CHAPTER XXIII.

PARTIAL FRACTIONS.

315. In elementary Algebra, a group of fractions connected by the signs of addition and subtraction is reduced to a more simple form by being collected into one single fraction whose denominator is the lowest common denominator of the given fractions. But the converse process of separating a fraction into a group of simpler, or *partial*, fractions is often required. For example, if we wish to expand $\dfrac{3-5x}{1-4x+3x^2}$ in a series of ascending powers of x, we might use the method of Art. 314, Ex. 1, and so obtain as many terms as we please. But if we wish to find the general term of the series this method is inapplicable, and it is simpler to express the given fraction in the equivalent form $\dfrac{1}{1-x}+\dfrac{2}{1-3x}$. Each of the expressions $(1-x)^{-1}$ and $(1-3x)^{-1}$ can now be expanded by the Binomial Theorem, and the general term obtained.

316. In the present chapter we shall give some examples illustrating the decomposition of a rational fraction into partial fractions. For a fuller discussion of the subject the reader is referred to Serret's *Cours d'Algèbre Supérieure*, or to treatises on the Integral Calculus. In these works it is proved that any rational fraction may be resolved into a series of partial fractions; and that to any linear factor $x-a$ in the denominator there corresponds a partial fraction of the form $\dfrac{A}{x-a}$; to any linear factor $x-b$ occurring *twice* in the denominator there correspond *two* partial fractions, $\dfrac{B_1}{x-b}$ and $\dfrac{B_2}{(x-b)^2}$. If $x-b$ occurs *three* times, there is an additional fraction $\dfrac{B_3}{(x-b)^3}$; and so on. To

any quadratic factor x^2+px+q there corresponds a partial fraction of the form $\dfrac{Px+Q}{x^2+px+q}$; if the factor x^2+px+q occurs twice, there is a second partial fraction $\dfrac{P_1x+Q_1}{(x^2+px+q)^2}$; and so on.

Here the quantities A_1, B_1, B_2, B_3,......P, Q, P_1, Q_1 are all independent of x.

We shall make use of these results in the examples that follow.

Example 1. Separate $\dfrac{5x-11}{2x^2+x-6}$ into partial fractions.

Since the denominator $2x^2+x-6=(x+2)(2x-3)$, we assume

$$\frac{5x-11}{2x^2+x-6}=\frac{A}{x+2}+\frac{B}{2x-3},$$

where A and B are quantities independent of x whose values have to be determined.

Clearing of fractions,

$$5x-11=A(2x-3)+B(x+2).$$

Since this equation is identically true, we may equate coefficients of like powers of x; thus

$$2A+B=5, \quad -3A+2B=-11;$$

whence $\qquad A=3, \quad B=-1.$

$$\therefore \quad \frac{5x-11}{2x^2+x-6}=\frac{3}{x+2}-\frac{1}{2x-3}.$$

Example 2. Resolve $\dfrac{mx+n}{(x-a)(x+b)}$ into partial fractions.

Assume $\qquad \dfrac{mx+n}{(x-a)(x+b)}=\dfrac{A}{x-a}+\dfrac{B}{x+b}.$

$$\therefore \quad mx+n=A(x+b)+B(x-a) \text{..................(1)}.$$

We might now equate coefficients and find the values of A and B, but it is simpler to proceed in the following manner.

Since A and B are independent of x, we may give to x any value we please.

In (1) put $x-a=0$, or $x=a$; then

$$A=\frac{ma+n}{a+b};$$

putting $x+b=0$, or $x=-b$, $\qquad B=\dfrac{mb-n}{a+b}.$

$$\therefore \quad \frac{mx+n}{(x-a)(x+b)}=\frac{1}{a+b}\left(\frac{ma+n}{x-a}+\frac{mb-n}{x+b}\right).$$

Example 3. Resolve $\dfrac{23x - 11x^2}{(2x-1)(9-x^2)}$ into partial fractions.

Assume $$\frac{23x - 11x^2}{(2x-1)(3+x)(3-x)} = \frac{A}{2x-1} + \frac{B}{3+x} + \frac{C}{3-x} \qquad \text{.........(1)};$$

$$\therefore \; 23x - 11x^2 = A(3+x)(3-x) + B(2x-1)(3-x) + C(2x-1)(3+x).$$

By putting in succession $2x-1=0$, $3+x=0$, $3-x=0$, we find that

$$A=1, \; B=4, \; C=-1.$$

$$\therefore \; \frac{23x-11x^2}{(2x-1)(9-x^2)} = \frac{1}{2x-1} + \frac{4}{3+x} - \frac{1}{3-x}.$$

Example 4. Resolve $\dfrac{3x^2+x-2}{(x-2)^2(1-2x)}$ into partial fractions.

Assume $$\frac{3x^2+x-2}{(x-2)^2(1-2x)} = \frac{A}{1-2x} + \frac{B}{x-2} + \frac{C}{(x-2)^2};$$

$$\therefore \; 3x^2+x-2 = A(x-2)^2 + B(1-2x)(x-2) + C(1-2x).$$

Let $1-2x=0$, then $\qquad A = -\dfrac{1}{3}$;

let $x-2=0$, then $\qquad C=-4$.

To find B, equate the coefficients of x^2; thus

$$3 = A - 2B; \text{ whence } B = -\frac{5}{3}.$$

$$\therefore \; \frac{3x^2+x-2}{(x-2)^2(1-2x)} = -\frac{1}{3(1-2x)} - \frac{5}{3(x-2)} - \frac{4}{(x-2)^2}.$$

Example 5. Resolve $\dfrac{42-19x}{(x^2+1)(x-4)}$ into partial fractions.

Assume $$\frac{42-19x}{(x^2+1)(x-4)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-4};$$

$$\therefore \; 42-19x = (Ax+B)(x-4) + C(x^2+1).$$

Let $x=4$, then $\qquad C=-2$;

equating coefficients of x^2, $\qquad 0 = A + C$, and $A=2$;

equating the absolute terms, $42 = -4B + C$, and $B=-11$,

$$\therefore \; \frac{42-19x}{(x^2+1)(x-4)} = \frac{2x-11}{x^2+1} - \frac{2}{x-4}.$$

317. The artifice employed in the following example will sometimes be found useful.

Example. Resolve $\dfrac{9x^3 - 24x^2 + 48x}{(x-2)^4(x+1)}$ into partial fractions.

Assume $$\frac{9x^3 - 24x^2 + 48x}{(x-2)^4(x+1)} = \frac{A}{x+1} + \frac{f(x)}{(x-2)^4},$$

where A is some constant, and $f(x)$ a function of x whose value remains to be determined.

$$\therefore \quad 9x^3 - 24x^2 + 48x = A\,(x-2)^4 + (x+1)f(x).$$

Let $x = -1$, then $\qquad A = -1.$

Substituting for A and transposing,

$$(x+1)f(x) = (x-2)^4 + 9x^3 - 24x^2 + 48x = x^4 + x^3 + 16x + 16;$$

$$\therefore f(x) = x^3 + 16.$$

To determine the partial fractions corresponding to $\dfrac{x^3+16}{(x-2)^4}$, put $x - 2 = z$;

then $$\frac{x^3+16}{(x-2)^4} = \frac{(z+2)^3+16}{z^4} = \frac{z^3+6z^2+12z+24}{z^4}$$

$$= \frac{1}{z} + \frac{6}{z^2} + \frac{12}{z^3} + \frac{24}{z^4}$$

$$= \frac{1}{x-2} + \frac{6}{(x-2)^2} + \frac{12}{(x-2)^3} + \frac{24}{(x-2)^4}.$$

$$\therefore \quad \frac{9x^3 - 24x^2 + 48x}{(x-2)^4(x+1)} = -\frac{1}{x+1} + \frac{1}{x-2} + \frac{6}{(x-2)^2} + \frac{12}{(x-2)^3} + \frac{24}{(x-2)^4}.$$

318. In all the preceding examples the numerator has been of lower dimensions than the denominator; if this is not the case, we divide the numerator by the denominator until a remainder is obtained which is of lower dimensions than the denominator.

Example. Resolve $\dfrac{6x^3+5x^2-7}{3x^2-2x-1}$ into partial fractions.

By division,

$$\frac{6x^3+5x^2-7}{3x^2-2x-1} = 2x+3+\frac{8x-4}{3x^2-2x-1};$$

and $$\frac{8x-4}{3x^2-2x-1} = \frac{5}{3x+1} + \frac{1}{x-1};$$

$$\therefore \quad \frac{6x^3+5x^2-7}{3x^2-2x-1} = 2x+3+\frac{5}{3x+1}+\frac{1}{x-1}.$$

319. We shall now explain how resolution into partial fractions may be used to facilitate the expansion of a rational fraction in ascending powers of x.

Example 1. Find the general term of $\dfrac{3x^2+x-2}{(x-2)^2(1-2x)}$ when expanded in a series of ascending powers of x.

By Ex. 4, Art. 316, we have

$$\frac{3x^2+x-2}{(x-2)^2(1-2x)} = -\frac{1}{3(1-2x)} - \frac{5}{3(x-2)} - \frac{4}{(x-2)^2}$$

$$= -\frac{1}{3(1-2x)} + \frac{5}{3(2-x)} - \frac{4}{(2-x)^2}$$

$$= -\frac{1}{3}(1-2x)^{-1} + \frac{5}{6}\left(1-\frac{x}{2}\right)^{-1} - \left(1-\frac{x}{2}\right)^{-2}.$$

Hence the general term of the expansion is

$$\left(-\frac{2^r}{3} + \frac{5}{6}\cdot\frac{1}{2^r} - \frac{r+1}{2^r}\right)x^r.$$

Example 2. Expand $\dfrac{7+x}{(1+x)(1+x^2)}$ in ascending powers of x and find the general term.

Assume $$\frac{7+x}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2};$$

$$\therefore \quad 7+x = A(1+x^2) + (Bx+C)(1+x).$$

Let $1+x=0$, then $A=3$;

equating the absolute terms, $7=A+C$, whence $C=4$;

equating the coefficients of x^2, $0=A+B$, whence $B=-3$.

$$\therefore \quad \frac{7+x}{(1+x)(1+x^2)} = \frac{3}{1+x} + \frac{4-3x}{1+x^2}$$

$$= 3(1+x)^{-1} + (4-3x)(1+x^2)^{-1}$$

$$= 3\{1-x+x^2-\ldots\ldots+(-1)^p x^p+\ldots\}$$

$$+ (4-3x)\{1-x^2+x^4-\ldots\ldots+(-1)^p x^{2p}+\ldots\}.$$

To find the coefficient of x^r:

(1) If r is even, the coefficient of x^r in the second series is $4(-1)^{\frac{r}{2}}$; therefore in the expansion the coefficient of x^r is $3+4(-1)^{\frac{r}{2}}$.

(2) If r is odd, the coefficient of x^r in the second series is $-3(-1)^{\frac{r-1}{2}}$, and the required coefficient is $3(-1)^{\frac{r+1}{2}}-3$.

EXAMPLES. XXIII.

Resolve into partial fractions:

1. $\dfrac{7x-1}{1-5x+6x^2}$. 2. $\dfrac{46+13x}{12x^2-11x-15}$. 3. $\dfrac{1+3x+2x^2}{(1-2x)(1-x^2)}$.

4. $\dfrac{x^2-10x+13}{(x-1)(x^2-5x+6)}$.

5. $\dfrac{2x^3+x^2-x-3}{x(x-1)(2x+3)}$.

6. $\dfrac{9}{(x-1)(x+2)^2}$.

7. $\dfrac{x^4-3x^3-3x^2+10}{(x+1)^2(x-3)}$.

8. $\dfrac{26x^2+208x}{(x^2+1)(x+5)}$.

9. $\dfrac{2x^2-11x+5}{(x-3)(x^2+2x-5)}$.

10. $\dfrac{3x^3-8x^2+10}{(x-1)^4}$.

11. $\dfrac{5x^3+6x^2+5x}{(x^2-1)(x+1)^3}$.

Find the general term of the following expressions when expanded in ascending powers of x.

12. $\dfrac{1+3x}{1+11x+28x^2}$.

13. $\dfrac{5x+6}{(2+x)(1-x)}$.

14. $\dfrac{x^2+7x+3}{x^2+7x+10}$.

15. $\dfrac{2x-4}{(1-x^2)(1-2x)}$.

16. $\dfrac{4+3x+2x^2}{(1-x)(1+x-2x^2)}$.

17. $\dfrac{3+2x-x^2}{(1+x)(1-4x)^2}$.

18. $\dfrac{4+7x}{(2+3x)(1+x)^2}$.

19. $\dfrac{2x+1}{(x-1)(x^2+1)}$.

20. $\dfrac{1-x+2x^2}{(1-x)^3}$.

21. $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$.

22. $\dfrac{3-2x^2}{(2-3x+x^2)^2}$.

23. Find the sum of n terms of the series

(1) $\dfrac{1}{(1+x)(1+x^2)}+\dfrac{x}{(1+x^2)(1+x^3)}+\dfrac{x^2}{(1+x^3)(1+x^4)}+\ldots\ldots$

(2) $\dfrac{x(1-ax)}{(1+x)(1+ax)(1+a^2x)}+\dfrac{ax(1-a^2x)}{(1+ax)(1+a^2x)(1+a^3x)}+\ldots\ldots$

24. When $x<1$, find the sum of the infinite series

$$\frac{1}{(1-x)(1-x^3)}+\frac{x^2}{(1-x^3)(1-x^5)}+\frac{x^4}{(1-x^5)(1-x^7)}+\ldots\ldots$$

25. Sum to n terms the series whose p^{th} term is

$$\frac{x^p(1+x^{p+1})}{(1-x^p)(1-x^{p+1})(1-x^{p+2})}.$$

26. Prove that the sum of the homogeneous products of n dimensions which can be formed of the letters a, b, c and their powers is

$$\frac{a^{n+2}(b-c)+b^{n+2}(c-a)+c^{n+2}(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}.$$

CHAPTER XXIV.

Recurring Series.

320. A series $u_0 + u_1 + u_2 + u_3 + \ldots\ldots,$
in which from and after a certain term each term is equal to the sum of a fixed number of the preceding terms multiplied respectively by certain constants is called a **recurring series.**

321. In the series

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots\ldots,$$

each term after the second is equal to the sum of the two preceding terms multiplied respectively by the *constants* $2x$, and $-x^2$; these quantities being called constants because they are the same for all values of n. Thus

$$5x^4 = 2x \,.\, 4x^3 + (-x^2) \,.\, 3x^2;$$

that is,

$$u_4 = 2xu_3 - x^2u_2;$$

and generally when n is greater than 1, each term is connected with the two that immediately precede it by the equation

$$u_n = 2xu_{n-1} - x^2u_{n-2},$$

or

$$u_n - 2xu_{n-1} + x^2u_{n-2} = 0.$$

In this equation the coefficients of u_n, u_{n-1}, and u_{n-2}, taken with their proper signs, form what is called the **scale of relation.**

Thus the series

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots\ldots$$

is a recurring series in which the scale of relation is

$$1 - 2x + x^2.$$

322. If the scale of relation of a recurring series is given, any term can be found when a sufficient number of the preceding

terms are known. As the method of procedure is the same however many terms the scale of relation may consist of, the following illustration will be sufficient.

If $$1 - px - qx^2 - rx^3$$

is the scale of relation of the series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$$

we have

$$a_nx^n = px \,.\, a_{n-1}x^{n-1} + qx^2 \,.\, a_{n-2}x^{n-2} + rx^3 \,.\, a_{n-3}x^{n-3},$$

or $$a_n = pa_{n-1} + qa_{n-2} + ra_{n-3};$$

thus any coefficient can be found when the coefficients of the three preceding terms are known.

323. Conversely, if a sufficient number of the terms of a series be given, the scale of relation may be found.

Example. Find the scale of relation of the recurring series

$$2 + 5x + 13x^2 + 35x^3 + \ldots\ldots$$

Let the scale of relation be $1 - px - qx^2$; then to obtain p and q we have the equations $13 - 5p - 2q = 0$, and $35 - 13p - 5q = 0$;
whence $p = 5$, and $q = -6$, thus the scale of relation is

$$1 - 5x + 6x^2.$$

324. If the scale of relation consists of 3 terms it involves 2 constants, p and q; and we must have 2 equations to determine p and q. To obtain the first of these we must know at least 3 terms of the series, and to obtain the second we must have one more term given. Thus to obtain a scale of relation involving two constants we must have at least 4 terms given.

If the scale of relation be $1 - px - qx^2 - rx^3$, to find the 3 constants we must have 3 equations. To obtain the first of these we must know at least 4 terms of the series, and to obtain the other two we must have two more terms given; hence to find a scale of relation involving 3 constants, at least 6 terms of the series must be given.

Generally, to find a scale of relation involving m constants, we must know at least $2m$ consecutive terms.

Conversely, if $2m$ consecutive terms are given, we may assume for the scale of relation

$$1 - p_1x - p_2x^2 - p_3x^3 - \ldots\ldots - p_mx^m.$$

325. *To find the sum of* n *terms of a recurring series.*

The method of finding the sum is the same whatever be the scale of relation; for simplicity we shall suppose it to contain only two constants.

Let the series be

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots\dots\dots\dots\dots\dots (1)$$

and let the sum be S; let the scale of relation be $1 - px - qx^2$; so that for every value of n greater than 1, we have

$$a_n - pa_{n-1} - qa_{n-2} = 0.$$

$$\begin{aligned}
\text{Now}\quad S &= a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}, \\
-px\, S &= \quad - pa_0 x - pa_1 x^2 - \dots - pa_{n-2} x^{n-1} - pa_{n-1} x^n, \\
-qx^2\, S &= \quad\quad - qa_0 x^2 - \dots - qa_{n-3} x^{n-1} - qa_{n-2} x^n - qa_{n-1} x^{n+1}.
\end{aligned}$$

$$\therefore\ (1 - px - qx^2)\, S = a_0 + (a_1 - pa_0)\, x - (pa_{n-1} + qa_{n-2})\, x^n - qa_{n-1} x^{n+1},$$

for the coefficient of every other power of x is zero in consequence of the relation

$$a_n - pa_{n-1} - qa_{n-2} = 0.$$

$$\therefore\ S = \frac{a_0 + (a_1 - pa_0)\, x}{1 - px - qx^2} - \frac{(pa_{n-1} + qa_{n-2})\, x^n + qa_{n-1} x^{n+1}}{1 - px - qx^2}.$$

Thus the sum of a recurring series is a fraction whose denominator is the scale of relation.

326. If the second fraction in the result of the last article decreases indefinitely as n increases indefinitely, the sum of an infinite number of terms reduces to $\dfrac{a_0 + (a_1 - pa_0)\, x}{1 - px - qx^2}$.

If we develop this fraction in ascending powers of x as explained in Art. 314, we shall obtain as many terms of the original series as we please; for this reason the expression

$$\frac{a_0 + (a_1 - pa_0)\, x}{1 - px - qx^2}$$

is called the *generating function* of the series.

327. From the result of Art. 325, we obtain

$$\frac{a_0 + (a_1 - pa_0)\, x}{1 - px - qx^2} = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n+1}$$
$$+ \frac{(pa_{n-1} + qa_{n-2})\, x^n + qa_{n-1} x^{n+1}}{1 - px - qx^2};$$

from which we see that although the generating function

$$\frac{a_0 + (a_1 - pa_0)\, x}{1 - px - qx^2}$$

may be used to obtain as many terms of the series as we please, it can be regarded as the true equivalent of the infinite series

$$a_0 + a_1 x + a_2 x^2 + \ldots\ldots,$$

only if the remainder

$$\frac{(pa_{n-1} + qa_{n-2})\, x^n + qa_{n-1} x^{n+1}}{1 - px - qx^2}$$

vanishes when n is indefinitely increased; in other words only when the series is convergent.

328. When the *generating function* can be expressed as a group of partial fractions the general term of a recurring series may be easily found. Thus, suppose the generating function can be decomposed into the partial fractions

$$\frac{A}{1 - ax} + \frac{B}{1 + bx} + \frac{C}{(1 - cx)^2}.$$

Then the general term is

$$\{Aa^r + (-1)^r Bb^r + (r+1)\, Cc^r\}\, x^r.$$

In this case the sum of n terms may be found without using the method of Art. 325.

Example. Find the generating function, the general term, and the sum to n terms of the recurring series

$$1 - 7x - x^2 - 43x^3 - \ldots\ldots$$

Let the scale of relation be $1 - px - qx^2$; then

$$-1 + 7p - q = 0, \quad -43 + p + 7q = 0;$$

whence $p = 1$, $q = 6$; and the scale of relation is

$$1 - x - 6x^2.$$

Let S denote the sum of the series; then

$$S = 1 - 7x - x^2 - 43x^3 - \ldots\ldots$$

$$-xS = \quad - x + 7x^2 + x^3 + \ldots\ldots$$

$$-6x^2 S = \quad -6x^2 + 42x^3 + \ldots\ldots$$

$$\therefore \quad (1 - x - 6x^2)\, S = 1 - 8x,$$

$$S = \frac{1 - 8x}{1 - x - 6x^2};$$

which is the generating function.

If we separate $\dfrac{1-8x}{1-x-6x^2}$ into partial fractions, we obtain $\dfrac{2}{1+2x} - \dfrac{1}{1-3x}$; whence the $(r+1)^{th}$ or general term is

$$\{(-1)^r 2^{r+1} - 3^r\} x^r.$$

Putting $\qquad r = 0, 1, 2, \ldots n-1,$

the sum to n terms

$$= \{2 - 2^2x + 2^3x^2 - \ldots + (-1)^{n-1} 2^n x^{n-1}\} - (1 + 3x + 3^2x^2 + \ldots + 3^{n-1} x^{n-1})$$

$$= \frac{2 + (-1)^{n-1} 2^{n+1} x^n}{1+2x} - \frac{1 - 3^n x^n}{1-3x}.$$

329. To find the general term and sum of n terms of the recurring series $a_0 + a_1 + a_2 + \ldots\ldots$, we have only to find the general term and sum of the series $a_0 + a_1x + a_2x^2 + \ldots\ldots$, and put $x = 1$ in the results.

Example. Find the general term and sum of n terms of the series

$$1 + 6 + 24 + 84 + \ldots\ldots$$

The scale of relation of the series $1 + 6x + 24x^2 + 84x^3 + \ldots$ is $1 - 5x + 6x^2$, and the generating function is $\dfrac{1+x}{1-5x+6x^2}$.

This expression is equivalent to the partial fractions

$$\frac{4}{1-3x} - \frac{3}{1-2x}.$$

If these expressions be expanded in ascending powers of x the general term is $\qquad (4 \cdot 3^r - 3 \cdot 2^r) x^r.$

Hence the general term of the given series is $4 \cdot 3^r - 3 \cdot 2^r$; and the sum of n terms is $\qquad 2(3^n - 1) - 3(2^n - 1).$

330. We may remind the student that in the preceding article the generating function cannot be taken as the sum of the series

$$1 + 6x + 24x^2 + 84x^3 + \ldots\ldots$$

except when x has such a value as to make the series convergent. Hence when $x = 1$ (in which case the series is obviously divergent) the generating function is not a true equivalent of the series. But the general term of

$$1 + 6 + 24 + 84 + \ldots\ldots$$

is independent of x, *and whatever value* x *may have* it will always be the coefficient of x^n in

$$1 + 6x + 24x^2 + 84x^3 + \ldots\ldots.$$

We therefore treat this as a convergent series and find its general term in the usual way, and then put $x = 1$.

EXAMPLES. XXIV.

Find the generating function and the general term of the following series:

1. $1+5x+9x^2+13x^3+\dots\dots$
2. $2-x+5x^2-7x^3+\dots\dots$
3. $2+3x+5x^2+9x^3+\dots\dots$
4. $7-6x+9x^2+27x^4+\dots\dots$
5. $3+6x+14x^2+36x^3+98x^4+276x^5+\dots\dots$

Find the n^{th} term and the sum to n terms of the following series:

6. $2+5+13+35+\dots\dots$
7. $-1+6x^2+30x^3+\dots\dots$
8. $2+7x+25x^2+91x^3+\dots\dots$
9. $1+2x+6x^2+20x^3+66x^4+212x^5+\dots\dots$
10. $-\frac{3}{2}+2+0+8+\dots\dots$

11. Shew that the series

$$1^2+2^2+3^2+4^2+\dots\dots+n^2,$$
$$1^3+2^3+3^3+4^3+\dots\dots+n^3,$$

are recurring series, and find their scales of relation.

12. Shew how to deduce the sum of the first n terms of the recurring series

$$a_0+a_1x+a_2x^2+a_3x^3+\dots\dots$$

from the sum to infinity.

13. Find the sum of $2n+1$ terms of the series

$$3-1+13-9+41-53+\dots\dots$$

14. The scales of the recurring series

$$a_0+a_1x+a_2x^2+a_3x^3+\dots\dots,$$
$$b_0+b_1x+b_2x^2+b_3x^3+\dots\dots,$$

are $1+px+qx^2$, $1+rx+sx^2$, respectively; shew that the series whose general term is $(a_n+b_n)x^n$ is a recurring series whose scale is

$$1+(p+r)x+(q+s+pr)x^2+(qr+ps)x^3+qsx^4.$$

15. If a series be formed having for its n^{th} term the sum of n terms of a given recurring series, shew that it will also form a recurring series whose scale of relation will consist of one more term than that of the given series.