Continuity and Differentiability

- Suppose f is a real function on a subset of the real numbers and c be a point in the domain of f. Then, f is continuous at c, if $\lim_{x \to c} f(x) = f(c)$

  More elaborately, we can say that f is continuous at c, if

  $$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

- If f is not continuous at c, then we say that f is discontinuous at c and c is called the point of discontinuity.
- A real function f is said to be continuous, if it is continuous at every point in the domain of f.

- If f and g are two continuous real functions, then
  - $(f + g)$, $(f - g)$, $f.g$ are continuous
  - $\frac{f}{g}$ is continuous provided g assumes non zero value.

- If f and g are two continuous functions, then fog is also continuous.

- Suppose f is a real function and c is a point in its domain. Then, the derivative of f at c is defined by, $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$
- Derivative of a function $f(x)$, denoted by $\frac{d}{dx}(f(x))$ of $f'(x)$, is defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Example:
Find derivative of sin $2x$.

Solution:

Let $f(x) = \sin 2x$

$$\therefore f'(x) = \lim_{h\to 0} \frac{\sin 2(x+h) - \sin 2x}{h}$$

$$= \lim_{h\to 0} \frac{2\cos(2x+h)\cdot \sin h}{h}$$

$$= 2\lim_{h\to 0} \cos(2x+h) \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$$= 2\times\cos 2x\times 1$$

$$= 2\cos 2x$$

- For two functions f and g, the rules of algebra of derivatives are as follows:
  - $(f+g)' = f' + g'$
  - $(f-g)' = f' - g'$
  - $(fg)' = f'g'$ [Leibnitz or product rule]
  - $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, where $g \neq 0$ [Quotient rule]
- Every differentiable function is continuous, but the converse is not true.

Example:

$f(x) = |x|$ is continuous at all points on real line, but it is not differentiable at $x = 0$.

Since L.H.S $= \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \frac{-h}{h} = -1$

R.H.S $= \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \frac{h}{h} = 1$

$\therefore$L.H.S $\neq$ R.H.S.

Therefore, $f'(x)$ does not exist at $x = 0$; i.e., f is not differentiable at $x = 0$.

The derivatives of some useful functions are as follows:

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

- $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

- **Chain rule:** This rule is used to find the derivative of a composite function. Let $f = v \circ u$. Suppose $t = u(x)$; and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, then $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$
  Similarly, if $f = (w \circ u) \circ v$, and if $t = v(x)$, $s = u(t)$, then
  $\frac{df}{dx} = \frac{d(w \circ u)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$

  Example: Find the derivative of $\sin^2(\log x + \cos^2 x)$.

  Solution:

$$\frac{d}{dx}\left[\sin^2\left(\log x + \cos^2 x\right)\right] = 2\sin\left(\log x + \cos^2 x\right) \times \frac{d}{dx}\left[\sin\left(\log x + \cos^2 x\right)\right]$$

$$= 2\sin\left(\log x + \cos^2 x\right) \cdot \cos\left(\log x + \cos^2 x\right) \times \frac{d}{dx}\left(\log x + \cos^2 x\right)$$

$$= \sin 2\left(\log x + \cos^2 x\right) \cdot \left[\frac{1}{x} + 2\cos x \times \frac{d}{dx}(\cos x)\right]$$

$$= \sin\left(\log x^2 + 2\cos^2 x\right) \times \left(\frac{1}{x} - 2\sin x \cos x\right)$$

$$= \left(\frac{1}{x} - \sin 2x\right)\sin\left(\log x^2 + 2\cos^2 x\right)$$

The derivatives of exponential functions are as follows:

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(e^{ax}) = ae^{ax}$

- **Mean value theorem:**
  If f: $[a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

  Example: Verify Mean Value Theorem for the function:

$f(x) = 2x^2 - 17x + 30$ in the interval $\left[\frac{5}{2}, 6\right]$.

Solution:

$f(x) = 2x^2 - 17x + 30$

$\therefore f'(x) = 4x - 17$

The function $f(x)$ being a polynomial, is continuous on $\left[\frac{5}{2}, 6\right]$ and is differentiable on $\left(\frac{5}{2}, 6\right)$.

Also, $f\left(\frac{5}{2}\right) = 2\left(\frac{5}{2}\right)^2 - 17\left(\frac{5}{2}\right) + 30 = 0$

and, $f(6) = 2(6)^2 - 17 \times 6 + 30 = 0$

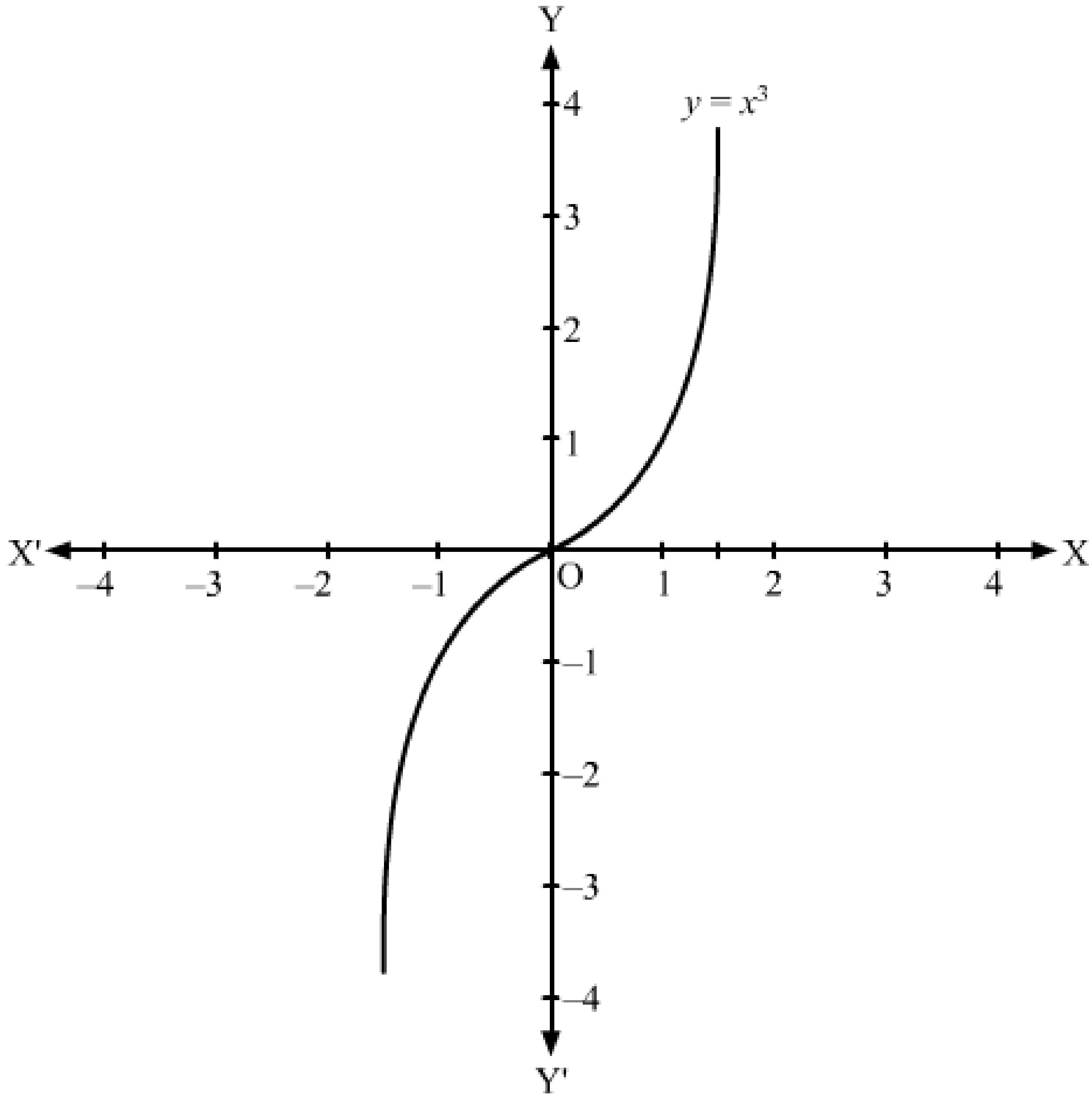
$\therefore f\left(\frac{5}{2}\right) = f(6)$

Now, $\dfrac{f(6) - f\left(\frac{5}{2}\right)}{6 - \frac{5}{2}} = 0$

According to Mean Value Theorem (MVT), there exists $c \in \left(\frac{5}{2}, 6\right)$ such that $f'(c) = 0$.

$\therefore 4c - 17 = 0$

$\Rightarrow c = \frac{17}{4} \in \left(\frac{5}{2}, 6\right)$



Therefore, M.V.T is verified.

- Derivative of a function $f(x) = [u(x)]^{v(x)}$ can be calculated by taking logarithm on both the sides, i.e. $\log f(x) = v(x) \log [u(x)]$, and then differentiating both sides with respect to *x*.

Example: If $y = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$, find $\frac{dy}{dx}$

Solution:

Let If $y = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = x^y$

$\therefore \log y = y \log x$

$$\Rightarrow \frac{d}{dx}(\log y) = \frac{d}{dx}(y \log x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\log x + \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{1}{y} - \log x\right] = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}}{\frac{1}{y} - \log x} = \frac{y^2}{x - xy\log x}$$

- If the variables x and y are expressed in the form of $x = f(t)$ and $y = g(t)$, then they are said to be in parametric form. In this case, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{g'(t)}{f'(t)}$, provided $f'(t) \neq 0$

- If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ and $\frac{d^2y}{dx^2}$ or $f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

  Here, $f''(x)$ or $\frac{d^2y}{dx^2}$ is called the second order derivative of y with respect to x.

- **Rolle's Theorem:**
  If f: $[a, b] \rightarrow \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, where a and b are some real numbers, then there exists some $c \in (a, b)$ such that $f'(c) = 0$

  Example: Verify Rolle's Theorem for the function:

$f(x) = 2x^2 - 17x + 30$ in the interval $\left[\frac{5}{2}, 6\right]$.

Solution:

$f(x) = 2x^2 - 17x + 30$

$\therefore f'(x) = 4x - 17$

The function $f(x)$ being a polynomial, is continuous on $\left[\frac{5}{2}, 6\right]$ and is differentiable on $\left(\frac{5}{2}, 6\right)$.

Also, $f\left(\frac{5}{2}\right) = 2\left(\frac{5}{2}\right)^2 - 17\left(\frac{5}{2}\right) + 30 = 0$

And, $f(6) = 2(6)^2 - 17\times6 + 30 = 0$

$\therefore f\left(\frac{5}{2}\right) = f(6)$

Therefore, we can apply Rolle's Theorem for $f(x)$.

According to this theorem, there exists $c \in \left(\frac{5}{2}, 6\right)$ such that $f'(c) = 0$

We have $f'(x) = 4x - 17$

$\therefore f'(c) = 0$

$\Rightarrow 4c - 17 = 0$

$\Rightarrow c = \frac{17}{4} \in \left(\frac{5}{2}, 6\right)$

Therefore, Rolle's Theorem is verified.

- **Mean value theorem:**

  If f: $[a, b] \rightarrow \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

  Example: Verify Mean Value Theorem for the function:

  $f(x) = 2x^2 - 17x + 30$ in the interval $\left[\frac{5}{2}, 6\right]$.

  Solution:

  $f(x) = 2x^2 - 17x + 30$

  $\therefore f'(x) = 4x - 17$

The function $f(x)$ being a polynomial, is continuous on $\left[\frac{5}{2}, 6\right]$ and is differentiable on $\left(\frac{5}{2}, 6\right)$.

Also, $f\left(\frac{5}{2}\right) = 2\left(\frac{5}{2}\right)^2 - 17\left(\frac{5}{2}\right) + 30 = 0$

And, $f(6) = 2(6)^2 - 17\times 6 + 30 = 0$

$\therefore f\left(\frac{5}{2}\right) = f(6)$

Now, $\dfrac{f(6) - f\left(\frac{5}{2}\right)}{6 - \frac{5}{2}} = 0$

According to Mean Value Theorem (MVT), there exists $c \in \left(\frac{5}{2}, 6\right)$ such that $f'(c) = 0$

$\therefore 4c - 17 = 0$

$\Rightarrow c = \frac{17}{4} \in \left(\frac{5}{2}, 6\right)$

Therefore, M.V.T is verified.