CHAPTER 09

Slope Deflection Method of Analysis

9.1 Introduction

This method was given by G.A. Maney. This method is based on stiffness approach and basic unknowns are taken as joint displacements (θ and Δ). To find unknowns (joint displacements), joint moment equilibrium conditions and shear equations are written and the joint moment in members are found by force displacement relations called slope-deflection equations. In this method, deformations due to bending are only considered and axial deformation are neglected.

9.2 Sign Convention

(a) **End moments:** Clockwise end moments are taken as positive and anticlockwise end moments are taken as negative.

Here, M_{AB} = −ve (Anticlockwise)

M_{BA} = +ve (Clockwise)

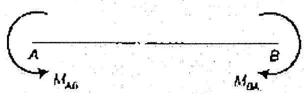


Fig. 9.1

(b) **Slope (Rotations):** Clockwise rotations are taken as positive and anticlockwise rotations are taken as negative.

Here, θ_A = +ve (Clockwise)

θ_B = −ve (Anticlockwise)

θ_C = +ve (Clockwise)

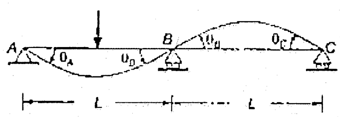


Fig. 9.2

(c) **Deflection (Settlement)**

(i) Positive deflections: Those displacement (deflections) will be positive which produces clockwise rotation to the member.

Here, Δ = +ve

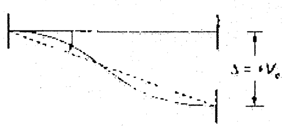


Fig. 9.3 Clockwise rotation to the member

(*ii*) **Negative deflections:** Those displacements (deflections) will be negative which produces anticlockwise rotation to the member.
Here, $\Delta = -ve$

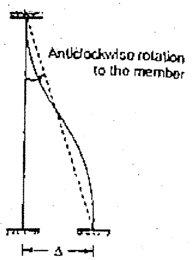


Fig.9.4

9.3 Derivation of Slope Deflection Equations

Consider a segment AB of a continuous beam as shown in figure. The slope deflection equations are derived by superimposing the end moments developed due to

(i) Applied load
(ii) Rotation of joint A (θ_A)
(iii) Rotation of joint B (θ_B)
(iv) Settlement of support B w.r.t support A (Δ)

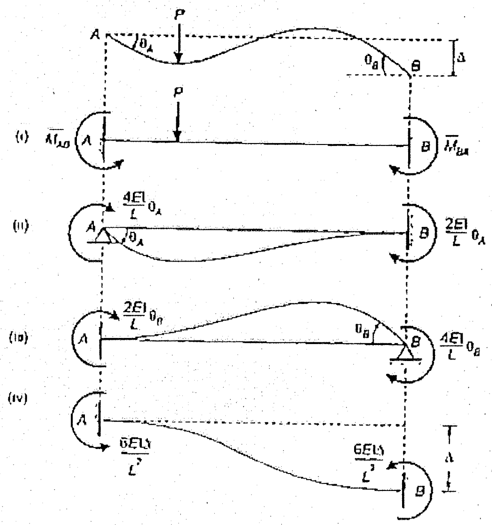


Fig.9.5

The span AB can be subjected to combination of multiple loading as shown above.

(*i*) In beam shown in figure (*i*), the fixed end moment at ends A and B will be $\overline{M}_{AB}$ and $\overline{M}_{BA}$ respectively.

(*ii*) In beam shown in figure (*ii*), if joint A rotates by angle θ_A, then fixing moments at end A and B will be

$$\overline{M}_{AB} = \frac{4EI}{L}\theta_A, \quad \overline{M}_{BA} = \frac{2EI}{L}\theta_A$$

(*iii*) In beam shown in figure (*iii*), if joint B rotates by angle θ_B then fixing moments at ends B and A will be

$$\overline{M}_{BA} = \frac{4EI}{L}\theta_B$$

$$\overline{M}_{AB} = \frac{2EI}{L}\theta_B$$

(*iv*) If support B settles down Δ with respect to support A causing rotation to member BA in clockwise direction, then fixing moment produce at B and A will be

$$\overline{M}_{BA} = -\frac{6EI\Delta}{L^2}$$

$$\overline{M}_{AB} = -\frac{6EI\Delta}{L^2}$$

Thus the final moment at end A and B due to above multiple effect will be

$$M_{AB} = \overline{M}_{AB} + \frac{4EI}{L}\theta_A + \frac{2EI\theta_B}{L} - \frac{6EI\Delta}{L^2}$$

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right) \quad ...(i)$$

Similarly,

$$M_{BA} = \overline{M}_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right) \quad ...(ii)$$

Equations (*i*) and (*ii*) known as slope deflection equations.

9.4 Procedure of Analysis

Step-1: Consider each span fixed and find fixed end moment for each span due to given loading.

Note: If any support settles then do not find fixing end moment separately because we have already consider its effect in derivation of slope-deflection equation.

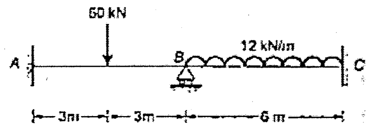
Step-2: Take θ and Δ as unknown and write slope deflection equation for end moments.

Step-3: Find joint displacement (θ and Δ) by using moment equilibrium conditions and shear equations.
(*i*) Number of moment equilibrium conditions = Number of rotational displacement components
(*ii*) Number of shear equations = Number of translational displacement components

Step-4: Substituting the values of unknowns in slope deflection equations and determine the end moments.

Step-5: Draw bending moment diagram by superimposing end moment BMD and free BMD for each span.

Consider a beam shown in figure 9.6:



EI is constant
Fig.9.6

Step-1: Fixed end moments:
Span BC:

$$\overline{M}_{AB} = -\frac{PL}{8} = -\frac{60\times6}{8} = -45 \text{ kN-m}$$

$$\overline{M}_{BA} = +\frac{PL}{8} = +45 \text{ kN-m}$$

Span BC:

$$\overline{M}_{BC} = -\frac{wL^2}{12} = -\frac{12\times6^2}{12} = -36 \text{ kN-m}$$

$$\overline{M}_{CB} = +\frac{wL^2}{12} = +36 \text{ kN-m}$$

Step-2: Slope deflection equations:

Member AB:

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

Here, $\overline{M}_{AB} = -45$ kN-m, $\theta_A = 0$, and $\Delta = 0$

$$M_{AB} = -45 + \frac{2EI}{6}\theta_B \quad \text{...(i)}$$

and $$M_{BA} = \overline{M}_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

Here, $\overline{M}_{BA} = +45$ kN-m, $\theta_A = 0$ and $\Delta = 0$

$\therefore$ $$M_{BA} = +45 + \frac{2EI}{6}(2\theta_B - 0)$$

$$M_{BA} = 45 + \frac{4EI}{6}\theta_B \quad \text{...(ii)}$$

Member BC:

$$M_{BC} = \overline{M}_{BC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

Here, $\overline{M}_{BC} = -36$ kN-m, $\theta_C = 0$ and $\Delta = 0$

$$M_{BC} = -36 + \frac{2EI}{6}(2\theta_B + 0 - 0)$$

$$M_{BC} = -36 + \frac{4EI}{6}\theta_B \quad \text{...(iii)}$$

$$M_{CB} = \overline{M}_{CB} + \frac{2EI}{L}\left(2\theta_C + \theta_B - \frac{3\Delta}{L}\right)$$

Here, $\overline{M}_{CB} = +36$ kN-m, $\theta_C = 0$ and $\Delta = 0$

$$M_{CB} = +36 + \frac{2EI}{6}(0 + \theta_B - 0)$$

$$M_{CB} = +36 + \frac{2EI}{6}\theta_B \quad \text{...(iv)}$$

Step-3: Joint Equilibrium Conditions

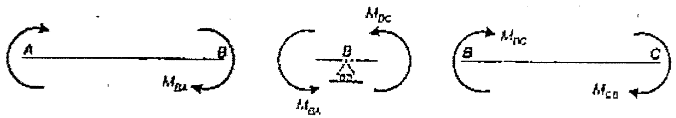


Fig.9.7

Here we have unknown θ_B i.e. rotational displacement. Hence one joint equilibrium condition is required. Consider joint B,

$$-M_{BA} - M_{BC} = 0$$

$\Rightarrow$ $$M_{BA} + M_{BC} = 0 \quad \text{...(A)}$$

$\Rightarrow$ $$\left[45 + \frac{4EI\theta_B}{6}\right] + \left[-36 + \frac{4EI\theta_B}{6}\right] = 0$$

$\Rightarrow$ $$9 + \frac{8EI\theta_B}{6} = 0$$

$\Rightarrow$ $$\theta_B = -\frac{9 \times 6}{8EI} = -\frac{6.75}{EI}$$

Step-4: Final End Moments

Substituting value of θ_B in slope deflection equation, we get

$$M_{AB} = -45 + \frac{2EI}{6} \times -\frac{6.75}{EI} = -47.25 \text{ kNm}$$

$$M_{BA} = +45 + \frac{4EI}{6} \times -\frac{6.75}{EI} = +40.5 \text{ kNm}$$

$$M_{BC} = -36 + \frac{4EI}{6} \times -\frac{6.75}{EI} = -40.5 \text{ kNm}$$

$$M_{CB} = +36 + \frac{2EI}{6} \times -\frac{6.75}{EI} = +33.75 \text{ kNm}$$

Step-5: Support Reactions:

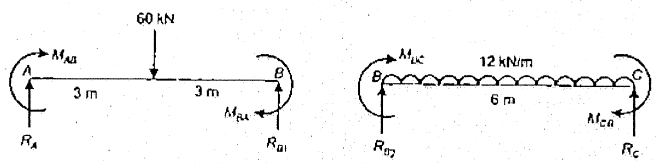


Fig.9.8

$\Sigma M_B = 0$ (for span AB)

$\Rightarrow$ $$R_A \times 6 + M_{AB} + M_{BA} - 60 \times 3 = 0$$

$\Rightarrow$ $$R_A = \frac{1}{6}\left[180 - (M_{AB} + M_{BA})\right]$$

$\Rightarrow$ $$R_A = \frac{1}{6}\left[180 - (-47.25 + 40.5)\right] = 31.125 \text{ kN } (\uparrow)$$

$\Sigma M_B = 0$ (for span BC)

$\Rightarrow$ $$-R_C \times 6 + M_{BC} + M_{CB} + 12 \times 6 \times 3 = 0$$

$\Rightarrow$ $$R_C = \frac{1}{6}\left[216 + M_{BC} + M_{CB}\right]$$

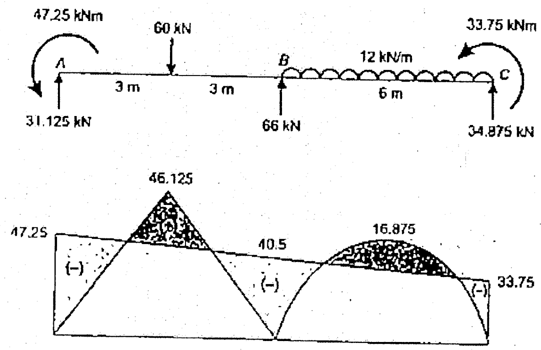
$\Rightarrow$ $$R_C = \frac{1}{6}\left[216 + (-40.5 + 33.75)\right] = 34.875 \text{ kN } (\uparrow)$$

Also, $\Sigma F_y = 0$ (for entire beam)

⇒ $R_A + R_B + R_C = 60 + 12 \times 6 = 132$ kN

⇒ $R_B = 132 - R_A - R_C = 132 - 31.125 - 34.875 = 66$ kN (↑)

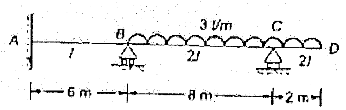
Step-6: Bending Moment Diagram



BMD

Fig. 9.9

Example 9.1 For the beam shown in figure determine support reaction for all member of beam using slope deflection method.



Solution:

Fixed end moments:

$$\overline{M}_{AB} = \overline{M}_{BA} = 0$$

$$\overline{M}_{BC} = -\frac{wL^2}{12} = -\frac{3\times 8^2}{12} = -16 \text{ t-m}$$

$$\overline{M}_{CB} = +\frac{wL^2}{12} = +16 \text{ t-m}$$

$$\overline{M}_{CD} = M_{CD} = \frac{3\times 2^2}{2} = -6 \text{ t-m}$$

Slope deflection equations:

Member *AB*:

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

Here, $\overline{M}_{AB} = 0$, $\theta_A = 0$ and $\Delta = 0$

∴ $$M_{AB} = 0 + \frac{2EI}{6}[0 + \theta_B - 0]$$

$$M_{AB} = \frac{2EI\theta_B}{6} \quad \text{...(i)}$$

$$M_{BA} = \overline{M}_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

Here, $\overline{M}_{BA} = 0$, $\theta_A = 0$ and $\Delta = 0$

$$M_{BA} = 0 + \frac{2EI}{6}[2\theta_B + 0 - 0]$$

$$M_{BA} = \frac{4EI\theta_B}{6} \quad \text{...(ii)}$$

Member *BC*:

$$M_{BC} = \overline{M}_{BC} + \frac{2\times 2EI}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

Here, $\overline{M}_{BC} = -16$ t-m and $\Delta = 0$

$$M_{BC} = -16 + \frac{4EI}{8}(2\theta_B + \theta_C - 0)$$

$$M_{BC} = -16 + EI\theta_B + \frac{EI\theta_C}{2} \quad \text{...(iii)}$$

$$M_{CB} = \overline{M}_{CB} + \frac{2\times 2EI}{L}\left(2\theta_C + \theta_B - \frac{3\Delta}{L}\right)$$

$$M_{CB} = +16 + \frac{4EI}{8}(2\theta_C + \theta_B)$$

$$M_{CB} = +16 + EI\theta_C + \frac{EI\theta_B}{2} \quad \text{...(iv)}$$

Member CD:

$$M_{CD} = -6 \text{ t-m} \quad \text{...(v)}$$

Equilibrium equations:

Consider joint equilibrium at joint *B*,

$$M_{BA} + M_{BC} = 0$$

$$\frac{2}{3}EI\theta_B + EI\theta_B + \frac{EI\theta_C}{2} - 16 = 0$$

$$\frac{5}{3}EI\theta_B + \frac{EI\theta_C}{2} - 16 = 0$$

$$10EI\theta_B + 3EI\theta_C - 96 = 0 \quad \text{...(A)}$$

Consider joint equilibrium at joint *C*,

$$M_{CB} + M_{CD} = 0$$

$$16 + EI\theta_C + \frac{EI\theta_B}{2} - 6 = 0$$

$$EI\theta_B + 2EI\theta_C = -20 \quad \text{...(B)}$$

On solving equation (*A*) and (*B*) we get

$$EI\theta_C = -17.41$$

and

$$EI\theta_B = +14.82$$

Final end moment:

On putting values of $EI\theta_C$ and $EI\theta_B$ in slope deflection equations we get,

$$M_{AB} = \frac{EI\theta_B}{3} = \frac{14.82}{3} = +4.94 \text{ t-m}$$

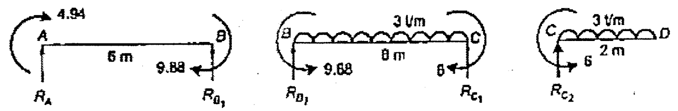
$$M_{BA} = \frac{2}{3}EI\theta_B = 9.88 \text{ t-m}$$

$$M_{BC} = -16 + EI\theta_B + \frac{EI\theta_C}{2} = -16 + (14.82) + \left(-\frac{17.41}{2}\right) = -9.88 \text{ t-m}$$

$$M_{CB} = 16 + EI\theta_C + \frac{EI\theta_B}{2} = 1 + (-17.41) + 7.41 = +6 \text{ t-m}$$

Support reactions:

Consider free body equilibrium of each span subjected to end moments and given loading.



$\Sigma M_B = 0$ (For span *AB*)

$$R_A \times 6 + 4.94 + 9.88 = 0$$

$$R_A = -\frac{1}{6}(4.94 + 9.88) = -2.47\,(\uparrow) = 2.47\text{ t}(\downarrow)$$

$$R_{B1} = -R_A = 2.47\text{ t}(\uparrow)$$

$\Sigma M_B = 0$ (For span *BC*)

$$R_{C1} \times 8 - 6 - 3 \times 8 \times 4 + 9.88 = 0$$

$$R_{C1} = \frac{1}{8}[6 + 96 - 9.88] = 11.515\text{ t}(\uparrow)$$

$\Sigma M_C = 0$ (For span *BC*)

$$R_{B2} \times 8 - 9.88 - 3 \times 8 \times 4 + 6 = 0$$

$$R_{B2} = \frac{1}{8}[9.88 + 90] = 12.485\text{ t}(\uparrow)$$

$\Sigma M_D = 0$ (For span *CD*)

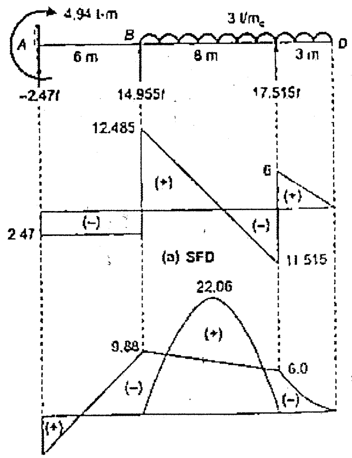
$$R_{C2} \times 2 - 6 - 2 \times 3 = 0 = 6\text{ t}(\uparrow)$$

Thus the reactions are

$$R_A = 2.47\text{ t}(\downarrow)$$

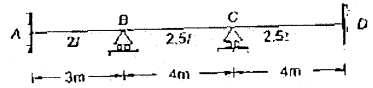
$$R_B = R_{B1} + R_{B2} = 2.47 + 12.485 = 14.955\text{ t}(\uparrow)$$

$$R_C = R_{C1} + R_{C2} = 11.515 + 6 = 17.515\text{ t}(\uparrow)$$

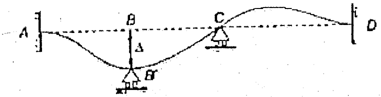


SFD and BMD

Example 9.2 Analyse the continuous beam shown in figure below by slope deflection method. Support *B* settles down by 5 mm. Take $E = 2 \times 10^5$ N/mm² and $I = 36 \times 10^6$ mm⁴.



Solution:



Slope deflection equations:

Member *AB*:

$$M_{AB} = \overline{M}_{AB} + \frac{2E(2I)}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

Here, $\overline{M}_{AB} = 0$, $\theta_A = 0$ and $\Delta = +0.005$ m (Produce clockwise rotation)

$$\therefore \quad M_{AB} = \frac{4EI}{3}\left(0 + \theta_B - \frac{3 \times 0.005}{3}\right)$$

$$M_{AB} = 1.33\,EI\theta_B - 6.67 \times 10^{-3}\,EI \qquad \text{...(i)}$$

$$M_{BA} = \overline{M}_{BA} + \frac{2E(2I)}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

$$\therefore \quad M_{BA} = 0 + \frac{4EI}{3}\left(2\theta_B - \frac{3 \times 0.005}{3}\right)$$

$$M_{BA} = 2.67\,EI\theta_B - 6.67 \times 10^{-3}\,EI \qquad \text{...(ii)}$$

Member *BC*:

$$M_{BC} = \overline{M}_{BC} + \frac{2EI(2.5I)}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

Here, $\overline{M}_{BC} = 0$, and $\Delta = -0.005$ m (Produce anticlockwise rotation)

$$\therefore \quad M_{BC} = 0 + \frac{5EI}{4}\left(2\theta_B + \theta_C + \frac{3 \times 0.005}{4}\right)$$

$$M_{BC} = 2.5\,EI\theta_B + 1.25\,EI\theta_C + 4.6875 \times 10^{-3}\,EI \qquad \text{...(iii)}$$

$$M_{CB} = \overline{M}_{CB} + \frac{2E(2.5I)}{L}\left(2\theta_C + \theta_B - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{5EI}{4}\left(2\theta_C + \theta_B + \frac{3 \times 0.005}{4}\right)$$

$$M_{CB} = 2.5\,EI\theta_C + 1.25\,EI\theta_B + 4.6875 \times 10 \qquad \text{...(iv)}$$

Member *CD*:

$$M_C = \overline{M}_{CD} + \frac{2E(2.5I)}{L}\left(2\theta_C + \theta_D - \frac{3\Delta}{L}\right)$$

Here. $\overline{M}_{CD} = 0$, $\theta_D = 0$ and $\Delta = 0$

$\therefore$
$$M_{CD} = 0 + \frac{5EI}{4}(2\theta_C)$$

$$M_{CD} = 2.5\,EI\,\theta_C \qquad \text{...(v)}$$

$$M_{DC} = \overline{M}_{DC} + \frac{2E(2.5I)}{L}\left(2\theta_D + \theta_C - \frac{3\Delta}{L}\right)$$

$$M_{DC} = 0 + \frac{5EI}{4}(\theta_C)$$

$$M_{DC} = 1.25\,EI\theta_C \qquad \text{...(vi)}$$

Equilibrium equations:

Consider joint equilibrium of joint B,

$$M_{BA} + M_{BC} = 0$$

$$2.67\,EI\,\theta_B - 6.67 \times 10^{-3}\,EI + 2.5\,EI\,\theta_B + 1.25\,EI\,\theta_C + 4.6875 \times 10^{-3}\,EI$$

$$\Rightarrow \quad 5.17\,EI\,\theta_B + 1.25\,EI\,\theta_C - 1.9825 \times 10^{-3}\,EI = 0 \qquad \text{...(A)}$$

Now, consider joint equilibrium of joint C,

$$M_{CB} + M_{CD} = 0$$

$$2.5\,EI\,\theta_C - 1.25\,EI\,\theta_B + 4.6875 \times 10^{-3}\,EI + 2.5\,EI\theta_c = 0 \qquad \text{...(B)}$$

On solving equation (A) and (B), we get

$$\theta_B = 6.49 \times 10^{-4}$$
$$\theta_C = -1.099 \times 10^{-3}$$

Final end moments:

On putting the values of θ_B and θ_C in slope deflection equations, we get

$$M_{AB} = 1.33\,EI \times 6.49 \times 10^{-4} - 6.67 \times 10^{-3}\,EI$$
$$= -5.80\,EI$$
$$= -5.80 \times 2 \times 10^5 \times 36 \times 10^9 \times 10^{-6}$$
$$= -41.76 \text{ kN-m}$$

$$M_{BA} = 2.67\,EI \times 6.49 \times 10^{-4} - 6.67 \times 10^{-3}\,EI$$
$$= -4.93 \times 10^{-3} \times 2 \times 10^5 \times 36 \times 10^5 \times 10^{-6}$$
$$= -35.49 \text{ kN-m}$$

$$M_{BC} = 2.5\,EI \times 6.49 \times 10^{-4} + 1.25\,EI \times (-1.099 \times 10^{-3}) + 4.6875 \times 10^{-3}$$
$$= 4.98 \times 10^{-3}\,EI$$
$$= +35.49 \text{ kN-m}$$

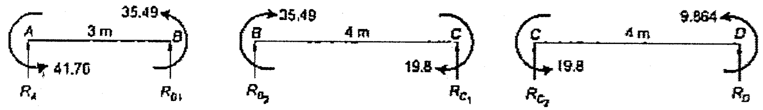
$$M_{CB} = 2.5\,EI \times (-1.099 \times 10^{-3}) + 1.25\,EI \times 6.49 \times 10^{-4} + 4.6875 \times 10^{-3}\,EI$$
$$= 2.751 \times 10^{-3}\,EI$$
$$= 19.8 \text{ kN-m}$$

$$M_{CD} = -19.8 \text{ kN-m}$$

$$M_{DC} = 1.25\,EI \times (-1.099 \times 10^{-3})$$
$$= -1.37 \times 10^{-3}\,EI$$
$$= -9.864 \text{ kN-m}$$

Support reactions:

Consider free body equilibrium of span AB, BC and CD separately.



Span AB:

$\Sigma M_B = 0$ (For span AB)

$$R_A \times 3 - 41.76 - 35.49 = 0$$

$$R_A = \frac{1}{3}(41.76 + 35.49) = 25.75 \text{ kN}(\uparrow)$$

$\Sigma F_y = 0$ (For span AB)

$$R_A + R_{B1} = 0$$
$$R_{B1} = -R_A = -25.75 \text{ kN} = 25.75 \text{ kN}(\downarrow)$$

Span BC:

$\Sigma M_C = 0$

$$R_{B2} \times 4 + 35.49 + 19.8 = 0$$

$$R_{B2} = -\frac{1}{4}(35.49 + 19.8) = -13.82 \text{ kN} = 13.82 \text{ kN}(\downarrow)$$

Span CD:

$$-R_D \times 4 - 19.8 - 9.864 = 0$$

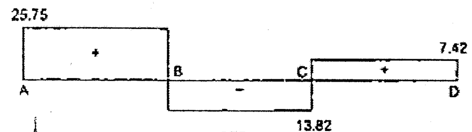
$$R_D = -\frac{1}{4}(19.8 + 9.864) = -7.416 = 7.416 \text{ kN}(\downarrow)$$

Thus the reactions are

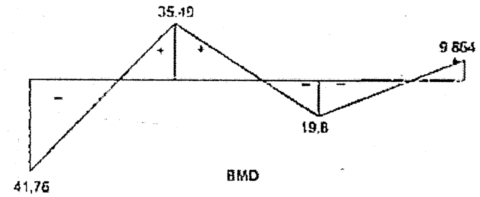
$$R_A = 25.75 \text{ kN}(\uparrow)$$
$$R_B = R_{B1} + R_{B2} = 25.75 + 13.82 = 39.57 \text{ kN}(\downarrow)$$
$$R_D = 7.416 \text{ kN}(\downarrow)$$

Since, $\quad R_A + R_B + R_C + R_D = 0$

$$R_C = -(R_A + R_B + R_D) = -(25.75 - 39.57 - 7.416) = 21.236 \text{ kN}(\uparrow)$$

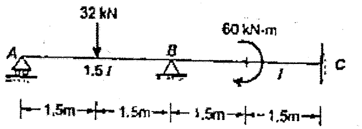


SFD



BMD

Example 9.3 Analyse the beam loaded as shown in figure below using slope deflection method. Draw the *BMD* and *SFD*.



Solution:

Fixed end moments:

$$\overline{M}_{AB} = -\frac{PL}{8} = -\frac{32\times 3}{8} = -12 \text{ kN-m}$$

$$\overline{M}_{BA} = +\frac{PL}{8} = +12 \text{ kN-m}$$

$$\overline{M}_{BC} = +\frac{M_0}{4} = +\frac{60}{4} = +15 \text{ kN-m}$$

$$\overline{M}_{CB} = +\frac{M_0}{4} = +15 \text{ kN-m}$$

Slope deflection equations:

Member AB:

$$M_{AB} = \overline{M}_{AB} + \frac{2E(1.5I)}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$M_{AB} = -12 + \frac{3EI}{3}(2\theta_A + \theta_B)$$

$$M_{AB} = -12 + 2\,EI\theta_A + EI\theta_B \quad ...(i)$$

and

$$M_{BA} = \overline{M}_{BA} + \frac{2E(1.5I)}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

$$M_{BA} = +12 + \frac{3EI}{3}(2\theta_B + \theta_A)$$

$$M_{BA} = +12 + 2\,EI\theta_B + EI\theta_A \quad ...(ii)$$

Member *BC*:

$$M_{BC} = \overline{M}_{BC} + \frac{2E(I)}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right) = +15 + \frac{2EI}{3}(2\theta_B - 0)$$

$$M_{BC} = 15 + \frac{4EI\theta_B}{3} \quad ...(iii)$$

$$M_{CB} = \overline{M}_{CB} + \frac{2EI}{L}\left(2\theta_C + \theta_B - \frac{3\Delta}{L}\right) = +15 + \frac{2EI}{3}(0 + \theta_B)$$

$$M_{CB} = +15 + \frac{2}{3}EI\theta_B \quad ...(iv)$$

Equilibrium equations:

There are two rotational unknowns θ_A and θ_B. Hence two joint equilibrium conditions are required.

$$M_A = 0 \quad \text{(Hinged)}$$

$$\therefore \quad -12 + 2\,EI\theta_A + EI\theta_B = 0 \quad ...(A)$$

Consider joint equilibrium at *B*,

$$M_{BA} + M_{BC} = 0$$

$$+12 + 2EI\theta_B + EI\theta_A + 15 + \frac{4}{3}EI\theta_B = 0$$

$$27 + 3.33\,EI\theta_B + EI\theta_A = 0$$

$$EI\theta_A + 3.33\,EI\theta_B = -27 \quad ..(B)$$

On solving equation (*A*) and (*B*), we get

$$EI\theta_A = 11.823$$

$$EI\theta_B = -11.647$$

Final end moments

$$M_{AB} = 0$$

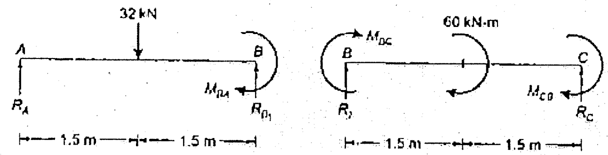
$$M_{BA} = 12 + 2EI\theta_B + EI\theta_A$$

$$= 12 + 2(-11.647) + 11.823 = 0.529 \text{ kNm}$$

$$M_{BC} = 15 + \frac{4}{3}EI\theta_B = 15 + \frac{4}{3}(-11.647) = -0.529 \text{ kNm}$$

$$M_{CB} = 15 + \frac{2}{3}EI\theta_B = 15 + \frac{2}{3}(-11.647) = 7.235 \text{ kNm}$$

Support reactions:



Span AB:

$\Sigma M_B = 0$

$$R_A \times 3 + M_{BA} - 32 \times 1.5 = 0$$

$$R_A = \frac{1}{3}[48 - M_{BA}] = \frac{1}{3}[48 - 0.529] = 15.82 \text{ kN } (\uparrow)$$

Span BC:

$\Sigma M_B = 0$

$$-R_C \times 3 + 60 + M_{BC} + M_{CB} = 0$$

$$R_C = \frac{1}{3}[60 + M_{BC} + M_{CB}]$$

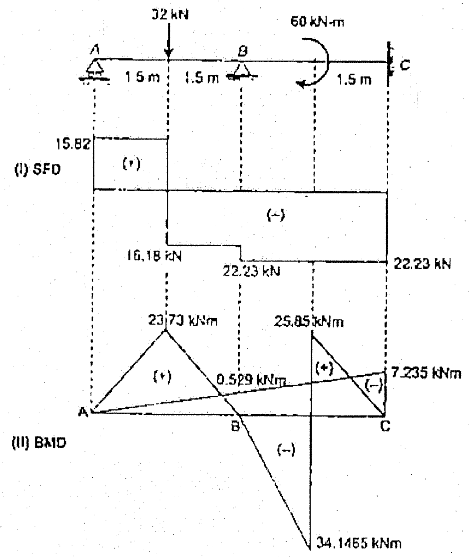
$$= \frac{1}{3}[60 - 0.529 + 7.235] = 22.23 \text{ kN}(\uparrow)$$

$\Sigma F_y = 0$ (For entire beam)

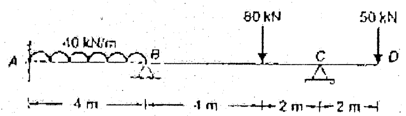
$$R_A + R_B + R_C = 32$$

$$\therefore \quad R_B = 32 - R_A - R_C$$
$$= 32 - 15.82 - 22.23$$
$$= -6.05 \text{ kN or } 6.05 \text{ kN } (\downarrow)$$

SFD and BMD



Example 9.4 Analyse the continuous beam shown in figure below by slope deflection method. Draw the SFD and BMD. Also sketch the deflected shape.



EI is constant.

Solution:

Fixed end moments:

$$\overline{M}_{AB} = -\frac{wL^2}{12} = -\frac{40 \times 4^2}{12} = -53.33 \text{ kN-m}$$

$$\overline{M}_{BA} = +\frac{wL^2}{12} = +53.33 \text{ kN-m}$$

$$\overline{M}_{BC} = -\frac{Pab^2}{L^2} = \frac{-80 \times 4 \times 2^2}{6^2} = -35.56 \text{ kN-m}$$

$$\overline{M}_{CB} = +\frac{Pa^2b}{L^2} = \frac{+80 \times 4^2 \times 2}{6^2} = +71.11 \text{ kN-m}$$

$$\overline{M}_{CD} = -50 \times 2 = -100 \text{ kN-m}$$

Slope deflection equations:

Members AB:

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$= -53.33 + \frac{2EI}{4}(0 + \theta_B - 0)$$

$$M_{AB} = -53.33 + 0.5\,EI\,\theta_B \quad \text{...(i)}$$

$$M_{BA} = \overline{M}_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

$$= +53.33 + \frac{2EI}{4}(2\theta_B + 0 - 0)$$

$$M_{BA} = 53.33 + EI\theta_B \quad \text{...(ii)}$$

Member BC:

$$M_{BC} = \overline{M}_{BC} + \frac{2EI}{6}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

$$M_{BC} = -35.56 + 0.67\,EI\theta_B + 0.33\,EI\theta_C \quad \text{...(iii)}$$

$$M_{CB} = \overline{M}_{CB} + \frac{2EI}{6}(2\theta_C + \theta_B - 0)$$

$$M_{CB} = 71.11 + 0.67\,EI\theta_C + 0.33\,EI\theta_B \quad \text{...(iv)}$$

Member CD:

$$M_{CD} = \overline{M}_{CD}$$

$$M_{CD} = -100 \quad \text{...(v)}$$

Equilibrium conditions:

Consider the equilibrium as given by

$$M_{BA} + M_{BC} = 0$$

$$(53.33 + EI\theta_B) + (-35.56 + 0.67\,EI\theta_B + 0.33\,EI\theta_C) = 0$$

$$1.67\,EI\theta_B + 0.33\,EI\theta_C = -17.77 \quad \text{...(A)}$$

Now, consider joint equilibrium at joint C.

$$M_{CB} + M_{CD} = 0$$

$$71.11 + 0.67\,EI\theta_C + 0.33\,EI\theta_B - 100 = 0$$

$$0.33\,EI\theta_B + 0.67\,EI\theta_C = 28.89 \quad \text{...(B)}$$

On solving equation (A) and (B), we get

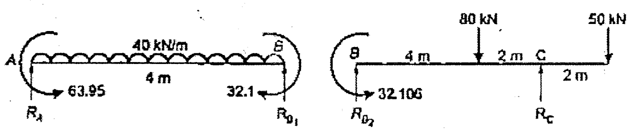
$$EI\theta_B = -21.23$$

$$EI\theta_C = 53.57$$

Final and moments:

$$M_{AB} = -53.33 + 0.5 \times (-21.23)$$
$$= -63.955 \text{ kN-m}$$
$$M_{BA} = +53.33 + (-21.23)$$
$$= 32.1 \text{ kN-m}$$
$$M_{BC} = -35.56 + 0.67\, EI\theta_B + 0.33\, EI\theta_C$$
$$= -35.56 + 0.67 \times (-21.23) + 0.33 \times 53.57$$
$$= -32.10 \text{ kN-m}$$
$$M_{CB} = 71.11 + 0.67\, EI\theta_C + 0.33\, EI\theta_B$$
$$= 71.11 + 0.67 \times 53.57 - 0.33 \times 21.23$$
$$= +100 \text{ kN-m}$$
$$M_{CD} = -100 \text{ kN-m}$$

Support reactions:



Span AB:

$R_A \times 4 - 63.95 + 32.1 - 40 \times 4 \times 2 = 0$

$$R_A = \frac{1}{4}(63.95 - 32.1 + 40 \times 4 \times 2)$$
$$R_A = 87.96 \text{ kN}(\uparrow)$$

Span BC:

$\Sigma M_B = 0$

$-32.10 + 80 \times 4 - R_C \times 6 + 50 \times 8 = 0$

$$R_C = \frac{1}{6}[-32.10 + 80 \times 4 + 50 \times 8]$$
$$R_C = 114.64 \text{ kN}(\uparrow)$$

For vertical reaction at B,

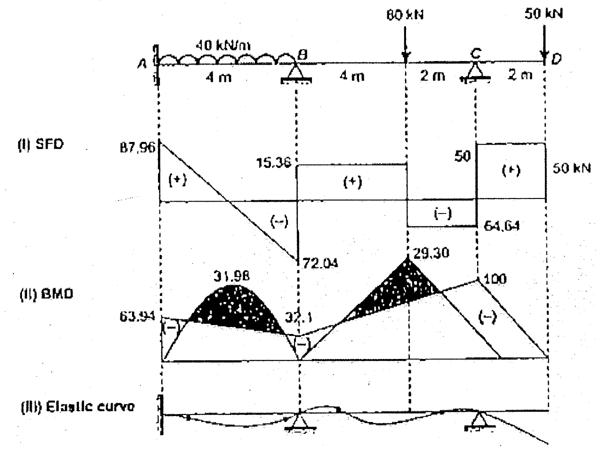
$$R_A + R_B + R_C = 40 \times 4 + 80 + 50$$
$$\therefore \quad R_B = 290 - R_A - R_C$$
$$R_B = 290 - 87.96 - 114.64$$
$$R_B = 87.40 \text{ kN}(\uparrow)$$

SFD and BMD:

Maximum simply supported moment in AB,

$$= \frac{40 \times 4^2}{8} = 80 \text{ kN-m}$$

Maximum simply supported moment in $BC = \dfrac{80 \times 4 \times 2}{6} = 106.67$ kN-m



9.5 Analysis of Frames without Sway

Frames do not undergo any sway when

(i) Columns are at same level and net horizontal force is zero.

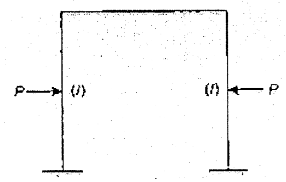
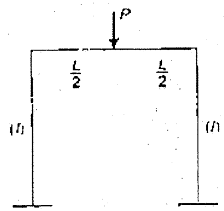


Fig 9.10(i)

(ii) Stiffness of column is same and loading is symmetrical in vertical plane.



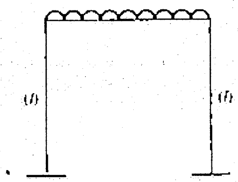


Fig.9.10(ii) *Fig.9.10(iii)*

(*iii*) Sway are prevented by unyielding support at beam level.

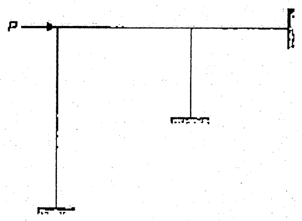


Fig. 9.10(iv)

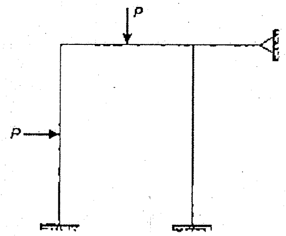
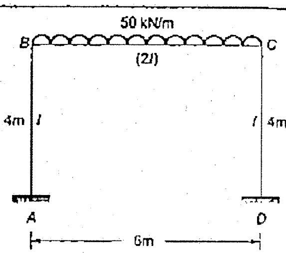


Fig. 9.10(v)

Example 9.5 Analyse the frame shown in figure by slope deflection method and draw bending moment diagram:



Solution:

Since load is symmetrical and stiffness of both column is same. Hence there is no sway in the frame.

Fixed end moments:

$$\overline{M}_{AB} = \overline{M}_{BA} = 0$$

$$\overline{M}_{BC} = -\frac{wL^2}{12} = -\frac{50\times 6^2}{12} = -150 \text{ kN-m}$$

$$\overline{M}_{CB} = +\frac{wL^2}{12} = +150 \text{ kN-m}$$

$$\overline{M}_{CD} = \overline{M}_{DC} = 0$$

Slope deflection equations:

Member *AB*:

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}(0 + \theta_B - 0)$$

$$M_{AB} = 0.5\, EI\theta_B \quad \text{...(i)}$$

$$M_{BA} = \overline{M}_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right) = 0 + \frac{2EI}{4}(2\theta_B + 0 - 0)$$

$$M_{BA} = EI\theta_B \quad \text{...(ii)}$$

Member *BC*:

$$M_{BC} = \overline{M}_{BC} + \frac{2E(2I)}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

$$= -150 + \frac{4EI}{6}(2\theta_B + \theta_C)$$

$$M_{BC} = -150 + 1.33\, EI\theta_B + 0.67\, EI\theta_C \quad \text{...(iii)}$$

$$M_{CB} = \overline{M}_{CB} + \frac{2E(2I)}{L}\left(2\theta_C + \theta_B - \frac{3\Delta}{L}\right)$$

$$= +150 + \frac{4EI}{6}(2\theta_C + \theta_B - 0)$$

$$M_{CB} = 150 + 0.67\, EI\theta_B + 1.33\, EI\theta_C \quad \text{...(iv)}$$

Member *CD*:

$$M_{CD} = \overline{M}_{CD} + \frac{2EI}{L}\left(2\theta_C + \theta_D - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}(2\theta_C + 0 - 0)$$

$$M_{CD} = EI\theta_C \quad \text{...(v)}$$

$$M_{DC} = \overline{M}_{DC} + \frac{2EI}{L}\left(2\theta_D + \theta_C - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}(0 + \theta_C - 0)$$

$$M_{DC} = 0.5\, EI\theta_C \quad \text{...(vi)}$$

Equilibrium equations:

Consider equilibrium of joint *B*,

$$M_{BA} + M_{BC} = 0$$

$$EI\theta_B + 1.33\, EI\theta_B + 0.67\, EI\theta_C - 150 = 0$$

$$2.33\, EI\theta_B + 0.67\, EI\theta_C = 150 \quad \text{...(A)}$$

Consider equilibrium of joint *C*,

$$M_{CB} + M_{CD} = 0$$

$$150 + 0.67\, EI\theta_B + 1.33\, EI\theta_C + EI\theta_C = 0$$

$$0.67\, EI\theta_B + 2.33\, EI\theta_C = -150 \quad \text{...(B)}$$

$$EI\theta_B = 90.36$$

$$EI\theta_C = -90.36$$

Final end moments:

$$M_{AB} = 0.5\, EI\theta_B = 0.5 \times (90.36)$$

$$M_{AB} = 45.18 \text{ kN-m}$$

$$M_{BA} = EI\theta_B$$

$$M_{BA} = 90.36 \text{ kN-m}$$

$$M_{BC} = -150 + 1.33 \times 90.36 + 0.67 \times (90.36) = -90.36 \text{ kN-m}$$

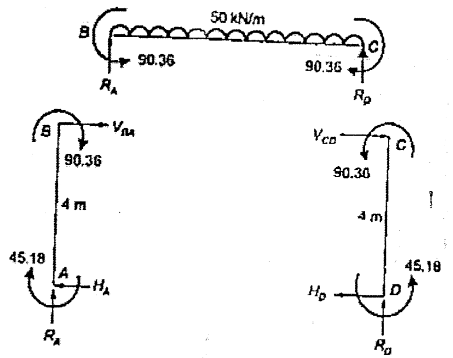
$$M_{CB} = 150 + 0.67\, EI\theta_B + 1.33\, EI\theta_C$$

$$= 150 + 0.67 \times 90.36 + 1.33 \times (-90.36) = 90.36 \text{ kN-m}$$

$$M_{CD} = EI\theta_C = -90.36 \text{ kN-m}$$

$$M_{DC} = 0.5\, EI\theta_C = -45.18 \text{ kN-m}$$

Support reaction:



Member *AB*:

$$H_A \times 4 + 90.36 + 45.18 = 0$$

$$H_A = -\frac{1}{4}(90.36 + 45.18) = -33.88 \text{ kN} = 33.88 \text{ kN } (\rightarrow)$$

Member *BC*:

$\Sigma M_C = 0$;

$$R_A \times 6 + 50 \times 6 \times 3 - 90.36 + 90.36 = 0$$

$$R_A = 150 \text{ kN } (\uparrow)$$

$\Sigma F_y = 0$; $\quad R_A + R_D = 50 \times 6$ ...(*i*)

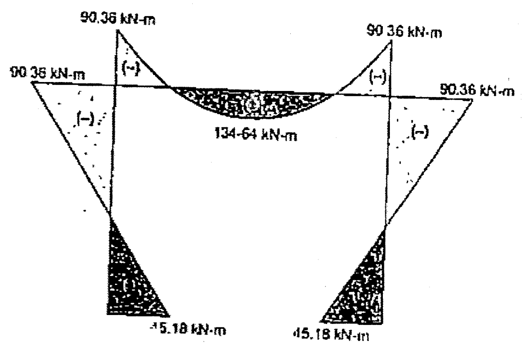
$\therefore \quad R_D = 50 \times 6 - 150 = 150 \text{ kN } (\uparrow)$

Member *CD*:

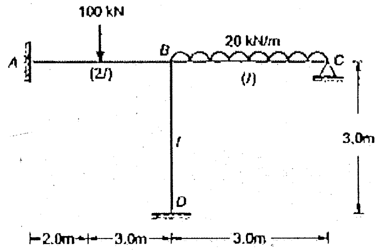
$\Sigma M_C = 0$; $\quad H_D \times 4 - 90.36 - 45.18$

$$H_D = \frac{1}{4}(90.36 + 45.18) = 33.88 \text{ kN } (\leftarrow)$$

Bending moment diagram:



Example 9.6 Analyse the frame shown in figure below by slope deflection method. Draw the *BMD*. The second moment of inertia are indicated in the figure.



Solution:

Fixed end moments:

$$\overline{M}_{AB} = -\frac{Pab^2}{L^2} = -\frac{100 \times 2 \times 3^2}{5^2} = -72 \text{ kN-m}$$

$$\overline{M}_{BA} = +\frac{Pa^2b}{L^2} = -\frac{100 \times 2^2 \times 3}{5^2} = +48 \text{ kN-m}$$

$$\overline{M}_{BC} = -\frac{wL^2}{12} = -\frac{20 \times 3^2}{12} = -15 \text{ kN-m}$$

$$\overline{M}_{CB} = +\frac{wL^2}{12} = +\frac{20 \times 3^2}{12} = +15 \text{ kN-m}$$

$$\overline{M}_{BD} = \overline{M}_{DB} = 0$$

Slope deflection equations:

Member *AB*:

$$M_{AB} = \overline{M}_{AB} + \frac{2E(2I)}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$= -72 + \frac{4EI}{5}(0 + \theta_B - 0)$$

$$M_{AB} = -72 + 0.8\, EI\theta_B \quad \text{...(i)}$$

$$M_{BA} = \overline{M}_{BA} + \frac{2E(2I)}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

$$= +48 + \frac{4EI}{5}(2\theta_B + 0 - 0)$$

$$M_{BA} = 48 + 1.6\, EI\theta_B \quad \text{...(ii)}$$

Member *BC*:

$$M_{BC} = \overline{M}_{BC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

$$= -15 + \frac{2EI}{3}(2\theta_B + \theta_C)$$

$$M_{BC} = -15 + 1.33\,EI\theta_B + 0.67\,EI\theta_C \qquad ...(iii)$$

$$M_{CB} = \overline{M}_{CB} + \frac{2EI}{L}\left(2\theta_C + \theta_B - \frac{3\Delta}{L}\right)$$

$$= +15 + \frac{2EI}{3}(2\theta_C + \theta_B - 0)$$

$$M_{CB} = 15 + 0.67\,EI\theta_B + 1.33\,EI\theta_C \qquad ...(iv)$$

Member BD:

$$M_{BD} = \overline{M}_{BD} + \frac{2EI}{L}\left(2\theta_B + \theta_D - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{3}(2\theta_B + 0 - 0)$$

$$M_{BD} = 1.33\,EI\theta_B \qquad ...(v)$$

$$M_{DB} = \overline{M}_{BD} + \frac{2}{3}EI\left(2\theta_D + \theta_B - \frac{3\Delta}{L}\right) \qquad ...(vi)$$

$$= 0.67EI\theta_B$$

Equilibrium equations:

Since end C is hinge. Hence $M_{CB} = 0$

$15 + 0.67\,EI\theta_B + 1.33\,EI\theta_C = 0$

$$0.67\,EI\theta_B + 1.33\,EI\theta_C = -15 \qquad ...(A)$$

Also consider joint equilibrium of joint B.

$$M_{BA} + M_{BD} + M_{BC} = 0$$

$48 + 1.6\,EI\theta_B + 1.33\,EI\theta_B - 15 + 1.33\,EI\theta_B + 0.67\,EI\theta_C = 0$

$$4.26\,EI\theta_B + 0.67\,EI\theta_C = -33 \qquad ...(B)$$

On solving equations (A) and (B) we get

$$EI\theta_B = -6.487$$

$$EI\theta_C = -8.01$$

Final end moments:

$$M_{AB} = -72 + 0.8\,EI\theta_B$$

$$= -72 + 0.8\,(-6.487) = -77.189 \text{ kN-m}$$

$$M_{BA} = 48 + 1.6\,EI\theta_B$$

$$= 48 + 1.6\,(-6.487) = 37.62 \text{ kN-m}$$

$$M_{BC} = -15 + 1.33\,EI\theta_B + 0.67\,EI\theta_C$$

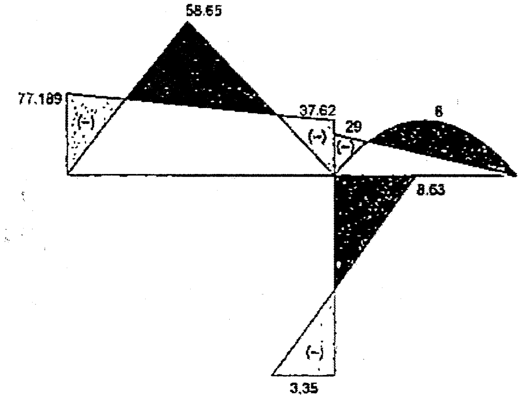
$$= -15 + 1.33\,(-6.487) + 0.67\,(-8.01) = -29 \text{ kN-m}$$

$$M_{CB} = 0$$

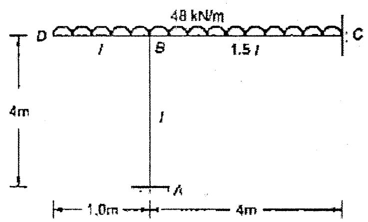
$$M_{BD} = 1.33\,EI\theta_B = 1.33\,(-6.487) = -8.63 \text{ kN-m}$$

$$M_{DB} = 0.67\,EI\theta_B = 0.67\,(-6.487) = -4.35 \text{ kN-m}$$

Bending moment diagram:



Example 9.7 A horizontal member DBC is rigidly jointed at B with a vertical member AB (as shown below) having the supports A and C fixed and D free. The members AB and BC are 4 m in length each and moments of inertia I and $1.5\,I$ respectively, the over hanging portion BD is 1.2 m long with moment of inertia I. The member DBC carries a UDL of 48 kN/m. Draw the B.M. diagram of the frame. Calculate the vertical deflection of the free end D in terms of EI.



Solution:

Fixed end moments:

$$\overline{M}_{AB} = \overline{M}_{BA} = 0$$

$$\overline{M}_{BC} = -\frac{wL^2}{12} = -\frac{48 \times 4^2}{12} = -64 \text{ kN-m}$$

$$\overline{M}_{CB} = +\frac{wL^2}{12} = +64 \text{ kN-m}$$

$$\overline{M}_{BD} = -48 \times 1 \times 0.5 = -24 \text{ kN-m}$$

Slope deflection equations:

Member *AB*:

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}(0 + \theta_B - 0)$$

$$M_{AB} = 0.5\ EI\theta_B \qquad \ldots(i)$$

$$M_{BA} = \overline{M}_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}(2\theta_B + 0 - 0)$$

$$M_{BA} = EI\theta_B \qquad \ldots(ii)$$

Member BC:

$$M_{BC} = \overline{M}_{BC} + \frac{2E(1.5I)}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

$$= -64 + \frac{3EI}{4}(2\theta_B + 0 - 0)$$

$$M_{BC} = -64 + 1.5\ EI\theta_B \qquad \ldots(iii)$$

$$M_{CB} = \overline{M}_{CB} + \frac{2E(1.5I)}{L}\left(2\theta_C + \theta_B - \frac{3\Delta}{L}\right)$$

$$= +64 + \frac{3EI}{4}(0 + \theta_B - 0)$$

$$M_{CB} = 64 + 0.75\ EI\theta_B \qquad \ldots(iv)$$

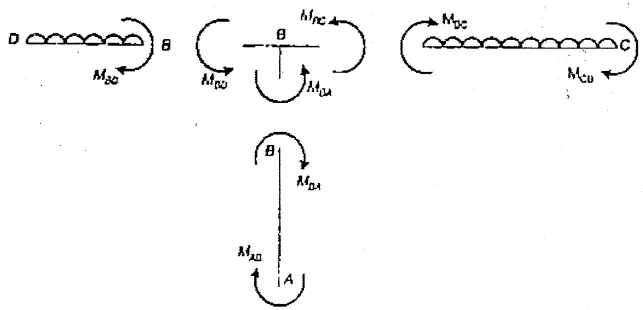
Member *BD*:

$$M_{BD} = \overline{M}_{BD}$$

$$M_{BD} = -24\ \text{kN-m} \qquad \ldots(v)$$

Equilibrium equations:

Consider joint equilibrium of joint *B*.



$$-M_{BA} - M_{BC} - M_{BD} = 0$$

$$M_{BA} + M_{BC} + M_{BD} = 0$$

$$EI\theta_B - 64 + 1.5\ EI\theta_B - 24 = 0$$

$$2.5\ EI\theta_B = 88$$

$$EI\theta_B = 35.2$$

Final end moments:

$$M_{AB} = -0.5\ EI\theta_B$$

$$= 0.5 \times 35.2 = 17.6\ \text{kN-m}$$

$$M_{BA} = EI\theta_B$$

$$= 35.2\ \text{kN-m}$$

$$M_{BC} = -64 + 1.5\ EI\theta_B$$

$$= -64 + 1.5 \times (35.2)$$

$$= -11.2\ \text{kN-m}$$

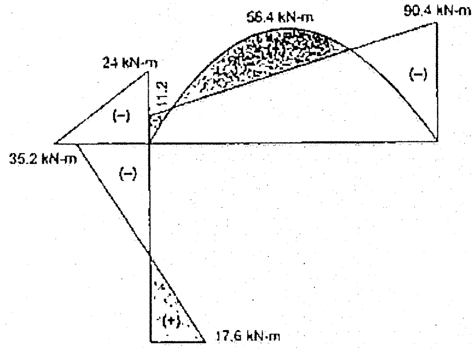
$$M_{CB} = 64 + 0.75\ EI\theta_B$$

$$= 64 + 0.75 \times (35.2)$$

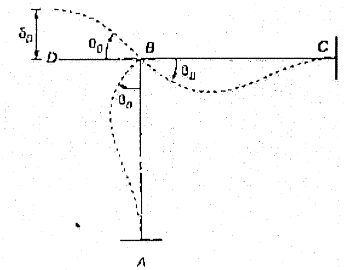
$$= 90.4\ \text{kN-m}$$

$$M_{BD} = -24\ \text{kN-m}$$

(*i*) Bending moment diagram:



(*ii*) Deflection of free end *D*:



$$\delta_D = +(\theta_B \times L_{BD}) - \frac{wL^4_{BD}}{8EI}$$

$$\delta_D = +\frac{35.2}{EI} \times 1 - \frac{48 \times 1^4}{8EI}$$

$$\delta_D = -\frac{29.9}{EI}\ \text{(Upward)}$$

9.6 Analysis of Frames with Sway

Consider a frame shown in figure below.

In this case there is a unbalance horizontal force. Therefore sway will take place in the direction of applied force.

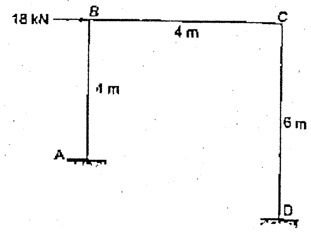


Fig. 9.11

Step-1: Draw approximate sway shape of frame as shown in figure.
Here, Δ = positive (rotations are clockwise)

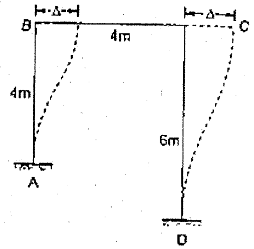


Fig. 9.12

Step-2: Apply slope deflection equations for each member
Here,

Member *AB*:

$$\Delta = +ve$$

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_B - \frac{3\Delta}{4}\right)$$

$$M_{AB} = 0.5\,EI\theta_B - 0.375\,EI\Delta \qquad ...(i)$$

$$M_{BA} = \overline{M}_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_B + 0 - \frac{3\Delta}{4}\right)$$

$$M_{BA} = EI\theta_B - 0.375\,EI\Delta \qquad ...(ii)$$

Member *BC*:
Here, $\Delta = 0$

$$M_{BC} = \overline{M}_{BC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}(2\theta_B + \theta_C - 0)$$

$$M_{BC} = EI\theta_B + 0.5\,EI\theta_C \qquad ...(iii)$$

$$M_{CB} = \overline{M}_{CB} + \frac{2EI}{L}\left(2\theta_C + \theta_B - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}(2\theta_C + \theta_B - 0)$$

$$M_{CB} = 0.5\,EI\theta_B + EI\theta_C \qquad ...(iv)$$

Member *CD*:

$$M_{CD} = \overline{M}_{CD} + \frac{2EI}{L}\left(2\theta_C + \theta_D - \frac{3\Delta}{L}\right) = 0 + \frac{2EI}{6}\left(2\theta_C + 0 - \frac{3\Delta}{6}\right)$$

$$M_{CD} = 0.67\,EI\theta_C - 0.167\,EI\Delta \qquad ...(v)$$

$$M_{DC} = \overline{M}_{DC} + \frac{2EI}{L}\left(2\theta_D + \theta_C - \frac{3\Delta}{L}\right) = 0 + \frac{2EI}{6}\left(0 + \theta_C - \frac{3\Delta}{6}\right)$$

$$= 0.33EI\theta_C - 0.167EI\Delta \qquad ...(vi)$$

Step-4: Apply joint equilibrium equations for each joint.
Here,

(*i*) At joint *B*:

$$M_{BA} + M_{BC} = 0$$

$$EI\theta_B - 0.375\,EI\Delta + EI\theta_B + 0.5\,EI\theta_C = 0$$

$$2\,EI\theta_B + 0.5\,EI\theta_C - 0.375\,EI\Delta = 0 \qquad ...(A)$$

(*ii*) At joint *C*:

$$M_{CB} + M_{CD} = 0$$

$$0.5\,EI\theta_B + EI\theta_C + 0.67\,EI\theta_C - 0.167\,EI\Delta = 0$$

$$0.5\,EI\theta_B + 1.67\,EI\theta_C - 0.167\,EI\Delta = 0 \qquad ...(B)$$

Now there are three unknown (Δ, θ_B, θ_C) but only two equations of equilibrium are available, so we need one extra equilibrium equation that can be obtain by shear equation in the direction of unknown displacement.

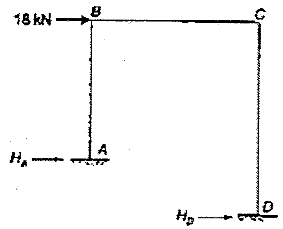


Fig. 9.13

Step-5: Write shear equation

when Δ is horizontal, shear equation is $\Sigma F_x = 0$ and

when Δ is vertical, shear equation is $\Sigma F_y = 0$

In this case, Δ is horizontal.

Therefore the shear equation is obtained by

$\Sigma F_x = 0$

$$H_A + H_D + 18 = 0 \qquad ...(C)$$

Horizontal reactions can be found by free body equilibrium of vertical members *AB* and *CD*

For member *AB*,

$\Sigma M_B = 0$ $\qquad -H_A \times 4 + M_{AB} + M_{BA} = 0$

$$H_A = \frac{M_{AB} + M_{BA}}{4}$$

and For member *CD*

$\Sigma M_C = 0$ $\qquad -H_D \times 6 + M_{CD} + M_{DC} = 0$

$$H_D = \frac{M_{CD} + M_{DC}}{6}$$

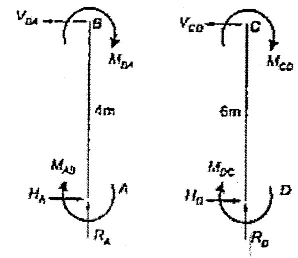


Fig. 9.14

On putting values of H_A and H_D in equation (*C*), we get

$$\frac{M_{AB} + M_{BA}}{4} + \frac{M_{CD} + M_{DC}}{6} + 18 = 0$$

$$3(M_{AB} + M_{BA}) + 2(M_{CD} + M_{DC}) + 216 = 0$$

$$3(0.5\,EI\theta_B - 0.375\,EI\Delta + EI\theta_B - 0.375\,EI\Delta) + 2(0.67\,EI\theta_C - 0.167\,EI\Delta + 0.33\,EI\theta_C - 0.167\,EI\Delta) = 0$$

$$4.5\,EI\theta_B + 2\,EI\theta_C - 2.918\,EI\Delta = -216 \qquad ...(D)$$

On solving equations A, B, and D we get

$$EI\theta_B = 18.6785$$
$$EI\theta_C = 5.0288$$
$$EI\Delta = 106.323$$

(Assumed direction is correct)

NOTE: If $EI\Delta$ value is negative, the sway is occurring in the opposite of assumed direction.

Step-6: Final end moments and BMD

$$M_{AB} = -30.53 \text{ kN-m}$$
$$M_{BA} = -21.19 \text{ kN-m}$$
$$M_{BC} = +21.19 \text{ kN-m}$$
$$M_{CB} = 14.37 \text{ kN-m}$$
$$M_{CD} = -14.37 \text{ kN-m}$$
$$M_{DC} = -16.04 \text{ kN-m}$$

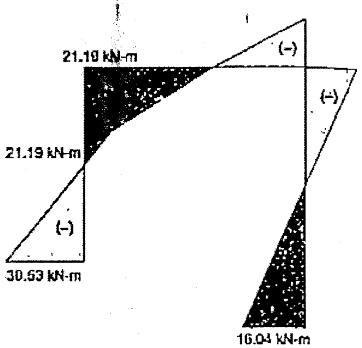


Fig. 9.15

Horizontal reactions,

$$H_A = \frac{-30.53 - 21.19}{4} = -12.93 \text{ kN} (\rightarrow)$$

$$H_A = 12.93 \text{ kN} (\leftarrow)$$

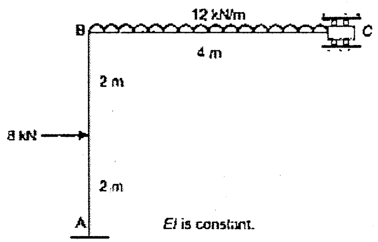
$$H_D = \frac{-14.37 - 16.04}{6} = -5.07 \text{ kN} (\rightarrow)$$

$$H_D = 5.07 \text{ kN} (\leftarrow)$$

$$R_D = \frac{21.19 + 14.37}{4} = 8.89 \text{ kN} (\uparrow)$$

$$R_A = -R_D = -8.89 \text{ kN} (\uparrow)$$

Example 9.8 Analyse the frame shown in figure and draw bending moment diagram.



Solution:

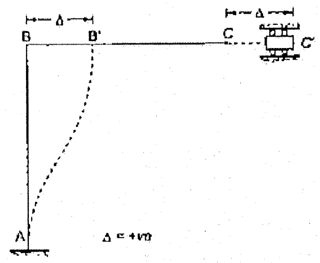
Fixed end Moments:

$$\bar{M}_{AB} = -\frac{8 \times 4}{8} = -4 \text{ kNm}$$

$$\bar{M}_{BA} = +\frac{8 \times 4}{8} = +4 \text{ kNm}$$

$$\bar{M}_{BC} = -\frac{12 \times 4^2}{12} = -16 \text{ kNm}$$

$$\bar{M}_{CB} = +\frac{12 \times 4^2}{12} = +16 \text{ kNm}$$



Let Δ be the sway displacement from left to right as shown in figure

Slope deflection equation:

Member AB:

$$M_{AB} = \bar{M}_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$= -4 + \frac{2EI}{4}\left(0 + \theta_B - \frac{3\Delta}{4}\right)$$

$$= -4 + \frac{2EI\theta_B}{4} - \frac{3EI\Delta}{8} \qquad ...(i)$$

$$M_{BA} = \bar{M}_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

$$= +4 + \frac{2EI}{4}\left(2\theta_B + 0 - \frac{3\Delta}{4}\right)$$

$$= +4 + \frac{4EI\theta_B}{4} - \frac{3EI\Delta}{8} \qquad ...(ii)$$

Member BC:

Here Δ = 0,

$$M_{BC} = \bar{M}_{BC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

$$= -16 + \frac{2EI}{4}(2\theta_B + 0 - 0)$$

$$= -16 + \frac{4EI\theta_B}{4} \qquad ...(iii)$$

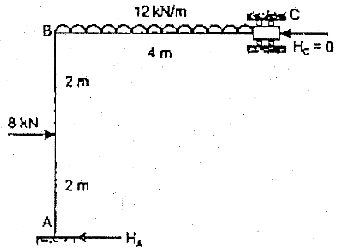
$$M_{CB} = \bar{M}_{CB} + \frac{2EI}{L}\left(2\theta_C + \theta_B - \frac{3\Delta}{L}\right)$$

$$= +16 + \frac{2EI}{4}(0 + \theta_B - 0)$$

$$= +16 + \frac{2EI\theta_B}{4} \qquad ...(iv)$$

Equilibrium equations

Consider joint equilibrium of B,



$$M_{BA} + M_{BC} = 0$$

$$\left(4 + \frac{4EI\theta_B}{4} - \frac{3EI\Delta}{8}\right) + \left(-16 + \frac{4EI\theta_B}{4}\right) = 0$$

$$2EI\theta_B - \frac{3EI\Delta}{8} = 12 \quad ...(A)$$

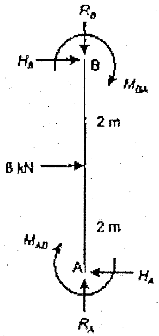
Shear equation:

Consider horizontal equilibrium of member frame

$\Sigma F_x = 0;$

$$\therefore \quad H_A = 8 \text{ kN} \quad ...(i)$$

Consider equilibrium of member AB.



$\Sigma M_B = 0;$

$$H_A \times 4 - 8 \times 2 + M_{AB} + M_{BA} = 0$$

$$H_A = \frac{1}{4}\left[16 - (M_{AB} + M_{BA})\right]$$

From eq. (i), shear equation becomes,

$$\frac{1}{4}\left[16 - (M_{AB} + M_{BA})\right] = 8$$

$$M_{AB} + M_{BA} + 16 = 0$$

$$-4 + \frac{2EI\theta_B}{4} - \frac{3EI\Delta}{8} + 4 + \frac{4EI\theta_B}{4} - \frac{3EI\Delta}{8} + 16 = 0$$

$$\frac{6EI\theta_B}{4} - \frac{6EI\Delta}{8} + 16 = 0$$

$$12EI\theta_B - 6EI\Delta + 128 = 0 \quad ...(B)$$

On solving equation (A) and (B) we get

$$EI\theta_B = 16$$

and

$$EI\Delta = \frac{160}{3}$$

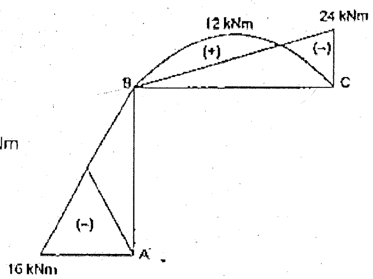
Final end moments:

$$M_{AB} = -4 + \frac{2}{4} \times 16 - \frac{3}{8} \times \frac{160}{3} = -16 \text{ kNm}$$

$$M_{BA} = 4 + 16 - \frac{3}{8} \times \frac{160}{3} = 0$$

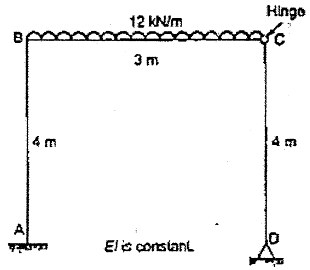
$$M_{BC} = -16 + 16 = 0$$

$$M_{CB} = 16 + \frac{2}{4} \times 16 = +24 \text{ kNm}$$



BMD

Example 9.9 Analyse the portal frame shown in figure. Also draw bending moment diagram



Solution:

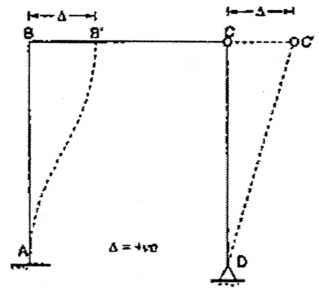
Fixed end moments:

$$\bar{M}_{AB} = \bar{M}_{BA} = \bar{M}_{CD} = \bar{M}_{DC} = 0$$

$$\bar{M}_{BC} = -\frac{12 \times 3^2}{12} = -9 \text{ kNm}$$

$$\bar{M}_{CB} = +\frac{12 \times 3^2}{12} = +9 \text{ kNm}$$

Let Δ be the sway displacement from left to right as shown in figure



Member AB:

$$M_{AB} = \bar{M}_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_B - \frac{3\Delta}{4}\right)$$

$$= \frac{2}{4}EI\theta_B - \frac{3}{8}EI\Delta$$

$$M_{BA} = \bar{M}_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right) = 0 + \frac{2EI}{4}\left(2\theta_B + 0 - \frac{3\Delta}{4}\right)$$

$$= EI\theta_B - \frac{3}{8}EI\Delta$$

Member BC:

Here, $\Delta = 0$

$$M_{BC} = \bar{M}_{BC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right) = -9 + \frac{2EI}{3}(2\theta_B + \theta_C)$$

$$= -9 + \frac{4}{3}EI\theta_B + \frac{2}{3}EI\theta_C$$

$$M_{CB} = \bar{M}_{CB} + \frac{2EI}{L}\left(2\theta_C + \theta_B - \frac{3\Delta}{L}\right) = +9 + \frac{2EI}{3}(2\theta_C + \theta_B)$$

$$= +9 + \frac{2}{3}EI\theta_B + \frac{4}{3}EI\theta_C$$

Member CD:

$M_{CD} = 0, \quad M_{DC} = 0$

Equilibrium equations:

Consider equilibrium of joint B,

$$M_{BA} + M_{BC} = 0$$

$$\left(EI\theta_B - \frac{3}{8}EI\Delta\right) + \left(-9 + \frac{4}{3}EI\theta_B + \frac{2}{3}EI\theta_C\right) = 0$$

$$2.333\,EI\theta_B + 0.6666\,EI\theta_C - 0.375\,EI\Delta = 9 \quad \text{...(i)}$$

Since there is a hinge at joint C,

$$\therefore \quad M_{CB} = 0$$

$$+9 + \frac{2}{3}EI\theta_B + \frac{4}{3}EI\theta_C = 0$$

$$0.6666\,EI\theta_B + 1.3333\,EI\theta_C = -9 \quad \text{...(ii)}$$

Shear equation:

$\Sigma F_x = 0$ (for entire frame)

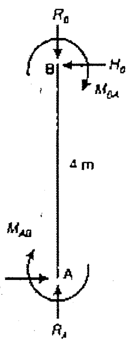
$$H_A + H_D = 0$$

$$\therefore \quad H_A = -H_D \quad \text{...(A)}$$

Consider equilibrium of member AB,

$$-H_A \times 4 + M_{AB} + M_{BA} = 0$$

$$\therefore \quad H_A = \frac{1}{4}(M_{AB} + M_{BA})$$



From eq. (i)

$$M_{AB} + M_{BA} = 0$$

$$\left(\frac{2}{4}EI\theta_B - \frac{3}{8}EI\Delta\right) + \left(EI\theta_B - \frac{3}{8}EI\Delta\right) = 0$$

$$1.5\,EI\theta_B - 0.75\,EI\Delta = 0 \quad \text{...(ii)}$$

On solving eq. (i), (ii) and (iii), we get

$$EI\theta_B = 10.8$$
$$EI\theta_C = -12.15$$
$$EI\Delta = 21.6$$

Final end moments:

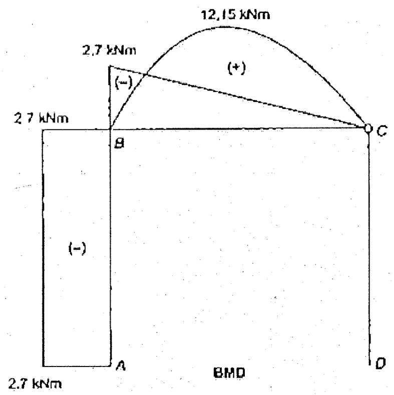
$$M_{AB} = \frac{2}{4}(10.8) - \frac{3}{8}(21.6) = -2.7 \text{ kNm}$$

$$M_{BA} = 10.8 - \frac{3}{8}(21.6) = +2.7 \text{ kNm}$$

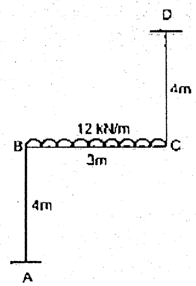
$$M_{BC} = -9 + \frac{4}{3}(10.8) + \frac{2}{3}(-12.15) = -2.7 \text{ kNm}$$

$M_{CB} = 0, \quad M_{CD} = 0, \quad M_{DC} = 0$

Bending moment diagram:



Example 9.10 Analyse the portal frame shown in figure. Also draw BM diagram. *EI* is constant.



Solution:

Fixed end moments:

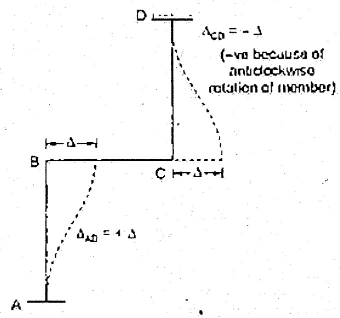
$$\bar{M}_{AB} = \bar{M}_{BA} = 0$$

$$\bar{M}_{BC} = -\frac{wL^2}{12} = \frac{-12 \times 3^2}{12} = -9 \text{ kN·m}$$

$$\bar{M}_{CB} = +\frac{wL^2}{12} = +\frac{12 \times 3^2}{12} = +9 \text{ kN·m}$$

$$\bar{M}_{CD} = \bar{M}_{DC} = 0$$

Let portal sway from left to right and Δ be the sway displacement in the direction of sway.



Slope deflection equations:

Member *AB*:

$$M_{AB} = \bar{M}_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_B - \frac{3\Delta}{4}\right)$$

$$M_{AB} = \frac{EI\theta_B}{2} - \frac{3EI\Delta}{8} \quad \text{...(i)}$$

and $\quad M_{BA} = \bar{M}_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$

$$= 0 + \frac{2EI}{4}\left(2\theta_B + 0 - \frac{3\Delta}{4}\right)$$

$$M_{BA} = EI\theta_B - \frac{3EI\Delta}{8} \quad \text{...(ii)}$$

Member *BC*:

Here, $\Delta_{BC} = 0$

$$M_{BC} = \overline{M}_{BC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

$$= -9 + \frac{2EI}{3}(2\theta_B + \theta_C - 0)$$

$$M_{BC} = -9 + \frac{4EI\theta_B}{3} + \frac{2EI\theta_C}{3} \quad \text{...(iii)}$$

and

$$M_{CB} = \overline{M}_{CB} + \frac{2EI}{L}\left(2\theta_C + \theta_B - \frac{3\Delta}{L}\right)$$

$$= +9 + \frac{2EI}{3}(2\theta_C + \theta_B - 0)$$

$$M_{CB} = 9 + \frac{4EI\theta_C}{3} + \frac{2EI\theta_B}{3} \quad \text{...(iv)}$$

Member *CD*:

Here, $\Delta_{CD} = -\Delta$

$$M_{CD} = \overline{M}_{CD} + \frac{2EI}{L}\left(2\theta_C + \theta_D + \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_C + 0 + \frac{3\Delta}{4}\right)$$

$$M_{CD} = EI\theta_C + \frac{3EI\Delta}{8} \quad \text{...(v)}$$

and

$$M_{DC} = \overline{M}_{DC} + \frac{2EI}{L}\left(2\theta_D + \theta_C + \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_C + \frac{3\Delta}{4}\right)$$

$$M_{DC} = \frac{EI\theta_C}{2} + \frac{3EI\Delta}{8} \quad \text{...(vi)}$$

Equilibrium equations:

Consider equilibrium of joint *B*.

$$M_{BA} + M_{BC} = 0$$

$$EI\theta_B - \frac{3EI\Delta}{8} - 9 + \frac{4EI\theta_B}{3} + \frac{2EI\theta_C}{3} = 0$$

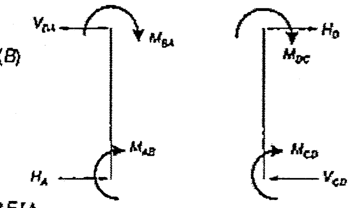
$$\frac{7}{3}EI\theta_B + \frac{2EI\theta_C}{3} - \frac{3EI\Delta}{8} = 9 \quad \text{...(A)}$$

Consider equilibrium of joint *C*.

$$M_{CB} + M_{CD} = 0$$

$$9 + \frac{4EI\theta_C}{3} + \frac{2EI\theta_B}{3} + EI\theta_C + \frac{3EI\Delta}{8} = 0$$

$$\frac{2EI\theta_B}{3} + \frac{7}{3}EI\theta_C + \frac{3}{8}EI\Delta = -9 \quad \text{...(B)}$$



Shear equation, $\Sigma F_x = 0$

$$H_A + H_D = 0 \quad (\rightarrow)$$

$$\left(\frac{M_{AB} + M_{BA}}{4}\right) - \left(\frac{M_{CD} + M_{DC}}{4}\right) = 0$$

$$\frac{EI\theta_B}{2} - \frac{3EI\Delta}{8} + EI\theta_B - \frac{3EI\Delta}{8} - EI\theta_C - \frac{3EI\Delta}{8} - \frac{EI\theta_C}{2} - \frac{3EI\Delta}{8} = 0$$

$$1.5\,EI\theta_B - 1.5\,EI\theta_C - 1.5\,EI\Delta = 0$$

$$EI\theta_B - EI\theta_C - EI\Delta = 0 \quad \text{...(C)}$$

Solving (*A*), (*B*) and (*C*), we get

$$EI\theta_B = 9.81$$

$$EI\theta_C = -9.81$$

$EI\Delta = 19.63$ (Assumed direction in correct)

Final end moments:

$$M_{AB} = \frac{EI\theta_B}{2} - \frac{3EI\Delta}{8}$$

$$= 0.5 \times 9.81 - 0.375 \times 19.63 = -2.46 \text{ kN-m}$$

$$M_{BA} = EI\theta_B - \frac{3EI\Delta}{8}$$

$$= 9.81 - 0.375 \times 19.63 = 2.46 \text{ kN-m}$$

$$M_{BC} = -9 + \frac{4}{3}EI\theta_B + \frac{2EI\theta_C}{3}$$

$$= -9 + \frac{4}{3}\times(9.81) - \frac{2}{3}(9.81) = 2.46 \text{ kN-m}$$

$$M_{CB} = 9 + \frac{4}{3}EI\theta_C + \frac{2}{3}EI\theta_B = 9 - \frac{4}{3}\times 9.81 + \frac{2}{3}\times 9.81 = 2.46 \text{ kNm}$$

$$M_{CD} = EI\theta_C + \frac{3EI\Delta}{8} = -9.81 + \frac{3}{8}\times(19.63) = -2.46$$

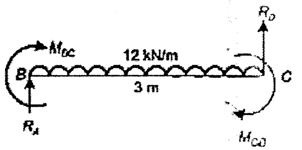
$$M_{DC} = \frac{EI\theta_C}{2} + \frac{3EI\Delta}{8} = 0.5\times(9.81) + \frac{3}{8}(19.63) = 2.46 \text{ kN-m}$$

Support reactions:

$$H_A = \left(\frac{M_{AB} + M_{BA}}{4}\right) = \frac{-2.46 + 2.46}{4} = 0$$

$$H_D = -\left(\frac{M_{CD} + M_{DC}}{4}\right) = \frac{-(-2.46 + 2.46)}{4} = 0$$

For vertical reactions,



$\Sigma M_C = 0$

$$R_A \times 3 + M_{BC} + M_{CB} - 12 \times 3 \times 1.5 = 0$$

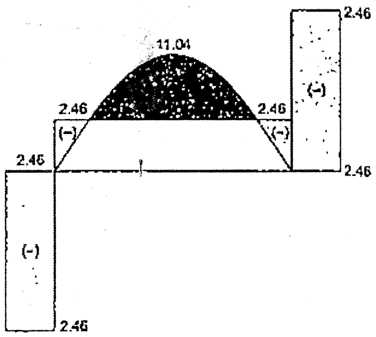
$$R_A \times 3 - 2.46 + 2.46 = 12 \times 3 \times 1.5$$

$$R_A = 18 \text{ kN } (\uparrow)$$

$$R_D = 12 \times 3 - R_A$$

$$= 36 - 18$$

$$R_D = 18 \text{ kN } (\uparrow)$$



BMD

9.7 Beam Sway (Joint Displacement)

Occurs when

(i) Properties of cross-section changed at a joint.

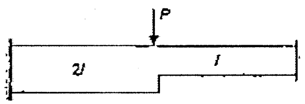


Fig. 9.16

(ii) When internal hinges are provided within the span

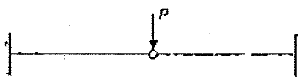


Fig. 9.17

Procedure

Consider a beam as shown in figure below.

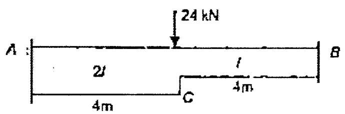


Fig. 9.18

Fixed end moments:

$$\overline{M}_{AC} = \overline{M}_{CA} = \overline{M}_{CB} = M_{BC} = 0$$

Step-1. Draw approximate deflected shape and apply slope deflection equations for all members.

Here,

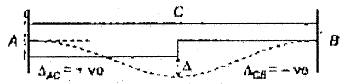


Fig. 9.19

$$\Delta_{AC} = +\text{ve}$$

Slope deflection equations:

Member *AC*:

Here, Δ = +ve

$$M_{AC} = \overline{M}_{AC} + \frac{2E(2I)}{L}\left(2\theta_A + \theta_C - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{4EI}{4}\left(0 + \theta_C - \frac{3\Delta}{4}\right)$$

$$M_{AC} = EI\theta_C - \frac{3}{4}EI\Delta \qquad ...(i)$$

and

$$M_{CA} = \overline{M}_{CA} + \frac{2E(2I)}{L}\left(2\theta_C + \theta_A - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{4EI}{4}\left(2\theta_C + 0 - \frac{3\Delta}{4}\right)$$

$$M_{CA} = 2EI\theta_C - \frac{3}{4}EI\Delta \qquad ...(ii)$$

Member *CB*:

Here, Δ = −ve

$$M_{CB} = \overline{M}_{BC} + \frac{2EI}{L}\left(2\theta_C + \theta_B + \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_C + 0 + \frac{3\Delta}{4}\right)$$

$$M_{CB} = EI\theta_C + \frac{3}{8}EI\Delta \qquad ...(iii)$$

and

$$M_{BC} = \overline{M}_{BC} + \frac{2EI}{L}\left(2\theta_B + \theta_C + \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_C + \frac{3\Delta}{L}\right)$$

$$M_{BC} = \frac{1}{2}EI\theta_C + \frac{3}{8}EI\Delta \qquad ...(iv)$$

Step-2. Apply joint equilibrium conditions

Here, consider equilibrium of join *C*,

$$M_{CA} + M_{CB} = 0$$

$\Rightarrow 2EI\theta_C - \frac{3}{4}EI\Delta + EI\theta_C + \frac{3}{8}EI\Delta = 0$

$\Rightarrow \quad 3EI\theta_C - \frac{3}{8}EI\Delta = 0$

$\Rightarrow \quad 8EI\theta_C - EI\Delta = 0 \qquad ...(A)$

Step-3. Write shear equation

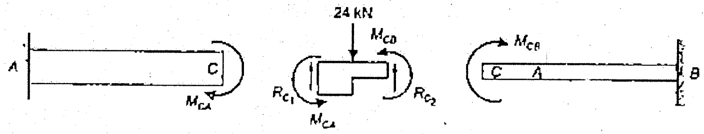


Fig. 9.20

Shear equation at joint C is,

$$\Sigma F_y = 0$$

$$R_{C_1} + R_{C_2} = 24 \text{ kN}$$

R_{C_1} and R_{C_2} can be found by free body equilibrium of member AC and BC as shown below.

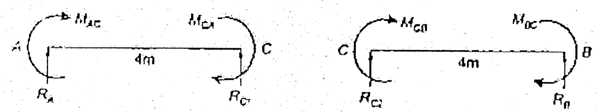


Fig. 9.21

Member AC,

$\Sigma M_A = 0$: $\quad M_{AC} + M_{CA} - R_{C_1} \times 4 = 0$

$$R_{C_1} = \frac{(M_{AC} + M_{CA})}{4}$$

Member CB,

$\Sigma M_B = 0$: $\quad M_{CB} + M_{BC} + R_{C_2} \times 4 = 0$

$$R_{C_2} = \frac{-(M_{CB} + M_{BC})}{4}$$

Substituting value of R_{C_1} and R_{C_2} in shear equation, we get

$$\frac{(M_{AC} + M_{CA})}{4} - \frac{(M_{CB} + M_{BC})}{4} = 24$$

$$M_{AC} + M_{CA} - M_{CB} - M_{BC} = 96$$

$$\therefore \quad EI\theta_C - \frac{3}{4}EI\Delta + 2EI\theta_C - \frac{3}{4}EI\Delta - \frac{1}{2}EI\theta_C - \frac{3}{8}EI\Delta - \frac{EI\theta_C}{2} - \frac{3}{8}EI\Delta = 96$$

$$\Rightarrow \quad -\frac{9}{2}EI\theta_c + \frac{3}{4}EI\Delta = 96 \qquad ...(B)$$

Solving equations (A) and (B), we get

$$EI\theta_C = -5.818$$
$$EI\Delta = -46.54$$

Step-4. Find final end moments

$$M_{AC} = EI\theta_C - \frac{3}{4}EI\Delta = 9 - \frac{4}{3}\times 9.81 + \frac{2}{3}\times 9.81 = 29.09 \text{ kNm}$$

$$M_{CA} = 2EI\theta_C - \frac{3}{4}EI\Delta = 2\times(-5.818) - \frac{3}{4}(-46.545) = 23.27 \text{ kNm}$$

$$M_{CB} = EI\theta_C + \frac{3}{8}EI\Delta = -5.818 + \frac{3}{8}(46.545) = 23.27 \text{ kN-m}$$

$$M_{BC} = \frac{1}{2}EI\theta_C + \frac{3}{8}EI\Delta = \frac{1}{2}\times(-5.818) + \frac{3}{8}\times(-46.545) = 20.36 \text{ kN-m}$$

Step-5. Draw resulting BM diagram

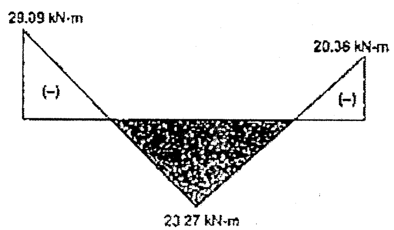
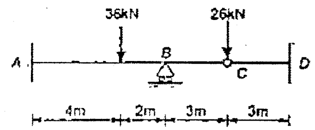


Fig. 9.22

Example 9.11 For the beam shown in figure draw BM diagram using slope deflection method.



Solution:

$$D_K = 3j - r_e + r_r - 2$$

Here, $j = 4$, $r_e = 7$ and $r_r = 1$

$\therefore \quad D_K = 3 \times 4 - 7 + 1 - 2$

$\quad\quad = 4$

Unknowns are, θ_B, θ_{C1} (just to left of C), θ_{C2} (just to right of C) and Δ_C

Fixed end moments:

$$\overline{M}_{AB} = -\frac{Pab^2}{L^2} = -\frac{36\times4\times2^2}{6^2} = -16 \text{ kN-m}$$

$$\overline{M}_{BA} = +\frac{Pa^2b}{L^2} = +\frac{36\times4^2\times2}{6^2} = +32 \text{ kN-m}$$

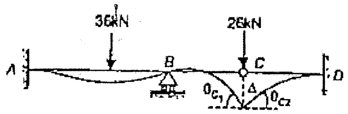
$$\overline{M}_{BC} = \overline{M}_{CB} = 0$$

$$\overline{M}_{CD} = \overline{M}_{DC} = 0$$

Slope deflection equations:

$\Delta_{BC} = +\text{ve } \Delta$

$\Delta_{CD} = -\text{ve } \Delta$



Member *AB*:

Here, $\Delta_{AB} = 0$

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right) = -16 + \frac{2EI}{6}(0 + \theta_B - 0)$$

$$M_{AB} = -16 + \frac{EI}{3}\theta_B \quad \text{...(i)}$$

and

$$M_{BA} = \overline{M}_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

$$= +32 + \frac{2EI}{6}(2\theta_B + 0 - 0)$$

$$M_{BA} = 32 + \frac{2EI\theta_B}{3} \quad \text{...(ii)}$$

Member *BC*:

Here, $\Delta_{BC} = \Delta$

$$M_{BC} = \overline{M}_{BC} + \frac{2EI}{L}\left(2\theta_B + \theta_{C_1} - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{3}\left(2\theta_B + \theta_{C_1} - \frac{3\Delta}{3}\right)$$

$$M_{BC} = \frac{4EI}{3}\theta_B + \frac{2EI\theta_{C_1}}{3} - \frac{2EI\Delta}{3} \quad \text{...(iii)}$$

and

$$M_{CB} = \overline{M}_{CB} + \frac{2EI}{L}\left(2\theta_{C_1} + \theta_B - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{3}\left(2\theta_{C_1} + \theta_B - \frac{3\Delta}{3}\right)$$

$$M_{CB} = \frac{4EI\theta_{C_1}}{3} + \frac{2EI\theta_B}{3} - \frac{2EI\Delta}{3} \quad \text{...(iv)}$$

Member *CD*:

$$M_{CD} = \overline{M}_{CD} + \frac{2EI}{L}\left(2\theta_{C_2} + \theta_D + \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{3}\left(2\theta_{C_2} + 0 + \frac{3\Delta}{3}\right)$$

$$M_{CD} = \frac{4EI\theta_{C_2}}{3} + \frac{2EI\Delta}{3} \quad \text{...(v)}$$

and

$$M_{DC} = \overline{M}_{DC} + \frac{2EI}{L}\left(2\theta_D + \theta_{C_2} + \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{3}\left(0 + \theta_{C_2} + \frac{3\Delta}{3}\right)$$

$$M_{DC} = \frac{2EI\theta_{C_2}}{3} + \frac{2EI\Delta}{3} \quad \text{...(vi)}$$

Equilibrium equations:

Consider equilibrium of joint *B*,

$$M_{BA} + M_{BC} = 0$$

$$32 + \frac{2EI\theta_B}{3} + \frac{4EI\theta_B}{3} + \frac{2EI\theta_{C_1}}{3} - \frac{2EI\Delta}{3} = 0$$

$$2EI\theta_B + \frac{2EI\theta_{C_1}}{3} - \frac{2EI\Delta}{3} = -32 \quad \text{...(A)}$$

Also at joint *C*,

$$M_{CB} = 0$$

$$\therefore \quad \frac{4EI\theta_{C_1}}{3} + \frac{2EI\theta_B}{3} - \frac{2EI\Delta}{3} = 0$$

$$\Rightarrow \quad \frac{2EI\theta_B}{3} + \frac{4EI\theta_{C_1}}{3} - \frac{2EI\Delta}{3} = 0 \quad \text{...(B)}$$

Also at joint *C*,

$$M_{CD} = 0$$

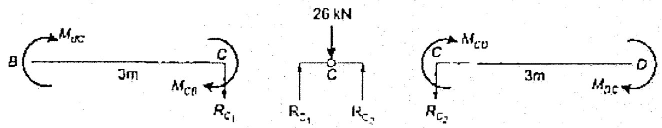
$$\therefore \quad \frac{4EI\theta_{C_2}}{3} + \frac{2EI\Delta}{3} = 0 \quad \text{...(C)}$$

Shear equation:

Shear equation at joint *C* is given by

$\Sigma F_y = 0$

$$\therefore \quad R_{C_1} + R_{C_2} = 26$$



$\Sigma M_B = 0$, for *BC* member

$$\Rightarrow \quad R_{C_1}\times3 + M_{CB} + M_{BC} = 0$$

$$\Rightarrow \quad R_{C_1}\times 3+0+M_{BC} = 0$$

$$\Rightarrow \quad R_{C_1} = -\frac{M_{BC}}{3}$$

and $\Sigma M_D = 0$, for member CD

$$-R_{C_2}\times 3+M_{CD}+M_{DC} = 0$$

$$-R_{C_2}\times 3+0+M_{DC} = 0$$

$$R_{C_2} = \frac{M_{DC}}{3}$$

Hence, shear equation is

$$-\frac{M_{BC}}{3}+\frac{M_{DC}}{3} = 26$$

$$\Rightarrow \quad -M_{BC}+M_{DC} = 78$$

$$\Rightarrow \quad -\frac{4EI\theta_B}{3}-\frac{2EI\theta_{C_1}}{3}+\frac{2EI\Delta}{3}+\frac{2EI\theta_{C_2}}{3}+\frac{2EI\Delta}{3} = 0$$

$$\Rightarrow \quad -\frac{4EI\theta_B}{3}-\frac{2EI\theta_{C_1}}{3}+\frac{2EI\theta_{C_2}}{3}+\frac{4EI\Delta}{3} = 0 \qquad ...(D)$$

On solving (*A*), (*B*), (*C*) and (*D*) we get

$$EI\theta_B = -16.69$$

$$EI\theta_{C_1} = 14.60$$

$$EI\theta_{C_2} = -25.04$$

$$EI\Delta = 12.52$$

Final end moments:

$$M_{AB} = -16+\frac{EI\theta_B}{3}$$

$$= -16+\frac{1}{3}\times(-16.69) = -21.56 \text{ kN-m}$$

$$M_{BA} = 32+\frac{2}{3}\times(-16.69)$$

$$M_{BA} = 20.57 \text{ kN-m}$$

$$M_{BC} = \frac{4EI\theta_B}{3}+\frac{2EI\theta_{C_1}}{3}-\frac{2EI\Delta}{3}$$

$$= \frac{4}{3}\times(-16.69)+\frac{2}{3}\times 14.60-\frac{2}{3}\times 12.52$$

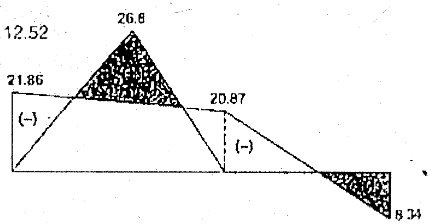
$$= 20.87 \text{ kN-m}$$

$$M_{CB} = 0$$

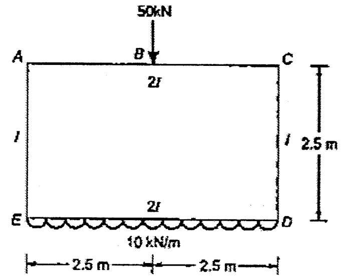
$$M_{CD} = 0$$

$$M_{DC} = \frac{2}{3}\times[(25.04)+12.52]$$

$$= -8.34 \text{ kN-m}$$



Example 9.12 Analyze the plane box frame shown in figure using the slope deflection method and making use of symmetry. Also, draw bending moment diagram.



Solution:

Slope deflection equations:

Due to symmetry,

$$\theta_C = -\theta_A$$

and

$$\theta_D = -\theta_E$$

Member AC:

$$M_{AC} = \bar{M}_{AC}+\frac{2E(2I)}{L}\left(2\theta_A+\theta_C-\frac{3\Delta}{L}\right)$$

$$= -31.25+\frac{4EI}{5}(2\theta_A-\theta_A-0) = -31.25+\frac{4}{5}EI\theta_A$$

$$M_{CA} = \bar{M}_{CA}+\frac{2E(EI)}{L}\left(2\theta_C+\theta_A-\frac{3\Delta}{L}\right)$$

$$= +31.25+\frac{4EI}{5}(-2\theta_A+\theta_A-0) = +31.25-\frac{4}{5}EI\theta_A$$

Member AE:

$$M_{AE} = \bar{M}_{AE}+\frac{2EI}{L}\left(2\theta_A+\theta_E-\frac{3\Delta}{L}\right)$$

$$= 0+\frac{2EI}{2.5}(2\theta_A+\theta_E) = \frac{8}{5}EI\theta_A+\frac{4}{5}EI\theta_E$$

$$M_{EA} = \bar{M}_{EA}+\frac{2EI}{L}\left(2\theta_E+\theta_A-\frac{3\Delta}{L}\right)$$

$$= 0+\frac{2EI}{2.5}(2\theta_E+\theta_A) = \frac{4}{5}EI\theta_A+\frac{8}{5}EI\theta_e$$

Member ED:

$$M_{ED} = \bar{M}_{ED}+\frac{2E(2I)}{L}\left(2\theta_E+\theta_D-\frac{3\Delta}{L}\right)$$

$$= +20.83+\frac{4EI}{8}(2\theta_E-\theta_E) = 20.83+\frac{4}{5}EI\theta_E$$

$$M_{DE} = \bar{M}_{DE}+\frac{2EI(2I)}{L}\left(2\theta_D+\theta_E-\frac{3\Delta}{L}\right)$$

$$= -20.83-\frac{4}{5}EI\theta_E$$

Equilibrium equations:

Consider equilibrium of joint A,

$$M_{AC} + M_{AE} = 0$$

$$\left(-31.25 + \frac{4}{5}EI\theta_A\right) + \left(\frac{8}{5}EI\theta_A + \frac{4}{5}EI\theta_E\right) = 0$$

$$\frac{12}{5}EI\theta_A + \frac{4}{5}EI\theta_E = 31.25 \quad \text{...(i)}$$

Consider equilibrium of joint E,

$$M_{EA} + M_{ED} = 0$$

$$\frac{4}{5}EI\theta_A + \frac{8}{5}EI\theta_E + 20.83 + \frac{4}{5}EI\theta_E = 0$$

$$\frac{4}{5}EI\theta_A + \frac{12}{5}EI\theta_E = -20.83 \quad \text{...(ii)}$$

On solving eq. (i) and (ii), we get

$$EI\theta_A = 17.903$$
$$EI\theta_E = -14.646$$

Final end moments:

$$M_{AC} = -31.25 + \frac{4}{5}\times(17.903) = -16.927 \text{ kNm}$$

$$M_{CA} = +31.25 - \frac{4}{5}\times(17.903) = +16.927 \text{ kNm}$$

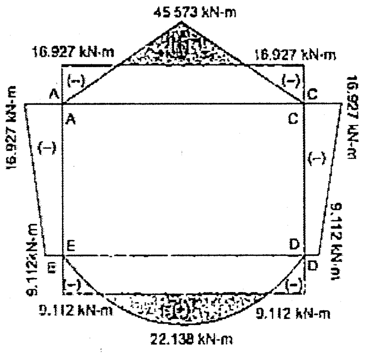
$$M_{AE} = \frac{8}{5}\times(17.903) + \frac{4}{5}\times(-14.646) = 16.927 \text{ kNm}$$

$$M_{EA} = \frac{4}{5}\times(17.903) + \frac{8}{5}\times(-14.646) = -9.112 \text{ kNm}$$

$$M_{ED} = 20.83 + \frac{4}{5}\times(-14.646) = +9.112 \text{ kNm}$$

$$M_{DE} = -20.83 - \frac{4}{5}\times(-14.646) = -9.112 \text{ kNm}$$

Bending moment diagram



BMD

Summary

- Slope deflection method is a displacement method. In this method unknowns are taken as joint displacements.
- Slope deflection method is used when $D_K < D_S$.
- The slope deflection equation for a member AB is given by

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

and

$$M_{BA} = \overline{M}_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

where, $\overline{M}_{AB}$ and $\overline{M}_{BA}$ are fixing moments at end A and B respectively.

- In slope deflection method all joints are considered rigid i.e. angles between member at the joints do not changes with the application of load.
- In slope deflection method, to find unknown displacement (θ and Δ) joint equilibrium equations and shear equations are written.

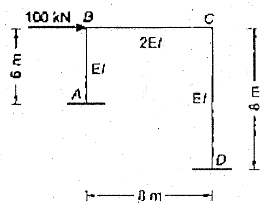
Objective Brain Teasers

Q.1 The moment required to rotate the near end of a prismatic beam through unit angle without translation, when the far end is fixed, is

(a) $\frac{EI}{L}$ (b) $\frac{2EI}{L}$

(c) $\frac{3EI}{L}$ (d) $\frac{4EI}{L}$

Q.2 The slope-deflection equation at the end B of member BC for the frame shown in the given figure will be



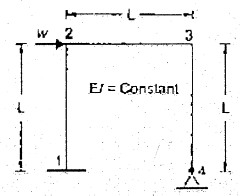
(a) $M_{BC} = \frac{4EI}{8}(2\theta_C - \theta_B)$

(b) $M_{BC} = \frac{4EI}{8}(2\theta_B - \theta_C)$

(c) $M_{BC} = \frac{4EI}{8}(2\theta_B + \theta_C)$

(d) $M_{BC} = \frac{4EI}{8}(2\theta_C + \theta_B)$

Q.3 The slope deflection equation at end 2 of the member 1-2 for the frame shown in the figure is given by



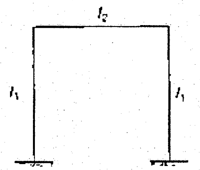
(a) $M_{21} = \frac{2EI}{L}(2\theta_1 + 2\theta_2) - WL$

(b) $M_{21} = \frac{2EI}{L}\left(2\theta_1 - \frac{3\delta}{L}\right)$

(c) $M_{21} = \frac{2EI}{L}\left(2\theta_2 - \frac{3\delta}{L}\right)$

(d) $M_{21} = \frac{2EI}{L}\left(\theta_1 + 2\theta_2 - \frac{3\delta}{L}\right) + WL$

Q.4 The rigid portal frame shown in the given figure will not have any side sway if



I_1 = the moment of inertia of the column cross-section

I_2 = the moment of inertia of the beam cross-section

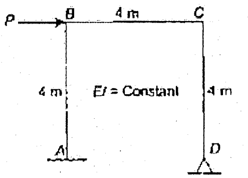
(a) it is subjected to vertical loading only

(b) $I_2 = 2I_1$

(c) the loading is symmetrical about its centre line

(d) loaded in any manner

Q.5 For the portal frame shown in the given figure, the shear equation is



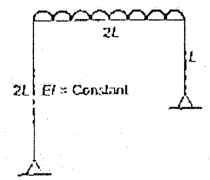
(a) $\frac{M_{BC}+M_{CB}}{4}+P=0$

(b) $\frac{M_{BA}+M_{AB}}{4}+P=0$

(c) $\frac{M_{BA}+M_{AB}}{4}+\frac{M_{CD}}{4}+P=0$

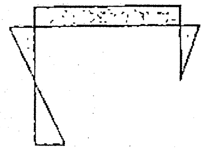
(d) $\frac{M_{CD}}{4}+P=0$

Q.6 The given figure shows a portal frame with loads.

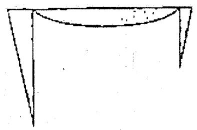


The bending moment diagram for this frame will be

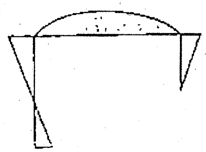
(a)



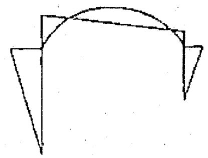
(b)



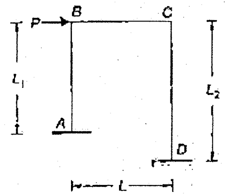
(c)



(d)



Q.7 The shear equation for the portal frame shown in the figure below will be



(a) $\left(\frac{M_{AB}+M_{BA}}{L_1}\right)+\left(\frac{M_{CD}+M_{DC}}{L_2}\right)+P=0$

(b) $\left(\frac{M_{AB}+M_{BA}}{L_1}\right)+\left(\frac{M_{BC}+M_{CB}}{L}\right)+P=0$

(c) $\left(\frac{M_{BC}+M_{CB}}{L}\right)+\left(\frac{M_{CD}+M_{DC}}{L_2}\right)+P=0$

(d) $\left(\frac{M_{AB}+M_{CD}}{L_1}\right)+\left(\frac{M_{BA}+M_{DC}}{L}\right)-P=0$

Answers

1. (a) 2. (c) 3. (c) 4. (c) 5. (c) 6. (d) 7. (a)

Hints and Explanations:

2. (c)

$$M_{BC}=\frac{2(2EI)}{8}(2\theta_B+\theta_C)=\frac{4EI}{8}(2\theta_B+\theta_C)$$

3. (c)

$$\theta_1=0.\ \text{So}\ M_{p_1}=\frac{2EI}{L}\left(2\theta_2-\frac{3\delta}{L}\right)$$

4. (c)

The frame is symmetrical in shape and size about a central vertical line. Therefore symmetrical loading about the centre line will not cause any side sway.

5. (c)

Shear in column $AB = \frac{M_{AB}+M_{BA}}{4}$

Shear in column $CD = \frac{M_{CD}}{4}$

Shear equation will be

$$\frac{M_{AB}+M_{BA}}{4}+\frac{M_{CD}}{4}+P=0$$

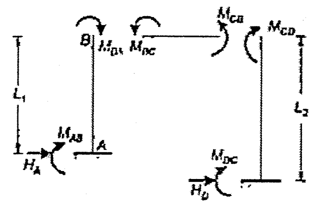
6. (d)

Due to unsymmetrical nature of frame, the lateral translation will occur towards left. Therefore horizontal force of equal and opposite nature will develop on the supports. So correct bending moment diagram will be (d).

7. (a)

Let horizontal reactions at A and D be $H_A(\rightarrow)$ and $H_D(\rightarrow)$ respectively.

Taking moment about B for column AB.



$$\therefore\quad H_A\times L_1 = M_{AB}+M_{BA}$$

$$\Rightarrow\quad H_A=\frac{M_{AB}+M_{BA}}{L_1}$$

Similarly, $H_D=\frac{M_{CD}+M_{DC}}{L_2}$

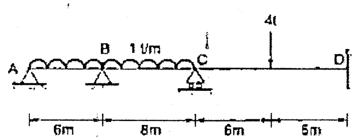
For horizontal equilibrium of the structure,

$$H_A+H_D+P=0$$

$$\Rightarrow\left(\frac{M_{CD}+M_{DC}}{L_2}\right)+\left(\frac{M_{AB}+M_{BA}}{L_1}\right)+P=0$$

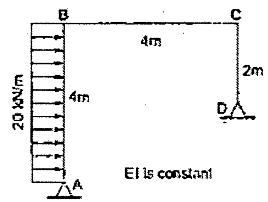
Conventional Practice Questions

Q.1 By using slope deflection method analyse the beam shown in figure and draw the B.M. diagram.



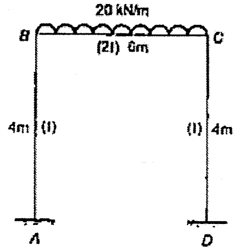
Ans. $M_{AB}=0$, $M_{BA}=-4.839$ kN-m, $M_{CB}=-5.8$ kN-m, $M_{DC}=-6.1$ kN-m

Q.2 Analyse the portal frame shown in figure.



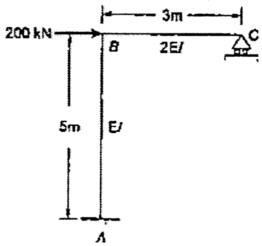
Ans. $M_{AB}=0$, $M_{BA}=-20.8$ kN-m, $M_{BC}=+20.8$ kNm, $M_{CB}=-69.6$ kN-m, $M_{DC}=0$, $H_A=-45.2$ kN, $H_D=-34.8$ kN, $R_A=-22.6$ kN, $R_D=22.6$ kN

Q.3 Using slope deflection method analyse the frame shown in figure. Also draw the BM diagram.



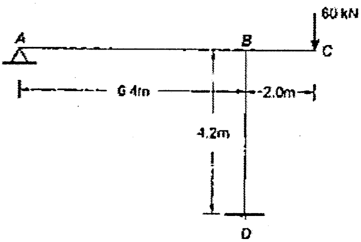
Ans. $M_{AB} = 18$ kN-m, $M_{BA} = 36$ kN-m
$M_{BC} = -36$ kN-m, $M_{CB} = 36$ kN-m
$M_{CD} = -36$ kN-m, $M_{DC} = 18$ kN-m,

Q.4 Determine the movement of the roller *C* of the frame shown in figure by slope deflection method.



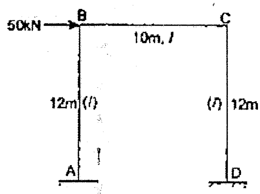
Ans. $\Delta = \dfrac{262.25}{EI}$

Q.5 For the given frame as shown in figure, calculate the slope and deflection at point. Sketch deflected shape indicating the points of contraflexure if any



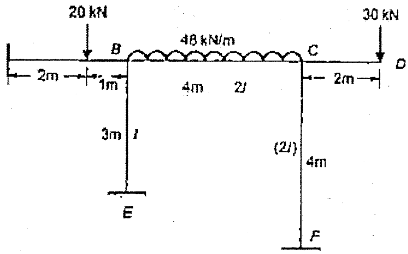
Ans. $\theta_C = 0.02129$ radian,
deflection at $C = 34.258$ mm (↓)

Q.6 Calculate the joint movement for the frame shown in figure. Also find end moments for all members.



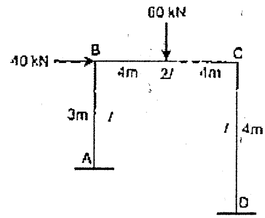
Ans. $M_{AB} = -16.829$, $M_{BA} = -13.1707$
$M_{BC} = 13.1706$, $M_{CB} = 13.1706$
$M_{CD} = -13.1706$, $M_{DC} = -16.829$

Q.7 Analyse the frame shown in figure by slope deflection method and draw BM diagram.



Ans. $M_{AB} = 4.02$ kN-m, $M_{BA} = 25.82$ kN-m
$M_{BC} = -42.77$ kN-m, $M_{CB} = -68.35$ kN-m,
$M_{BE} = 16.93$ kN-m, $M_{EB}' = 8.46$ kN-m
$M_{CF} = -8.35$ kN-m, $M_{FC} = -4.176$ kN-m

Q.8 Analyse the portal frame shown in figure by slope deflection method.



Ans. $M_{AB} = -14.286$ kN-m, $M_{BA} = 14.286$ kN-m
$M_{BC} = -14.286$ kN-m, $M_{CB} = 65.715$ kN-m
$M_{CD} = -65.175$ kN-m, $M_{DC} = -54.28$ kN-m

■■■■