6. Definite Integration

EXERCISE 6.1

Evaluate the following definite intergrals:

1. $\int_4^9 \frac{1}{\sqrt{x}}dx$

Solution:

$$\int_4^9 \frac{1}{\sqrt{x}}\,dx = \int_4^9 x^{-\frac{1}{2}}\,dx$$

$$= \left[\frac{x^{\frac{1}{2}}}{1/2}\right]_4^9 = 2\left[\sqrt{x}\right]_4^9$$

$$= 2(\sqrt{9} - \sqrt{4})$$

$$= 2(3 - 2) = 2.$$

2. $\int_{-2}^3 \frac{1}{x+5}dx$

Solution:

$$\int_{-2}^3 \frac{1}{x+5}\,dx$$

$$= \left[\log|x+5|\right]_{-2}^3$$

$$= \log 8 - \log 3$$

$$= \log\left(\frac{8}{3}\right).$$

3. $\int_2^3 \frac{x}{x^2-1}dx$

Solution:

$$\int_2^3 \frac{x}{x^2-1}\,dx$$

$$= \frac{1}{2}\int_2^3 \frac{2x}{x^2-1}\,dx$$

$$= \frac{1}{2}\left[\log|x^2-1|\right]_2^3 \quad \ldots \left[\because \frac{d}{dx}(x^2-1) = 2x \text{ and } \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c\right]$$

$$= \frac{1}{2}\left[\log(9-1) - \log(4-1)\right] = \frac{1}{2}\log\left(\frac{8}{3}\right).$$

4. $\int_0^1 \frac{x^2+3x+2}{\sqrt{x}}dx$

Solution:

$$\int_0^1 \frac{x^2+3x+2}{\sqrt{x}}\,dx$$

$$= \int_0^1 \left(\frac{x^2}{\sqrt{x}} + \frac{3x}{\sqrt{x}} + \frac{2}{\sqrt{x}}\right)dx$$

$$= \int_0^1 \left(x^{\frac{3}{2}} + 3x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}\right)dx$$

$$= \left[\frac{x^{\frac{5}{2}}}{5/2} + 3\left(\frac{x^{\frac{3}{2}}}{3/2}\right) + 2\left(\frac{x^{\frac{1}{2}}}{1/2}\right)\right]_0^1$$

$$= \left[\frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 4x^{\frac{1}{2}}\right]_0^1$$

$$= \left[\frac{2}{5}(1)^{\frac{5}{2}} + 2(1)^{\frac{3}{2}} + 4(1)^{\frac{1}{2}}\right] - (0+0+0)$$

$$= \frac{2}{5} + 2 + 4 = \frac{32}{5}.$$

5. $\int_{2}^{3} \frac{x}{(x+2)(x+3)} dx$

Solution:

$$\text{Let } I = \int_{2}^{3} \frac{x}{(x+2)(x+3)} dx$$

$$\text{Let } \frac{x}{(x+2)(x+3)} = \frac{A}{x+3} + \frac{B}{x+2}$$

$\therefore x = A(x+2) + B(x+3)$

Put $x+3=0$, i.e. $x=-3$, we get

$-3 = A(-1) + B(0) \quad \therefore A = 3$

Put $x+2=0$, i.e. $x=-2$, we get

$-2 = A(0) + B(1) \quad \therefore B = -2$

$$\therefore \frac{x}{(x+2)(x+3)} = \frac{3}{x+3} + \frac{(-2)}{x+2}$$

$$\therefore I = \int_{2}^{3} \left[\frac{3}{x+3} + \frac{(-2)}{x+2}\right] dx$$

$$= \left[3\log(x+3) - 2\log(x+2)\right]_2^3$$

$$= [3\log(3+3) - 2\log(3+2)] - [3\log(2+3) - 2\log(2+2)]$$

$$= 3\log 6 - 5\log 5 + 2\log 4$$

$$= \log 6^3 - \log 5^5 + \log 4^2$$

$$= \log 216 - \log 3125 + \log 16$$

$$= \log\left(\frac{216 \times 16}{3125}\right) = \log\left(\frac{3456}{3125}\right).$$

6. $\int_{1}^{2} \frac{dx}{x^2+6x+5} dx$

Solution:

$$\int_{1}^{2} \frac{dx}{x^2+6x+5}$$

$$= \int_{1}^{2} \frac{dx}{(x^2+6x+9)-4}$$

$$= \int_{1}^{2} \frac{1}{(x+3)^2-(2)^2} dx$$

$$= \frac{1}{2(2)}\left[\log\left|\frac{x+3-2}{x+3+2}\right|\right]_1^2 = \frac{1}{4}\left[\log\left|\frac{x+1}{x+5}\right|\right]_1^2$$

$$= \frac{1}{4}\left[\log\frac{3}{7} - \log\frac{2}{6}\right]$$

$$= \frac{1}{4}\log\left(\frac{3}{7} \times \frac{6}{2}\right) = \frac{1}{4}\log\left(\frac{9}{7}\right).$$

7. *If* $\int_{0}^{a} (2x+1)\, dx = 2,$

find the real value of a.

Solution:

$$\text{Let } I = \int_{0}^{a} (2x+1)\, dx$$

$$= \left[2 \cdot \frac{x^2}{2} + x\right]_0^a$$

$$= a^2 + a - 0 = a^2 + a$$

$\therefore I = 2$ gives $a^2 + a = 2$

$\therefore a^2 + a - 2 = 0$

$\therefore (a+2)(a-1) = 0$

$\therefore a + 2 = 0$ or $a - 1 = 0$

$\therefore a = -2$ or $a = 1.$

8. *If* $\int_{1}^{a} (3x^2 + 2x + 1) dx = 11,$ *find a.*

Solution:

$$\text{Let } I = \int_{1}^{a} (3x^2 + 2x + 1)\, dx$$

$$= \left[3\left(\frac{x^3}{3}\right) + 2\left(\frac{x^2}{2}\right) + x\right]_1^a$$

$$= \left[x^3 + x^2 + x\right]_1^a$$

$$= (a^3 + a^2 + a) - (1+1+1)$$

$$= a^3 + a^2 + a - 3$$

$\therefore I = 11$ gives $a^3 + a^2 + a - 3 = 11$

$\therefore a^3 + a^2 + a - 14 = 0$

$\therefore (a^3 - 8) + (a^2 + a - 6) = 0$

$\therefore (a-2)(a^2 + 2a + 4) + (a+3)(a-2) = 0$

$\therefore (a-2)(a^2 + 2a + 4 + a + 3) = 0$

$\therefore (a-2)(a^2 + 3a + 7) = 0$

$\therefore a - 2 = 0$ or $a^2 + 3a + 7 = 0$

$\therefore a = 2$ or $a = \frac{-3 \pm \sqrt{9-28}}{2}$

The latter two roots are not real.

$\therefore$ they are rejected.

$\therefore a = 2.$

9. $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}dx$

Solution:

$$\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx$$

$$=\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}\times\frac{\sqrt{1+x}-\sqrt{x}}{\sqrt{1-x}-\sqrt{x}}\,dx$$

$$=\int_0^1 \frac{\sqrt{1+x}-\sqrt{x}}{1+x-x}\,dx$$

$$=\int_0^1 (\sqrt{1+x}-\sqrt{x})\,dx$$

$$=\int_0^1 (1+x)^{\frac{1}{2}}dx-\int_0^1 x^{\frac{1}{2}}dx$$

$$=\left[\frac{(1+x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1-\left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1$$

$$=\frac{2}{3}\left[(1+x)^{\frac{3}{2}}\right]_0^1-\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1$$

$$=\frac{2}{3}(2^{\frac{3}{2}}-1)-\frac{2}{3}(1-0)$$

$$=\frac{2}{3}(2^{\frac{3}{2}}-1-1)=\frac{2}{3}(2\sqrt{2}-2)$$

$$=\frac{4}{3}(\sqrt{2}-1).$$

10. $\int_1^2 \frac{3x}{(9x^2-1)}dx$

Solution:

$$\text{Let } I=\int_1^2 \frac{3x}{9x^2-1}\,dx=\int_1^2 \frac{3x}{(3x)^2-1}\,dx$$

Put $3x=t$ $\quad\therefore\ 3dx=dt$ $\quad\therefore\ dx=\frac{dt}{3}$

When $x=1,\ t=3\times 1=3$

When $x=2,\ t=3\times 2=6$

$$\therefore\ I=\int_3^6 \frac{t}{t^2-1}\cdot\frac{dt}{3}=\frac{1}{6}\int_3^6 \frac{2t}{t^2-1}\,dt$$

$$=\frac{1}{6}\Big[\log|t^2-1|\Big]_3^6 \qquad \ldots\left[\because \frac{d}{dt}(t^2-1)=2t\right]$$

$$=\frac{1}{6}[\log 35-\log 8]$$

$$=\frac{1}{6}\log\left(\frac{35}{8}\right).$$

11. $\int_1^3 \log x\, dx$

Solution:

$$\int_1^3 \log x\,dx=\int_1^3 (\log x)\cdot 1\,dx$$

$$=[(\log x)]\int 1dx]_1^3-\int_1^3\left[\frac{d}{dx}(\log x)\int 1\,dx\right]dx$$

$$=[(\log x)x]_1^3-\int_1^3 \frac{1}{x}\times x\,dx$$

$$=(3\log 3-\log 1)-\int_1^3 1\,dx$$

$$=3\log 3-[x]_1^3 \qquad \ldots[\because \log 1=0]$$

$$=\log 3^3-(3-1)$$

$$=\log 27-2.$$

[Note : *Answer in the textbook is incorrect.*]

EXERCISE 6.2

Evaluate the following intergrals:

1) $\int_{-9}^{9} \frac{x^3}{4-x^2}dx$

Solution:

$$\text{Let } I=\int_{-9}^{9}\frac{x^3}{4-x^2}\,dx$$

Let $f(x)=\dfrac{x^3}{4-x^2}$

$$\therefore\ f(-x)=\frac{(-x)^3}{4-(-x)^2}=\frac{-x^3}{4+x^2}=-f(x)$$

$\therefore$ f is an odd function.

$$\therefore\ \int_{-9}^{9}f(x)\,dx=0$$

i.e. $\displaystyle\int_{-9}^{9}\frac{x^3}{4-x^2}\,dx=0.$

2) $\displaystyle\int_0^a x^2(a-x)^{3/2}dx$

Solution:

We use the property

$$\int_0^a f(x)dx=\int_0^a f(a-x)dx$$

$$\therefore\ \int_0^a x^2(a-x)^{\frac{3}{2}}dx=\int_0^a (a-x)^2(a-a+x)^{\frac{3}{2}}dx$$

$$=\int_0^a (a^2-2ax+x^2)x^{\frac{3}{2}}\,dx$$

$$=\int_0^a (a^2x^{\frac{3}{2}}-2ax^{\frac{5}{2}}+x^{\frac{7}{2}})\,dx$$

$$=a^2\int_0^a x^{\frac{3}{2}}\,dx-2a\int_0^a x^{\frac{5}{2}}\,dx+\int_0^a x^{\frac{7}{2}}\,dx$$

$$=a^2\left[\frac{x^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}\right]_0^a-2a\left[\frac{x^{\frac{7}{2}}}{\left(\frac{7}{2}\right)}\right]_0^a+\left[\frac{x^{\frac{9}{2}}}{\left(\frac{9}{2}\right)}\right]_0^a$$

$$=\frac{2a^2}{5}[a^{\frac{5}{2}}-0]-\frac{4a}{7}[a^{\frac{7}{2}}-0]+\frac{2}{9}[a^{\frac{9}{2}}-0]$$

$$=\frac{2}{5}a^{\frac{9}{2}}-\frac{4}{7}a^{\frac{9}{2}}+\frac{2}{9}a^{\frac{9}{2}}=\left(\frac{2}{5}-\frac{4}{7}+\frac{2}{9}\right)a^{\frac{9}{2}}$$

$$=\left(\frac{126-180+70}{315}\right)a^{\frac{9}{2}}=\frac{16}{315}a^{\frac{9}{2}}.$$

3) $\displaystyle\int_1^3\frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}dx$

Solution:

$$\text{Let } I=\int_1^3\frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx \qquad \dots (1)$$

We use the property, $\displaystyle\int_a^b f(x)dx=\int_a^b f(a+b-x)\,dx$

Hence in I, we replace x by $1+3-x$.

$$\therefore\ I=\int_1^3\frac{\sqrt[3]{1+3-x+5}}{\sqrt[3]{1+3-x+5}+\sqrt[3]{9-1-3+x}}\,dx$$

$$=\int_1^3\frac{\sqrt[3]{9-x}}{\sqrt[3]{9-x}+\sqrt[3]{x+5}}\,dx \qquad \dots (2)$$

Adding (1) and (2), we get

$$2I=\int_1^3\frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx+\int_1^3\frac{\sqrt[3]{9-x}}{\sqrt[3]{9-x}+\sqrt[3]{x+5}}\,dx$$

$$=\int_1^3\frac{\sqrt[3]{x+5}+\sqrt[3]{9-x}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx$$

$$=\int_1^3 1dx=[x]_1^3$$

$$=3-1=2$$

$\therefore\ I=1$

Hence, $\displaystyle\int_1^3\frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx=1.$

4) $\displaystyle\int_2^5\frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}dx$

Solution:

$$\text{Let } I=\int_2^5\frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}\,dx \qquad \dots (1)$$

We use the property, $\displaystyle\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx.$

Hence in I, we change x by $2+5-x$.

$$\therefore\ I=\int_2^5\frac{\sqrt{2+5-x}}{\sqrt{2+5-x}+\sqrt{7-2-5+x}}\,dx$$

$$=\int_2^5\frac{\sqrt{7-x}}{\sqrt{7-x}+\sqrt{x}}\,dx \qquad \dots (2)$$

Adding (1) and (2), we get

$$2I=\int_2^5\frac{\sqrt{7-x}}{\sqrt{x}+\sqrt{7-x}}\,dx+\int_2^5\frac{\sqrt{x}}{\sqrt{7-x}+\sqrt{x}}\,dx$$

$$= \int_2^5 \frac{\sqrt{x}+\sqrt{7-x}}{\sqrt{x}+\sqrt{7-x}}\,dx$$

$$= \int_2^5 1\,dx = [x]_2^5 = 5-2 = 3$$

$$\therefore\ I = \frac{3}{2}$$

Hence, $\int_2^5 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}\,dx = \frac{3}{2}$.

5) $\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}dx$

Solution:

Let $I = \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}} \cdot dx$ ...(i)

$=$

$\int_1^2 \frac{\sqrt{1+2-x}}{\sqrt{3-(1+2-x)}+\sqrt{1+2-x}} \cdot dx$ ...$\left[\because \int_a^b f(x)\cdot dx = \int_a^b f(a+b-x\right.$

$\therefore I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}} \cdot dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}} \cdot dx + \int_1^2 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}} \cdot dx$

$= \int_1^2 \frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}} \cdot dx$

$= \int_1^2 1 \cdot dx$

$= [x]_1^2$

$\therefore 2I = 2-1 = 1$

$\therefore I = \frac{1}{2}$.

6) $\int_2^7 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}dx$

Solution:

Let $I = \int_2^7 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}} \cdot dx$ ...(i)

$=$

$\int_2^7 \frac{\sqrt{2+7-x}}{\sqrt{2+7-x}+\sqrt{9-(2+7-x)}} \cdot dx$ ...$\left[\because \int_a^b f(x)\cdot dx = \int_a^b f(a+b-x\right.$

$\therefore I = \int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{x}} \cdot dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_2^7 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}} \cdot dx + \int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{x}} \cdot dx$

$= \int_2^7 \frac{\sqrt{x}+\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}} \cdot dx$

$= \int_2^7 1 \cdot dx$

$= [x]_2^7$

$\therefore 2I = 7-2 = 5$

$\therefore I = \frac{5}{2}$.

7) $\int_0^1 \boldsymbol{log}\left(\frac{1}{x}-1\right)dx$

Solution:

Let $I = \int_0^1 \log\left(\frac{1}{x}-1\right)dx = \int_0^1 \log\left(\frac{1-x}{x}\right)dx$

We use the property, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$\therefore\ I = \int_0^1 \log\left[\frac{1-(1-x)}{1-x}\right]dx = \int_0^1 \log\left(\frac{x}{1-x}\right)dx$$

$$= \int_0^1 -\log\left(\frac{1-x}{x}\right)dx = -\int_0^1 \log\left(\frac{1-x}{x}\right)dx$$

$$\therefore\ I = -I$$

$$\therefore\ 2I = 0 \qquad \therefore\ I = 0$$

Hence, $\int_0^1 \log\left(\frac{1}{x}-1\right)dx = 0$.

8) $\int_0^1 x(1-x)^5dx$

Solution:

We use the property,

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

$$\therefore \int_0^1 x(1-x)^5 dx = \int_0^1 (1-x)(1-1+x)^5\ dx$$

$$= \int_0^1 (1-x)x^5 dx = \int_0^1 (x^5 - x^6)\ dx$$

$$= \int_0^1 x^5 dx - \int_0^1 x^6 dx$$

$$= \left[\frac{x^6}{6}\right]_0^1 - \left[\frac{x^7}{7}\right]_0^1$$

$$= \frac{1}{6}(1-0) - \frac{1}{7}(1-0)$$

$$= \frac{1}{6} - \frac{1}{7} = \frac{1}{42}.$$

[**Note :** *Answer in the textbook is incorrect.*]