MATHEMATICS

CHAPTER - LINEAR ALGEBRA

Properties of Determinant of a matrix

1. Defination : The sum of products of elements of row or column with the corresponding cofactor is known as determinat value of square matrix.

2. $|A^T| = |A|$

   - $|AB| = |A|\,|B|$
   - $|A+B| \neq |A| + |B|$

3. In a square matrix, if each element of a raw or column is zero than value of its determinant is zero.

4. In a square matrix, if two raw or column are identical or in proportion, then determinant is zero.

5. The determinant value of a skew symmetrix matrix of odd order is always zero.

   - Symmetric matrix if $A^T = A$
   - Skew symmetric matrix if $A^T = -A$
   - orthogonal matrix if $A^T A = I$

6. Every matrix can be expressed as sum of symmetric and skew symmetric matrix

$$A = \underbrace{\left(\frac{A + A^T}{2}\right)}_{\text{Symmetric}} + \underbrace{\left(\frac{A - A^T}{2}\right)}_{\text{Skew symmetric}}$$

7. Determinant value of nonzero skew symmetric matrix of even order have perfect square as its determinant.

8. The determinant value of an orthogonal matrix is always either 1 or -1.

9. If A is a square matrix of order 'n' and K is any scalar then, $|KA| = K^n |A|$

10. If A is a non singular [i.e. $|A| \neq 0$] matrix of order 'n' then
   - $A(\text{adj.}A) = |A| I$ [$\because$ adj.A = transpose of cofactor matrix]
   - $A^{-1} = \dfrac{\text{Adj.}A}{|A|}$
   - $|\text{adj.}A| = |A|^{n-1}$
   - $|[\text{Adj.}(\text{adj}A)]| = |A|^{(n-1)^2}$
   - $|A^{-1}| = \dfrac{1}{|A|}$

11. if $A_{3\times4}\, B_{4\times5} = P_{3\times5}$
   then, no. of multiplications $= 3\times4\times5$
   no. of additions $= 3\times5\times(4-1)$

12. If DABEC = I then A^{-1} = BECD

Properties of Rank of a matrix

1. Defination: The order of highest ordered non-zero minor is called rank of matrix.
2. Rank of null matrix is zero
3. Rank of non-singular matrix is its order
4. If A is m×n matrix, $\rho(A) \leq \min\{m,n\}$
5. If A & B are two matrices of same order then
   $\rho(A+B) \leq \rho(A) + \rho(B)$
6. $\rho(A) = \rho(A^T)$
7. $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$

8. In a matrix if all rows or columns are identical or proportional then rank is 1.

9. A is $n \times n$ matrix with rank n then $\rho(\text{Adj } A) = n$

10. A is $n \times n$ matrix & $\rho(A) = n-1$ then $\rho(\text{adj } A) = 1$

11. A is $n \times n$ matrix & $\rho(A) \leq n-2$ then $\rho(\text{adj} A) = 0$

Echelon form

- The number of zeros before non-zero elements in a row are less than such no. of zeros in next row.
- zero rows (if any) must follow non-zero rows.

→ The no. of non-zero rows in above form is known as its rank.

* only row operations are allowed to get row echelon form.

System of Linear Equations

Homogenous

$AX = 0$ [always consistent]

- Trivial soln → zero soln
  - $\rho(A) = n$
  - $|A| \neq 0$
- non-trivial soln → non-zero soln
  - $\rho(A) < n$
  - $|A| = 0$

non-Homogenous

$AX = B$

- Inconsistent → no soln
  - $\rho(A) \neq \rho(A|B)$
- Consistent
  - Unique soln
    - $\rho(A) = \rho(AB) = n$
    - $|A| \neq 0$
  - Infinite soln
    - $\rho(A) = \rho(AB) < n$
    - $|A| = 0$

- Free variable
- no. of independent variable
- Dimension of null space
- Dimension of space of soln
- nulity

= Total no. of variables − Rank

OR

Total no. of columns − Rank

* Rank gives no. of independent vectors
* nullity gives no. of independent variables

Linearly dependent & independent vectors

- 2 vectors X_1 & X_2 are said to be linearly dependent vector if one can be expressed as scaler multiple of other. otherwise they are linearly independent.
- If $\rho(A) = n$ or $|A| \neq 0$ then set of vector are linearly independent.
- If $\rho(A) < n$ & $|A| = 0$ $\Rightarrow$ vectors are dependent.
- if $X_1^T . X_2 = 0$ $\Rightarrow$ X_1 & X_2 are orthogonal vectors
- if $\rightarrow X_1^T . X_2 = 0$; $\rightarrow ||X_1|| = ||X_2|| = 1$ $\Big]\Rightarrow$ orthonormal vectors
- normalised vector $X_1 \Rightarrow$ given by $\dfrac{X_1}{||X_1||}$

Basis of vector space

- The Set $S = \{X_1, X_2, X_3, X_4 --- X_n\}$ is said to be Basis of vector Space V if it satisfies below 2 conditions
  1. $X_1, X_2, X_3 --- X_n$ are linearly independent
  2. S spans vector space V.

* if determinant of vector set S is non zero then both conditions will be automatically satisfied.

* $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is standard basis of R^2

$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is standard basis of R^3

Properties of Eigen Values & Eigen Vectors.

Defination :- The roots of characteristic equation $|A-\lambda I|=0$ are known as Eigen values.

- To each value (Eigen) there exists a non-zero vector X, such that $AX=\lambda X$, Then X is called Eigen vector. [use $(A-\lambda I)X=0$ for questions]

1. Eigen vector corresponding to distinct eigen values of a real symmetric matrix are always orthogonal
2. If all the leading minors of a real symmetric matrix are positive, then all its eigen values are positive.
3. The Eigen vector corresponing to distinct Eigen values of any square matrix are always linearly independent.
4. Eigen vectors corresponding to repeated eigen values may or may not be independent.
   - Algebric multiplicity (AM) : no. of times a Eigen value is repeated
   - Geometric multiplicity (GM) : no. of linearly independent Eigen vectors correspond to repeated Eigen value.
5. Sum of Eigen values = Trace of matrix
6. Product of Eigen values = $|A|$
7. Eigen values of A = Eigen values of A^T
8. Eigen values of a diagonal matrix or triangular matric are its diagonal elements.
9. If λ is eigen value of A.
   - $\frac{1}{\lambda}$ is eigen value of A^{-1}
   - $\frac{|A|}{\lambda}$ is eigen value of (adj. A)
10. If λ is eigen value of A then eigen value of
    - $A^2 = \lambda^2$

- $A^n \Rightarrow \lambda^n$
- $KA \Rightarrow K\lambda$
- $A + KI \Rightarrow \lambda + K$
- $A - KI \Rightarrow \lambda - K$
- $A^2 + C_1 A + C_2 \Rightarrow \lambda^2 + C_1\lambda + C_2$

11. In a square matrix if n rows are identical then (n-1) eigen values will be zero.

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow$ eigen values $0, 2$

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow$ eigen values $0, 0, 3$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow$ eigen values $0, 0, 0, 4$

If, in a matrix there are sets of identical rows, then igen values will be calculated considering seperately

12. If matrix is 4×4, 6×6, 8×8 ....

$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ if matrix is in such forms, calculate eigen values of B & C only.

$\begin{bmatrix} B & D \\ 0 & C \end{bmatrix}$

$\begin{bmatrix} B & 0 \\ D & C \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right] \equiv \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$$

<u>Tranjugate of matrix</u> (A^θ)

<u>Defination</u>: transpose of conjugate of matrix is knows as tranjugate of a matrix.

\# Square matrix is said to be

1. Hermitian matrix if $A^\theta = A$
2. Skew Hermitian if $A^\theta = -A$
3. Unitary matrix if $AA^\theta = I$

- Eigen values of Hermitian or Symmetric matrix are always real.
- Eigen values of Skew Hermitian or Skew symmetric are either zero or purely imaginary.
- The eigen values of unitary matrix or orthogonal matrix have absolute value i.e. $|\lambda| = 1$

Cayley Hamilton theorem

Every square matrix satisfies its characteristic equation.

$|A - \lambda I| = 0 \Rightarrow a\lambda^2 + b\lambda + c = 0 \longrightarrow \frac{c}{a} = |A| = \lambda_1\lambda_2$ always

$$\Downarrow$$

$$aA^2 + bA + c = 0$$

CHAPTER-2 PROBABILITY & STATISTICS

- two events A & B are equally likely if $P(A) = P(B)$
- two events A & B are mutually exclusive if $P(A \cap B) = 0$
- two events A & B are independent if $P(A \cap B) = P(A) \cdot P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional Probability

The conditional Probability of a event B, given A, is denoted by $P\left(\frac{B}{A}\right)$ and is given by

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}$$

Random Variable

Discrete RV

- it takes countable values
- Probability are given by Probability mass funⁿ.

$\Sigma P(x) = 1$

Continous R.V.

- It takes values in interval
- Probability are given by probability density funⁿ

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

mean (μ) or Expectation [$E(x)$]

$$\mu = E(x) = \begin{cases} \sum x . P(x) & \text{if } x \text{ is Discrete.} \\ \int_{-\infty}^{\infty} x . f(x) dx & \text{if } x \text{ is continous.} \end{cases}$$

- $E(c) = c$
- $E(cx) = c E(x)$
- $E(X+Y) = E(X) + E(Y)$
- $E(XY) = E(X) . E(Y)$ if X & Y are independent

Variance (Var)

$$Var(x) = E(x-\mu)^2 = \begin{cases} \sum (x-\mu)^2 . P(x) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} (x-\mu)^2 . f(x) dx & \text{if } x \text{ is continous} \end{cases}$$

- $Var(x) = E(x^2) - (E(x))^2$
- $Var(c) = 0$
- $Var(cx) = c^2 Var(x)$
- $Var(-x) = (-1)^2 Var(x) = Var(x)$
- Standard deviation $= \sqrt{\text{variance}}$
- $Var.(X+Y) = Var(X) + Var(Y)$ if X & Y are independent

1. Binomial Distribution

Probability mass function $= P(x) = {}^nC_x p^x q^{n-x}$

μ = mean = np

var = npq

here, p = prob. of success

q = prob. of failure

$p + q = 1$

- $(1-t)^{-2} = 1 + 2t + 3t^2 + 4t^3 \ldots$

2. Poisson Distribution

Probability mass function $= P(x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$ $\quad [x \geq 0]$

$\mu = \text{mean} = \lambda$

$\text{var} = \lambda$

- Poisson Distribution is the limiting case of Binomial Distribution i.e. if $n =$ large, $p =$ small then $\lambda = np$

3. Uniform Distribution

If X follows uniform Distribution in interval $[\alpha, \beta]$

$f(x) = \dfrac{1}{\beta - \alpha}$ = Probability Density funcⁿ.

$\mu = \text{mean} = \dfrac{\alpha + \beta}{2}$

$\text{var} = \dfrac{(\beta - \alpha)^2}{12}$

4. Exponential Distribution

Probability Density function $= f(x) = \lambda e^{-\lambda x}$ $\quad [x \geq 0]$

$\mu = \text{mean} = \dfrac{1}{\lambda}$

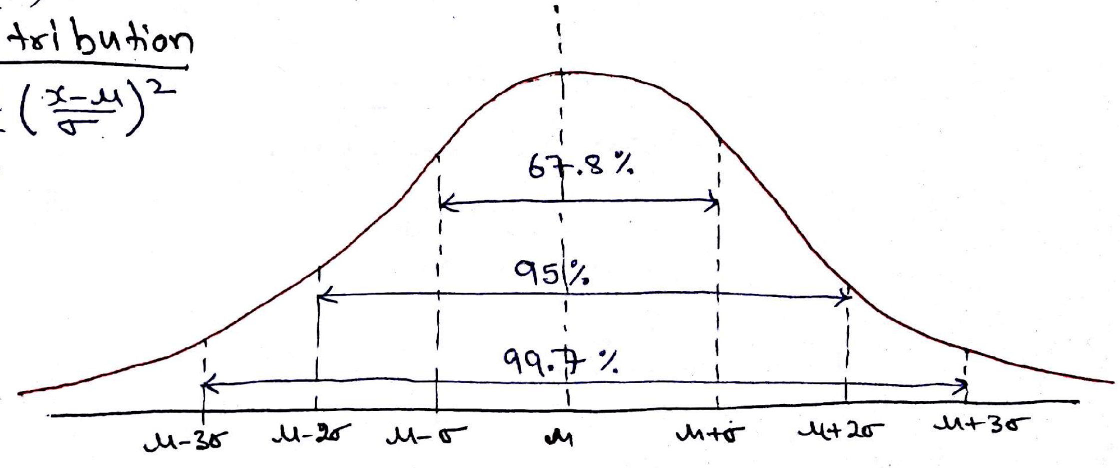
$\text{var} = \left(\dfrac{1}{\lambda}\right)^2$

5. Normal Distribution

$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\mu = \text{mean} = \mu$

$\text{var} = \sigma^2$



Coefficient of co-relation (r)

$$r = \frac{S_{xy}}{\sqrt{S_{xx} \cdot S_{yy}}}$$

here,
$$S_{xy} = \Sigma(x-\bar{x}).(y-\bar{y})$$
$$S_{xx} = \Sigma(x-\bar{x})^2$$
$$S_{yy} = \Sigma(y-\bar{y})^2$$

- regression line of y on x

$$(y-\bar{y}) = r . \frac{\sigma_y}{\sigma_x}(x-\bar{x}) \Rightarrow (y-\bar{y}) = b_{yx}(x-\bar{x})$$

here, $b_{yx} = r \frac{\sigma_y}{\sigma_x}$

- regression line of x on y

$$(x-\bar{x}) = r . \frac{\sigma_x}{\sigma_y}(y-\bar{y}) \Rightarrow (x-\bar{x}) = b_{xy}(y-\bar{y})$$

here, $b_{xy} = r . \frac{\sigma_x}{\sigma_y}$

- $r = \sqrt{b_{xy} . b_{yx}}$

CHAPTER - 3 CALCULUS

Trignometric formulas

1. $\sin(x+y) = \sin x \cos y + \sin y \cos x$
2. $\sin(x-y) = \sin x \cos y - \sin y \cos x$
3. $\cos(x+y) = \cos x \cos y - \sin x \sin y$
4. $\cos(x-y) = \cos x \cos y + \sin x \sin y$
5. $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
6. $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$
7. $\cot(x+y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}$
8. $\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$

9. $\sin(2x) = 2\sin x \cos x = \dfrac{2\tan x}{1+\tan^2 x}$

10. $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

11. $\tan(2x) = \dfrac{2\tan x}{1-\tan^2 x}$

12. $\sin(3x) = 3\sin x - 4\sin^3 x$

13. $\cos(3x) = 4\cos^3 x - 3\cos x$

14. $\tan(3x) = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

15. $2\cos x \cos y = \cos(x+y) + \cos(x-y)$

16. $-2\sin x \sin y = \cos(x+y) - \cos(x-y)$

17. $2\sin x \cos y = \sin(x+y) + \sin(x-y)$

18. $2\cos x \sin y = \sin(x+y) - \sin(x-y)$

19. $\sin^2 x + \cos^2 x = 1$

20. $\sec^2 x - \tan^2 x = 1$

21. $\operatorname{cosec}^2 x - \cot^2 x = 1$

Some standard limits

1. $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

2. $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

3. $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$

4. $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$

5. $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log a$

6. $\lim\limits_{x \to 0} \dfrac{1 - \cos mx}{x^2} = \dfrac{m^2}{2}$

7. $\lim\limits_{x \to 0} \dfrac{1 - \cos mx}{1 - \cos nx} = \dfrac{m^2}{n^2}$

8. $\lim\limits_{x \to 0} (1+x)^{1/x} = e$

9. $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e$

10. $\lim\limits_{x \to \infty} \left(1 + \dfrac{a}{x}\right)^{x/a} = e$

11. $\lim\limits_{x \to \infty} \dfrac{\sin x}{x} = 0$

Some Standard differentials

$x^n \rightarrow nx^{n-1}$

$a^x \rightarrow a^x \log a$

$e^x \rightarrow e^x$

$uv \rightarrow u'v + v'u$

$\frac{g}{f} \rightarrow \frac{g'f - gf'}{f^2}$

$\sinh x \rightarrow \cosh x$

$\cosh x \rightarrow \sinh x$

$\sin x \rightarrow \cos x$

$\cos x \rightarrow -\sin x$

$\tan x \rightarrow \sec^2 x$

$\operatorname{cosec} x \rightarrow -\operatorname{cosec} x \cot x$

$\sec x \rightarrow \sec x \tan x$

$\cot x \rightarrow -\operatorname{cosec}^2 x$

$\sin^{-1} x \rightarrow \frac{1}{\sqrt{1-x^2}}$

$\cos^{-1} x \rightarrow \frac{-1}{\sqrt{1-x^2}}$

$\tan^{-1} x \rightarrow \frac{1}{1+x^2}$

$\operatorname{cosec}^{-1} x \rightarrow \frac{-1}{x\sqrt{x^2-1}}$

$\sec^{-1} x \rightarrow \frac{1}{x\sqrt{x^2-1}}$

$\cot^{-1} x \rightarrow \frac{-1}{1+x^2}$

Some standard integral formula

$x^n \Rightarrow \frac{x^{n+1}}{n+1}$

$a^x \log a \rightarrow a^x$

$e^x \rightarrow e^x$

$uv \rightarrow u\int v - \int[u' . \int v]$

$\sin x \rightarrow -\cos x$

$\cos x \rightarrow +\sin x$

$\tan x \rightarrow \ln(\sec x)$

$\operatorname{cosec} x \rightarrow \ln(\operatorname{cosec} x - \cot x)$

$\sec x \rightarrow \ln(\sec x + \tan x)$

$\cot x \rightarrow \ln(\sin x)$

$\frac{1}{\sqrt{1-x^2}} \rightarrow \sin^{-1} x$

$\frac{-1}{\sqrt{1-x^2}} \rightarrow \cos^{-1} x$

$\frac{1}{1+x^2} \rightarrow \tan^{-1} x$

$\frac{-1}{x\sqrt{x^2-1}} \rightarrow \operatorname{cosec}^{-1} x$

$\frac{1}{x\sqrt{x^2-1}} \rightarrow \sec^{-1} x$

$\frac{-1}{1+x^2} \rightarrow \cot^{-1} x$

$\frac{1}{\sqrt{a^2-x^2}} \rightarrow \sin^{-1}\left(\frac{x}{a}\right)$

$\frac{1}{a^2+x^2} \rightarrow \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$\frac{1}{a^2-x^2} \rightarrow \frac{1}{2a} \ln\left(\frac{x+a}{x-a}\right)$

$\frac{1}{x^2-a^2} \rightarrow \frac{1}{2a} \ln\left(\frac{x-a}{x+a}\right)$

$\frac{1}{\sqrt{x^2+a^2}} \Rightarrow \ln\left(x + \sqrt{x^2+a^2}\right)$

$\frac{1}{\sqrt{x^2-a^2}} \rightarrow \ln\left(x + \sqrt{x^2-a^2}\right)$

- If f & g are continous functions then,

  $f+g \longrightarrow$ is continous

  $f-g \longrightarrow$ is continous

  $f \times g \longrightarrow$ is continous

  $\frac{f}{g} \longrightarrow$ is continous $[g \neq 0]$

- every const. function is continous
- every polynomial is continous
- ~~every~~ $\sin x$, $\cos x$, e^{-x}, e^{x} are always continous

Rolles theorem

1. $f(x)$ is continous on $[a,b]$
2. $f(x)$ is derivable on (a,b)
3. $f(a) = f(b)$

then, there exists atleast one point, such that $f'(c) = 0$

$c \in (a,b)$

Lagranges mean value theorem

1. $f(x)$ is continous on $[a,b]$
2. $f(x)$ is derivable on (a,b)

then there exists atleast 1 point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Cauchy's mean value theorem

1. $f(x)$ & $g(x)$ are continous on $[a,b]$
2. $f(x)$ & $g(x)$ are derivable on (a,b)
3. $g'(x) \neq 0$

then, there exists atleast one point $(c) \in (a,b)$ such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Taylor's Series Expansion

If $f(x)$ is continously differentiable at a point $x=a$ then,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

If $a=0$ in Taylor's Series

$\Downarrow$

Maclaurin's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$
- $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \cdots$
- $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \cdots$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$
- $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} \cdots$
- $\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} \cdots$
- $\ln(1-x) = -\left[x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots\right.$

Some important Expansions

- $(1-t)^{-1} = 1 + t + t^2 + t^3 + t^4 \cdots$
- $(1+t)^{-1} = 1 - t + t^2 - t^3 + t^4 \cdots$
- $(1-t)^{-2} = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + \cdots$
- $1 - t^2 = (1-t)(1+t)$

- $1-t^3 = (1-t)(1+t+t^2)$
- $1-t^4 = (1-t)(1+t+t^2+t^3)$

Maxima & minima [Single variable]

- **Relative minima**

$\frac{dy}{dx} = 0$ & $\frac{d^2y}{dx^2} > 0$ point of minima

⇒ The absolute or global minima of $f(x)$ in $[a, b]$ is defined as mini. $\{f(a), f(b)$, all its relative minima in $[a,b]\}$

- **Relative maxima**

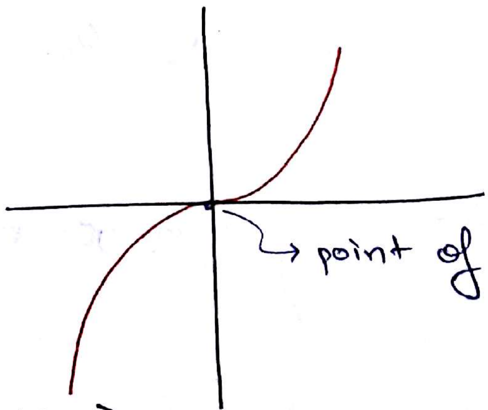
$\frac{dy}{dx} = 0$ & $\frac{d^2y}{dx^2} < 0$ point of maxima

⇒ absolute or global maxima = max. $\{f(a), f(b)$, all its relative maxima in $[a,b]\}$

- **Point of inflexion**

$\frac{dy}{dx} = 0$ & $\frac{d^2y}{dx} = 0$

& $\frac{d^3y}{dx^3} \neq 0$



point of inflexion

Maxima & minima [2 variables]

- find $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$
- solve $\frac{\partial u}{\partial x} = 0$ & $\frac{\partial u}{\partial y} = 0$ to get stationary points
- find $r = \frac{\partial^2 u}{\partial x^2}$

$s = \frac{\partial^2 u}{\partial x \partial y}$

$t = \frac{\partial^2 u}{\partial y^2}$

at the stationary point

if $rt - s^2 > 0$ & $r > 0$ then that stationary point is relative minima

if $rt - s^2 > 0$ & $r < 0$ then that stationary point is relative maxima

if $rt - s^2 < 0$ point of inflextion at that stationary point.

if $rt - s^2 = 0$ no conclusion can be drawn

Some Integration Properties

1. $\int (uv)\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\cdot\int v\,dx\right]dx$ [∴ ILATE]

2. $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(-x) = f(x)\text{; even fun}^n \\ 0 & \text{if } f(-x) = -f(x)\text{; odd func}^n \end{cases}$$

3. $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

4. $$\int_0^a \frac{f(x)}{f(x) + f(a-x)}\,dx = \frac{a}{2}$$

5. $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

6. $$\int_0^a x\cdot f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx \quad \text{if } f(a-x) = f(x)$$

7. $$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \frac{(n-1)(n-3)\cdots\cdots}{n.(n-2)(n-4)\cdots\cdots} \times K$$

$K = \frac{\pi}{2}$ if n is even

$K = 1$ if n is odd

8. $$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{(m-1)(m-3)\cdots\cdots(n-1)(n-3)\cdots\cdots}{(m+n)(m+n-2)(m+n-4)\cdots\cdots} \times K$$

$K = \frac{\pi}{2}$ if m & n both even

$K = 1$ otherwise.

Gamma function (Γ)

$$\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt$$

$$\Gamma(n+1) = n\Gamma n = n!$$

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$

* Gamma funcⁿ is not defined for −ve integers

Beta function (β)

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

$$\beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)}$$

Length of arc $y = f(x)$ b/w $x = a$ & $x = b$

$$l = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Length of arc $x = f(y)$ b/w $y = c$ & $y = d$

$$l = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Length of arc $r = f(\theta)$ b/w $\theta = \theta_1$ & $\theta = \theta_2$

$$l = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Length of arc $x = \phi(t)$ & $y = \psi(t)$ b/w $t = t_1$ & $t = t_2$

$$l = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Volume of solid revolution of arc $y = f(x)$ arround x-axis b/w $x = a$ & $x = b$

$$\text{Vol.} = \int_a^b \pi y^2\, dx$$

Volume of solid revolution of arc $x = f(y)$ arround y-axis b/w $y = c$ & $y = d$

$$\text{Vol.} = \int_c^d \pi x^2\, dy$$

Change of variables [General concept]

Jacobin: The transforming equations are

$$x = \phi(u,v) \quad \& \quad y = \psi(u,v)$$

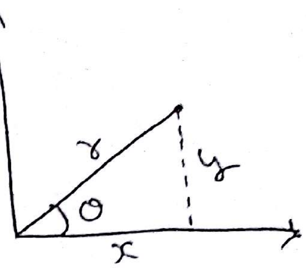
$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\Rightarrow \iint f(x,y)\, dx\, dy = \iint f(u,v)\, |J|\, du\, dv$$

$$\Rightarrow x = \phi(u,v,w)\ ;\ y = \psi(u,v,w)\ ;\ z = f(u,v,w)$$

$$\iiint f(x,y,z)\, dx\, dy\, dz = \iiint f(u,v,w)\, |J|\, du\, dv\, dw$$

A) Cartesian to polar coordinates
$[x, y]$ $[r, \theta]$



$$J\left(\frac{x, y}{r, \theta}\right) = r$$

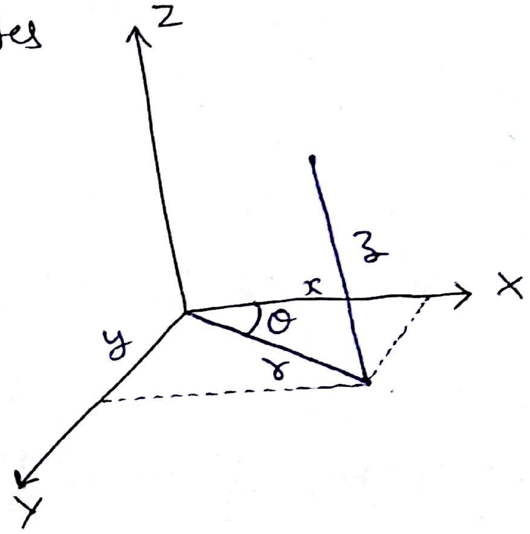
$$\iint f(x,y)\,dx\,dy = \iint f(r,\theta)\, r\,dr\,d\theta$$

eg:- for a circle

$r \longrightarrow 0$ to a
$\theta \longrightarrow 0$ to 2π
polar limits.

B) Cartesian to cylindrical coordinates
$[x, y, z]$ $[r, \theta, z]$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$



$$J\left(\frac{x, y, z}{r, \theta, z}\right) = r$$

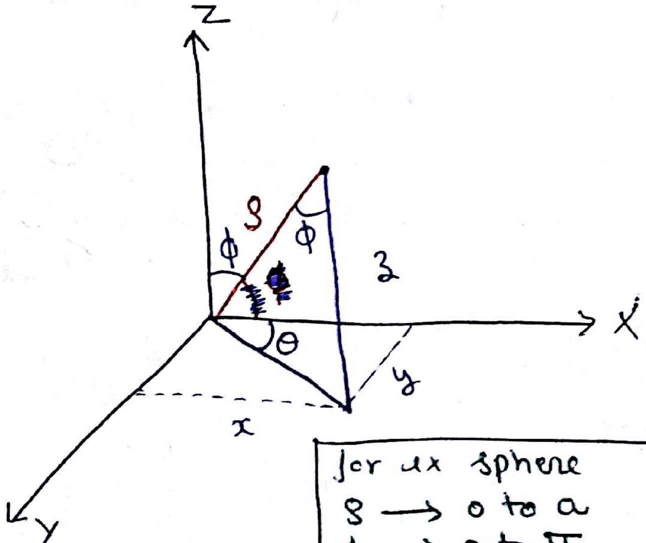
$$\iiint f(x,y,z)\,dx\,dy\,dz = \iiint f(r,\theta,z)\, r\,dr\,d\theta\,dz$$

eg:- for a cylinder

$r \longrightarrow 0$ to a
$\theta \longrightarrow 0$ to 2π
$z \longrightarrow 0$ to b

c) Cartesian to spherical cordinates

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$



$$J\left(\frac{x, y, z}{\rho, \phi, \theta}\right) = \rho^2\sin\phi$$

for ex sphere
$\rho \longrightarrow 0$ to a
$\phi \longrightarrow 0$ to π
$\theta \longrightarrow 0$ to 2π

$$\iiint f(x,y,z)\,dx\,dy\,dz = \iiint f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Total derivative

If $u = f(x,y)$ & $x = \phi(t)$ & $y = \psi(t)$

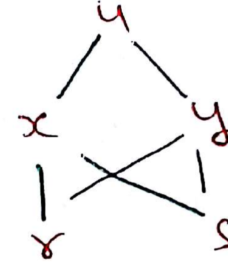
Then total derivative of u w.r.t. t

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Chain rule in Partial derivative

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$



$$|||$$

$$u_r = u_x \cdot x_r + u_y \cdot y_r$$

$$u_s = u_x \cdot x_s + u_y \cdot y_s$$

Homogenous function

if $f(Kx, Ky) = K^n f(x,y)$, then $f(x,y)$ is homogenous func^n of degree 'n'.

Euler's theorem

(I) if $u = f(x,y)$ with degree 'n' [homogenous]

then,
- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$
- $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

(II) if $u = f(x,y) + g(x,y)$ are homogenous with m & n degree

then,
- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = mf + ng$
- $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = m(m-1)f + n(n-1)g$

III if $u = f(x,y)$ is not homogenous but $F(u)$ is homogenous with degee 'n'.

Then,
- $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n.\dfrac{F(u)}{F'(u)} \Rightarrow g(u)$ say
- $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

CHAPTER-4 VECTOR CALCULUS

dot product $\Rightarrow$ $\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}| \cos\theta$

Cross product $\Rightarrow$ $\vec{a} \times \vec{b} = |\vec{a}|.|\vec{b}| \sin\theta . \hat{\eta}$

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

∇
- $\rightarrow$ Scalar $\phi(x,y,z)$ $\rightarrow$ $\nabla\phi$ grad. ϕ $\Rightarrow$ vector
- $\rightarrow$ Vector $\vec{F}$
  - $\rightarrow$ $\nabla.\vec{F}$ div. $\vec{F}$ $\Rightarrow$ Scalar
  - $\rightarrow$ $\nabla \times \vec{F}$ curl $\vec{F}$ $\Rightarrow$ Vector

- $\vec{r}$ position vector ; $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

  $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

- For a surface $\phi(x,y,z) = c$; $\nabla\phi$ & $\dfrac{d\vec{r}}{dt}$ are Lar

  $\nabla\phi \Downarrow$ normal ; $\dfrac{d\vec{r}}{dt} \Downarrow$ tangent

- grad $\phi = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right).(\phi) = \dfrac{\partial\phi}{\partial x}\hat{i} + \dfrac{\partial\phi}{\partial y}\hat{j} + \dfrac{\partial\phi}{\partial z}\hat{k}$

⊙ The unit vector normal to surface $\phi(x,y,z)$ is given by $\hat{\eta} = \dfrac{\nabla.\phi}{|\nabla.\phi|}$

- If θ is the angle b/w two surfaces $f(x,y,z) = c_1$ & $g(x,y,z) = c_2$ at Point P then,

$$\cos\theta = \frac{(\nabla f)_{at\,P} \cdot (\nabla g)_{at\,P}}{|(\nabla f)_{at\,P}| \cdot |(\nabla g)_{at\,P}|}$$

- If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ & $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

then,

$\rightarrow$ $\nabla f(r) = \frac{f'(r)}{r} \cdot \vec{r}$

$\rightarrow$ $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

- Directional Derivative of surface $\phi(x,y,z)$ at P in direction of vector $\vec{a}$ is given by

$$= (\nabla\phi)_{at\,P} \cdot \frac{\vec{a}}{|\vec{a}|}$$

- maximum value of directional derivative to surface $\phi(x,y,z) = c$ at Point P is given by.

$$= |\nabla\phi|_{at\,P}$$

- $$\text{div.}\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$\text{div.}\vec{F} = \nabla\cdot\vec{F} = 0 \iff \vec{F}$ is solenoidal Vector

- $$\text{Curl.}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$\text{curl}\,\vec{F} = \nabla\times\vec{F} = 0 \iff \vec{F}$ is irrotational

- $\text{div}(\text{curl } \vec{F}) = 0$ always
  i.e. $\nabla \cdot (\nabla \times \vec{F}) = 0$
- $\text{curl}(\text{grad } \phi) = 0$ always
  i.e. $\nabla \times (\nabla \phi) = 0$
- $\vec{F} = r^{-3} \cdot \vec{r}$ is standard solenoid.
  i.e. $\nabla \cdot \vec{F} = 0$ for above condition

Area of Δ when $\vec{a}$ & $\vec{b}$ are given as two sides.

$Ar = \frac{1}{2}(\vec{a} \times \vec{b})$

Green's theorem

If $F_1(x,y)$ & $F_2(x,y)$ are two differentiable function of x, y defined by Region R bounded by simple closed curve c then,

$$\text{work done} = \oint_C F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$$

Stokes theorem

If $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$ is differentible vector point function defined on an open surface S bounded by simple closed curve C then,

$$WD = \oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{\eta} \, ds$$

$\oint_C \vec{F} \cdot d\vec{r}$ ⇓ use for open surface

$\iint_S (\nabla \times \vec{F}) \cdot \hat{\eta} \, ds$ ⇓ use for simple closed curve

$ds = \frac{dx dy}{\lvert \hat{n} \cdot \hat{k} \rvert}$	x, y plane
$ds = \frac{dy dz}{\lvert \hat{n} \cdot \hat{i} \rvert}$	y, z plane
$ds = \frac{dz dx}{\lvert \hat{n} \cdot \hat{j} \rvert}$	z, x plane

Gauss Divergence theorem

If $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$ is differentible vector point function defined on closed surface S enclosing volume V then

$$\text{flux} = \oint_S \vec{F} \cdot \hat{n} \, ds = \int_V \nabla \cdot \vec{F} \, dv$$

Note: If line integral is asked then non of above theorem will be used, simple Integration with x, y, z relations are substituted & calculated

CHAPTER-5 DIFFERENTIAL EQUATION

Order :- The order of highest ordered derivative occuring in differential equation is known as order of D.E.

Degree :- The degree of highest ordered derivative, when derivative are free from fractional powers.

Solutions to differential equation [Ist order]

1. Variable seperable

$$\frac{dy}{dx} = f(x) \cdot f(y)$$

2. $$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$$

when f_1 & f_2 are homogenous of same degree

put $y = vx$; $\frac{dy}{dx} = v + x\frac{dv}{dx}$

3. Linear differential eqn

$$\frac{dy}{dx} + Py = Q \qquad \text{where } P = f(x)$$
$$Q = g(x)$$

then, $I.F = e^{\int P dx}$

Soln is $y.(I.F) = \int Q.(I.F)\, dx$

1. Exact differential Equation

If D.E. is in form $Mdx + Ndy = 0$

& $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then equation is exact DE.

Soln is

$$\int M dx + \int (\text{Terms of N free from 'x'})\, dy = c$$

integration factor if above eqn is not Exact.

Case-1 N & M are homogenous of same degree

$$I.F. = \frac{1}{Mx + Ny}$$

Case-2 when M & N are not homogenous but $M = yf(x,y)$ & $N = x\,g(x,y)$

$$I.F. = \frac{1}{Mx - Ny}$$

Case-3 If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ or const. then $I.F. = e^{\int f(x)dx}$

Case 4 If, $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$ or const. then $I.F. = e^{\int f(y)dy}$

Orthogonal Trajectory

1. Find the diff. Eqn of given family of curve
2. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ & $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$
3. Solve the resultant to get orthogonal trajectory.

Soln to differential equation [Higher order]

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\ a_n y = f(x)$$

1. if $f(x) = 0$ then LDE is homogenous $\therefore$ GS = CF
2. if $f(x) \neq 0$ then LDE is non-homogenous $\therefore$ GS = CF + PI

Roots of Auxillary Eqn	Corresponding Complimentary fun (CF)
1. real & Distinct m_1, m_2, m_3	$C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$
2. real & equal m, m, m	$(C_1 x^2 + C_2 x + C_3) e^{mx}$
3. imaginary $\alpha \pm i\beta$	$e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

method to find P.I. for non homogenous LDE

1. Type - I when $f(x) = e^{\alpha x}$

Hint :- Replace D by α in $f(D)$

i.e. $PI = \dfrac{e^{\alpha x}}{f(D)}$

2. Type II when $f(x) = \cos\alpha x$ or $\sin\alpha x$

$PI = \dfrac{f(x)}{f(D)}$ Hint : Replace D^2 by $-\alpha^2$ in $f(D)$

3. Type III when $f(x) = x^k$

Hint : write $f(D)$ in the form of

$(1+t)^{-1} = 1 - t + t^2 - t^3 + t^4 - - - - -$

$(1-t)^{-1} = 1 + t + t^2 + t^3 + t^4 - - - -$

$(1-t)^{-2} = 1 + 2t + 3t^2 + 4t^3 - - - -$

4. Type IV when $f(x) = e^{\alpha x} \cos\beta x$ or $e^{\alpha x} \sin\beta x$ or $e^{\alpha x} \cdot x^k$

here, $V = \cos\beta x$ or $\sin\beta x$ or x^k

Hint :-

$$PI = \frac{e^{\alpha x} \cdot V}{f(D)} = e^{\alpha x} \cdot \left[\frac{1}{f(D+\alpha)} \cdot V\right]$$

5. Type - V when $f(x) = x \cdot V$ here $V = \cos\alpha x$ or $\sin\alpha x$

Hint :

$$PI = \frac{1}{f(D)} x \cdot V = x \cdot \frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2} \cdot V$$

6. Type-6 when $f(x) = R(x)$ i.e. Trignometric func^n other than $\sin x$ & $\cos x$

$f(D)\,y = R(x)$ [method of variation of parameter]

$CF = c_1\,u(x) + c_2\,v(x)$

$PI = A\,u(x) + B\,v(x)$

here, $A = -\int \frac{v(x)\,R(x)\,dx}{u\frac{dv}{dx} - v\frac{du}{dx}}$ → wronskian

$B = \int \frac{u(x)\,R(x)\,dx}{u\frac{dv}{dx} - v\frac{du}{dx}}$

7. Type-7 Differential eqn^n with variable coefficient

Cauchy's - Euler differential Eqn^n.

$(x^n D^n + a_1 x^{n-1} D^{n-1} + a_2 x^{n-2} D^{n-2} \dots a_n)y = f(x)$

1. put $x = e^z$ and put $z = \log x$ in solution

2. $x\frac{d}{dx} = xD = \theta = \frac{d}{dz}$

$x^2 D^2 = \theta(\theta-1)$

$x^3 D^3 = \theta(\theta-1)(\theta-2)$

Standard Partial Differential Equation:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} = F(u)$$

- if $B^2 - 4AC = 0$ then PDE is parabolic
- if $B^2 - 4AC < 0$ then PDE is elliptic
- if $B^2 - 4AC > 0$ then PDE is hyperbolic

CHAPTER - 6 LAPLACE TRANSFORMS

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

- unit step funⁿ ⇒ $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$
- unit impulse funⁿ ⇒ $\delta(t)$
- $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} \times (a \sin bx - b \cos bx)$

Some importent laplace formula

$L(1) = \frac{1}{s}$

$L(t) = \frac{1}{s^2}$

$L(t^n) = \frac{n!}{s^{n+1}}$ or $\frac{\Gamma n+1}{s^{n+1}}$

$L(e^{at}) = \frac{1}{s-a}$

$L(e^{-at}) = \frac{1}{s+a}$

$L(\sin(at)) = \frac{a}{s^2+a^2}$

$L(\cos(at)) = \frac{s}{s^2+a^2}$

$L\{\sinh(at)\} = \frac{a}{s^2-a^2}$

$L(\cosh(at)) = \frac{s}{s^2-a^2}$

$L(u(t-a)) = \frac{e^{-as}}{s}$

$L(\delta(t)) = 1$

$$\underbrace{L(f(t)) = \frac{1}{1-e^{-Ts}} \times \int_0^T e^{-st} f(t)\, dt}$$

⇓

when $f(t)$ is periodic with period 'T'.

Properties of laplace

1. If $L(f(t)) = F(s)$ then, $L(e^{at} f(t)) = F(s-a)$

2. If $L(f(t)) = F(s)$ then,

$$L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} F(s)$$

3. If $L(f(t)) = F(s)$ then, $L\left(\frac{f(t)}{t}\right) = \int_s^{\infty} F(s)\,ds$

4. $L\left(\frac{\sin t}{t}\right) = \cot^{-1}(s)$ & $L\left(\frac{\sin at}{t}\right) = \cot^{-1}\left(\frac{s}{a}\right)$

5. $L(f(t)) = F(s)$

then, $L\left[\underbrace{\int_0^t\int_0^t\int_0^t \cdots \int_0^t}_{n \text{ times}} f(t)\,dt^n\right] = \frac{1}{s^n} F(s)$

6. If $L(f(t)) = L(y) = F(s)$ then,

- $L(y') = s\,L(y) - y(0)$
- $L(y'') = s^2 L(y) - s\,y(0) - y'(0)$.
- $L(y''') = s^3 L(y) - s^2 y(0) - s\,y'(0) - y''(0)$

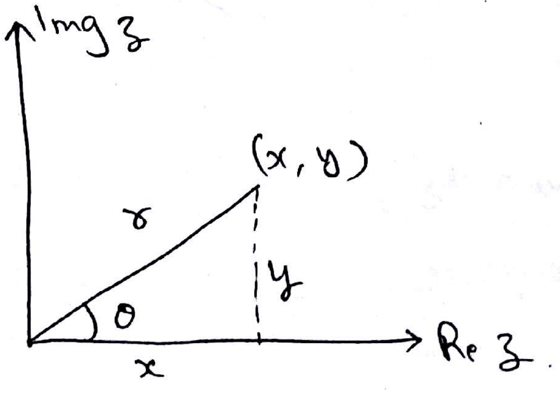
7. If $L\,f(t) = F(s)$ then,

- $\lim_{t \to 0} f(t) = \lim_{s \to \infty} s.F(s)$
- $\lim_{t \to \infty} f(t) = \lim_{s \to 0} s.F(s)$

Note

$\sin(i\theta) = i \sinh\theta$

CHAPTER – 7 COMPLEX VARIABLE



$z = x + iy$

$z = r\cos\theta + i\,r\sin\theta$

$z = r e^{i\theta}$

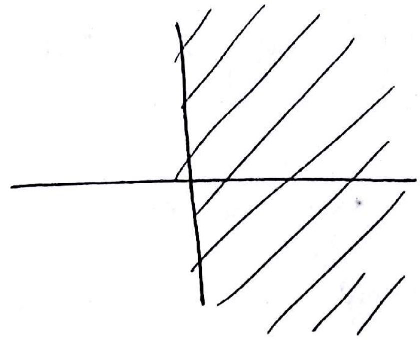
$r = |z| = \sqrt{x^2 + y^2}$

- Argument of $z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Cube root of unity

$\boxed{1 + \omega + \omega^2 = 0}$

- If two complex numbers z_1 & z_2 lies on Right half plane then,



→ $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

→ $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

- Analytic function
  - → $u_x = v_y$ & $u_y = -v_x$
  - → u_x, v_x, u_y, v_y are continous → Analytic.

- if $f(z) = u + iv$ is analytic function then $u(x,y)$ & $v(x,y)$ are orthogonal & u & v are Harmonic funⁿ of x, y

  i.e. $u_{xx} + u_{yy} = 0$ & $v_{xx} + v_{yy} = 0$

- Milne thomson method to find $f(z)$

Case-1 when $u(x,y)$ is given

- $f'(z) = u_x + i v_x$

  $= u_x - i u_y \quad [\because u_y = -v_x]$

- integrate $f'(z)$ by replacing $x = z$ & $y = 0$

Case-2 when $v(x,y)$ is given

- $f'(z) = u_x + i v_x$

  $= v_y + i v_x \quad [\because u_x = v_y]$

- put $x = z$ & $y = 0$ and integrate

- Taylor's Series Expansion

If $f(z)$ is analytic at point z_0 then,

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \text{------}$$

- **Laurent Series**

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{analytic Part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{Principle part}}$$

\# If Principle part is abscent $\Rightarrow$ z_0 is removable singular

\# If there are infinite terms of Principle part $\Rightarrow$ z_0 is essential

\# if there are finite terms of Principle part $\Rightarrow$ z_0 is a pole.

z_0 is Pole $\Leftrightarrow$ $f(z)$ is not analytic at z_0

- **Residue**: In Laurent Series Expansion of $f(z)$, the coefficient of $\frac{1}{z-a}$ is called residue of $f(z)$ at $z=a$

\# If $z=a$ is a simple pole of $f(z)$ then

$$\operatorname*{Res.}_{z=a} f(z) = \lim_{z \to a} (z-a) f(z)$$

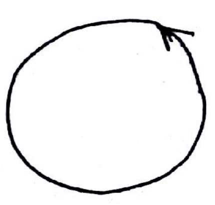
\# If $z=a$ is a pole with degree 'm'

$$\operatorname*{Resi}_{z=a} f(z) = \lim_{z \to a} \frac{1}{(m-1)!} \times \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$$

- **Cauchy's integral theorem**

If $f(z)$ is analytic and $f'(z)$ is continous within and on the boundary of a simple closed curve then

$$\oint_C f(z)\, dz = 0$$



C

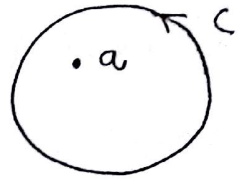
i.e. no poles inside curve & on curve

- Cauchy's integral formula

If f(z) is analytic within and on boundary of simple closed curve C,

\# 'a' is a pole inside with degree 1.

$$\oint_C \frac{f(z)}{(z-a)}\,dz = \frac{2\pi i}{1} \cdot f(a)$$



\# 'a' is pole inside with degree (n+1)

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} \cdot \frac{d^n}{dz^n} \cdot f(z)\Big|_{z=a}$$

- Cauchy's Residual theorem

If there are more than 1 but finite no. of poles inside the closed curve where f(z) is analytic.

$$\oint_C f(z)\,dz = 2\pi i \times [\text{Sum of Residues of all poles inside}]$$

CHAPTER-8 NUMERICAL METHOD

1. LU factorization method

$A = LU$

$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$$

example

2. Gauss elimination method

- rearrange the given eqn in form x_1, x_2, x_3 --- from 1st, 2nd, 3rd, --- equations.
- put $x_1 = x_2 = x_3 ---- = 0$ to get initial values of x_1, x_2, x_3 in expression [Ist itteration]

- put lattest updated values in Expressions of x_1, x_2, x_3 --- to get next itteration values.

3. Gauss Seidal method

- Seauence of given eauation will alter the result So, arrange in manar in which question is asked.
- find Expressions $x_1 =$ ______ from eqn ①
  $x_2 =$ ______ from eqn ②
  $x_3 =$ ______ from eqn ③
- Ist itteration ⇒ put $x_2 = x_3 = 0$ get $(x_1)_1$ in ①
  put $x_1 = (x_1)_1$ & $x_3 = 0$ get $(x_2)_1$ in ②
  put $x_1 = (x_1)_1$ & $x_2 = (x_2)_1$ gte $(x_3)_1$ in ③

Similarly continue, by using latest updated values.

4. Newton Bisection method

$$x = \frac{a+b}{2} \quad \text{where } f(a) \cdot f(b) < 0$$

$$\frac{|b-a|}{2^n} < \epsilon$$

where $b =$ upper limit
$a =$ Lower Limit
$n =$ no. of itteration reauired
$\epsilon =$ error

5. False position method or reegula falsi method

the itteration formula is

$$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

here $f(a) \cdot f(b) < 0$

$$\begin{vmatrix} a & b \\ f(a) & f(b) \end{vmatrix}$$

6. Secant method

the itteration formula is

$$x_{n+1} = \frac{x_{n-1} \cdot f(x_n) - x_n \cdot f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$\begin{vmatrix} x_{n-1} & x_n \\ f(x_{n-1}) & f(x_n) \end{vmatrix}$$

- **Newton - Raphson method**

The iteration formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

i.e. $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

method	Speed	Convergence	order of Convergence	no. of guesses	Type
Newton Bisection	Slower	converges	1	2	Closed
False Position	Slower	converges	1	2	Closed
Secant method	~~fast~~ medium	may or may not	1.62	1	open
Newton Raphson	fast	may or may not	2	1	open

⇓

Error at present is square of previous.

- **Euler's method** $\left[\dfrac{dy}{dx} = f(x,y)\right]$

iteration formula is $y_{n+1} = y_n + hf(x_n, y_n)$

$$x_{n+1} = x_n + h$$

i.e. $y_1 = y_0 + hf(x_0, y_0)$

$$y_2 = y_1 + hf(x_1, y_1)$$

- **Backward Euler's method** $\left[\dfrac{dy}{dx} = f(x,y)\right]$

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$$

⇓

Solve eqn for y_{n+1} by substituting h & x_{n+1}

modified Euler's method or Runge Kutta 2nd order

$$K_1 = hf(x_0, y_0)$$
$$K_2 = hf(x_0+h, y_0+K_1)$$
$$K = \frac{K_1+K_2}{2}$$

itteration formula $\Rightarrow$ $y_1 = y_0 + K$ & $x_1 = x_0 + h$

Runge Kutta 4th order $\left[\frac{dy}{dx} = f(x,y)\right]$

$$K_1 = hf(x_0, y_0)$$
$$K_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right)$$
$$K_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right)$$
$$K_4 = hf(x_0+h, y_0+K_3)$$
$$K = \frac{K_1+2K_2+2K_3+K_4}{6}$$

itteration formula is $y_1 = y_0 + K$ & $x_1 = x_0 + h$

Numerical Integration

A) Trapezoidal rule [Exact for 1st degree polynomial] accuracy $O(h^2)$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2} \cdot \left[(y_0+y_n) + 2(y_1+y_2+ \cdots y_{n-1})\right]$$

B) Simpson's $\frac{1}{3}$ rule [correct for 2nd degree polynomial] accuracy $(O(h^4))$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3} \cdot \left[(y_0+y_n) + 4(y_1+y_3+y_5 \cdots) + 2(y_2+y_4+y_6 \cdots)\right]$$

c) Simpson's $\frac{3}{8}$ rule [Correct for 3rd degree polynomial] accuracy $O(h^5)$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3}{8}h\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5 \cdots) + 2(y_3+y_6+y_9 \cdots)\right]$$