CBSE Board
Class XII Mathematics
Sample Paper 4

Time: 3 hours **Total Marks: 100**

General Instructions:

1. **All** the questions are **compulsory**.
2. The question paper consists of **37** questions divided into **three parts** A, B, and C.
3. **Part A** comprises of **20** questions of **1 mark** each. **Part B** comprises of **11** questions of **4 marks** each. **Part C** comprises of **6** questions of **6 marks** each.
4. There is no overall choice. However, an internal choice has been provided in **three questions of 4 marks** each, **four questions of 6 marks** each. You have to attempt only one of the alternatives in all such questions.
5. Use of calculator is **not** permitted.

Part A

Q1 – Q20 are multiple choice type questions. Select the correct option.

1. The value of the determinant $\begin{vmatrix} a^2 & a & 1 \\ \cos nx & \cos(n+1)x & \cos(n+2)x \\ \sin nx & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$ is independent of

A. n
B. a
C. x
D. none of these

2. If $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, then find (A + 2B)'

A. $\begin{bmatrix} -4 & 3 \\ 3 & 6 \end{bmatrix}$

B. $\begin{bmatrix} -4 & 1 \\ 5 & 6 \end{bmatrix}$

C. $\begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$

D. $\begin{bmatrix} -4 & -3 \\ -3 & 6 \end{bmatrix}$

3. The value of $\hat{i}\cdot\left(\hat{j}\times \hat{k}\right)+\hat{j}\cdot\left(\hat{i}\times \hat{k}\right)+\hat{k}\cdot\left(\hat{i}\times \hat{j}\right)$, is

A. 0
B. –1
C. 1
D. 3

4. A and B are two events such that P (A) = 0.25 and P (B) = 0.50. The probability of both happening together is 0.14. The probability of both A and B not happening is

A. 0.16
B. 0.25
C. 0.11
D. 0.39

5. The solution of the differential equation $\frac{dy}{dx}+\frac{2y}{x}=0$ with y(1) = 1 is given by

A. $y=\frac{1}{x^2}$

B. $x=\frac{1}{y^2}$

C. $x=\frac{1}{y}$

D. $y=\frac{1}{x}$

6. If 4 $\cos^{-1}x + \sin^{-1} x = \pi$, then the value of x is

A. $\frac{3}{2}$

B. $\frac{1}{\sqrt{2}}$

C. $\frac{\sqrt{3}}{2}$

D. $\frac{2}{\sqrt{3}}$

7. The area bounded by the parabola $x = 4 - y^2$ and y-axis, in square units, is

A. $\frac{3}{32}$

B. $\frac{32}{3}$

C. $\frac{33}{2}$

D. $\frac{16}{3}$

8. The interval of increase of the function f(x) = x − e^x + tan (2π/7) is

A. (0, ∞)

B. (−∞, 0)

C. (1, ∞)

D. (−∞, 1)

9. The intercept cut off by the plane 2x + y − z = 5 on x-axis is

A. $\frac{5}{2}$

B. $\frac{2}{5}$

C. 2

D. 5

10. The distance between the point (3, 4, 5) and the point where the line $\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2}$ meets the plane $x + y + z = 17$, is

A. 1

B. 2

C. 3

D. 4

11. Consider the binary operation * defined on Q − {1} by the rule a * b = a + b − ab for all a, b ∈ Q − {1}. The identity element in Q − {1} is

A. 1

B. 0

C. a

D. −1

12. The value of k, for which the function $f(x) = \begin{cases} kx^2, & x \le 2 \\ 3, & x > 2 \end{cases}$ is continuous at x = 2 is

A. $\frac{3}{4}$

B. $\frac{3}{2}$

C. $\frac{4}{3}$

D. 12

13. Find the slope of the tangent to the curve $y = x^3 - x + 1$ at the point where the curve cuts the y-axis.

A. 1

B. −1

C. 2

D. 0

14. What is the range of the function $f(x) = \dfrac{|x-1|}{(x-1)}$?

A. $R - \{1, -1\}$

B. R

C. $R - \{1\}$

D. $\{-1, 1\}$

15. If $[2\vec{a}+4\vec{b}\ \ \vec{c}\ \ \vec{d}] = \lambda[\vec{a}\ \ \vec{c}\ \ \vec{d}] + \mu[\vec{b}\ \ \vec{c}\ \ \vec{d}]$, then $\lambda + \mu$ is equal to

A. 2

B. −2

C. 6

D. 8

16. If $\sin^{-1} x = y$, then

A. $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$

B. $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

C. $0 \le y \le \dfrac{\pi}{2}$

D. $-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$

17. The differential equation which represents the family of curves $y = e^{Cx}$ is

A. $\ln y = \dfrac{dy}{dx}$

B. $y \ln y = x\dfrac{dy}{dx}$

C. $x \ln y = y\dfrac{dy}{dx}$

D. $y \ln y = \dfrac{dy}{dx}$

18. The value of $\int \frac{1}{5 + 3\cos x} dx$ is

A. $\tan^{-1}\left(\frac{x}{2}\right) + c$

B. $\frac{1}{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{2}\right) + c$

C. $\tan^{-1}\left(\frac{\tan\frac{x}{2}}{2}\right) + c$

D. $\frac{x}{4} + c$

19. Evaluate: $\int \frac{x^2}{1 + x^3} dx$

A. $\frac{1}{3}\log\left|1 + x^3\right| + c$

B. $\log\left|1 + x^3\right| + c$

C. $\log\left|x\right| + c$

D. $\log\left|x^3\right| + c$

20. The differential equation obtained on eliminating A and B from y = A cos ωt + B sin ωt, is

A. $y'' = -\omega^2 y'$

B. $y' = \omega^2 y$

C. $y'' = -\omega^2 y$

D. $y' = \omega^2 y''$

Part B

21. Prove that $\tan^{-1}\left(\dfrac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right)=\dfrac{\pi}{4}-\dfrac{x}{2}$ if $\pi < x < \dfrac{3\pi}{2}$

22. Show that the differential equation $2ye^{x/y}\,dx + (y - 2x\,e^{x/y})\,dy = 0$ is homogeneous.

OR

Obtain the differential equation of all the circles touching the x-axis at the origin.

23. Given that $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$, such that the scalar product of $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and unit vector along the sum of the given two vectors $\vec{b}$ and $\vec{c}$ is unity. Find λ.

24. Let A = Q × Q, where Q is the set of all rational numbers and * is a binary operation on A defined by (a, b) * (c, d) = (ac, b + ad) for (a, b), (c, d) ϵ A. Then find:

(i) The identity element of * in A.

(ii) Invertible element of A, and hence write the inverse of elements (5, 3) and $\left(\frac{1}{2}, 4\right)$.

25. Evaluate the integral: $\displaystyle\int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}\,dx$

26. Prove that: $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

27. Differentiate the following function w.r.t. x: $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

OR

Find $\dfrac{dy}{dx}$ if $(x^2 + y^2)^2 = xy$.

28. Prove that $\displaystyle\int_0^{\pi} \frac{x}{1+\sin x}\,dx = \pi$

OR

Evaluate: $\displaystyle\int_0^{\frac{\pi}{2}} 2\sin x\cos x\tan^{-1}(\sin x)\,dx$.

29. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter, a car and a truck are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

30. Find the Cartesian equation of the plane passing through the points A(0, 0, 0) and B(3, -1, 2) and parallel to the line $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$

31. Find the value of 'a' for which the function f defined as

$$f(x) = \begin{cases} a \sin \frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$

is continuous at x = 0.

Part C

32. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftsman's time in its making while a cricket bat takes 3 hours of machine time and 1 hour of craftsman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time. If the profit on a racket and on a bat is Rs20 and Rs10 respectively, find the number of tennis rackets and crickets bats that the factory must manufacture to earn the maximum profit. Make it as an L.P.P and solve graphically.

33. Suppose 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.

OR

A girl throws a cie. If she get a 5 OR 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 OR 4, she tosses a coin two times and notes the number of heads obtained. If she obtained exactly two heads, what is the probability that she threw 1, 2, 3 OR 4 with the die?

34. Using the method of method of integration, find the area of the region bounded by the following lines:

$3x - y - 3 = 0$,

$2x + y - 12 = 0$,

$x - 2y - 1 = 0$

OR

Using integration, find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

35. Show that a closed right circular cylinder of a given total surface area and maximum volume is such that its height is equal to the diameter of the base.

OR

Show that the semi-vertical angle of a cone of maximum volume and given slant height is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

36. Find the equation of plane passing through points (1, 2, 3), (0, –1, 0) and parallel to the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z}{-3}$

OR

Find the equation of the plane passing through the line of intersection of the planes 2x + y – z = 3, 5x - 3y + 4z + 9 = 0 and parallel to the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$.

37. Two schools A and B want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school A wants to award x each, y each and z each for the three respective values to 3, 2 and 1 students respectively with a total award money of 1,600. School B wants to spend 2,300 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as before). If the total amount for one prize on each value is 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award.

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Sample Paper 4 – Solution

Part A

1. **Correct option: A**
   Explanation:-

$$\begin{vmatrix} a^2 & a & 1 \\ \cos nx & \cos(n+1)x & \cos(n+2)x \\ \sin nx & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$$

Let $nx = u, (n+1)x = v, (n+2)x = w$

$$\Rightarrow \begin{vmatrix} a^2 & a & 1 \\ \cos u & \cos v & \cos w \\ \sin u & \sin v & \sin w \end{vmatrix}$$

$\Rightarrow a^2 \sin(w - v) - a\sin(w - u) + \sin(v - u)$

$\Rightarrow a^2 \sin x - a\sin 2x + \sin x$

$\Rightarrow$ It is independent of n.

2. **Correct option: C**
   Explanation:-

$A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$

$A = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$

$B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$

$(A + 2B) = \begin{bmatrix} -4 & 1 \\ 5 & 6 \end{bmatrix}$

$(A + 2B)' = \begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$

3. **Correct option: C**
   Explanation:-

$\hat{i}\cdot(\hat{j}\times k) + j\cdot(\hat{i}\times k) + k\cdot(\hat{i}\times\hat{j})$

$= \hat{i}\cdot\hat{i} + \hat{j}\cdot(-\hat{j}) + k\cdot k$

$= 1 - 1 + 1$

$= 1$

4. **Correct option: D**

Explanation:-

$$P(A) = 0.25,\ P(B) = 0.5,\ P(A \cap B) = 0.14$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = 0.25 + 0.5 - 0.14$$

$$P(A \cup B) = 0.61$$

$$P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$$

$$P(\overline{A} \cap \overline{B}) = 1 - 0.61$$

$$P(\overline{A} \cap \overline{B}) = 0.39$$

5. **Correct option: A**

Explanation:-

The given differential equation is $\frac{dy}{dx} + \frac{2y}{x} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{2y}{x}$$

$$\Rightarrow -\int \frac{dy}{2y} = \int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2}\log y = \log x + c$$

$\therefore y = x^{-2} + C$ ... (i)

When $x = 1, y = 1$

$$\Rightarrow 1 = 1 + C$$

$$\Rightarrow C = 0$$

$$\Rightarrow y = \frac{1}{x^2}$$

6. **Correct option: C**

Explanation:-

$$4\cos^{-1}x + \sin^{-1}x = \pi$$

$$\Rightarrow 3\cos^{-1}x + \cos^{-1}x + \sin^{-1}x = \pi$$

$$\Rightarrow 3\cos^{-1}x + \frac{\pi}{2} = \pi$$

$$\Rightarrow \cos^{-1}x = \frac{\pi}{6}$$

$$\Rightarrow x = \cos\frac{\pi}{6}$$

$$\Rightarrow x = \frac{\sqrt{3}}{2}$$

7. **Correct option: B**

Explanation:-

The parabola touches the y-axis at x = 0

i.e. y = 2 and y = −2

So, the required area is

$$A = \int_{-2}^{2} x\,dy$$

$$= \int_{-2}^{2} \left(4 - y^2\right) dy$$

$$= \left[4y - \frac{y^3}{3}\right]_{-2}^{2}$$

$$= 4(2+2) - \left(\frac{8}{3} + \frac{8}{3}\right)$$

$$= 16 - \frac{16}{3}$$

$$= \frac{32}{3} \text{ sq. units}$$

8. **Correct option: B**

Explanation:-

$f(x) = x - e^x + \tan(2\pi/7)$

$f'(x) = 1 - e^x$

The function f increases when $f'(x) > 0$

$\Rightarrow 1 - e^x > 0$

$\Rightarrow e^x < 1$

$\Rightarrow e^x < e^0$

$\Rightarrow x < 0$

Thus, $x \in (-\infty, 0)$

Hence, the interval of increase of the function is $(-\infty, 0)$.

9. **Correct option: A**

Explanation:-

$2x + y - z = 5$

Dividing both sides by 5,

$$\frac{2x}{5} + \frac{y}{5} - \frac{z}{5} = 1$$

$$\frac{x}{\frac{5}{2}} + \frac{y}{5} + \frac{z}{-5} = 1$$

It is known that the equation of a plane in intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, where a, b and c are the intercepts cut off by the plane at x, y, and z-axes respectively.

Thus, the intercept cut off by the given plane on the x-axis is $\frac{5}{2}$.

10. Correct option: C

Explanation:-

$$\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}=\lambda$$

$$\Rightarrow x=3+\lambda,\ y=4+2\lambda,\ z=5+2\lambda$$

$\Rightarrow (3+\lambda, 4+2\lambda, 5+2\lambda)$ point is on the $x+y+z=17$ plane.

$$\Rightarrow 3+\lambda+4+2\lambda+5+2\lambda=17$$

$$\Rightarrow \lambda=1$$

$$\Rightarrow (3+\lambda, 4+2\lambda, 5+2\lambda)=(4,6,7)$$

Distance between $(4,6,7)$ and $(3,4,5)$

$$=\sqrt{1+4+4}=3$$

11. Correct option: B

Explanation:-

Given $a*b=a+b-ab$

Let the identity element be e, then

$$a*e=a$$

$$\Rightarrow a+e-ae=a$$

$$\Rightarrow e=0$$

12. Correct option: A

Explanation:-

We have,

$$f(x)=\begin{cases}kx^2, & x\le 2\\ 3, & x>2\end{cases}$$

As f is continuous at x = 2

$$\Rightarrow \lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)$$

$$\Rightarrow \lim_{x\to 2} kx^2=3$$

$$\Rightarrow k(2)^2=3$$

$$\Rightarrow 4k=3$$

$$\Rightarrow k=\frac{3}{4}$$

13. Correct option: B

Explanation:-

$y = x^3 - x + 1$

$\frac{dy}{dx} = 3x^2 - 1$

At a point where the curve cuts y-axis, $x = 0$

$\left.\frac{dy}{dx}\right]_{x=0} = 3x^2 - 1\Big]_{x=0} = -1$

14. Correct option: D

Explanation:-

Given: $f(x) = \frac{|x-1|}{(x-1)}$

$|x-1| = \begin{cases} (x-1), & \text{if } x > 1 \\ -(x-1), & \text{if } x < 1 \end{cases}$

(i) For $x > 1$, $f(x) = \frac{x-1}{x-1} = 1$

(ii) For $x < 1$, $f(x) = \frac{-(x-1)}{x-1} = -1$

Thus, the range of f(x) is {–1, 1}.

15. Correct option: C

Explanation:-

$\left[2\vec{a}+4\vec{b}\ \vec{c}\ \vec{d}\right] = \left[2\vec{a}\ \vec{c}\ \vec{d}\right] + \left[4\vec{b}\ \vec{c}\ \vec{d}\right]$

$\left[2\vec{a}+4\vec{b}\ \vec{c}\ \vec{d}\right] = 2\left[\vec{a}\ \vec{c}\ \vec{d}\right] + 4\left[\vec{b}\ \vec{c}\ \vec{d}\right]$

$\lambda = 2, \mu = 4$

$\Rightarrow \lambda + \mu = 6$

16. Correct option: A

Explanation:-

Given: $\sin^{-1} x = y$

We know that the range of the principal value of sin is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

So, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Hence, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

17. **Correct option**: B

Explanation:-

Given: $y = e^{Cx}$

$\ln y = Cx$ … (i)

Differentiating both sides of (i) w.r.t x, we get

$$\frac{1}{y}\frac{dy}{dx} = C$$

Substituting this value in (i), we get

$$\ln y = \left(\frac{1}{y}\frac{dy}{dx}\right)x$$

$$y \ln y = x\frac{dy}{dx}$$

18. **Correct option: B**

Explanation:-

$$I = \int \frac{1}{5 + 3\cos x}dx$$

$$I = \int \frac{1}{5 + 3\left(\dfrac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right)}dx$$

$$I = \int \frac{\sec^2\frac{x}{2}}{8 + 2\tan^2\frac{x}{2}}dx$$

$$\text{Put } \tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}dx = 2dt$$

$$I = \int \frac{dt}{4 + t^2}$$

$$I = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + c$$

$$I = \frac{1}{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{2}\right) + c$$

19. **Correct option: A**

Explanation:-

We have,

$$I = \int \frac{x^2}{1 + x^3} dx$$

Put $1 + x^3 = t$

$3x^2 dx = dt$

$$x^2 dx = \frac{dt}{3}$$

$$I = \int \frac{dt}{3t}$$

$$I = \frac{1}{3} \int \frac{dt}{t}$$

$$I = \frac{1}{3} \log |t| + c$$

$$I = \frac{1}{3} \log \left|1 + x^3\right| + c$$

20. Correct option: C

Explanation:-

$y = A \cos \omega t + B \sin \omega t$

$$\frac{dy}{dx} = -A\omega \sin \omega t + B\omega \cos \omega t$$

$$\frac{d^2 y}{dx^2} = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t$$

$$\frac{d^2 y}{dx^2} = -\omega^2 \left(A \cos \omega t + B \sin \omega t\right)$$

$y'' = -\omega^2 y$

Part B

21. We have

$$\tan^{-1}\left(\frac{\sqrt{1 + \cos x} + \sqrt{1 - \cos x}}{\sqrt{1 + \cos x} - \sqrt{1 - \cos x}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2 x/2} + \sqrt{2\sin^2 x/2}}{\sqrt{2\cos^2 x/2} - \sqrt{2\sin^2 x/2}}\right)$$

$$= \tan^{-1}\left(\frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2}\right)$$

$$= \tan^{-1}\left(\frac{-\cos x/2 + \sin x/2}{-\cos x/2 - \sin x/2}\right) \qquad \left(\because \pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}\right)$$

$$= \tan^{-1}\left(\frac{\cos x/2 - \sin x/2}{\cos x/2 + \sin x/2}\right)$$

$$= \tan^{-1}\left(\frac{1-\tan x/2}{1+\tan x/2}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)$$

$$= \frac{\pi}{4}-\frac{x}{2}$$

22. $2ye^{x/y}dx + \left(y - 2x\,e^{x/y}\right)dy = 0$

$$\frac{dx}{dy} = \frac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}} \qquad \ldots 1$$

Let

$$F(x,y) = \frac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}}$$

Then,

$$F(\lambda x, \lambda y) = \frac{\lambda\left(2xe^{\frac{x}{y}} - y\right)}{\lambda\left(2ye^{\frac{x}{y}}\right)} = \lambda^{o}\left[F(x,y)\right]$$

Thus, F(x, y) is a homogeneous function of degree zero.
Therefore, the given differential equation is a homogeneous differential equation.

OR

The equation of the family of circles touching x-axis at the origin is

$(x-0)^2 + (y-a)^2 = a^2$, where a is a parameter.

$x^2 + y^2 - 2ay = 0$ ... (i)

This equation contains only one arbitrary constant, thus we differentiate it only once, we get

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$$

$$\Rightarrow a\frac{dy}{dx} = x + y\frac{dy}{dx}$$

$$\Rightarrow a = \frac{\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx}}$$

Substituting the value of 'a' in equation (i), we get

$$\Rightarrow x^2 + y^2 = 2y\left\{\frac{\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx}}\right\}$$

$$\Rightarrow \left(x^2 + y^2\right)\frac{dy}{dx} = 2y\left(x + y\frac{dy}{dx}\right)$$

This is the required differential equation of all the circles touching the x-axis at the origin.

23. Given that

$$\vec{b} = 2\hat{i} + 4\hat{j} - 5k$$

$$\vec{c} = \lambda\hat{i} + 2\hat{j} + 3k$$

Now consider the sum of the vectors $\vec{b} + \vec{c}$:

$$\vec{b} + \vec{c} = \left(2\hat{i} + 4\hat{j} - 5k\right) + \left(\lambda\hat{i} + 2\hat{j} + 3k\right)$$

$$\Rightarrow \vec{b} + \vec{c} = (2 + \lambda)\hat{i} + 6\hat{j} - 2k$$

Let $\hat{n}$ be the unit vector along the sum of vectors $\vec{b} + \vec{c}$:

$$\hat{n} = \frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2k}{\sqrt{(2 + \lambda)^2 + 6^2 + 2^2}}$$

The scalar product of a and $\vec{n}$ is 1. Thus,

$$\vec{a} \cdot \hat{n} = \left(\hat{i} + \hat{j} + k\right) \cdot \left(\frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2k}{\sqrt{(2 + \lambda)^2 + 6^2 + 2^2}}\right)$$

$$\Rightarrow 1 = \frac{1(2 + \lambda) + 1 \cdot 6 - 1 \cdot 2}{\sqrt{(2 + \lambda)^2 + 6^2 + 2^2}}$$

$$\Rightarrow \sqrt{(2 + \lambda)^2 + 6^2 + 2^2} = 2 + \lambda + 6 - 2$$

$$\Rightarrow \sqrt{(2 + \lambda)^2 + 6^2 + 2^2} = \lambda + 6$$

$$\Rightarrow (2 + \lambda)^2 + 40 = (\lambda + 6)^2$$

$$\Rightarrow \lambda^2 + 4\lambda + 4 + 40 = \lambda^2 + 12\lambda + 36$$

$$\Rightarrow 4\lambda + 44 = 12\lambda + 36$$

$$\Rightarrow 8\lambda = 8$$

$$\Rightarrow \lambda = 1$$

24. * is a binary operation on A defined by (a, b) * (c, d) = (ac, b + ad) for (a, b), (c, d) ϵ A

(i) Let (x, y) ϵ A is an identity element of *

Therefore, (x, y) * (a, b) = (a, b) = (a, b) * (x, y) for any (a, b) ϵ A

i.e. (xa, y + xb) = (a, b) = (ax, b + ay)

Equating the corresponding elements, we get

$xa = a \Rightarrow x = 1$

Also, $y + xb = b \Rightarrow y = 0$

So, we have (x, y) = (1, 0)

Thus, (1, 0) is the identity element of *.

(ii) Let (p, q) ϵ A is an inverse element of *

⇒ (p, q) * (a, b) = (1, 0) = (a, b) * (p, q) for any (a, b) ϵ A

⇒ (pa, q + pb) = (1, 0) = (ap, b + aq)

Equating the corresponding elements, we get

$pa = 1 \Rightarrow p = \frac{1}{a}$

$q + pb = 0 \Rightarrow q = -\frac{b}{a}$

Therefore, $(p, q) = \left(\frac{1}{a}, -\frac{b}{a}\right)$

Thus, the inverse of (a, b) is $\left(\frac{1}{a}, -\frac{b}{a}\right)$ where $a \neq 0$

Inverse of (5, 3) is $\left(\frac{1}{5}, -\frac{3}{5}\right)$

Inverse of $\left(\frac{1}{2}, 4\right)$ is $(2, -8)$

25. Using $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ in the given integral we get,

$$I = \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}} dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - \int dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - x + C$$

Let $I_1 = \int \sin^{-1}\sqrt{x}\ dx$

Put $\sin^{-1}\sqrt{x} = \theta$

$\Rightarrow \sqrt{x} = \sin\theta$

$\Rightarrow x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

$I_1 = \int \theta.\, 2\sin\theta\cos\theta\, d\theta$

$= \int \theta.\, \sin 2\theta\, d\theta$

$$= \theta\left(-\frac{\cos 2\theta}{2}\right) - \int 1.\left(-\frac{\cos 2\theta}{2}\right) d\theta$$

$$= \frac{-\theta\cos 2\theta}{2} + \frac{\sin 2\theta}{4}$$

$$= \frac{-\theta\left(1-2\sin^2\theta\right)}{2} + \frac{2\sin\theta\sqrt{1-\sin^2\theta}}{4}$$

$$= \frac{-\sin^{-1}\sqrt{x}\,(1-2x)}{2} + \frac{\sqrt{x}\sqrt{1-x}}{2}$$

Hence given integral is

$$= \frac{4}{\pi}\left[\frac{-\sin^{-1}\sqrt{x}\,(1-2x)}{2} + \frac{\sqrt{x-x^2}}{2}\right] - x + C$$

$$= \frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2\sqrt{x-x^2}}{\pi} - x + C$$

26. $$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we have:

$$\Delta = \begin{vmatrix} 1-1 & 1-1 & 1 \\ a-c & b-c & c \\ a^3-c^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ a-c \quad a^2+ac+c^2 & b-c \quad b^2+bc+c^2 & c^3 \end{vmatrix}$$

$$= c-a \quad b-c \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & c \\ -\quad a^2+ac+c^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$, we have:

$$\Delta = c-a \quad b-c \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ b^2-a^2+bc-ac & b^2+bc+c^2 & c^3 \end{vmatrix}$$

$$= b-c \quad c-a \quad a-b \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ -\ a+b+c & b^2+bc+c^2 & c^3 \end{vmatrix}$$

$$= a-b \quad b-c \quad c-a \quad a+b+c \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ -1 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = a-b \quad b-c \quad c-a \quad a+b+c \quad -1 \begin{vmatrix} 0 & 1 \\ 1 & c \end{vmatrix}$$

$$= a-b \quad b-c \quad c-a \quad a+b+c$$

Hence proved.

27. Given: $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

Now y = u + v

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \ldots(i)$$

Consider u = $(\sin x)^x$

Taking logarithms on both the sides, we have,

$$\log u = x\log(\sin x)$$

Differentiating with respect to x, we have,

$$\frac{1}{u}.\frac{du}{dx} = \log(\sin x) + \frac{x}{\sin x}.\cos x$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\left(\log(\sin x) + x\cot x\right) \ \ldots (ii)$$

Consider $v = \sin^{-1}\sqrt{x}$

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} \ldots(iii)$$

From (i), (ii) and (iii), we get

$$\frac{dy}{dx} = (\sin x)^x\left(\log(\sin x) + x\cot x\right) + \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

OR

We have

$$\left(x^2 + y^2\right)^2 = xy \text{ ____(i)}$$

Differentiating with respect to x, we have,

$$2\left(x^2 + y^2\right)\left(2x + 2y.\frac{dy}{dx}\right) = y + \frac{xdy}{dx}$$

$$\Rightarrow 4x\left(x^2 + y^2\right) + 4y\left(x^2 + y^2\right).\frac{dy}{dx} = y + \frac{xdy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(4x^2y + 4y^3 - x\right) = y - 4x^3 - 4xy^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - 4x^3 - 4xy^2}{4x^2y + 4y^3 - x}$$

28. Let I = $\int_0^{\pi} \frac{x}{1 + \sin x} dx$ ……………..(i)

[By property of definite integrals]

$$I = \int_0^{\pi} \frac{\pi - x}{1 + \sin(\pi - x)} dx \quad \text{using } \int_0^{a} f(x)dx = \int_0^{a} f(a - x)dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{\pi - x}{1 + \sin x} dx$$

$$\Rightarrow I = \pi\int_0^{\pi} \frac{1}{1 + \sin x} dx - \int_0^{\pi} \frac{x}{1 + \sin x} dx$$

$$\Rightarrow I = \pi\int_0^{\pi} \frac{1 - \sin x}{1 - \sin^2 x} dx - I \ldots \left(\text{using (i)}\right)$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \left(\sec^2 x - \tan x.\sec x\right)dx$$

$$\Rightarrow 2I = \pi\left[\tan x - \sec x\right]_0^{\pi}$$

2I=π [(tanπ – secπ) – (tan0 – sec0)]

2I= π [0 – (– 1) – (0 – 1)] = 2π

I = π

OR

Consider the given integral

$$I = \int_0^{\frac{\pi}{2}} 2\sin x\cos x\tan^{-1}\left(\sin x\right)dx$$

Let t = sin x

$\Rightarrow$ dt = cos x dx

When $x = \frac{\pi}{2}$, $t = 1$

When $x = 0$, $t = 0$

Now, $\int 2\sin x\cos x\tan^{-1}(\sin x)\,dx$

$= \int 2t\tan^{-1}t\,dt$

$= \left[\tan^{-1}t\right]\int 2t\,dt - \int\left[\frac{d}{dt}.\left(\tan^{-1}t\right)\int 2t\,dt\right]dt$

$= \left[\tan^{-1}t\right]\left[2.\frac{t^2}{2}\right] - \int\left(\frac{1}{1+t^2}\times 2.\frac{t^2}{2}\right)dt$

$= t^2\tan^{-1}t - \int\frac{t^2}{1+t^2}dt$

$= t^2\tan^{-1}t - \int\left[1 - \frac{1}{1+t^2}\right]dt$

$= t^2\tan^{-1}t - t + \tan^{-1}t$

$\therefore I = \int_0^{\frac{\pi}{2}} 2\sin x\cos x\tan^{-1}(\sin x)\,dx$

$= \left[t^2\tan^{-1}t - t + \tan^{-1}t\right]_0^1$

$= \left[1^2\tan^{-1}1 - 1 + \tan^{-1}1\right] - \left[0^2\tan^{-1}0 - 0 + \tan^{-1}0\right]$

$= \left[1\times\frac{\pi}{4} - 1 + \frac{\pi}{4}\right] - 0$

$= \frac{\pi}{4} - 1 + \frac{\pi}{4}$

$= \frac{\pi}{2} - 1$

29. Let E_1, E_2 and E_3 be the events of a driver being a scooter driver, car driver and truck driver respectively. Let A be the event that the person meets with an accident.

There are 2000 insured scooter drivers, 4000 insured car drivers and 6000 insured truck drivers.

Total number of insured vehicle drivers = 2000 + 4000 + 6000 = 12000

$\therefore P\left(E_1\right) = \frac{2000}{12000} = \frac{1}{6}, P\left(E_2\right) = \frac{4000}{12000} = \frac{1}{3}, P\left(E_3\right) = \frac{6000}{12000} = \frac{1}{2}$

Also, we have:

$P(A|E_1) = 0.01 = \frac{1}{100}$

$P(A|E_2) = 0.03 = \frac{3}{100}$

$P(A|E_3) = 0.15 = \frac{15}{100}$

Now, the probability that the insured person who meets with an accident is a scooter driver is $P(E_1|A)$.
Using Bayes' theorem, we obtain:

$$P(E_1|A) = \frac{P(E_1) \times P(A|E_1)}{P(E_1) \times P(A|E_1) + P(E_2) \times P(A|E_2) + P(E_3) \times P(A|E_3)}$$

$$= \frac{\frac{1}{6} \times \frac{1}{100}}{\frac{1}{6} \times \frac{1}{100} + \frac{1}{3} \times \frac{3}{100} + \frac{1}{2} \times \frac{15}{100}}$$

$$= \frac{\frac{1}{6}}{\frac{1}{6} + 1 + \frac{15}{2}}$$

$$= \frac{1}{6} \times \frac{6}{52}$$

$$= \frac{1}{52}$$

30. Let the equation of plane be $ax + by + cz + d = 0$ .... (1)

Since the plane passes through the point A (0, 0, 0) and B(3, −1, 2), we have

$a \times 0 + b \times 0 + c \times 0 + d = 0$

$\Rightarrow d = 0$ .... (2)

Similarly for point B (3, −1, 2), $a \times 3 + b \times (-1) + c \times 2 + d = 0$

$3a - b + 2c = 0$ (Using, $d = 0$) ...(3)

Given equation of the line is $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$

We can also write the above equation as $\frac{x-4}{1} = \frac{y-(-3)}{-4} = \frac{x-(-1)}{7}$

The required plane is parallel to the above line.

Therefore, $a \times 1 + b \times (-4) + c \times 7 = 0$

$\Rightarrow a - 4b + 7c = 0$ ... (4)

Cross multiplying equations (3) and (4), we obtain:

$$\frac{a}{(-1)\times 7 - (-4)\times 2} = \frac{b}{2 \times 1 - 3 \times 7} = \frac{c}{3 \times (-4) - 1 \times (-1)}$$

$$\Rightarrow \frac{a}{-7+8} = \frac{b}{2-21} = \frac{c}{-12+1}$$

$$\Rightarrow \frac{a}{1} = \frac{b}{-19} = \frac{c}{-11} = k$$

$\Rightarrow a = k, b = -19k, c = -11k$

Substituting the values of a, b and c in equation (1), we obtain the equation of plane as:

$kx - 19ky - 11kz + d = 0$

$\Rightarrow k(x - 19y - 11z) = 0$ (From equation(2))

$\Rightarrow x - 19y - 11z = 0$

So, the equation of the required plane is x – 19y – 11z = 0

31. $f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1), x \le 0 \\ \frac{\tan x - \sin x}{x^3}, x > 0 \end{cases}$

The given function f is defined for all x ∈ **R**.

It is known that a function f is continuous at x = 0, if $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\left[a\sin\frac{\pi}{2}(x+1)\right] = a\sin\frac{\pi}{2} = a(1) = a$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}\frac{\tan x - \sin x}{x^3} = \lim_{x\to 0}\frac{\frac{\sin x}{\cos x} - \sin x}{x^3}$$

$$= \lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3\cos x} = \lim_{x\to 0}\frac{\sin x . 2\sin^2\frac{x}{2}}{x^3\cos x}$$

$$= 2\lim_{x\to 0}\frac{1}{\cos x} \times \lim_{x\to 0}\frac{\sin x}{x} \times \lim_{x\to 0}\left[\frac{\sin\frac{x}{2}}{x}\right]^2$$

$$= 2 \times 1 \times 1 \times \frac{1}{4} \times \lim_{\frac{x}{2}\to 0}\left[\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]^2$$

$$= 2 \times 1 \times 1 \times \frac{1}{4} \times 1 = \frac{1}{2}$$

Now, $f(0) = a\sin\frac{\pi}{2}(0+1) = a\sin\frac{\pi}{2} = a \times 1 = a$

Since f is continuous at x = 0, $a = \frac{1}{2}$

Part C

32. Let the number of rackets and the number of bats to be made be x and y respectively.

The given information can be tabulated as below:

	Tennis Racket	Cricket Bat
Machine Time (h)	1.5	3
Craftsman's Time (h)	3	1

In a day, the machine time is not available for more than 42 hours.

$\therefore$ $1.5x + 3y \le 42$

In a day, the craftsman's time cannot be more than 24 hours.

$\therefore$ $3x + y \le 24$

Let the total profit be Rs. Z.

The profit on a racket is Rs. 20 and on a bat is Rs. 10.

$\therefore$ $Z = 20x + 10y$

Thus, the given linear programming problem can be stated as follows:

Maximise $Z = 20x + 10y$ ... (1)

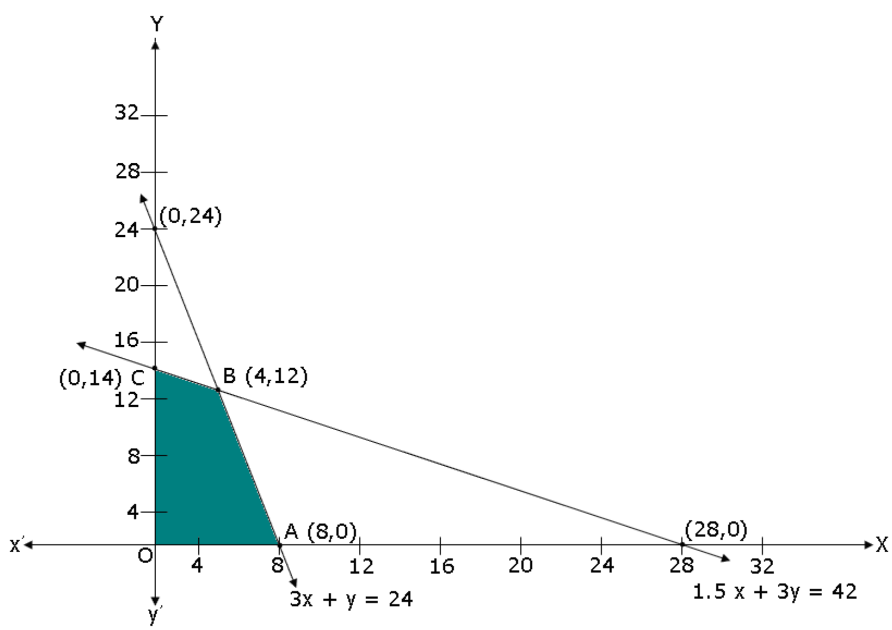
Subject to

$1.5x + 3y \le 42$ ... (2)

$3x + y \le 24$ ... (3)

$x, y \ge 0$ ... (4)

The feasible region can be shaded in the graph as below:



The corner points are A(8,0), B(4,12), C(0,14) and 0(0,0).

The values of Z at these corner points are tabulated as follows:

Corner point	Z = 20x + 10y
A(8,0)	160
B(4, 12)	200
C(0,14)	140
O(0,0)	0

→ Maximum

The maximum value of Z is 200, which occurs at x = 4 and y = 12.
Thus, the factory must produce 4 tennis rackets and 12 cricket bats to earn the maximum profit of Rs. 200.

33. Let the events M, F and G be defined as follows:
M: A male is selected
F: A female is selected
G: A person has grey hair
It is given that the number of males = the number of females

$\therefore$ P (M) = P(F) = $\frac{1}{2}$

Now, P (G/M) = Probability of selecting a grey haired person given that the person is a Male = 5% = $\frac{5}{100}$

Similarly, P (G/F) = 0.25% = $\frac{0.25}{100}$

A grey haired person is selected at random, the probability that this person is a male = P(M|G)

$$= \frac{P(M) \times P(G|M)}{P(M) \times P(G|M) + P(F) \times P(G|F)} \quad \text{[Using Baye's Theorem]}$$

$$= \frac{\frac{1}{2} \times \frac{5}{100}}{\frac{1}{2} \times \frac{5}{100} + \frac{1}{2} \times \frac{0.25}{100}}$$

$$= \frac{\frac{5}{100}}{\frac{5}{100} + \frac{0.25}{100}}$$

$$= \frac{5}{5.25}$$

$$= \frac{20}{21}$$

OR

Consider the following events:

E_1= Getting 5 OR 6 in a single throw of the die

E_2= Getting 1, 2, 3 OR 4 in a single throw of the die

A = Getting exactly 2 heads

We have to find, $P(E_2/A)$.

Since $P(E_2/A) = \dfrac{P(A/E_2)P(E_2)}{P(A/E_1)P(E_1) + P(A/E_2)P(E_2)}$

Now, $P(E_1) = \dfrac{2}{6} = \dfrac{1}{3}$ and $P(E_2) = \dfrac{4}{6} = \dfrac{2}{3}$

Also,

$P(A/E_1) =$ Probability of getting exactly 2 heads when a coin is tossed 3 times $= \dfrac{3}{8}$

And, $P(A/E_2) =$ Probability of getting 2 heads when a coin is tossed 2 times $= \dfrac{1}{4}$

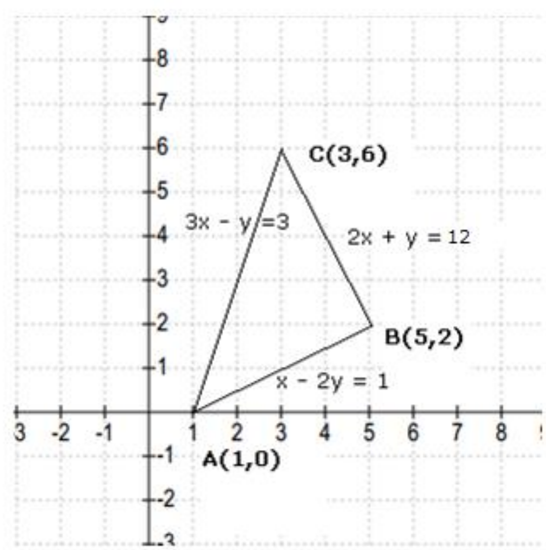
$$\therefore P(E_2/A) = \frac{\frac{1}{4}\times\frac{2}{3}}{\frac{3}{8}\times\frac{1}{3}+\frac{1}{4}\times\frac{2}{3}} = \frac{\frac{1}{6}}{\frac{1}{8}+\frac{1}{6}} = \frac{\frac{1}{6}}{\frac{3+4}{24}} = \frac{1}{6}\times\frac{24}{7} = \frac{4}{7}$$

34. Given equations are:

$3x - y = 3$ ... (1)

$2x + y = 12$ ... (2)

$x - 2y = 1$ ... (3)



To Solve (1) and (2),

(1) + (2) $\Rightarrow 5x = 15 \Rightarrow x = 3$

(2) $\Rightarrow y = 12 - 6 = 6$

Thus (1) and (2) intersect at C(3, 6).

To solve (2) and (3),

(2) – 2(3) $\Rightarrow 5y = 10 \Rightarrow y=2$

(2) $\Rightarrow 2x = 12 - 2 = 10 \Rightarrow x = 5$

Thus (2) and (3) intersect at B(5, 2).

To solve (3) and (1),

2(1) – (3) $\Rightarrow 5x = 5 \Rightarrow x = 1$

(3) $\Rightarrow 1 - 2y = 1 \Rightarrow y = 0$

Thus (3) and (1) intersect at A(1, 0).

$$\text{Area} = \int_1^3 (3x-3)dx + \int_3^5 (12-2x)dx - \int_1^5 \frac{1}{2}(x-1)dx$$

$$= 3\left[\frac{x^2}{2} - x\right]_1^3 + \left[12x - x^2\right]_3^5 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^5$$

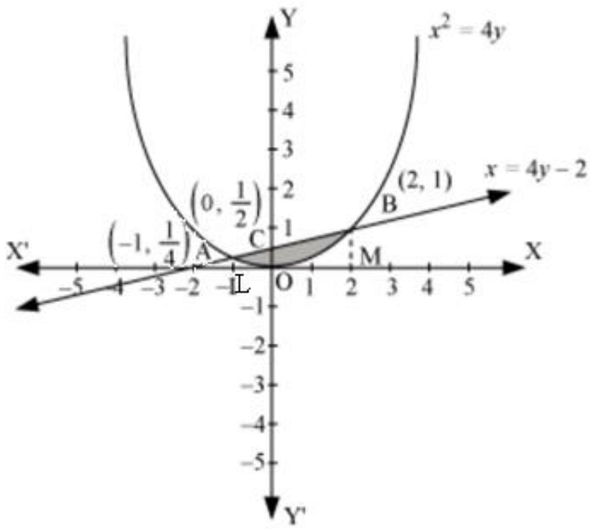
$$= 3\left[\left(\frac{9}{2}-3\right) - \left(\frac{1}{2}-1\right)\right] + \left[(60-25)-(36-9)\right] - \frac{1}{2}\left[\left(\frac{25}{2}-5\right)-\left(\frac{1}{2}-1\right)\right]$$

$$= 3\left[\frac{3}{2}+\frac{1}{2}\right] + [35-27] - \frac{1}{2}\left[\frac{15}{2}+\frac{1}{2}\right]$$

$$= 6 + 8 - 4 = 10 \text{ sq. units}$$

OR

The shaded area OBAO represents the area bounded by the curve $x^2 = 4y$ and line $x = 4y - 2$.



Let A and B be the points of intersection of the line and parabola.

Co-ordinates of point A are $\left(-1, \frac{1}{4}\right)$. Co-ordinates of point B are (2, 1).

Area OBAO = Area OBCO + Area OACO ... (1)

Area OBCO =

$$= \int_0^2 \frac{x+2}{4}dx - \int_0^2 \frac{x^2}{4}dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_0^2 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{4}\,2 + 4 - \frac{1}{4}\left[\frac{8}{3}\right]$$

$$= \frac{3}{2} - \frac{2}{3} = \frac{5}{6}$$

Area OACO =

$$= \int_{-1}^0 \frac{x+2}{4}dx - \int_{-1}^0 \frac{x^2}{4}dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^0 - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^0$$

$$= \frac{1}{4}\left[-\frac{-1^{\ 2}}{2} - 2\ -1\right] - \frac{1}{4}\left[-\left(\frac{-1^{\ 3}}{3}\right)\right]$$

$$= \frac{1}{4}\left[-\frac{1}{2} + 2\right] - \frac{1}{4}\left[\frac{1}{3}\right]$$

$$= \frac{3}{8} - \frac{1}{12} = \frac{7}{24}$$

Therefore, required area = $\left(\frac{5}{6}+\frac{7}{24}\right)=\frac{9}{8}$ sq. units

35. We have

$S = 2\pi rh + 2\pi r^2$ ....(i)

$h = \frac{S-2\pi r^2}{2\pi r}$

$V = \pi r^2 h = \pi r^2\left[\frac{S-2\pi r^2}{2\pi r}\right]=\frac{1}{2}\left[Sr-2\pi r^3\right]$

$\frac{dV}{dr}=\frac{1}{2}\left[S-6\pi r^2\right]$

$\frac{dV}{dr}=0 \Rightarrow \frac{1}{2}\left[S-6\pi r^2\right]=0 \Rightarrow S-6\pi r^2=0 \Rightarrow r=\sqrt{\frac{S}{6\pi}}$

$\frac{d^2V}{dr^2}=-6\pi r$

$\left.\frac{d^2V}{dr^2}\right]_{r=\sqrt{\frac{S}{6\pi}}}=-6\pi r\Big]_{r=\sqrt{\frac{S}{6\pi}}}=-6\pi\sqrt{\frac{S}{6\pi}}<0$

$\Rightarrow$ When $r=\sqrt{\frac{S}{6\pi}}$; V is maximum.

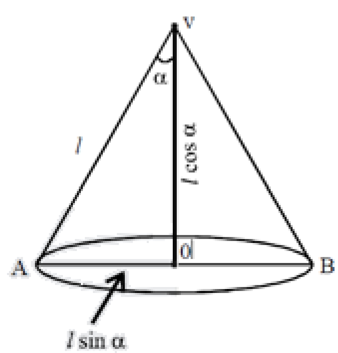
$\Rightarrow S = 6\pi r^2 = 2\pi rh + 2\pi r^2$

$\Rightarrow 4\pi r^2 = 2\pi rh$

$\Rightarrow h = 2r$

$\Rightarrow$ height = diameter of the base .

OR



Volume of cone

$=\frac{1}{3}\pi r^2 h$

$=\frac{1}{3}\pi(\ell\sin\alpha)^2(\ell\cos\alpha)$

$=\frac{1}{3}\pi\ell^3\sin^2\alpha\cos\alpha$

$$\frac{dv}{d\alpha} = \frac{\pi l^3}{3}[-\sin^3\alpha + 2\sin\alpha\cos\times\cos\alpha]$$

$$= \frac{\pi l^3 \sin\alpha}{3}(-\sin^2\alpha + 2\cos^2\alpha)$$

For maximum or minimum

$$\frac{dv}{d\alpha} = 0$$

$$\frac{\pi l^3 \sin\alpha}{3}(-\sin^2\alpha + 2\cos^2\alpha) = 0$$

$$\sin\alpha \neq 0$$

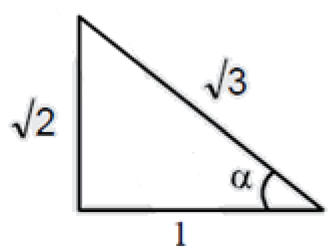
$$2\cos^2\alpha = \sin^2\alpha$$

$$\tan^2\alpha = 2$$

$$\tan\alpha = \sqrt{2}$$

$$\cos\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \cos^{-1}\frac{1}{\sqrt{3}}$$



Again diff. w.r.t. α, we get

$$\frac{d^2v}{d\alpha^2} = \frac{1}{3}\pi l^3\cos^3\alpha(2 - 7\tan^2\alpha)$$

$$\text{at} \quad \cos\alpha = \frac{1}{\sqrt{3}}$$

$$\frac{d^2v}{d\alpha^2} < 0$$

$$\text{V is maximum when } \cos\alpha = \frac{1}{\sqrt{3}} \text{ or } \alpha = \cos^{-1}\frac{1}{\sqrt{3}}$$

36. Let the equation of required plane be $\ell x + my + nz + p = 0$ ...... (1)

Plane passes through points (1, 2, 3) and (0, -1, 0)

∴ (1, 2, 3) and (0, -1, 0) satisfies the equation (1)

$\ell + 2m + 3n + p = 0$ ...... (2)

$-m + p = 0$

$\Rightarrow p = m$ ...... (3)

d.c.'s of line $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z}{-3}$ are 2, 3, −3

d.c.'s normal to plane are ℓ, m, and n normal to the plane will be $\perp$ to line
i.e., $2\ell + 3m - 3n = 0$

$\Rightarrow \ell = \frac{3}{2}(n-m)$ …… (4)

From (2) and (3) we have
$\ell + 3m + 3n = 0$
$\ell = -3(m+n)$ …… (5)
From (4) and (5)

$\frac{3}{2}(n-m) = -3(m+n)$

$n - m = -2m - 2n$
$\Rightarrow 3n = -m$ or $m = -3n$
$\ell = -3(-3n + n) = -3 \times -2n$
Using $\ell = 6n$, $m = -3n$ & $p = -3n$ in (1) we have required equation as
$6x - 3y + z - 3 = 0$

OR

Equation of any plane through the intersection of the planes
$2x + y - z = 3$ and $5x - 3y + 4z + 9 = 0$ is

$(2x + y - z - 3) + \lambda(5x - 3y + 4z + 9) = 0$ … (i)

$\Rightarrow (2+5\lambda)x + (1-3\lambda)y + (4\lambda - 1)z - (3 - 9\lambda) = 0$

This is parallel to the line

$\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-5}{5}$

$\therefore (2+5\lambda)\times 2 + (1-3\lambda)\times 4 + (4\lambda-1)\times 5 = 0$

$\Rightarrow 18\lambda + 3 = 0 \Rightarrow \lambda = -\frac{1}{6}$

Substitute for $\lambda = -\frac{1}{6}$ in Eq. (i)

$$\left(2-\frac{5}{6}\right)x + \left(1+\frac{1}{2}\right)y + \left(-\frac{2}{3}-1\right)z - \left(3+\frac{3}{2}\right) = 0$$

$\Rightarrow 7x + 9y - 10z - 27 = 0.$

Thus the required equation of plane is $7x + 9y - 10z - 27 = 0.$

37. From the given data, we write the following equations:

$$\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 1600$$

$$\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} = 2300$$

$$\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 900$$

From above system, we get:

$3x+2y+z=1600$

$4x+y+3z=2300$

$x+y+z=900$

Thus we get:

$$\begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1600 \\ 2300 \\ 900 \end{pmatrix}$$

This is of the form

$$AX=B, \text{ where } A=\begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B=\begin{pmatrix} 1600 \\ 2300 \\ 900 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 3(1-3)-2(4-3)+1(4-1) = -6-2+3 = -5 \neq 0$$

We need to find A^{-1}:

$C_{11} = -2; C_{12} = -1; C_{13} = 3$

$C_{21} = -1; C_{22} = 2; C_{23} = -1$

$C_{31} = 5; C_{32} = -5; C_{33} = -5$

$$\text{Therefore, adj } A = \begin{pmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{pmatrix}^T = \begin{pmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{pmatrix}$$

$$\text{Thus, } A^{-1} = \frac{\text{adj}A}{|A|} = -\frac{1}{5}\begin{pmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{pmatrix}$$

Therefore, $X = A^{-1}B$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{pmatrix}\begin{pmatrix} 1600 \\ 2300 \\ 900 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -2\times 1600 - 1\times 2300 + 5\times 900 \\ -1\times 1600 + 2\times 2300 - 5\times 900 \\ 3\times 1600 - 1\times 2300 - 5\times 900 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -1000 \\ -1500 \\ -2000 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 200 \\ 300 \\ 400 \end{pmatrix}$$

Awards can be given for discipline.