DEFINITE INTEGRAL

EXERCISE 7.8

Evaluate the following integrals as limit of sums.

Q.No. 1 $\int_a^b x\,dx$

Soln: $\int_a^b x\,dx = \int_a^b f(x)\,dx$ where $f(x) = x$.

$$= \underset{h\to 0}{Lt}\; h\left[f(a) + f(a+h) + f(a+2h) + \cdots f(a+\overline{n-1}\,h)\right];\ nh = b-a.$$

$$= \underset{h\to 0}{Lt}\; h\left[a + (a+h) + (a+2h) + \cdots (a+(n-1)h)\right]$$

$$= \underset{h\to 0}{Lt}\; h\left[n\times a + h\left(1+2+3\cdots(n-1)\right)\right]$$

$$= \underset{h\to 0}{Lt}\left[(nh)a + h^2\left(\frac{(n-1)(n)}{2}\right)\right] \quad \left[\because \sum n = \frac{n(n+1)}{2}\right]$$

$$= \underset{h\to 0}{Lt}\left[(b-a)a + \frac{(nh-h)(nh)}{2}\right] \quad \because nh = b-a$$

$$= \underset{h\to 0}{Lt}\left[(b-a)a + \frac{[(b-a)-h][b-a]}{2}\right] = \underset{h\to 0}{Lt}\left[(b-a)\left(a + \frac{b-a-h}{2}\right)\right]$$

$$= (b-a)\left(a + \frac{b-a-0}{2}\right) = (b-a)\left(\frac{2a+b-a}{2}\right)$$

$$= \frac{(b-a)(b+a)}{2} = \frac{b^2-a^2}{2} = \frac{1}{2}\left(b^2-a^2\right)$$

Q.No 2 $\int_0^5 (x+1)\,dx$

Soln: $\int_0^5 (x+1)\,dx = \int_a^b f(x)\,dx$ where $f(x) = x+1$, $a=0$, $b=5$

and $nh = b-a = 5-0 = 5$

and $f(a) = f(0) = 0+1 = 1$

$f(a+h) = f(0+h) = h+1$

$f(a+2h) = f(0+2h) = 2h+1$

$\cdots\cdots$

$f(a+(n-1)h) = f((n-1)h) = (n-1)h+1$

$$\therefore \int_0^5 (x+1)\,dx = \underset{h\to 0}{Lt}\, h\left[f(a)+f(a+h)+f(a+2h)\cdots f(a+(n-1)h)\right]$$

$$= \underset{h\to 0}{Lt}\, h\left[(1)+(h+1)+(2h+1)\cdots((n-1)h+1)\right]$$

$$= \underset{h\to 0}{Lt}\, h\left[n\times 1 + h(1+2+3\cdots(n-1))\right]$$

$$= \underset{h\to 0}{Lt}\left[nh + h^2\frac{(n-1)(n)}{2}\right] \quad \left[\because \sum n = \frac{n(n+1)}{2}\right.$$

$$= \underset{h\to 0}{Lt}\left[(nh) + \frac{(nh-h)(nh)}{2}\right] = \underset{h\to 0}{Lt}\left[5+\frac{(5-h)(5)}{2}\right] \left[\because nh = 5\right]$$

$$= 5 + \frac{5\times 5}{2} = \frac{35}{2}.$$

QNo 3 $\int_2^3 x^2\,dx$

Sol. $\int_2^3 x^2\,dx = \int_a^b f(x)\,dx$ where $f(x) = x^2$, $a=2$, $b=3$, $nh = b-a = 3-2 = 1$

and $f(a) = a^2 = 2^2$

$f(a+h) = (a+h)^2 = a^2+2ah+h^2 = 2^2+4h+h^2$

$f(a+2h) = (a+2h)^2 = a^2+2\cdot 2ah+h^2 = 2^2+8h+2^2h^2$

$\cdots\cdots$

$f(a+(n-1)h) = [a+(n-1)h]^2 = a^2+2a\cdot(n-1)h+h^2(n-1)^2$
$= 2^2+4h(n-1)+h^2(n-1)^2$

$\therefore$ By def. $\int_a^b f(x)\,dx = \underset{h\to 0}{Lt}\ h\left[f(a) + f(a+h) + \cdots f(a+(n-1)h)\right]$

$= \underset{h\to 0}{Lt}\ h\left[2^2 + (2^2+h^2+4h) + (2^2+h^2+8h) + \cdots (2^2+(n-1)^2h^2+4(n-1))\right]$

$= \underset{h\to 0}{Lt}\ h\left[2^2 \times n + h^2(1^2+2^2+\cdots(n-1)^2) + 4h(1+2+3\cdots(n-1))\right]$

$= \underset{h\to 0}{Lt}\left[4nh + h^3\dfrac{(n-1)(n)(2(n-1)+1)}{6} + 4h^2\dfrac{(n-1)(n)}{2}\right]$ $\left[\because \Sigma n = \dfrac{n(n+1)}{2},\ \Sigma n^2 = \dfrac{n(n+1)(2n+1)}{6}\right]$

$= \underset{h\to 0}{Lt}\left[4\times(nh) + \dfrac{(nh-h)(nh)(2nh-h)}{6} + 2(nh-h)(nh)\right]$

$= \underset{h\to 0}{Lt}\left[4 + \dfrac{(1-h)(1)(2-h)}{6} + 2(1-h)(1)\right]$ $[\because nh=1]$

$= 4 + \dfrac{(1)(1)(2)}{6} + 2(1)(1) = 4 + \dfrac{1}{3} + 2 = \dfrac{19}{3}$

Q No 4 $\int_1^4 (x^2-x)\,dx$

Sol: $\int_1^4 (x^2-x)\,dx = \int_a^b f(x)\,dx$ where $a=1$, $b=4$, $nh=b-a=3$

and $f(x) = x^2 - x = x(x-1)$

$\therefore\ f(a) = f(1) = 1(1-1) = 0.$

$f(a+h) = f(1+h) = (1+h)(1+h-1) = (1+h)h.$

$f(a+2h) = f(1+2h) = (1+2h)(1+2h-1) = (1+2h)(2h)$

$\cdots\cdots$

$f[a+(n-1)h] = f(1+(n-1)h) = (1+(n-1)h)(1+(n-1)h-1)$

$= (1+(n-1)h)((n-1)h)$

$\therefore$ By. $\int_a^b f(x) = \underset{h\to 0}{Lt}\; h\left[f(a) + f(a+h) + \cdots f(a+(n-1)h)\right]$

$$\int_1^4 (x^2-x)\,dx = \underset{h\to 0}{Lt}\; h\left[0 + (1+h)h + (1+2h)(2h) + \cdots (1+(n-1)h)[(n-1)h]\right]$$

$$= \underset{h\to 0}{Lt}\; h^2\left[(1+h) + 2(1+2h) + 3(1+3h) + \cdots (n-1)[1+(n-1)h]\right]$$

$$= \underset{h\to 0}{Lt}\; h^2\left[[1+2+3\cdots(n-1)] + h\left(1^2+2^2+3^2\cdots(n-1)^2\right)\right]$$

$$= \underset{h\to 0}{Lt}\; h^2\left[\frac{(n-1)(n)}{2} + h\left(\frac{(n-1)(n)(2n-1)}{6}\right)\right]$$

$$= \underset{h\to 0}{Lt}\left[\frac{(nh)(nh-h)}{2} + \frac{(nh-n)(nh)(2nh-h)}{6}\right]$$

$$= \underset{h\to 0}{Lt}\left[\frac{3(3-h)}{2} + \frac{(3-h)(3)(6-h)}{6}\right] = \frac{3\times 3}{2} + \frac{(3)(3)(6)}{6} = \frac{27}{2}.$$

QNo 5 $\int_{-1}^{1} e^x dx.$

Sol: $\int_{-1}^{1} e^x dx. = \int_a^b f(x)\,dx$ where $a=-1$, $b=1$, $nh = b-a = 2$

$f(x) = e^x.$

$\therefore\; f(a) = f(-1) = e^{-1}$

$f(a+h) = f(-1+h) = e^{-1+h} = e^{-1}e^{h}.$

$f(a+2h) = f(-1+2h) = e^{-1+2h} = e^{-1}\cdot e^{2h}.$

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$f(a+(n-1)h) = f[-1+(n-1)h] = e^{-1+(n-1)h} = e^{-1}\cdot e^{(n-1)h}.$

$\therefore$ By def $\int_a^b f(x)\,dx = \underset{h\to 0}{Lt}\; h\left[f(a) + f(a+h) + \cdots f(a+(n-1)h)\right]$

$$\int_{-1}^{1} e^x dx \; \underset{h\to 0}{Lt}\, h\left[e^{-1} + e^{-1}e^{h} + e^{-1}e^{2h} + \cdots e^{-1}e^{(n-1)h}\right]$$

$$= \underset{h\to 0}{Lt}\; he^{-1}\left[1 + e^{h} + e^{2h} + \cdots e^{(n-1)h}\right]$$

$$= \underset{h\to 0}{Lt}\; he^{-1}\left[\frac{(1)\left[(e^{h})^{n}-1\right]}{e^{h}-1}\right] \quad \left[\begin{array}{c}\text{Sum of } n \text{ terms of G.P.} \\ = \dfrac{a(r^{n}-1)}{r-1}\end{array}\right]$$

$$= \underset{h\to 0}{Lt}\; \frac{h}{e}\left(\frac{e^{nh}-1}{e^{h}-1}\right) = \frac{1}{e}\,\underset{h\to 0}{Lt}\; \frac{e^{nh}-1}{\frac{e^{h}-1}{h}} = \frac{1}{e}\cdot\frac{e^{2}-1}{\underset{h\to 0}{Lt}\frac{e^{h}-1}{h}}$$

$$= \frac{1}{e}\left(\frac{e^{2}-1}{1}\right) = \frac{e^{2}-1}{e}.$$

Q No 6 $\int_{0}^{4} (x + e^{2x})\, dx$

Sol: $\int_{0}^{4} (x + e^{2x})\, dx = \int_{a}^{b} f(x)dx$ where $a = 0$, $b = 4$, $nh = b - a = 4$

$$f(x) = x + e^{2x}.$$

$$\therefore\; f(a) = f(0) = 0 + e^{2\times 0} = 0 + 1 = 1$$

$$f(a+h) = f(0+h) = h + e^{2h}$$

$$f(a+2h) = f(2h) = 2h + e^{4h}$$

$$\cdots\cdots\cdots$$

$$f(a+(n-1)h) = f((n-1)h) = (n-1)h + e^{2(n-1)h}$$

$$\therefore \text{By } \int_{a}^{b} f(x)\,dx = \underset{h\to 0}{Lt}\; h\left[f(a) + f(a+h) + f(a+2h) + \cdots f(a+(n-1)h)\right]$$

$$= \underset{h\to 0}{Lt}\; h\left[1 + (h + e^{2h}) + (2h + e^{4h}) + \cdots \left[(n-1)h + e^{2(n-1)h}\right]\right]$$

$$= \underset{h\to 0}{Lt}\; h\left[\left[h + 2h + \cdots (n-1)h\right] + \left(1 + e^{2h} + e^{4h} + \cdots e^{2(n-1)h}\right)\right]$$

$$= \underset{h \to 0}{Lt}\ h\left[h\left(1+2+3+\cdots(n-1)\right) + \left(1 + e^{2h} + e^{4h} + \cdots e^{2nh-2h}\right)\right]$$

$$= \underset{h \to 0}{Lt}\ h\left[h\frac{(n-1)(n)}{2} + 1 \cdot \frac{(e^{2h})^{n}-1}{e^{2h}-1}\right] = \underset{h \to 0}{Lt}\left[\frac{(nh)(nh-h)}{2} + \frac{e^{2(nh)}-1}{\frac{e^{2h}-1}{2h}\times 2}\right]$$

$$= \underset{h \to 0}{Lt}\left[\frac{(4-h)(4)}{2} + \frac{e^{8}-1}{\frac{e^{2h}-1}{2h}\times 2}\right] = \frac{4\times 4}{2} + \frac{e^{8}-1}{2} = \left(8-\frac{1}{2}\right) + \frac{e^{8}}{2}$$

$$= \frac{15+e^{8}}{2}.$$

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