-\theta) = sec\theta)$

$= 3sec\left(\frac{8\pi-\pi}{4}\right)$

$= 3sec\left(2\pi-\frac{\pi}{4}\right)$

$= 3sec\left(\frac{-\pi}{4}\right)$ (2π is a period of *sec*.)

$= 3sec\frac{\pi}{4} = 3\sqrt{2}$

$\therefore\ 3sec\left(\frac{-7\pi}{4}\right) = 3\sqrt{2}$

Example 2 : Evaluate : (1) $\frac{sin\left(\theta-\frac{\pi}{2}\right)}{cos(\theta-\pi)} + \frac{tan\left(\frac{\pi}{2}+\theta\right)}{cot(3\pi+\theta)} + \frac{cosec(2\pi+\theta)}{sec\left(\frac{3\pi}{2}-\theta\right)}$

(2) $sin\frac{10\pi}{3} \cdot cos\frac{11\pi}{6} + cos\frac{2\pi}{3} \cdot sin\frac{5\pi}{6}$

(3) $cos^2\frac{\pi}{8} + cos^2\frac{3\pi}{8} + cos^2\frac{5\pi}{8} + cos^2\frac{7\pi}{8}$

Solution : (1) $\dfrac{sin\left(\theta-\frac{\pi}{2}\right)}{cos(\theta-\pi)} + \dfrac{tan\left(\frac{\pi}{2}+\theta\right)}{cot(3\pi+\theta)} + \dfrac{cosec(2\pi+\theta)}{sec\left(\frac{3\pi}{2}-\theta\right)}$

$= \dfrac{-sin\left(\frac{\pi}{2}-\theta\right)}{cos(\pi-\theta)} + \dfrac{-cot\,\theta}{cot\,\theta} + \dfrac{cosec\,\theta}{-cosec\,\theta}$

$= \dfrac{-cos\,\theta}{-cos\,\theta} + (-1) + (-1)$

$= 1 - 1 - 1 \ = -1$

(2) $sin\frac{10\pi}{3} \cdot cos\frac{11\pi}{6} + cos\frac{2\pi}{3} \cdot sin\frac{5\pi}{6}$

$= sin\left(\frac{9\pi+\pi}{3}\right) \cdot cos\left(\frac{12\pi-\pi}{6}\right) + cos\left(\frac{3\pi-\pi}{3}\right) \cdot sin\left(\frac{6\pi-\pi}{6}\right)$

$= sin\left(3\pi+\frac{\pi}{3}\right) \cdot cos\left(2\pi-\frac{\pi}{6}\right) + cos\left(\pi-\frac{\pi}{3}\right) \cdot sin\left(\pi-\frac{\pi}{6}\right)$

$= -sin\frac{\pi}{3} \cdot cos\frac{\pi}{6} + \left(-cos\frac{\pi}{3}\right) \cdot sin\frac{\pi}{6}$

$= \frac{-\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2}$

$= \frac{-3}{4} - \frac{1}{4} \ = -1$

(3) $cos^2\frac{\pi}{8} + cos^2\frac{3\pi}{8} + sin^2\left(\frac{\pi}{2} - \frac{5\pi}{8}\right) + sin^2\left(\frac{\pi}{2} - \frac{7\pi}{8}\right)$

$= cos^2\frac{\pi}{8} + cos^2\frac{3\pi}{8} + sin^2\left(\frac{-\pi}{8}\right) + sin^2\left(\frac{-3\pi}{8}\right)$

$= \left(cos^2\frac{\pi}{8} + sin^2\frac{\pi}{8}\right) + \left(cos^2\frac{3\pi}{8} + sin^2\frac{3\pi}{8}\right)$

$= 1 + 1 \ = 2$

Example 3 : Decide whether following numbers are positive or negative.

(1) $sin110^o + cos110^o$ (2) $cosec\frac{17\pi}{12} - sec\frac{17\pi}{12}$

Solution : (1) $sin110^o + cos110^o = sin(180^o - 70^o) + cos(90^o + 20^o)$

$= sin70^o - sin20^o$

Now *sine* is an increasing function in the first quadrant.

$\therefore$ 70 > 20. Hence $sin70^o > sin20^o$

$\therefore$ $sin70^o - sin20^o > 0$

$\therefore$ $sin110^o + cos110^o$ is positive.

(2) $cosec\frac{17\pi}{12} - sec\frac{17\pi}{12}$

$= cosec\left(\frac{12\pi+5\pi}{12}\right) - sec\left(\frac{18\pi-\pi}{12}\right)$

$= cosec\left(\pi+\frac{5\pi}{12}\right) - sec\left(\frac{3\pi}{2}-\frac{\pi}{12}\right)$

$= -cosec\frac{5\pi}{12} + cosec\frac{\pi}{12}$

Now, as *sine* is increasing and so *cosec* is a decreasing function in the first quadrant and

$\frac{\pi}{12} < \frac{5\pi}{12}$

$\therefore \quad cosec\frac{\pi}{12} > cosec\frac{5\pi}{12}$

$\therefore \quad \left(cosec\frac{\pi}{12} - cosec\frac{5\pi}{12}\right) > 0$

$\therefore \quad cosec\frac{17\pi}{12} - sec\frac{17\pi}{12}$ is positive.

Exercise 4.1

1. Evaluate :

(1) $cos135^{\circ}$ (2) $tan\left(\frac{-23\pi}{6}\right)$ (3) $cos\left(\frac{-50\pi}{3}\right)$

(4) $sec690^{\circ}$ (5) $cosec\ \frac{15\pi}{4}$ (6) $cot\left(\frac{-7\pi}{3}\right)$

Prove : **(2 to 11)**

2. $cos\left(\frac{\pi}{2}+\theta\right) \cdot sec(-\theta) \cdot tan(\pi - \theta) + sec(2\pi + \theta) \cdot sin(\pi + \theta) \cdot cot\left(\frac{\pi}{2}-\theta\right) = 0$

3. $\frac{sin(\pi-\theta)}{sin(\pi+\theta)} \cdot \frac{cosec(\pi+\theta)}{cosec(-\pi+\theta)} \cdot \frac{cosec(2\pi+\theta)}{sin(3\pi-\theta)} = -cosec^2\theta$

4. $\frac{sin(-\theta) \cdot tan\left(\frac{\pi}{2}-\theta\right) \cdot sin(\pi-\theta) \cdot sec\left(\frac{3\pi}{2}+\theta\right)}{sin(\pi+\theta) \cdot cos\left(\frac{3\pi}{2}-\theta\right) \cdot cosec(\pi-\theta) \cdot cot(2\pi-\theta)} = 1$

5. $sin(n + 1)A \cdot cos(n + 2)A - cos(n + 1)A \cdot sin(n + 2)A = -sinA$

6. $sin^2(40^{\circ} + \theta) + sin^2(50^{\circ} - \theta) = 1$

7. $\frac{cot\ 333^{\circ} - cos\ 567^{\circ}}{tan\ 297^{\circ} + sin\ 477^{\circ}} = 1$

8. $\frac{sec^2 129^{\circ} - cosec^2 31^{\circ}}{cosec 39^{\circ} - sec 121^{\circ}} = cosec39^{\circ} - sec59^{\circ}$

9. $cos(A + B + C) = cosA\ cosB\ cosC - sinA \cdot sinB \cdot cosC - sinA\ cosB\ sinC - cosA\ sinB\ sinC$

10. $sin\alpha \cdot sin(\beta - \gamma) + sin\beta \cdot sin(\gamma - \alpha) + sin\gamma \cdot sin(\alpha - \beta) = 0$

11. $(sin\alpha - cos\alpha) \cdot (sin\beta + cos\beta) = sin(\alpha - \beta) - cos(\alpha + \beta)$

12. For ΔABC, prove following results :

(1) $sin(B + C) = sinA$ (2) $cos(A + B) = -cosC$

(3) $sin\left(\frac{B+C}{2}\right) = cos\frac{A}{2}$ (4) $tan(A - B - C) = tan2A$

(5) $\frac{sin(B+C) \cdot cos(B+C) \cdot sin\frac{A}{2} \cdot cos\frac{A}{2}}{sin\left(\frac{B+C}{2}\right) \cdot cos\left(\frac{B+C}{2}\right) \cdot sin(\pi+A) \cdot cos(2\pi-A)} = 1$

(6) If $cosA = cosB\ cosC$, then prove that $2cotB\ cotC = 1$.

13. For a convex quadrilateral ABCD, prove that

(1) $sin(A + B) + sin(C + D) = sin(B + C) + sin(A + D)$

(2) $cot(A + B + C) + cotD = 0$

14. For cyclic quadrilateral ABCD, prove that

(1) $\cos A + \cos B + \cos C + \cos D = 0$

(2) $\sin A + \sin B = \sin C + \sin D$

15. If $\alpha - \beta = \frac{\pi}{6}$, then prove that $2\sin\alpha - \cos\beta = \sqrt{3}\sin\beta$.

16. If $\theta = \frac{19\pi}{4}$, then prove that $\cos^2\theta - \sin^2\theta - 2\tan\theta + \sec^2\theta - 4\cot^2\theta = 0$.

17. Evaluate : (1) $\sin^2\frac{\pi}{12} + \sin^2\frac{3\pi}{12} + \sin^2\frac{5\pi}{12} + \sin^2\frac{7\pi}{12} + \sin^2\frac{9\pi}{12} + \sin^2\frac{11\pi}{12}$

(2) $\sin x + \sin(\pi + x) + \sin(2\pi + x) + \ldots 2n$ terms.

(3) $\cos x + \cos(\pi - x) + \cos(2\pi - x) + \cos(3\pi - x) + \ldots (2n + 1)$ terms, if $x = \frac{\pi}{3}$

(4) $\cot\frac{\pi}{20} \cdot \cot\frac{3\pi}{20} \cdot \cot\frac{5\pi}{20} \cdot \cot\frac{7\pi}{20} \cdot \cot\frac{9\pi}{20}$

18. Determine whether each of the following is positive or negative :

(1) $\sin 155^\circ + \cos 155^\circ$ (2) $\tan\frac{6\pi}{7} + \cot\left(\frac{-6\pi}{7}\right)$

(3) $\tan 111^\circ - \cot 111^\circ$ (4) $\operatorname{cosec}\frac{7\pi}{12} + \sec\frac{7\pi}{12}$

19. If $\tan\theta = \left(\frac{-3}{4}\right)$ and $\frac{\pi}{2} < \theta < \pi$, then find the value of $\dfrac{\sin(\pi - \theta) + \tan(\pi + \theta) + \tan(4\pi - \theta)}{\sin\left(\frac{3\pi}{2} + \theta\right) + \cos\left(\frac{5\pi}{2} - \theta\right)}$.

20. Prove that $\sin(n\pi + (-1)^n\,\theta) = \sin\theta$, for all $n \in$ N.

*

4.4 Some Important Results

(1) We have already obtained values of trigonometric functions for $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$. With the help of $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$, we will obtain values of $\sin\frac{\pi}{12}$ and $\cos\frac{\pi}{12}$.

Let $\alpha = \frac{\pi}{3}, \beta = \frac{\pi}{4}$ or $\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{6}$

$\therefore \quad \alpha - \beta = \frac{\pi}{12}$

$$\therefore \quad \sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\therefore \quad \sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Similarly, we can show that $\cos\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$

Also, $sin\frac{5\pi}{12} = sin\left(\frac{\pi}{2}-\frac{\pi}{12}\right) = cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$

$$cos\frac{5\pi}{12} = cos\left(\frac{\pi}{2}-\frac{\pi}{12}\right) = sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(2) **(i)** $sin(\alpha + \beta) \cdot sin(\alpha - \beta) = sin^2\alpha - sin^2\beta = cos^2\beta - cos^2\alpha$

(ii) $cos(\alpha + \beta) \cdot cos(\alpha - \beta) = cos^2\alpha - sin^2\beta = cos^2\beta - sin^2\alpha$

(i) $sin(\alpha + \beta) \cdot sin(\alpha - \beta) = (sin\alpha\ cos\beta + cos\alpha\ sin\beta)(sin\alpha\ cos\beta - cos\alpha\ sin\beta)$

$$= sin^2\alpha \cdot cos^2\beta - cos^2\alpha \cdot sin^2\beta$$

$$= sin^2\alpha\ (1 - sin^2\beta) - (1 - sin^2\alpha) \cdot sin^2\beta$$

$$= sin^2\alpha - sin^2\alpha\ sin^2\beta - sin^2\beta + sin^2\alpha\ sin^2\beta$$

$$= sin^2\alpha - sin^2\beta$$

$\therefore$ $sin(\alpha + \beta) \cdot sin(\alpha - \beta) = sin^2\alpha - sin^2\beta$

Now, $sin(\alpha + \beta) \cdot sin(\alpha - \beta) = sin^2\alpha - sin^2\beta$

$$= (1 - cos^2\alpha) - (1 - cos^2\beta)$$

$$= cos^2\beta - cos^2\alpha$$

$\therefore$ $sin(\alpha + \beta) \cdot sin(\alpha - \beta) = cos^2\beta - cos^2\alpha$

Similarly, it can be proved that

$$cos(\alpha + \beta) \cdot cos(\alpha - \beta) = cos^2\alpha - sin^2\beta = cos^2\beta - sin^2\alpha$$

4.5 The Range of $f(\alpha) = acos\alpha + bsin\alpha$, $\alpha \in$ R, $a, b \in$ R, $a^2 + b^2 \neq 0$

As $a^2 + b^2 \neq 0$, we consider three cases :

(1) $a = 0, b \neq 0$ (2) $a \neq 0, b = 0$ (3) $a \neq 0, b \neq 0$

Case (1) : $a = 0, b \neq 0$

Then, $f(\alpha) = bsin\alpha$. Range of $sin\alpha$ is $[-1, 1]$.

$-1 \leq sin\alpha \leq 1$

$\Leftrightarrow$ $-b \leq bsin\alpha \leq b$ $(b > 0)$

$\therefore$ For $b > 0$, the range of $bsin\alpha$ is $[-b, b] = [-|b|, |b|]$. $(|b| = b)$

Now; for $b < 0$, $-1 \leq sin\alpha \leq 1 \Leftrightarrow -b \geq bsin\alpha \geq b$

$$\Leftrightarrow b \leq bsin\alpha \leq -b$$

$\therefore$ For $b < 0$, the range is $[b, -b] = [-|b|, |b|]$. $(|b| = -b)$

$\therefore$ The range of $f(\alpha) = bsin\alpha$ is $[-|b|, |b|]$.

Case (2) : $a \neq 0, b = 0$

Then, $f(\alpha) = acos\alpha$. Its range is $[-|a|, |a|]$ as before.

Case (3) : $a \neq 0, b \neq 0$

In this case, we shall express $acos\alpha + bsin\alpha$ in the form $r\,cos(\theta - \alpha)$.

As $r\,cos(\theta - \alpha) = r\,cos\theta\ cos\alpha + r\,sin\theta\ sin\alpha$, we shall find r and θ such that $a = r\,cos\theta$, $b = r\,sin\theta$. $(r > 0)$

These equations imply that $a^2 + b^2 = r^2$ and $tan\theta = \frac{b}{a}$.

$\therefore$ $r = \sqrt{a^2+b^2}$. As the range of *tan* function is R, corresponding to real number $\frac{b}{a}$ we can find $\theta \in R - \left\{(2n-1)\frac{\pi}{2} \mid n \in Z\right\}$ such that $tan\theta = \frac{b}{a}$.

Hence, for given a and b (both non-zero), we can select $r = \sqrt{a^2+b^2}$ and θ such that $tan\theta = \frac{b}{a}$. We can select θ, so that $rcos\theta = a$, $rsin\theta = b$.

Thus, $f(\alpha) = acos\alpha + bsin\alpha$

$= rcos\theta\, cos\alpha + r\, sin\theta\, sin\alpha$

$= r\,(cos\theta\, cos\alpha + sin\theta\, sin\alpha)$

$= r\, cos(\theta - \alpha)$

$f(\alpha) = r\, cos(\theta - \alpha)$

$-1 \le cos(\theta - \alpha) \le 1 \Leftrightarrow -r \le r\, cos(\theta - \alpha) \le r$ **($r > 0$)**

$\therefore$ The range of $f(\alpha)$ is $[-r, r]$.

Hence, range of $f(\alpha)$ is $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$.

This means that the maximum value attained by $f(\alpha)$ is $\sqrt{a^2+b^2}$ and the minimum value is $-\sqrt{a^2+b^2}$.

4.6 Addition Formulae for *tan* and *cot*

(1) If α, β and $\alpha + \beta \in R - \left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\}$, then

$$tan\,(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha \cdot tan\beta}$$

and if α, β and $\alpha - \beta \in R - \left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\}$, then

$$tan\,(\alpha - \beta) = \frac{tan\alpha - tan\beta}{1 + tan\alpha \cdot tan\beta}$$

Proof : $tan\,(\alpha+\beta) = \frac{sin\,(\alpha+\beta)}{cos\,(\alpha+\beta)} = \frac{sin\,\alpha\, cos\beta + cos\alpha\, sin\beta}{cos\alpha\, cos\beta - sin\,\alpha\, sin\beta}$ $\left(\alpha + \beta \in R - \left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\}\right)$

As, α, $\beta \in R - \left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\}$, $cos\alpha \ne 0$, $cos\beta \ne 0$

Hence, dividing both numerator and denominator by $cos\alpha \cdot cos\beta$, we get

$$tan\,(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha \cdot tan\beta}$$

Similarly, we can get, $tan(\alpha - \beta) = \frac{tan\alpha - tan\beta}{1 + tan\alpha \cdot tan\beta}$

(2) If α, β and $\alpha + \beta \in R - \{k\pi \mid k \in Z\}$, then

$$cot\,(\alpha + \beta) = \frac{cot\,\alpha \cdot cot\,\beta - 1}{cot\,\beta + cot\,\alpha}$$

and if α, β and $\alpha - \beta \in R - \{k\pi \mid k \in Z\}$, then

$$cot\,(\alpha - \beta) = \frac{cot\,\alpha \cdot cot\,\beta + 1}{cot\,\beta - cot\,\alpha}$$

Proof : $cot\,(\alpha + \beta) = \dfrac{cos(\alpha + \beta)}{sin(\alpha + \beta)} = \dfrac{cos\,\alpha \cdot cos\,\beta - sin\,\alpha \cdot sin\,\beta}{sin\,\alpha \cdot cos\,\beta + cos\,\alpha \cdot sin\,\beta}$ **($\alpha + \beta \in R - \{k\pi \mid k \in Z\}$)**

As, $\alpha, \beta \in R - \{k\pi \mid k \in Z\}$, $sin\alpha \neq 0$, $sin\beta \neq 0$.

Hence, dividing both numerator and denominator by $sin\alpha \cdot sin\beta$, we get

$$cot\,(\alpha + \beta) = \frac{cot\,\alpha \cdot cot\,\beta - 1}{cot\,\beta + cot\,\alpha}.$$

Similarly, we can prove that, $cot(\alpha - \beta) = \dfrac{cot\,\alpha \cdot cot\,\beta + 1}{cot\,\beta - cot\,\alpha}$.

4.7 Value of $tan\,\frac{\pi}{12}$ and $cot\,\frac{\pi}{12}$

We have $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ or $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$

(1) $tan\frac{\pi}{12} = tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \dfrac{tan\frac{\pi}{3} - tan\frac{\pi}{4}}{1 + tan\frac{\pi}{3}\,tan\frac{\pi}{4}}$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{3 - 2\sqrt{3} + 1}{3 - 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

$\therefore$ $tan\frac{\pi}{12} = 2 - \sqrt{3}$

(2) $cot\frac{\pi}{12} = cot\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \dfrac{cot\frac{\pi}{3}\,cot\frac{\pi}{4} + 1}{cot\frac{\pi}{4} - cot\frac{\pi}{3}}$

$$= \frac{\frac{1}{\sqrt{3}} \cdot 1 + 1}{1 - \frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

$$= \frac{1 + \sqrt{3}}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$$

$$= \frac{3 + 2\sqrt{3} + 1}{3 - 1} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$$

$cot\frac{\pi}{12} = 2 + \sqrt{3}$

Also, $tan\frac{5\pi}{12} = tan\left(\frac{\pi}{2} - \frac{\pi}{12}\right) = cot\frac{\pi}{12} = 2 + \sqrt{3}$ and

$cot\frac{5\pi}{12} = cot\left(\frac{\pi}{2} - \frac{\pi}{12}\right) = tan\frac{\pi}{12} = 2 - \sqrt{3}$.

Example 4 : If $sin\alpha = \frac{4}{5}$, $\frac{\pi}{2} < \alpha < \pi$ and $tan\beta = \frac{-12}{5}$, $-\frac{\pi}{2} < \beta < 0$, then determine the quadrant of $P(\alpha + \beta)$ and $P(\alpha - \beta)$.

Solution : Here $\frac{\pi}{2} < \alpha < \pi$ and $-\frac{\pi}{2} < \beta < 0$. On addition, we get $0 < \alpha + \beta < \pi$.

$\therefore$ $P(\alpha + \beta)$ is in the first or in the second quadrant. As *cosine* takes $+ve$ value in the first quadrant and $-ve$ value in the second quadrant and *sine* takes $+ve$ value in the first and the second both quadrants, so to determine the quadrant of $P(\alpha + \beta)$, we must find $cos(\alpha + \beta)$.

$\therefore$ $cos\alpha = -\sqrt{1 - sin^2\alpha} = -\sqrt{1 - \frac{16}{25}} = \frac{-3}{5}$ **$\left(\frac{\pi}{2} < \alpha < \pi\right)$**

$tan\beta = \frac{-12}{5}$, $-\frac{\pi}{2} < \beta < 0$

$\therefore$ $sec\beta = \sqrt{1 + tan^2\beta} = \sqrt{1 + \frac{144}{25}} = \frac{13}{5}$ **$\left(-\frac{\pi}{2} < \beta < 0\right)$**

$\therefore$ $cos\beta = \frac{5}{13}$, $sin\beta = tan\beta \cdot cos\beta = \frac{-12}{5} \times \frac{5}{13} = \frac{-12}{13}$

$$cos(\alpha + \beta) = cos\alpha \cdot cos\beta - sin\alpha \cdot sin\beta$$

$$= \left(\frac{-3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{-12}{13}\right)$$

$$= \frac{-15}{65} + \frac{48}{65} = \frac{33}{65}$$

$\therefore$ $cos(\alpha + \beta) > 0$

$\therefore$ $P(\alpha + \beta)$ is in the first quadrant.

Second method for determining quadrant :

To determine the quadrant of $P(\alpha + \beta)$, we can use another method.

$$sin(\alpha + \beta) = sin\alpha\ cos\beta + cos\alpha\ sin\beta$$

$$= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{-3}{5}\right)\left(\frac{-12}{13}\right) = \frac{20 + 36}{65} = \frac{56}{65} > 0$$

$cos(\alpha + \beta) = cos\alpha\ cos\beta - sin\alpha\ sin\beta = \frac{33}{65}$ **(Method 1)**

As $sin(\alpha + \beta) > 0$ and $cos(\alpha + \beta) > 0$, $P(\alpha + \beta)$ is in the first quadrant.

Now, for $P(\alpha - \beta)$. $\frac{\pi}{2} < \alpha < \pi$ and $-\frac{\pi}{2} < \beta < 0$

$\therefore$ $\frac{\pi}{2} > -\beta > 0$

$\therefore$ $0 < -\beta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \alpha < \pi$ **(i)**

$\frac{\pi}{2} < \alpha - \beta < \frac{3\pi}{2}$ **(adding inequalities in (i))**

$\therefore$ $P(\alpha - \beta)$ is in the second or in the third quadrant. As *sine* takes +*ve* values in the second quadrant and −*ve* values in the third quadrant and *cosine* takes −*ve* values in the second and in the third both quadrants, so to determine the quadrant of $P(\alpha - \beta)$, we must find $sin(\alpha - \beta)$.

$$sin(\alpha - \beta) = sin\alpha\ cos\beta - cos\alpha\ sin\beta$$

$$= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{-3}{5}\right)\left(\frac{-12}{13}\right) = \frac{20-36}{65} = \frac{-16}{65}$$

$\therefore$ $sin(\alpha - \beta) < 0$

$\therefore$ $P(\alpha - \beta)$ is in the third quadrant.

Example 5 : Find the range of $sin\theta + cos\left(\theta + \frac{\pi}{3}\right)$.

Solution : Suppose $f(\theta) = sin\theta + cos\left(\theta + \frac{\pi}{3}\right)$

$$= sin\theta + cos\theta\ cos\frac{\pi}{3} - sin\theta\ sin\frac{\pi}{3}$$

$$= sin\theta + \frac{1}{2}cos\theta - \frac{\sqrt{3}}{2}sin\theta$$

$$f(\theta) = \frac{1}{2}cos\theta + \left(1 - \frac{\sqrt{3}}{2}\right)sin\theta = acos\theta + bsin\theta$$

Comparing $f(\theta)$ with $acos\theta + bsin\theta$, we get

$$a = \frac{1}{2},\ b = 1 - \frac{\sqrt{3}}{2}$$

Now, $r^2 = a^2 + b^2 = \frac{1}{4} + \left(1 - \frac{\sqrt{3}}{2}\right)^2$

$$= \frac{1}{4} + 1 - \sqrt{3} + \frac{3}{4}$$

$$r^2 = 2 - \sqrt{3}$$

$\therefore$ $r = \sqrt{2-\sqrt{3}} = \sqrt{\frac{4-2\sqrt{3}}{2}} = \sqrt{\frac{3-2\sqrt{3}+1}{2}} = \sqrt{\frac{(\sqrt{3}-1)^2}{2}}$

$\therefore$ $r = \frac{\sqrt{3}-1}{\sqrt{2}} = \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}$

$\therefore$ The range of $f(\theta)$ is $[-r, r] = \left[\frac{1}{\sqrt{2}} - \sqrt{\frac{3}{2}},\ \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}\right]$.

Example 6 : Determine whether the $sin110^o + cos110^o$ is positive or negative.

Solution : Suppose $f(\theta) = sin110^o + cos110^o$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}sin110^o + \frac{1}{\sqrt{2}}cos110^o\right)$$

$$= \sqrt{2}\left(cos45^o\ sin110^o + sin45^o\ cos110^o\right)$$

$$= \sqrt{2}\ sin(110^o + 45^o)$$

$$= \sqrt{2}\ sin155^o > 0 \qquad (90 < 155 < 180)$$

$\therefore$ $sin110^o + cos110^o$ is positive.

Note : The example 3 solved earlier in this chapter can be solved by this alternative method too.

Example 7 : Express $\sqrt{3}\,sin\alpha - cos\alpha$ in the form $rsin(\alpha - \theta)$ and find r and θ, where, $r > 0$, $0 \le \theta < 2\pi$.

Solution : Let $f(\alpha) = \sqrt{3}\,sin\alpha - cos\alpha$

Multiplying and dividing by $\sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$,

$$f(\alpha) = 2\left(\frac{\sqrt{3}}{2}sin\alpha - \frac{1}{2}cos\alpha\right)$$
$$= 2\left(sin\alpha\, cos\frac{\pi}{6} - cos\alpha\, sin\frac{\pi}{6}\right)$$
$$= 2sin\left(\alpha - \frac{\pi}{6}\right)$$
$$= rsin(\alpha - \theta)$$

$r = 2$, $\theta = \frac{\pi}{6}$. Here $\theta = \frac{\pi}{6}$ satisfies $0 \le \theta < 2\pi$.

Example 8 : If $\sqrt{3}\,cos\alpha - sin\alpha = rcos(\alpha - \theta)$, find r and θ. $r > 0$,

where (i) $0 < \theta < 2\pi$ (ii) $\frac{-\pi}{2} < \theta < 0$

Solution : Let $f(\alpha) = \sqrt{3}\,cos\alpha - sin\alpha$

Multiplying and dividing by $r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$,

$$f(\alpha) = 2\left(\frac{\sqrt{3}}{2}cos\alpha - \frac{1}{2}sin\alpha\right)$$
$$= 2\left(cos\frac{\pi}{6}\, cos\alpha - sin\frac{\pi}{6} sin\alpha\right)$$
$$= 2cos\left(\alpha + \frac{\pi}{6}\right)$$
$$= 2cos\left(\alpha - \left(-\frac{\pi}{6}\right)\right)$$

Now comparing with $rcos(\alpha - \theta)$, we get

$r = 2$, $\theta = \frac{-\pi}{6}$ and $\theta = \frac{-\pi}{6}$ satisfies $\frac{-\pi}{2} < \theta < 0$

$$2cos\left(\alpha + \frac{\pi}{6}\right) = 2cos\left(\alpha + \frac{\pi}{6} - 2\pi\right) = 2cos\left(\alpha - \frac{11\pi}{6}\right)$$

$\therefore$ $\theta = \frac{11\pi}{6}$ satisfies $0 < \theta < 2\pi$.

Example 9 : Prove that $sin^2A = cos^2(A - B) + cos^2B - 2cos(A - B)cosA\, cosB$.

Solution : R.H.S. $= cos^2(A - B) + cos^2B - 2cos(A - B)cosA\, cosB$.

$$= cos^2B + cos^2(A - B) - 2cos(A - B)\, cosA\, cosB$$
$$= cos^2B + cos(A - B)\,[cos(A - B) - 2cosA\, cosB]$$
$$= cos^2B + cos(A - B)\,[cosA\, cosB + sinA\, sinB - 2cosA\, cosB]$$
$$= cos^2B + cos(A - B)\,(sinA\, sinB - cosA\, cosB)$$
$$= cos^2B - cos(A - B)\, cos(A + B)$$
$$= cos^2B - (cos^2A - sin^2B)$$
$$= cos^2B + sin^2B - cos^2A$$
$$= 1 - cos^2A$$
$$= sin^2A = \text{L.H.S.}$$

Exercise 4.2

1. Evaluate :

(1) $sin^2 37\frac{1}{2}^o - sin^2 7\frac{1}{2}^o$ (2) $sin^2 52\frac{1}{2}^o - cos^2 7\frac{1}{2}^o$ (3) $cos^2 37\frac{1}{2}^o - sin^2 37\frac{1}{2}^o$

2. Prove that : $sin^2 A + sin^2 B + cos^2(A + B) + 2 sinA\, sinB\, cos(A + B) = 1$.

3. (1) If $cosA = \frac{1}{7}$, $cosB = \frac{13}{14}$ and $0 < A, B < \frac{\pi}{2}$, then prove that $A - B = \frac{\pi}{3}$.

(2) If $sinA = \frac{1}{\sqrt{5}}$, $cosB = \frac{3}{\sqrt{10}}$ and $0 < A, B < \frac{\pi}{2}$, then prove that $A + B = \frac{\pi}{4}$.

4. (1) Find the quadrant of $P(\alpha - \beta)$, if $cos\alpha = \frac{4}{5}$, $cos\beta = \frac{12}{13}$, $\frac{3\pi}{2} < \alpha, \beta < 2\pi$.

(2) Find the quadrant of $P(\alpha + \beta)$, if $cos\alpha = \frac{-5}{13}$, $\frac{\pi}{2} < \alpha < \pi$ and $tan\beta = \frac{4}{3}$, $\pi < \beta < \frac{3\pi}{2}$.

5. If $cot\alpha = \frac{1}{2}$, $sec\beta = \frac{-5}{3}$, where $\pi < \alpha < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$. Find the value of $tan(\alpha + \beta)$ and find the quadrant of $P(\alpha + \beta)$.

6. Determine the range of (1) $7sin\theta + 24cos\theta$ (2) $cos\theta + sin\left(\theta - \frac{\pi}{6}\right) + 1$

7. Prove that $5cos\theta + 3cos\left(\theta + \frac{\pi}{3}\right) + 7$ in $[0, 14]$.

8. Express $\sqrt{3}sin\theta + cos\theta$ in the form $rcos(\theta - \alpha)$, where $r > 0$ and $0 < \alpha < 2\pi$.

9. $\frac{-\pi}{2} < \theta < 0$ and $cos\alpha - \sqrt{3}sin\alpha = rcos(\alpha - \theta)$, find r and θ.

10. Prove :

(1) $tan\left(\frac{\pi}{3} - \alpha\right) = \frac{\sqrt{3}cos\,\alpha - sin\,\alpha}{cos\,\alpha + \sqrt{3}\,sin\,\alpha}$ (2) $tan39^o = \frac{\sqrt{3}cos\,21^o - sin\,21^o}{cos\,21^o + \sqrt{3}\,sin\,21^o}$

(3) $tan3A \cdot tan2A \cdot tanA = tan3A - tan2A - tanA$

(4) $cotA \cdot cot2A - cot2A \cdot cot3A - cot3A \cdot cotA = 1$

(5) $tan25^o \cdot tan15^o + tan15^o \cdot tan50^o + tan25^o \cdot tan50^o = 1$

11. If $A + B = \frac{\pi}{4}$, then prove that

(1) $(1 + tanA)(1 + tanB) = 2$

(2) $(cotA - 1)(cotB - 1) = 2$

12. (1) Prove that $A + B = \frac{\pi}{2} \Rightarrow tanA = tanB + 2tan(A - B)$

(2) Prove that $tan65^o = tan25^o + 2tan40^o$

13. If $A + B + C = (2k + 1)\frac{\pi}{2}$, $k \in Z$, then prove that

(1) $tanA\; tanB + tanB\; tanC + tanC\; tanA = 1$

(2) $cotA + cotB + cotC = cotA\; cotB\; cotC$

14. If A + B + C = $k\pi$, $k \in$ Z, then prove that

(1) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(2) $\cot B \cdot \cot C + \cot C \cdot \cot A + \cot A \cdot \cot B = 1$

15. If $\tan A = 3$, $\tan B = \frac{1}{2}$, $0 < A, B < \frac{\pi}{2}$, then prove that $A - B = \frac{\pi}{4}$.

16. If $\tan B = 2$ and $\tan C = 3$ in ΔABC, then prove that $\tan A = 1$.

17. If $0 < A, B < \frac{\pi}{2}$, $\tan A = \frac{a}{a+1}$ and $\tan B = \frac{1}{2a+1}$, prove that $A + B = \frac{\pi}{4}$.

18. If $\alpha + \beta = \theta$, $\alpha - \beta = \phi$ and $\frac{\tan\alpha}{\tan\beta} = \frac{x}{y}$, then prove that $\frac{\sin\theta}{\sin\phi} = \frac{x+y}{x-y}$.

19. If $\frac{\tan(A-B)}{\tan A} + \frac{\sin^2 C}{\sin^2 A} = 1$, then prove that $\tan A \cdot \tan B = \tan^2 C$.

20. If $\tan(A + B) = 3$ and $\tan(A - B) = 2$, then find $\tan 2A$ and $\tan 2B$.

21. If $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$, then prove that $\tan(\alpha - \beta) = (1 - n)\tan\alpha$.

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4.8 Expression of a Product in the Form of a Sum or a Difference

We have studied the following formulae valid for all real $\alpha, \beta \in R$:

$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$ **(i)**

$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$ **(ii)**

$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$ **(iii)**

$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$ **(iv)**

Taking sum and difference of (i) and (ii), we get,

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha \cos\beta$$

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha \sin\beta$$

that is,

$2\sin\alpha \cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$ **(v)**

$2\cos\alpha \sin\beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$ **(vi)**

In the same way, taking sum and difference of (iii) and (iv), we get

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha \cos\beta$$

$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin\alpha \sin\beta$$

that is,

$2\cos\alpha \cos\beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$ **(vii)**

$2\sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$ **(viii)**

In each of (v), (vi), (vii) and (viii), the left side is the product of trigonometric functions where as the right side is the sum or difference of a trigonometric functions with variable $\alpha + \beta$ or $\alpha - \beta$. It would therefore be easy to express product of trigonometric functions in terms of a sum or a difference.

For example, $2sin3\theta\ cos5\theta = sin(3\theta + 5\theta) + sin(3\theta - 5\theta)$

$= sin8\theta + sin(-2\theta)$

$= sin8\theta - sin2\theta$ **($sin(-\theta) = -sin\theta$)**

Now, if we take bigger angle first, then calculation become simpler

$$2cos3\theta \cdot sin5\theta = 2sin5\theta \cdot cos3\theta = sin(5\theta + 3\theta) + sin(5\theta - 3\theta)$$

$$= sin8\theta + sin2\theta$$

Example 11 : Express each of the following as a sum or a difference.

(1) $2sin5\theta\ cos\theta$ (2) $2cos\frac{5\theta}{2}\ sin\frac{3\theta}{2}$ (3) $2sin3\theta\ sin5\theta$ (4) $sin^2\theta$ (5) $2cos5\theta\ cos\frac{\theta}{2}$

Solution : (1) $2sin5\theta\ cos\theta = sin(5\theta + \theta) + sin(5\theta - \theta) = sin6\theta + sin4\theta$

(2) $2cos\frac{5\theta}{2}\ sin\frac{3\theta}{2} = sin\left(\frac{5\theta}{2} + \frac{3\theta}{2}\right) - sin\left(\frac{5\theta}{2} - \frac{3\theta}{2}\right) = sin4\theta - sin\theta$

(3) $2sin3\theta\ sin5\theta = cos(3\theta - 5\theta) - cos(3\theta + 5\theta) = cos(-2\theta) - cos8\theta$

$= cos2\theta - cos8\theta$

(4) $sin^2\theta = sin\theta\ sin\theta = \frac{1}{2}[2sin\theta\ sin\theta] = \frac{1}{2}\left[cos(\theta - \theta) - cos(\theta + \theta)\right]$

$= \frac{1}{2}[cos0 - cos2\theta] = \frac{1}{2}[1 - cos2\theta]$

(5) $2cos5\theta\ cos\frac{\theta}{2} = cos\left(5\theta + \frac{\theta}{2}\right) + cos\left(5\theta - \frac{\theta}{2}\right) = cos\frac{11\theta}{2} + cos\frac{9\theta}{2}$

Example 12 : Prove that $sin20^o \cdot sin40^o \cdot sin60^o \cdot sin80^o = \frac{3}{16}$.

Solution : L.H.S. $= sin20^o \cdot sin40^o \cdot sin60^o \cdot sin80^o$

$= sin60^o \cdot (sin20^o \cdot sin40^o) \cdot sin80^o$

$= \frac{\sqrt{3}}{2} \times \frac{1}{2}(2sin40^o \cdot sin20^o) \cdot sin80^o$

$= \frac{\sqrt{3}}{4}[cos(40^o - 20^o) - cos(40^o + 20^o)]\ sin80^o$

$= \frac{\sqrt{3}}{4}[cos(20^o) - cos60^o]\ sin80^o$

$= \frac{\sqrt{3}}{4}\left(cos20^o - \frac{1}{2}\right) sin80^o$

$= \frac{\sqrt{3}}{8}(2sin80^o\ cos20^o - sin80^o)$

$= \frac{\sqrt{3}}{8}[sin(80^o + 20^o) + sin(80^o - 20^o) - sin80^o]$

$= \frac{\sqrt{3}}{8}[sin100^o + sin60^o - sin80^o]$

$= \frac{\sqrt{3}}{8}\left[sin(180^o - 80^o) + \frac{\sqrt{3}}{2} - sin80^o\right]$

$= \frac{\sqrt{3}}{8}\left(sin80^o + \frac{\sqrt{3}}{2} - sin80^o\right)$

$= \frac{3}{16}$ = R.H.S.

Example 13 : If A + B = 90°, then find the maximum and minimum values of $sin A \cdot sin B$.

Solution : Let $y = sinA \cdot sinB = sinA\ sin(90^\circ - A) = sinA\ cosA$

Then, $y = \frac{1}{2}(2sinA \cdot cosA) = \frac{1}{2}[sin(A + A) - sin(A - A)]$

$= \frac{1}{2}sin2A$ **($sin0 = 0$)**

Now, $-1 \le sin2A \le 1 \Leftrightarrow \frac{-1}{2} \le \frac{1}{2}sin2A \le \frac{1}{2} \Leftrightarrow \frac{-1}{2} \le y \le \frac{1}{2}$

Hence, $\frac{1}{2}$ and $\frac{-1}{2}$ are respectively the maximum and minimum values of $sinA\ sinB$.

Exercise 4.3

1. Express as a sum or a difference :

(1) $2sin7\theta \cdot cos3\theta$ (2) $2sin\frac{\theta}{2} \cdot cos\frac{5\theta}{2}$ (3) $2cos5\theta \cdot sin3\theta$

(4) $2cos\frac{5\theta}{2} \cdot sin\frac{7\theta}{2}$ (5) $2cos11\theta \cdot cos3\theta$ (6) $2cos\frac{5\theta}{2} \cdot cos\frac{3\theta}{2}$

(7) $sin9\theta \cdot sin11\theta$ (8) $2sin\frac{7\theta}{2} \cdot sin\frac{9\theta}{2}$ (9) $2sin\theta \cdot cos\theta$

2. Find the value :

(1) $2sin\frac{5\pi}{12} \cdot sin\frac{\pi}{12}$ (2) $2sin\frac{5\pi}{12} \cdot cos\frac{7\pi}{12}$ (3) $2cos\frac{\pi}{12} \cdot sin\frac{5\pi}{12}$

(4) $2cos\frac{5\pi}{12} \cdot cos\frac{7\pi}{12}$ (5) $8cos15^\circ \cdot cos45^\circ \cdot cos75^\circ$ (6) $8sin10^\circ \cdot sin50^\circ \cdot sin70^\circ$

3. Prove :

(1) $sin\left(\frac{\pi}{4}+\theta\right)\ sin\left(\frac{\pi}{4}-\theta\right) = \frac{1}{2}cos2\theta$

(2) $sin\theta \cdot sin\left(\frac{\pi}{3}-\theta\right) \cdot sin\left(\frac{\pi}{3}+\theta\right) = \frac{1}{4}sin3\theta$

(3) $2cos\frac{\pi}{13} \cdot cos\frac{9\pi}{13} + cos\frac{3\pi}{13} + cos\frac{5\pi}{13} = 0$

(4) $cos20^\circ \cdot cos40^\circ \cdot cos60^\circ \cdot cos80^\circ = \frac{1}{16}$

(5) $4cos12^\circ \cdot cos48^\circ \cdot cos72^\circ = cos36^\circ$

4. Prove that $4cos\theta \cdot cos\left(\frac{\pi}{3}-\theta\right)\ cos\left(\frac{\pi}{3}+\theta\right) = cos3\theta$ and deduce that

$cos6^\circ\ cos42^\circ\ cos66^\circ\ cos78^\circ = \frac{1}{16}$.

5. Find the value of $\frac{1}{2sin\ 10^\circ} - 2sin70^\circ$.

6. Prove that $cos\frac{2\pi}{7} + cos\frac{4\pi}{7} + cos\frac{6\pi}{7} = \frac{-1}{2}$.

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4.9 Expressing the Sum or the Difference as a Product

We have seen the formula (v) to (viii) which are reproduced below :

$$2\sin\alpha \cdot \cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta) \quad \textbf{(v)}$$

$$2\cos\alpha \cdot \sin\beta = \sin(\alpha + \beta) - \sin(\alpha - \beta) \quad \textbf{(vi)}$$

$$2\cos\alpha \cdot \cos\beta = \cos(\alpha + \beta) + \cos(\alpha - \beta) \quad \textbf{(vii)}$$

$$2\sin\alpha \cdot \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta) \quad \textbf{(viii)}$$

Let us substitute $\alpha + \beta = C$ and $\alpha - \beta = D$ in these formulae.

Then, $\alpha = \frac{C+D}{2}$ and $\beta = \frac{C-D}{2}$. We get

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\cos D - \cos C = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \quad \text{or}$$

$$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right).$$

These formulae are useful as they express sums or differences as products.

Example 14 : Express the following as products :

(1) $\sin6\theta + \sin4\theta$ (2) $\sin6\theta - \sin2\theta$ (3) $\cos5\theta + \cos2\theta$

(4) $\cos6\theta - \cos10\theta$ (5) $\sin\theta - 1$ (6) $\cos\theta + 1$

Solution : (1) $\sin6\theta + \sin4\theta = 2\sin\left(\frac{6\theta+4\theta}{2}\right)\cos\left(\frac{6\theta-4\theta}{2}\right) = 2\sin5\theta\cos\theta$

(2) $\sin6\theta - \sin2\theta = 2\cos\left(\frac{6\theta+2\theta}{2}\right)\sin\left(\frac{6\theta-2\theta}{2}\right) = 2\cos4\theta\sin2\theta$

(3) $\cos5\theta + \cos2\theta = 2\cos\left(\frac{5\theta+2\theta}{2}\right)\cos\left(\frac{5\theta-2\theta}{2}\right) = 2\cos\frac{7\theta}{2}\cos\frac{3\theta}{2}$

(4) $\cos6\theta - \cos10\theta = -2\sin\left(\frac{6\theta+10\theta}{2}\right)\sin\left(\frac{6\theta-10\theta}{2}\right)$

$$= -2\sin8\theta\sin(-2\theta) = 2\sin8\theta\sin2\theta$$

(5) $\sin\theta - 1 = \sin\theta - \sin\frac{\pi}{2} = 2\cos\left(\frac{\theta+\frac{\pi}{2}}{2}\right)\sin\left(\frac{\theta-\frac{\pi}{2}}{2}\right)$

$$= 2\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$$

(6) $\cos\theta + 1 = \cos\theta + \cos 0 = 2\cos\left(\frac{\theta+0}{2}\right)\cos\left(\frac{\theta-0}{2}\right) = 2\cos\frac{\theta}{2}\cos\frac{\theta}{2}$

$= 2\cos^2\frac{\theta}{2}$

Example 15 : Prove that

(1) $\cos 20^\circ + \cos 60^\circ + \cos 100^\circ + \cos 140^\circ = \frac{1}{2}$

(2) $1 + \cos 2A + \cos 4A + \cos 6A = 4\cos A \cdot \cos 2A \cdot \cos 3A$

(3) $\sqrt{3}\sin 10^\circ + \sqrt{2}\sin 55^\circ = \cos 80^\circ + 2\cos 50^\circ$

Solution : (1) L.H.S. $= \cos 20^\circ + \cos 60^\circ + \cos 100^\circ + \cos 140^\circ$

$= \cos 20^\circ + \frac{1}{2} + 2\cos\left(\frac{100^\circ + 140^\circ}{2}\right)\cos\left(\frac{100^\circ - 140^\circ}{2}\right)$

$= \cos 20 + \frac{1}{2} + 2\cos 120^\circ \cos(20^\circ)$ **($\cos(-20^\circ) = \cos 20^\circ$)**

$= \cos 20^\circ + \frac{1}{2} + 2\cos(180^\circ - 60^\circ)\cos 20^\circ$

$= \frac{1}{2} + \cos 20^\circ - 2\cos 60^\circ \cos 20^\circ$

$= \frac{1}{2} + \cos 20^\circ - 2 \cdot \frac{1}{2}\cos 20^\circ$

$= \frac{1}{2} + \cos 20^\circ - \cos 20^\circ$

$= \frac{1}{2} =$ R.H.S.

(2) L.H.S. $= 1 + \cos 2A + \cos 4A + \cos 6A$

$= (\cos 0 + \cos 2A) + (\cos 4A + \cos 6A)$

$= 2\cos A \cdot \cos A + 2\cos 5A \cdot \cos A$

$= 2\cos A(\cos A + \cos 5A)$

$= 2\cos A(2\cos 3A \cdot \cos 2A)$

$= 4\cos A \cdot \cos 2A \cdot \cos 3A =$ R.H.S.

(3) L.H.S. $= \sqrt{3}\sin 10^\circ + \sqrt{2}\sin 55^\circ$

$= 2 \cdot \frac{\sqrt{3}}{2}\sin 10^\circ + 2 \cdot \frac{1}{\sqrt{2}}\sin 55^\circ$

$= 2\sin 60^\circ \sin 10^\circ + 2\sin 45^\circ \sin 55^\circ$

$= \cos 50^\circ - \cos 70^\circ + \cos 10^\circ - \cos 100^\circ$

$= \cos 50^\circ - \cos(180^\circ - 80^\circ) - (\cos 70^\circ - \cos 10^\circ)$ **(rearranging)**

$= \cos 50^\circ + \cos 80^\circ + 2\sin 40^\circ \sin 30^\circ$

$= \cos 50^\circ + \cos 80^\circ + 2\sin(90^\circ - 50^\circ) \cdot \frac{1}{2}$

$= \cos 50^\circ + \cos 80^\circ + \cos 50^\circ$

$= \cos 80^\circ + 2\cos 50^\circ =$ R.H.S.

Exercise 4.4

1. Convert into a form of product :

(1) $\sin 7\theta + \sin 3\theta$ (2) $\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}$ (3) $\sin 3\theta - \sin 5\theta$

(4) $\sin\frac{7\theta}{2} - \sin\frac{3\theta}{2}$ (5) $\cos 11\theta + \cos 9\theta$ (6) $\cos\frac{5\theta}{2} + \cos\frac{11\theta}{2}$

(7) $\cos 5\theta - \cos 11\theta$ (8) $\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}$ (9) $\cos\theta - 1$

(10) $\sin\theta + 1$ (11) $\cos\theta + \sin\theta$ (12) $\sin\theta - \cos\theta$

Prove : **(2 to 7)**

2. (1) $\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0$ (2) $\cos\frac{5\pi}{12} - \cos\frac{\pi}{12} = \frac{-1}{\sqrt{2}}$

(3) $\sin 65^\circ + \cos 65^\circ = \sqrt{2}\cos 20^\circ$ (4) $\dfrac{\sin\frac{5\pi}{12} - \cos\frac{5\pi}{12}}{\cos\frac{5\pi}{12} + \sin\frac{5\pi}{12}} = \frac{1}{\sqrt{3}}$

(5) $\dfrac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \cot A$

(6) $\cos 2\theta \cos\frac{\theta}{2} - \cos 3\theta \cos\frac{9\theta}{2} = \sin 5\theta \sin\frac{5\theta}{2}$

(7) $\sin\theta + \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{4\pi}{3}\right) = 0$

3. (1) $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\left(\frac{\alpha - \beta}{2}\right)$

(2) $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\frac{\alpha - \beta}{2}\right)$

4. (1) $\sin A + \sin B + \sin C - \sin(A + B + C) = 4\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}$

(2) $\cos A + \cos B + \cos C + \cos(A + B + C) = 4\cos\frac{A+B}{2}\cos\frac{B+C}{2}\cos\frac{C+A}{2}$

5. (1) $\dfrac{\sin(A+B) - 2\sin A + \sin(A-B)}{\cos(A+B) - 2\cos A + \cos(A-B)} = \tan A$

(2) $\dfrac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \cos 2A - \sin 2A \tan 3A$

6. (1) $\dfrac{1}{\sin 10^\circ} - \dfrac{\sqrt{3}}{\cos 10^\circ} = 4$ (2) $\sqrt{2}\sin 10^\circ + \sqrt{3}\cos 35^\circ = \sin 55^\circ + 2\cos 65^\circ$

7. (1) $\sin\theta = n\sin(\theta + 2\alpha) \Leftrightarrow \tan(\theta + \alpha) = \frac{1+n}{1-n}\tan\alpha$

(2) $\sin(2A + 3B) = 5\sin B \Rightarrow 2\tan(A + 2B) = 3\tan(A + B)$

*

Miscellaneous Problems :

Example 16 : Prove that $0 < \alpha, \beta < \frac{\pi}{2} \Rightarrow \sin(\alpha + \beta) < \sin\alpha + \sin\beta$ and deduce from this that $\sin 49^\circ + \sin 41^\circ > 1$.

Solution : $sin(\alpha + \beta) - sin\alpha - sin\beta$

$= sin\alpha\ cos\beta + cos\alpha\ sin\beta - sin\alpha - sin\beta$

$= sin\alpha\ (cos\beta - 1) + sin\beta\ (cos\alpha - 1)$ **(i)**

Now, as $0 < \alpha, \beta < \frac{\pi}{2}$, so $0 < sin\alpha < 1$, $0 < sin\beta < 1$ and

$0 < cos\alpha < 1$, $0 < cos\beta < 1$

$\therefore$ $cos\alpha - 1 < 0$, $cos\beta - 1 < 0$

$\therefore$ $sin\alpha(cos\beta - 1) < 0$ and $sin\beta(cos\alpha - 1) < 0$

$\therefore$ $sin\alpha(cos\beta - 1) + sin\beta(cos\alpha - 1) < 0$

$\therefore$ $sin(\alpha + \beta) - sin\alpha - sin\beta < 0$

$\therefore$ $sin(\alpha + \beta) < sin\alpha + sin\beta$

Now, taking $\alpha = 49^o$, $\beta = 41^o$

As $0 < 49 < 90$ and $0 < 41 < 90$

$sin(49^o + 41^o) < sin49^o + sin41^o$

$\therefore$ $sin90^o < sin49^o + sin41^o$

$\therefore$ $sin49^o + sin41^o > 1$

Example 17 : If $cos(\alpha + \beta) = \frac{4}{5}$, $sin(\alpha - \beta) = \frac{5}{13}$ and $0 < \alpha, \beta < \frac{\pi}{4}$, then prove that $tan2\alpha = \frac{56}{33}$.

Solution : We have, $0 < \alpha < \frac{\pi}{4}$, $0 < \beta < \frac{\pi}{4}$

$\therefore$ $0 < \alpha + \beta < \frac{\pi}{2}$ and $\frac{-\pi}{4} < \alpha - \beta < \frac{\pi}{4}$.

$\therefore$ $cos(\alpha - \beta)$ and $sin(\alpha + \beta)$ are positive.

Now, $sin(\alpha + \beta) = \sqrt{1 - cos^2(\alpha + \beta)} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$ $\left(0 < \alpha + \beta < \frac{\pi}{2}\right)$

$cos(\alpha - \beta) = \sqrt{1 - sin^2(\alpha - \beta)} = \sqrt{1 - \frac{25}{169}} = \frac{12}{13}$ $\left(-\frac{\pi}{2} < -\frac{\pi}{4} < \alpha - \beta < \frac{\pi}{4} < \frac{\pi}{2}\right)$

$\therefore$ $sin(\alpha + \beta) = \frac{3}{5}$ and $cos(\alpha - \beta) = \frac{12}{13}$

Now, $tan(\alpha + \beta) = \frac{sin(\alpha + \beta)}{cos(\alpha + \beta)} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$

and $tan(\alpha - \beta) = \frac{sin(\alpha - \beta)}{cos(\alpha - \beta)} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12}$

$tan2\alpha = tan[(\alpha + \beta) + (\alpha - \beta)]$

$= \frac{tan(\alpha + \beta) + tan(\alpha - \beta)}{1 - tan(\alpha + \beta)\, tan(\alpha - \beta)}$

$= \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \times \frac{5}{12}} = \frac{56}{33}$

$\therefore$ $tan2\alpha = \frac{56}{33}$

Example 18 : If α and β are roots of $acos\theta + bsin\theta = c$, then show that,

(1) $cos(\alpha + \beta) = \frac{a^2 - b^2}{a^2 + b^2}$ (2) $cos(\alpha - \beta) = \frac{2c^2 - (a^2 + b^2)}{a^2 + b^2}$

Solution : We have, $acos\theta + bsin\theta = c$ **(i)**

$\therefore \quad acos\theta = c - bsin\theta$

$\therefore \quad a^2cos^2\theta = (c - bsin\theta)^2$

$\therefore \quad a^2(1 - sin^2\theta) = c^2 - 2bcsin\theta + b^2 sin^2\theta$

$\therefore \quad (a^2 + b^2) sin^2\theta - 2bcsin\theta + (c^2 - a^2) = 0$ **(ii)**

Since α and β are roots of equation (i), $sin\alpha$ and $sin\beta$ are the roots of the equation (ii).

$\therefore \quad sin\alpha \; sin\beta = \frac{c^2 - a^2}{a^2 + b^2}$ **(iii)**

Again, $acos\theta + bsin\theta = c$

$\therefore \quad bsin\theta = c - acos\theta$

$\therefore \quad b^2(1 - cos^2\theta) = c^2 - 2accos\theta + a^2 cos^2\theta$

$\therefore \quad b^2 - b^2cos^2\theta = a^2 cos^2\theta - 2accos\theta + c^2$

$\therefore \quad (a^2 + b^2) cos^2\theta - 2ac\,cos\theta + (c^2 - b^2) = 0$ **(iv)**

Since α and β are roots of equation (i), $cos\alpha$, $cos\beta$ are the roots the equation (iv).

$\therefore \quad cos\alpha \; cos\beta = \frac{c^2 - b^2}{a^2 + b^2}$ **(v)**

Now, $cos(\alpha + \beta) = cos\alpha \; cos\beta - sin\alpha \; sin\beta$

$$= \frac{c^2 - b^2}{a^2 + b^2} - \frac{c^2 - a^2}{a^2 + b^2} = \frac{a^2 - b^2}{a^2 + b^2}$$

$\therefore \quad cos(\alpha + \beta) = \frac{a^2 - b^2}{a^2 + b^2}$

and $cos(\alpha - \beta) = cos\alpha \; cos\beta + sin\alpha \; sin\beta$

$$= \frac{c^2 - b^2}{a^2 + b^2} + \frac{c^2 - a^2}{a^2 + b^2} = \frac{2c^2 - (a^2 + b^2)}{a^2 + b^2}$$ **(from (iii) and (v))**

$\therefore \quad cos(\alpha - \beta) = \frac{2c^2 - (a^2 + b^2)}{a^2 + b^2}$

Example 19 : If $asin\theta = bsin\left(\theta + \frac{2\pi}{3}\right) = csin\left(\theta + \frac{4\pi}{3}\right)$, then prove that $ab + bc + ca = 0$. ($abc \neq 0$)

Solution : Let $asin\theta = bsin\left(\theta + \frac{2\pi}{3}\right) = csin\left(\theta + \frac{4\pi}{3}\right) = k$

It is clear that $k \neq 0$ (why ?)

$$\therefore \quad \frac{k}{a} + \frac{k}{b} + \frac{k}{c} = sin\theta + sin\left(\theta + \frac{2\pi}{3}\right) + sin\left(\theta + \frac{4\pi}{3}\right)$$

$$= sin\theta + sin\left(\theta + \frac{4\pi}{3}\right) + sin\left(\theta + \frac{2\pi}{3}\right)$$

$$= 2sin\left(\theta + \frac{2\pi}{3}\right) cos\frac{2\pi}{3} + sin\left(\theta + \frac{2\pi}{3}\right)$$

$$= 2sin\left(\theta + \frac{2\pi}{3}\right) \times \left(-\frac{1}{2}\right) + sin\left(\theta + \frac{2\pi}{3}\right)$$

$$= -sin\left(\theta + \frac{2\pi}{3}\right) + sin\left(\theta + \frac{2\pi}{3}\right) = 0$$

$$\therefore \quad \frac{k}{a} + \frac{k}{b} + \frac{k}{c} = 0$$

$$\therefore \quad k\left(\frac{bc + ca + ab}{abc}\right) = 0$$

$\therefore \quad ab + bc + ca = 0$ **($k \neq 0$)**

Exercise 4

1. Prove that :

(1) $\dfrac{cos^2 33^\circ - cos^2 57^\circ}{sin^2 \frac{21^\circ}{2} - sin^2 \frac{69^\circ}{2}} = -\sqrt{2}$ (2) $\dfrac{\sqrt{3}}{sin\, 20^\circ} - \dfrac{1}{cos\, 20^\circ} = 4$

2. Prove that $0 < \alpha, \beta < \frac{\pi}{4} \Rightarrow tan(\alpha + \beta) > tan\alpha + tan\beta$ and deduce that $tan35^\circ + tan25^\circ < \sqrt{3}$.

3. Prove that $2tan\beta + cot\beta = tan\alpha \Rightarrow 2tan(\alpha - \beta) = cot\beta$.

4. If $\theta + \beta = \alpha$ and $sin\theta = ksin\beta$, prove that $tan\theta = \dfrac{k\,sin\alpha}{1 + k\,cos\alpha}$ and $tan\beta = \dfrac{sin\,\alpha}{k + cos\alpha}$.

5. If $sinA + cosB = 0$ in ΔABC, prove that ΔABC is an obtuse angled triangle and that $0 < sinA < \frac{1}{\sqrt{2}}$.

6. If $cos(\beta - \gamma) + cos(\gamma - \alpha) + cos(\alpha - \beta) = \frac{-3}{2}$, prove that $sin\alpha + sin\beta + sin\gamma = 0$ and $cos\alpha + cos\beta + cos\gamma = 0$.

7. If $tan(\alpha + \theta) = ntan(\alpha - \theta)$, then prove that $(n + 1)\, sin2\theta = (n - 1)\, sin2\alpha$.

8. If α and β are the roots of the equation $atan\theta + bsec\theta = c$, then prove that $tan(\alpha + \beta) = \dfrac{2ac}{a^2 - c^2}$.

9. Find the maximum and minimum values of $3cos\theta + 5sin\left(\theta - \frac{\pi}{6}\right)$.

10. Prove that $sin10^\circ \cdot sin30^\circ \cdot sin50^\circ \cdot sin70^\circ = \frac{1}{16}$.

11. Prove : $cos\frac{\pi}{11} + cos\frac{3\pi}{11} + cos\frac{5\pi}{11} + cos\frac{7\pi}{11} + cos\frac{9\pi}{11} = \frac{1}{2}$

12. Prove that $\dfrac{cos8\theta\, cos5\theta - cos12\theta\, cos9\theta}{sin8\theta\, cos5\theta + cos12\theta\, sin9\theta} = tan4\theta$.

13. Prove that $m\,tan\left(\theta - \frac{\pi}{6}\right) = n\,tan\left(\theta + \frac{2\pi}{3}\right) \Rightarrow cos2\theta = \dfrac{m+n}{2(m-n)}$.

14. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The value of $\dfrac{cos\ 10^{\circ} + sin\ 10^{\circ}}{cos\ 10^{\circ} - sin\ 10^{\circ}}$ is ... ☐

(a) $tan25^{\circ}$ (b) $tan35^{\circ}$ (c) $tan55^{\circ}$ (d) $tan80^{\circ}$

(2) The value of $cos245^{\circ} + sin155^{\circ}$ is ... ☐

(a) 0 (b) $\dfrac{\sqrt{2}+1}{\sqrt{2}}$ (c) $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$ (d) $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$

(3) The value of $cos(270^{\circ} + \alpha)\ cos(90^{\circ} - \alpha) - sin(270^{\circ} - \alpha)\ cos\alpha$ is... ☐

(a) -1 (b) 0 (c) $\frac{1}{2}$ (d) 1

(4) The value of $2sin\left(\frac{\pi}{12}\right)\ sin\left(\frac{5\pi}{12}\right)$ is ... ☐

(a) $\frac{-1}{4}$ (b) 1 (c) $\frac{-\sqrt{3}}{2}$ (d) $\frac{1}{2}$

(5) If A = 125 and $x = sinA^{\circ} + cosA^{\circ}$, then ☐

(a) $x < 0$ (b) $x = 0$ (c) $x > 0$ (d) $x \geq 0$

(6) If $tan\alpha = \dfrac{n}{n+1}$ and $tan\beta = \dfrac{1}{2n+1}$, $(0 < \alpha, \beta < \frac{\pi}{4})$, then $\alpha + \beta$ is ... ☐

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

(7) The value of $\dfrac{tan\ 50^{\circ} - tan\ 40^{\circ}}{tan\ 10^{\circ}}$ is ... ☐

(a) 0 (b) 1 (c) 2 (d) 3

(8) $sin190^{\circ} + cos190^{\circ}$ ... ☐

(a) is negative (b) is zero

(c) is positive (d) is not defined.

(9) If $cot15^{\circ} = m$, then $\dfrac{tan\ 225^{\circ} + tan\ 345^{\circ}}{tan\ 195^{\circ} - tan\ 105^{\circ}}$ is ... ☐

(a) $\dfrac{m-1}{m^2+1}$ (b) $\dfrac{2m}{m^2+1}$ (c) $\dfrac{m^2-1}{m^2+1}$ (d) $\dfrac{m+1}{m^2+1}$

(10) The value of $\log tan1^{\circ} + \log tan2^{\circ} + ... + \log tan89^{\circ}$ is ... ☐

(a) 0 (b) 1 (c) 2 (d) 3

(11) The value of $\dfrac{1 - tan^2 15^{\circ}}{1 + tan^2 15^{\circ}}$ is ... ☐

(a) 1 (b) $\frac{\sqrt{3}}{2}$ (c) 2 (d) $\sqrt{3}$

(12) The value of $cos480^o\ sin150^o + sin600^o\ cos390^o$ is ...

(a) $\frac{-1}{2}$ (b) 0 (c) -1 (d) $\frac{1}{2}$

(13) $tan25^o + tan20^o + tan25^o\ tan20^o$ is equal to ...

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) 2

(14) In ΔABC, if $tanA = \frac{1}{2}$, $tanB = \frac{1}{3}$, then the measure of angle C is ...

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{3\pi}{4}$ (d) $\frac{2\pi}{3}$

(15) The value of $\sqrt{3}cosec20^o - sec20^o$ is ...

(a) -4 (b) 1 (c) 2 (d) 4

(16) The value of $\sqrt{3}sin75^o - cos75^o$ is ...

(a) $\frac{1}{\sqrt{2}}$ (b) 1 (c) $\sqrt{2}$ (d) $2\sqrt{2}$

(17) $cos^2\frac{\pi}{12} + cos^2\frac{3\pi}{12} + cos^2\frac{5\pi}{12} = ...$

(a) $\frac{-1}{2}$ (b) 0 (c) $\frac{1}{2}$ (d) $\frac{3}{2}$

(18) The value of $cos15^o - sin15^o$ is ...

(a) $\frac{-1}{\sqrt{2}}$ (b) 0 (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

(19) $cos^2 7\frac{1}{2}^o - cos^2 37\frac{1}{2}^o$ is equal to ...

(a) $\frac{3}{4}$ (b) $\frac{2}{\sqrt{2}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{2\sqrt{2}}$

*

Summary

We studied following points in this chapter :

1. $cos(\alpha - \beta) = cos\alpha\ cos\beta + sin\alpha\ sin\beta$
2. $cos(\alpha + \beta) = cos\alpha\ cos\beta - sin\alpha\ sin\beta$
3. $cos\left(\frac{\pi}{2} - \theta\right) = sin\theta$, $sin\left(\frac{\pi}{2} - \theta\right) = cos\theta$
4. $sin(\alpha + \beta) = sin\alpha\ cos\beta + cos\alpha\ sin\beta$
5. $sin(\alpha - \beta) = sin\alpha\ cos\beta - cos\alpha\ sin\beta$
6. $sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$, $cos\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$
7. $sin(\alpha + \beta) \cdot sin(\alpha - \beta) = sin^2\alpha - sin^2\beta$

   $sin(\alpha + \beta) \cdot sin(\alpha - \beta) = cos^2\beta - cos^2\alpha$

8. $\cos(\alpha + \beta) \cdot \cos(\alpha - \beta) = \cos^2\alpha - \sin^2\beta$

$\cos(\alpha + \beta) \cdot \cos(\alpha - \beta) = \cos^2\beta - \sin^2\alpha$

9. The range of $f(\alpha) = a\cos\alpha + b\sin\beta,\ \alpha,\ \beta \in \mathrm{R},\ a,\ b, \in \mathrm{R},\ a^2 + b^2 \neq 0$

is $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$.

In proper domain,

10. $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$

11. $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}$

12. $\cot(\alpha + \beta) = \dfrac{\cot\alpha \cdot \cot\beta - 1}{\cot\beta + \cot\alpha}$

13. $\cot(\alpha - \beta) = \dfrac{\cot\alpha \cdot \cot\beta + 1}{\cot\beta - \cot\alpha}$

14. $\tan\frac{\pi}{12} = 2 - \sqrt{3}, \quad \cot\frac{\pi}{12} = 2 + \sqrt{3}$

15. $2\sin\alpha \cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$

16. $2\cos\alpha \sin\beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$

17. $2\cos\alpha \cos\beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$

18. $2\sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$

19. $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right) \cos\left(\dfrac{C-D}{2}\right)$

20. $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right) \sin\left(\dfrac{C-D}{2}\right)$

21. $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right) \cos\left(\dfrac{C-D}{2}\right)$

22. $\cos C - \cos D = -2\sin\left(\dfrac{C+D}{2}\right) \sin\left(\dfrac{C-D}{2}\right)$.

Aryabhata is also known as **Aryabhata I** to distinguish him from the later mathematician of the same name who lived about 400 years later.

The surviving text is Aryabhata's masterpiece the *Aryabhatiya* which is a small astronomical treatise written in 118 verses giving a summary of Hindu mathematics up to that time. Its mathematical section contains 33 verses giving 66 mathematical rules without proof.

The mathematical part of the *Aryabhatiya* covers arithmetic, algebra, plane trigonometry and spherical trigonometry. It also contains continued fractions, quadratic equations, sums of power series and a table of *sines*.

Chapter **5**

VALUES OF TRIGONOMETRIC FUNCTIONS FOR MULTIPLES AND SUBMULTIPLES

Geometry is not true, it is advantageous.

– Henri Poincare

Since the mathematicians have invaded the theory of relativity, I do not understand it myself anymore.

– Albert Einstein

5.1 Introduction

In this chapter we shall use addition formulae to obtain values of trigonometric functions for multiples like 2α, 3α etc. of α and for sub-multiples like $\frac{\alpha}{2}$ of α. Then we will obtain the values of trigonometric functions for some standard particular numbers and finally, we will use them for proving some conditional identities.

5.2 Trigonometric Functions of 2α

(1) Formula for $sin2\alpha$: For $\alpha, \beta \in R$,

$sin(\alpha + \beta) = sin\alpha \, cos\beta + cos\alpha \, sin\beta$

Substituting $\beta = \alpha$ in this formula,

$sin(\alpha + \alpha) = sin\alpha \, cos\alpha + cos\alpha \, sin\alpha$

$\therefore \quad \mathbf{sin2\alpha = 2sin\alpha \, cos\alpha}$ **(i)**

(2) Formulae for $cos2\alpha$: For $\alpha, \beta \in R$,

$cos(\alpha + \beta) = cos\alpha \, cos\beta - sin\alpha \, sin\beta$

Putting $\beta = \alpha$ in this we see that,

$cos(\alpha + \alpha) = cos\alpha \, cos\alpha - sin\alpha \, sin\alpha$

$\therefore \quad \mathbf{cos2\alpha = cos^2\alpha - sin^2\alpha}$ **(ii)**

$\therefore \quad cos2\alpha = cos^2\alpha - (1 - cos^2\alpha)$

$\therefore \quad \mathbf{cos2\alpha = 2cos^2\alpha - 1}$ **(iii)**

Again, $cos2\alpha = cos^2\alpha - sin^2\alpha$

$= 1 - sin^2\alpha - sin^2\alpha$

$\therefore \quad \mathbf{cos2\alpha = 1 - 2sin^2\alpha}$ **(iv)**

So we have, $\mathbf{cos2\alpha = cos^2\alpha - sin^2\alpha = 1 - 2sin^2\alpha = 2cos^2\alpha - 1}$

Thus, once $sin\alpha$ and $cos\alpha$ for $\alpha \in R$ are known, we can obtain the values of $sin2\alpha$ and $cos2\alpha$ using above formulae. Also values of *sine* and *cosine* functions for numbers that are twice of the given numbers can be obtained.

From (iii) and (iv) we have,

$\mathbf{1 + cos2\alpha = 2cos^2\alpha, \; 1 - cos2\alpha = 2sin^2\alpha}$

These are quite useful forms.

If we replace 2α by α (and so α by $\frac{\alpha}{2}$), we get

$$sin\alpha = 2sin\frac{\alpha}{2} \cdot cos\frac{\alpha}{2}$$

$$cos\alpha = cos^2\frac{\alpha}{2} - sin^2\frac{\alpha}{2}$$

Also we have $1 + cos\alpha = 2cos^2\frac{\alpha}{2}$ and $1 - cos\alpha = 2sin^2\frac{\alpha}{2}$

(3) $sin2\alpha$, $cos2\alpha$ and $tan2\alpha$ in terms of $tan\alpha$.

$$sin2\alpha = 2sin\alpha \cdot cos\alpha$$

$$= \frac{2sin\alpha \cdot cos\alpha}{cos^2\alpha + sin^2\alpha} \qquad (cos^2\alpha + sin^2\alpha = 1)$$

If $\alpha \in R - \left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\}$, then $cos\alpha \neq 0$. So let us divide both numerator and denominator by $cos^2\alpha$. Then,

$$sin2\alpha = \frac{2tan\alpha}{1 + tan^2\alpha} \qquad \textbf{(v)}$$

$$cos2\alpha = cos^2\alpha - sin^2\alpha$$

$$= \frac{cos^2\alpha - sin^2\alpha}{cos^2\alpha + sin^2\alpha}$$

Again, taking $\alpha \in R - \left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\}$, $cos\alpha \neq 0$, we divide both numerator and denominator by $cos^2\alpha$, to get

$$cos2\alpha = \frac{1 - tan^2\alpha}{1 + tan^2\alpha} \qquad \textbf{(vi)}$$

Now suppose α and 2α both are in the domain of *tan*. Then

$$tan2\alpha = tan(\alpha + \alpha)$$

$$= \frac{tan\alpha + tan\alpha}{1 - tan\alpha\, tan\alpha} \quad \left(\alpha \in R - \left[\left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\} \cup \left\{(2k-1)\frac{\pi}{4} \mid k \in Z\right\}\right]\right)$$

That is $tan2\alpha = \frac{2tan\alpha}{1 - tan^2\alpha}$ **(vii)**

Finally, assuming that α and 2α are in the domain of *cot*, we can similarly prove that

$$cot2\alpha = \frac{cot^2\alpha - 1}{2cot\alpha} \quad \left(\alpha \in R - \left\{\frac{k\pi}{2} \mid k \in Z\right\}\right) \qquad \textbf{(viii)}$$

Note that if $\alpha \neq \frac{k\pi}{2}$ for all $k \in Z$, then certainly $\alpha \neq k\pi$ for all $k \in Z$, because $k\pi = \frac{2k\pi}{2}$, $2k \in Z$.

Thus, if $\alpha \neq \frac{k\pi}{2}$ for all $k \in Z$, then $cot2\alpha = \frac{cot^2\alpha - 1}{2cot\alpha}$

In the results (v), (vi) and (vii), if we replace 2α by α (and so replace α by $\frac{\alpha}{2}$), we get

$$sin\alpha = \frac{2tan\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}}, \quad cos\alpha = \frac{1-tan^2\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}} \text{ and } tan\alpha = \frac{2tan\frac{\alpha}{2}}{1-tan^2\frac{\alpha}{2}}.$$

If we put $tan\frac{\alpha}{2} = t$, then above formulae become

$$sin\alpha = \frac{2t}{1+t^2}, \; cos\alpha = \frac{1-t^2}{1+t^2} \text{ and } tan\alpha = \frac{2t}{1-t^2}.$$

5.3 Trigonometric Functions of 3α

(1) $sin3\alpha = sin(2\alpha + \alpha)$

$= sin2\alpha \cdot cos\alpha + cos2\alpha \cdot sin\alpha$

$= (2sin\alpha \cdot cos\alpha) \cdot cos\alpha + (1 - 2sin^2\alpha) \cdot sin\alpha$

$= 2sin\alpha \cdot cos^2\alpha + sin\alpha - 2sin^3\alpha$

$= 2sin\alpha\,(1 - sin^2\alpha) + sin\alpha - 2sin^3\alpha$

$= 2sin\alpha - 2sin^3\alpha + sin\alpha - 2sin^3\alpha$

$= 3sin\alpha - 4sin^3\alpha$

$\therefore \quad sin3\alpha = 3sin\alpha - 4sin^3\alpha$ **(ix)**

(2) $cos3\alpha = cos(\alpha + 2\alpha)$

$= cos\alpha \cdot cos2\alpha - sin\alpha \cdot sin2\alpha$

$= cos\alpha \cdot (2cos^2\alpha - 1) - sin\alpha\,(2sin\alpha\,cos\alpha)$

$= 2cos^3\alpha - cos\alpha - 2cos\alpha \cdot sin^2\alpha$

$= 2cos^3\alpha - cos\alpha - 2cos\alpha\,(1 - cos^2\alpha)$

$= 2cos^3\alpha - cos\alpha - 2cos\alpha + 2cos^3\alpha$

$= 4cos^3\alpha - 3cos\alpha$

$\therefore \quad cos3\alpha = 4cos^3\alpha - 3cos\alpha$ **(x)**

(3) Taking α, 2α, 3α in the domain of *tan*,

that is $\alpha \neq (2k-1)\frac{\pi}{2}$, $\alpha \neq (2k-1)\frac{\pi}{4}$ and $\alpha \neq (2k-1)\frac{\pi}{6}$, $k \in Z$

(Remember that every odd multiple of $\frac{\pi}{2}$ is an odd multiple of $\frac{\pi}{6}$, for example $\frac{3\pi}{2} = \frac{9\pi}{6}$.

i.e. $\left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\} \subset \left\{(2k-1)\frac{\pi}{6} \mid k \in Z\right\}$)

$tan3\alpha = tan(2\alpha + \alpha)$

$$= \frac{tan2\alpha + tan\alpha}{1 - tan2\alpha\,tan\alpha}$$

$$= \frac{\frac{2tan\,\alpha}{1-tan^2\alpha} + tan\,\alpha}{1 - \frac{2tan\,\alpha}{1-tan^2\alpha} \cdot tan\,\alpha}$$

$$= \frac{2tan\,\alpha + tan\,\alpha - tan^3\alpha}{1 - tan^2\alpha - 2tan^2\alpha}$$

$$= \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}, \ \alpha \in R - \left\{(2k-1)\frac{\pi}{6}, \ k \in Z\right\}$$

$$\therefore \quad \tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}, \ \alpha \in R - \left\{(2k-1)\frac{\pi}{6}, \ k \in Z\right\} \qquad \text{(xi)}$$

This formula remains true even if 2α is in the domain of *tan*. ($2\alpha \in D_{tan}$)

Similarly, we can prove that if α, 2α and $3\alpha \in D_{cot}$, then

$$\alpha \neq k\pi, \ \alpha \neq \frac{k\pi}{2}, \ \alpha \neq \frac{k\pi}{3}, \ k \in Z$$

$$\{k\pi \mid k \in Z\} \subset \left\{\frac{k\pi}{3} \mid k \in Z\right\}$$

$$\cot 3\alpha = \frac{\cot^3\alpha - 3\cot\alpha}{3\cot^2\alpha - 1}, \ \alpha \in R - \left\{\frac{k\pi}{3} \mid k \in Z\right\} \qquad \text{(xii)}$$

Indeed, this is true for all $\alpha = \frac{k\pi}{2}$, $k \in Z$.

Thus, for any $\alpha \in R$, we can calculate $\sin 3\alpha$, $\cos 3\alpha$ and $\tan 3\alpha$, if $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ are given. Also values of trigonometric functions of 4α, 5α, ... etc. can be expressed in terms of trigonometric functions of α.

Example 1 : Prove : (1) $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$ (2) $\frac{\sin\theta + \cos\frac{\theta}{2}}{1+\sin\frac{\theta}{2} - \cos\theta} = \cot\frac{\theta}{2}$

(3) $\frac{\cos 2\theta}{1+\sin 2\theta} = \tan\left(\frac{\pi}{4} - \theta\right)$ (4) $\sec\theta + \tan\theta = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

Solution : (1) L.H.S. $= \frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \tan\theta$ = R.H.S.

(2) L.H.S. $= \frac{\sin\theta + \cos\frac{\theta}{2}}{1+\sin\frac{\theta}{2} - \cos\theta}$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \cos\frac{\theta}{2}}{\sin\frac{\theta}{2} + (1-\cos\theta)}$$

$$= \frac{\cos\frac{\theta}{2}\left(2\sin\frac{\theta}{2} + 1\right)}{\sin\frac{\theta}{2} + 2\sin^2\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2}\left(2\sin\frac{\theta}{2} + 1\right)}{\sin\frac{\theta}{2}\left(1 + 2\sin\frac{\theta}{2}\right)}$$

$$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2} = \text{R.H.S.}$$

(3) L.H.S. $= \frac{\cos 2\theta}{1+\sin 2\theta} = \frac{\sin\left(\frac{\pi}{2} - 2\theta\right)}{1+\cos\left(\frac{\pi}{2} - 2\theta\right)}$ $\left(\cos A = \sin\left(\frac{\pi}{2} - A\right), \ \sin A = \cos\left(\frac{\pi}{2} - A\right)\right)$

$$= \frac{2\sin\left(\frac{\pi}{4} - \theta\right)\cos\left(\frac{\pi}{4} - \theta\right)}{2\cos^2\left(\frac{\pi}{4} - \theta\right)} = \tan\left(\frac{\pi}{4} - \theta\right) = \text{R.H.S.}$$

(4) L.H.S. $= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$

$$= \frac{1+\sin\theta}{\cos\theta}$$

$$= \frac{1-\cos\left(\frac{\pi}{2}+\theta\right)}{\sin\left(\frac{\pi}{2}+\theta\right)}$$

$$= \frac{2\sin^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}$$

$$= \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \text{R.H.S.}$$

Example 2 : Express $\cos 4\theta$ in terms of $\cos\theta$ and $\sin 5\theta$ in terms of $\sin\theta$.

Solution :
$$\begin{aligned}
\cos 4\theta &= \cos 2(2\theta)\\
&= 2\cos^2 2\theta - 1\\
&= 2(2\cos^2\theta - 1)^2 - 1\\
&= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1\\
&= 8\cos^4\theta - 8\cos^2\theta + 1\\
\sin 5\theta &= (\sin 5\theta + \sin\theta) - \sin\theta\\
&= 2\sin 3\theta \cos 2\theta - \sin\theta\\
&= 2(3\sin\theta - 4\sin^3\theta)(1 - 2\sin^2\theta) - \sin\theta\\
&= 6\sin\theta - 12\sin^3\theta - 8\sin^3\theta + 16\sin^5\theta - \sin\theta\\
\therefore \quad \sin 5\theta &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta
\end{aligned}$$

Example 3 : Prove that $\cos A \cdot \cos(60^\circ - A)\cos(60^\circ + A) = \frac{1}{4}\cos 3A$ and use it to find the value of $\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 60^\circ \cdot \cos 80^\circ$.

Solution : L.H.S. $= \cos A \cdot \cos(60^\circ - A)\cos(60^\circ + A)$
$$\begin{aligned}
&= \cos A(\cos^2 60^\circ - \sin^2 A)\\
&= \cos A\left(\tfrac{1}{4} - \sin^2 A\right)\\
&= \cos A\left(\tfrac{1}{4} - (1 - \cos^2 A)\right)\\
&= \cos A\left(-\tfrac{3}{4} + \cos^2 A\right)\\
&= \tfrac{1}{4}\left(4\cos^3 A - 3\cos A\right)\\
&= \tfrac{1}{4}\cos 3A = \text{R.H.S.}
\end{aligned}$$

Now, $\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 60^\circ \cdot \cos 80^\circ = \frac{1}{2}(\cos 20^\circ \cdot \cos(60^\circ + 20^\circ)\cos(60^\circ - 20^\circ))$

$$= \frac{1}{2}\left[\frac{1}{4}\cos 3(20^\circ)\right] \qquad \textbf{(A = 20°)}$$

$$= \frac{1}{8}\cos 60^\circ = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16}$$

Example 4 : Prove that $\cos^3\theta + \cos^3\left(\frac{2\pi}{3}+\theta\right) + \cos^3\left(\frac{4\pi}{3}+\theta\right) = \frac{3}{4}\cos 3\theta$.

Solution : We know that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. So, $\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$

$$\text{L.H.S.} = \cos^3\theta + \cos^3\left(\frac{2\pi}{3}+\theta\right) + \cos^3\left(\frac{4\pi}{3}+\theta\right)$$

$$= \frac{1}{4}[\cos3\theta + 3\cos\theta] + \frac{1}{4}\left[\cos(2\pi + 3\theta) + 3\cos\left(\frac{2\pi}{3}+\theta\right)\right]$$

$$+ \frac{1}{4}\left[\cos(4\pi + 3\theta) + 3\cos\left(\frac{4\pi}{3}+\theta\right)\right]$$

$$= \frac{1}{4}[\cos3\theta + 3\cos\theta] + \frac{1}{4}\left[\cos3\theta + 3\cos\left(\frac{2\pi}{3}+\theta\right)\right]$$

$$+ \frac{1}{4}\left[\cos3\theta + 3\cos\left(\frac{4\pi}{3}+\theta\right)\right]$$

$$= \frac{3}{4}\cos3\theta + \frac{3}{4}\left[\cos\theta + \cos\left(\frac{2\pi}{3}+\theta\right) + \cos\left(\frac{4\pi}{3}+\theta\right)\right]$$

$$= \frac{3}{4}\cos3\theta + \frac{3}{4}\left[\cos\theta + 2\cos(\pi + \theta)\cos\frac{\pi}{3}\right]$$

$$= \frac{3}{4}\cos3\theta + \frac{3}{4}\left[\cos\theta - 2\cos\theta \times \frac{1}{2}\right]$$

$$= \frac{3}{4}\cos3\theta + \frac{3}{4}(\cos\theta - \cos\theta) = \frac{3}{4}\cos3\theta = \text{R.H.S.}$$

Example 5 : Prove that $\cos A \cdot \cos 2A \cdot \cos 2^2A \cdot \cos 2^3A \cdot \ldots \cdot \cos 2^{n-1}A = \frac{\sin 2^n A}{2^n \cdot \sin A}$ and use it to find the value of $\cos\frac{2\pi}{15} \cdot \cos\frac{4\pi}{15} \cdot \cos\frac{8\pi}{15} \cdot \cos\frac{14\pi}{15}$.

Solution : $\sin2\theta = 2\sin\theta\cos\theta$

$$\therefore \quad \cos\theta = \frac{\sin2\theta}{2\sin\theta}$$

$$\text{L.H.S.} = \cos A \cdot \cos 2A \cdot \cos 2^2A \cdot \cos 2^3A \cdot \ldots \cdot \cos 2^{n-1}A$$

$$= \frac{\sin2A}{2\sin A} \cdot \frac{\sin2(2A)}{2\sin2A} \cdot \frac{\sin2(2^2A)}{2\sin2^2A} \cdot \frac{\sin2(2^3A)}{2\sin2^3A} \cdot \ldots \cdot \frac{\sin2(2^{n-1}A)}{2\sin2^{n-1}A}$$

$$= \frac{\sin2(2^{n-1}A)}{2^n \cdot \sin A} = \frac{\sin 2^n A}{2^n \cdot \sin A} = \text{R.H.S.}$$

$$\cos\frac{2\pi}{15} \cdot \cos\frac{4\pi}{15} \cdot \cos\frac{8\pi}{15} \cdot \cos\frac{14\pi}{15} = -\cos\frac{2\pi}{15} \cdot \cos\frac{4\pi}{15} \cdot \cos\frac{8\pi}{15} \cdot \cos\frac{\pi}{15}$$

$$\left(\cos\frac{14\pi}{15} = \cos\left(\pi - \frac{\pi}{15}\right) = -\cos\frac{\pi}{15}\right)$$

$$= -\frac{\sin\frac{16\pi}{15}}{16\sin\frac{\pi}{15}}$$

$$= -\frac{\sin\left(\pi + \frac{\pi}{15}\right)}{16\sin\frac{\pi}{15}}$$

$$= \frac{\sin\frac{\pi}{15}}{16\sin\frac{\pi}{15}} = \frac{1}{16}$$

Exercise 5.1

Prove **(1 to 19) :**

1. $\frac{\sin2\theta}{1-\cos2\theta} = \cot\theta$

2. $\frac{\cos2\theta}{1+\sin2\theta} = \cot\left(\frac{\pi}{4}+\theta\right)$

3. $\tan\frac{\theta}{2} + \cot\frac{\theta}{2} = 2\operatorname{cosec}\theta$

4. $\frac{\cos\theta}{1+\sin\theta} = \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

5. $\dfrac{1+sin2\theta+cos2\theta}{1+sin2\theta \quad cos2\theta} = cot\theta$

6. $tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = 2sec\theta$

7. $\dfrac{cot\theta - tan\theta}{1-2sin^2\theta} = sec\theta \cdot cosec\theta = 2cosec2\theta$

8. $sec2\theta - tan2\theta = tan\left(\frac{\pi}{4}-\theta\right)$

9. $\dfrac{sin\,5\theta - 2sin\,3\theta + sin\,\theta}{cos\,5\theta - cos\,\theta} = tan\theta$

10. $\dfrac{sin\theta - sin3\theta}{sin^2\theta - cos^2\theta} = 2sin\theta$

11. $\sqrt{3}cosec20^{\circ} - sec20^{\circ} = 4$

12. $2(cos^8\theta - sin^8\theta) = cos2\theta + cos^32\theta$

13. If $tan\alpha = \frac{1}{3}$ and $tan\frac{\beta}{2} = \frac{1}{2}$, then $tan(\alpha + \beta) = 3$.

14. If $cos\theta = \frac{1}{2}\left(x+\frac{1}{x}\right)$, then $cos2\theta = \frac{1}{2}\left(x^2+\frac{1}{x^2}\right)$ and $cos3\theta = \frac{1}{2}\left(x^3+\frac{1}{x^3}\right)$.

15. $\dfrac{sin^2A - sin^2B}{sin2A - sin2B} = \frac{1}{2}tan(A + B)$

16. $\dfrac{sin\,3\theta}{sin\,\theta} - \dfrac{cos3\theta}{cos\theta} = 2$

17. $\dfrac{cos3\theta}{cos\theta} + \dfrac{sin3\theta}{sin\theta} = 4cos2\theta$

18. $cos^3\theta\ sin3\theta + sin^3\theta\ cos3\theta = \frac{3}{4}sin4\theta$

19. $cos^3\theta\ cos3\theta + sin^3\theta\ sin3\theta = cos^32\theta$

20. If $sinA = \frac{3}{5}$, $0 < A < \frac{\pi}{2}$ then find the value of $sin2A$, $cos2A$, $tan2A$ and $sin4A$.

21. If $15\theta = \pi$, then prove that $cos\theta \cdot cos2\theta \cdot cos3\theta \cdot cos4\theta \cdot cos5\theta \cdot cos6\theta \cdot cos7\theta = \frac{1}{128}$.

22. Show that $\sqrt{2+\sqrt{2+\sqrt{2+2cos8\theta}}} = 2cos\theta$, where $0 < \theta < \frac{\pi}{8}$.

23. Prove that $tan\theta + tan\left(\frac{\pi}{3}+\theta\right) + tan\left(\frac{2\pi}{3}+\theta\right) = 3tan3\theta$ and deduce that

$tan20^{\circ} + tan80^{\circ} + tan140^{\circ} = 3\sqrt{3}$.

24. Prove that $tan\theta \cdot tan\left(\frac{\pi}{3}+\theta\right) \cdot tan\left(\frac{\pi}{3}-\theta\right) = tan3\theta$ and deduce that

$tan6^{\circ} \cdot tan42^{\circ} \cdot tan66^{\circ} \cdot tan78^{\circ} = 1$.

25. Prove : $cos6\theta = 32cos^6\theta - 48cos^4\theta + 18cos^2\theta - 1$

*

5.4 Trigonometric Functions of $\frac{\alpha}{2}$ in Terms of $cos\alpha$

(1) We know that $cos2\alpha = 1 - 2sin^2\alpha$. If we put α in place of 2α (and $\frac{\alpha}{2}$ in place of α), we get

$cos\alpha = 1 - 2sin^2\frac{\alpha}{2}$.

$\therefore \quad 2sin^2\frac{\alpha}{2} = 1 - cos\alpha$

$\therefore \quad sin^2\frac{\alpha}{2} = \frac{1 - cos\alpha}{2}$

(2) Similarly, substituting α in place of 2α (and $\frac{\alpha}{2}$ in place of α) in $cos2\alpha = 2cos^2\alpha - 1$

$\therefore \quad 2cos^2\frac{\alpha}{2} = 1 + cos\alpha$

$\therefore \quad cos^2\frac{\alpha}{2} = \frac{1 + cos\alpha}{2}$

(3) $tan^2\frac{\alpha}{2} = \frac{sin^2\frac{\alpha}{2}}{cos^2\frac{\alpha}{2}} = \frac{\frac{1-cos\alpha}{2}}{\frac{1+cos\alpha}{2}}$ **($\alpha \neq (2k - 1)\pi$; $k \in Z$)**

$\therefore \quad tan^2\frac{\alpha}{2} = \frac{1 - cos\alpha}{1 + cos\alpha}$

5.5 Values of Trigonometric Functions for Some Special Numbers

(1) $sin18^o$:

Suppose $\theta = 18^o$

$\therefore \quad 5\theta = 90^o$

$\therefore \quad 3\theta + 2\theta = 90^o$

$\therefore \quad 2\theta = 90^o - 3\theta$

$\therefore \quad sin2\theta = sin(90^o - 3\theta)$

$\therefore \quad sin2\theta = cos3\theta$

$\therefore \quad 2sin\theta\, cos\theta = 4cos^3\theta - 3cos\theta$

$\therefore \quad 2sin\theta = 4cos^2\theta - 3$ **($cos18^o \neq 0$)**

$\therefore \quad 2sin\theta = 4(1 - sin^2\theta) - 3$

$\therefore \quad 2sin\theta = 4 - 4sin^2\theta - 3$

$\therefore \quad 4sin^2\theta + 2sin\theta - 1 = 0$

$$\therefore \quad sin\theta = \frac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)}$$

$$= \frac{-2 \pm \sqrt{20}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

Here $\theta = 18^o$. Hence, $P(\theta)$ is in the first quadrant.

$\therefore \quad sin\theta > 0$

$$\therefore \quad sin18^o = \frac{\sqrt{5} - 1}{4}$$

(2) $cos18^o$:

Substituting $\theta = 18^o$ in $cos^2\theta = 1 - sin^2\theta$, we get

$cos^2 18^o = 1 - sin^2 18^o$

$$= 1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2 = \frac{16 - 5 + 2\sqrt{5} - 1}{16}$$

$\therefore \quad cos^2 18^o = \frac{10+2\sqrt{5}}{16}$

$\therefore \quad cos 18^o = \sqrt{\frac{10+2\sqrt{5}}{16}}$ **(0 < 18 < 90. So *cos*18^o > 0)**

(3) *cos*36^o :

Substituting $\theta = 18^o$ in $cos 2\theta = 1 - 2sin^2\theta$, we get

$$cos 36^o = 1 - 2sin^2 18^o$$

$$= 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= 1 - 2\left(\frac{5-2\sqrt{5}+1}{16}\right)$$

$$= \frac{8-5+2\sqrt{5}-1}{8} = \frac{2+2\sqrt{5}}{8} = \frac{\sqrt{5}+1}{4}$$

$$\therefore \quad cos 36^o = \frac{\sqrt{5}+1}{4}$$

(4) *sin*36^o :

Substituting $\theta = 36^o$ in $sin^2\theta = 1 - cos^2\theta$, we get

$$sin^2 36^o = 1 - cos^2 36^o$$

$$= 1 - \left(\frac{\sqrt{5}+1}{4}\right)^2$$

$$= 1 - \left(\frac{5+2\sqrt{5}+1}{16}\right) = \frac{16-6-2\sqrt{5}}{16} = \frac{10-2\sqrt{5}}{16}$$

$\therefore \quad sin 36^o = \sqrt{\frac{10-2\sqrt{5}}{16}}$ **(0 < 36 < 90. So *sin*36^o > 0)**

We can similarly get *sine*s and *cosine*s of multiples of 18 like 54, 72, 144 etc. In fact,

$$sin 72^o = sin(90^o - 18^o) = cos 18^o = \sqrt{\frac{10+2\sqrt{5}}{16}}$$

and $\quad sin 54^o = sin(90^o - 36^o) = cos 36^o = \frac{\sqrt{5}+1}{4}$

Similarly, $cos 72^o = sin 18^o$ and $cos 54^o = sin 36^o$

(5) *sin*$22\frac{1}{2}^o$ or *sin*$\frac{\pi}{8}$:

Putting $\theta = \frac{45}{2}^o$ in $sin^2\theta = \frac{1-cos 2\theta}{2}$, we get

$$sin^2 \frac{45}{2}^o = \frac{1-cos 45^o}{2}$$

$$= \frac{1-\frac{1}{\sqrt{2}}}{2}$$

$$= \frac{\sqrt{2}-1}{2\sqrt{2}}$$

$$= \frac{\sqrt{2}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2-\sqrt{2}}{4}$$

$\therefore \quad sin \frac{45}{2}^o = \frac{\sqrt{2-\sqrt{2}}}{2}$ **(0 < $22\frac{1}{2}$ < 90. So *sin*$22\frac{1}{2}^o$ > 0)**

(6) In the same way, we get $cos22\frac{1}{2}^{o} = \frac{\sqrt{2+\sqrt{2}}}{2}$

(7) $tan22\frac{1}{2}^{o}$:

$$tan^2 22\frac{1}{2}^{o} = tan^2\frac{45}{2}^{o} = \frac{1-cos45}{1+cos45}$$

$$= \frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{(\sqrt{2}-1)^2}{2-1}$$

Now, $0 < 22\frac{1}{2} < 90$. Hence, $tan22\frac{1}{2}^{o} > 0$

$\therefore \quad tan22\frac{1}{2}^{o} = \sqrt{2} - 1$

Similarly, we can show that $cot22\frac{1}{2}^{o} = \sqrt{2} + 1$. We can also get value of *sines* and *cosines* of $67\frac{1}{2}^{o}$ etc.

$cos67\frac{1}{2}^{o} = sin22\frac{1}{2}^{o}$, $sin67\frac{1}{2}^{o} = cos22\frac{1}{2}^{o}$ and $tan67\frac{1}{2}^{o} = cot22\frac{1}{2}^{o}$

Example 6 : If $cot\theta = \frac{-5}{12}$, $\frac{\pi}{2} < \theta < \pi$, then find the value of $sin\frac{\theta}{2} + cos\frac{\theta}{2}$.

Solution : Since $cot\theta = \frac{-5}{12}$, $tan\theta = \frac{-12}{5}$

$\therefore \quad sec^2\theta = 1 + tan^2\theta = 1 + \frac{144}{25} = \frac{169}{25}$

$\therefore \quad sec\theta = \pm\frac{13}{5}$. Since $\frac{\pi}{2} < \theta < \pi$, $sec\theta < 0$

$\therefore \quad sec\theta = -\frac{13}{5}$. Hence, $cos\theta = \frac{-5}{13}$

Now, $sin^2\frac{\theta}{2} = \frac{1-cos\,\theta}{2} = \frac{1+\frac{5}{13}}{2} = \frac{18}{26} = \frac{9}{13}$

Since $\frac{\pi}{2} < \theta < \pi$, $\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}$. So $sin\frac{\theta}{2} > 0$

$\therefore \quad sin\frac{\theta}{2} = \frac{3}{\sqrt{13}}$

$cos^2\frac{\theta}{2} = \frac{1+cos\,\theta}{2} = \frac{1-\frac{5}{13}}{2} = \frac{8}{26} = \frac{4}{13}$

$\therefore \quad cos\frac{\theta}{2} = \frac{2}{\sqrt{13}}$ $\quad \left(\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}\right)$

$\therefore \quad sin\frac{\theta}{2} + cos\frac{\theta}{2} = \frac{3}{\sqrt{13}} + \frac{2}{\sqrt{13}} = \frac{5}{\sqrt{13}}$

Example 7 : Prove that $sin^4\frac{\pi}{8} + sin^4\frac{3\pi}{8} + sin^4\frac{5\pi}{8} + sin^4\frac{7\pi}{8} = \frac{3}{2}$.

Solution : L.H.S. $= sin^4\frac{\pi}{8} + sin^4\frac{3\pi}{8} + sin^4\frac{5\pi}{8} + sin^4\frac{7\pi}{8}$

$$= sin^4\frac{\pi}{8} + sin^4\frac{3\pi}{8} + sin^4\left(\pi-\frac{3\pi}{8}\right) + sin^4\left(\pi-\frac{\pi}{8}\right)$$

$$= 2\left(sin^4\frac{\pi}{8} + sin^4\frac{3\pi}{8}\right)$$

$$= 2\left[\left(sin^2\frac{\pi}{8}\right)^2 + \left(sin^2\frac{3\pi}{8}\right)^2\right]$$

$$= 2\left[\left(\frac{1-cos\frac{\pi}{4}}{2}\right)^2 + \left(\frac{1-cos\frac{3\pi}{4}}{2}\right)^2\right] \quad \left(sin^2\theta = \frac{1-cos\,2\theta}{2}\right)$$

$$= \frac{2}{4}\left[\left(1 - \cos\frac{\pi}{4}\right)^2 + \left(1 - \cos\frac{3\pi}{4}\right)^2\right]$$

$$= \frac{1}{2}\left[\left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} - \sqrt{2} + 1 + \frac{1}{2} + \sqrt{2}\right]$$

$$= \frac{3}{2} = \text{R.H.S.}$$

Example 8 : If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$, prove that

(1) $\cos(\alpha - \beta) = \dfrac{a^2 + b^2 - 2}{2}$ (2) $\tan\left(\dfrac{\alpha-\beta}{2}\right) = \pm\sqrt{\dfrac{4 - a^2 - b^2}{a^2 + b^2}}$

Solution : (1) We have $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$

Squaring and adding,

$$(\sin\alpha + \sin\beta)^2 + (\cos\alpha + \cos\beta)^2 = a^2 + b^2$$

$\therefore$ $\sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta + \cos^2\alpha + 2\cos\alpha\cos\beta + \cos^2\beta = a^2 + b^2$

$\therefore$ $2 + 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = a^2 + b^2$

$\therefore$ $2 + 2\cos(\alpha - \beta) = a^2 + b^2$

$\therefore$ $\cos(\alpha - \beta) = \dfrac{a^2 + b^2 - 2}{2}$

(2) Now, $\tan^2\left(\dfrac{\alpha-\beta}{2}\right) = \dfrac{1 - \cos(\alpha-\beta)}{1 + \cos(\alpha-\beta)}$

$$\tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1 - \frac{a^2+b^2-2}{2}}{1 + \frac{a^2+b^2-2}{2}}$$

$$\tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{4 - a^2 - b^2}{a^2 + b^2}$$

$\therefore$ $\tan\left(\dfrac{\alpha-\beta}{2}\right) = \pm\sqrt{\dfrac{4 - a^2 - b^2}{a^2 + b^2}}$

Example 9 : Prove $\sin^4\theta \cdot \cos^4\theta = \frac{1}{128} : (3 - 4\cos4\theta + \cos8\theta)$

Solution : $\sin^4\theta \cdot \cos^4\theta = (\sin\theta\cos\theta)^4$

$$= \frac{1}{16}(2\sin\theta\cos\theta)^4$$

$$= \frac{1}{16}(\sin2\theta)^4$$

$$= \frac{1}{16}(\sin^2 2\theta)^2$$

$$= \frac{1}{16}\left(\frac{1-\cos4\theta}{2}\right)^2$$

$$= \frac{1}{64}(1 - 2\cos4\theta + \cos^2 4\theta)$$

$$= \frac{1}{64}\left(1 - 2\cos4\theta + \frac{1+\cos8\theta}{2}\right)$$

$$= \frac{1}{128}(2 - 4\cos4\theta + 1 + \cos8\theta)$$

$$= \frac{1}{128}(3 - 4\cos4\theta + \cos8\theta)$$

Exercise 5.2

1. If $\tan x = \frac{3}{4}$, $\pi < x < \frac{3\pi}{2}$, find the values of $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$.

2. If $\cos\alpha = \frac{3}{5}$, $\cos\beta = \frac{5}{13}$, $0 < \alpha, \beta < \frac{\pi}{2}$, then find the values of $\sin^2\left(\frac{\alpha-\beta}{2}\right)$ and $\cos^2\left(\frac{\alpha-\beta}{2}\right)$.

Prove : **(3 to 12)**

3. $\cos^6\theta - \sin^6\theta = \frac{1}{4}(\cos^3 2\theta + 3\cos 2\theta)$

4. $\cos^2 A + \cos^2\left(A + \frac{2\pi}{3}\right) + \cos^2\left(A - \frac{2\pi}{3}\right) = \frac{3}{2}$

5. $\sin^2 A + \sin^2\left(A + \frac{2\pi}{3}\right) + \sin^2\left(A + \frac{4\pi}{3}\right) = \frac{3}{2}$. Deduce this from example 4.

6. $\left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right)\left(1 + \cos\frac{7\pi}{8}\right) = \frac{1}{8}$

7. $\sin^4\theta \cdot \cos^2\theta = \frac{1}{32}\,[2 - \cos 2\theta - 2\cos 4\theta + \cos 6\theta]$

8. $\sin^6\theta = \frac{1}{32}\,[10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta]$

9. $\sin 6^\circ \cdot \sin 42^\circ \cdot \sin 66^\circ \cdot \sin 78^\circ = \frac{1}{16}$

10. $\cos 6^\circ \cdot \cos 42^\circ \cdot \cos 66^\circ \cdot \cos 78^\circ = \frac{1}{16}$

11. $16\cos\frac{2\pi}{15} \cdot \cos\frac{4\pi}{15} \cdot \cos\frac{8\pi}{15} \cdot \cos\frac{14\pi}{15} = 1$

12. $\left(1 + \cos\frac{\pi}{10}\right)\left(1 + \cos\frac{3\pi}{10}\right)\left(1 + \cos\frac{7\pi}{10}\right)\left(1 + \cos\frac{9\pi}{10}\right) = \frac{1}{16}$

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5.6 Conditional Identities

Now we shall discuss some identities satisfying certain conditions.

e.g. $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$ for $A + B + C = \pi$. This identity is true for all A, B, C satisfying the condition $A + B + C = \pi$. Therefore, this identity is called a **conditional identity**. On the other hand $\sin^2 A + \cos^2 A = 1$ is true for every A without any condition. This is an example of an unconditional identity.

Most of the relations, relating to the angles of a triangle are of the type of conditional identities. They are useful in understanding the properties of a triangle. Here we need to keep the following in mind.

$A + B + C = \pi$

$\therefore \quad A + B = \pi - C \quad \text{and} \quad \frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$

$\therefore \quad \sin(A + B) = \sin(\pi - C) \quad \text{and} \quad \sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right)$

$\therefore \quad \sin(A + B) = \sin C \quad \text{and} \quad \sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}$

In the same way,

$$\cos(A + B) = -\cos C \quad \text{and} \quad \cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$$