2.1 Functions

If A and B are two non-empty sets, then a rule f which associated to each $x \in A$, a unique number $y \in B$, is called a function from A to B and we write, $f : A \to B$.

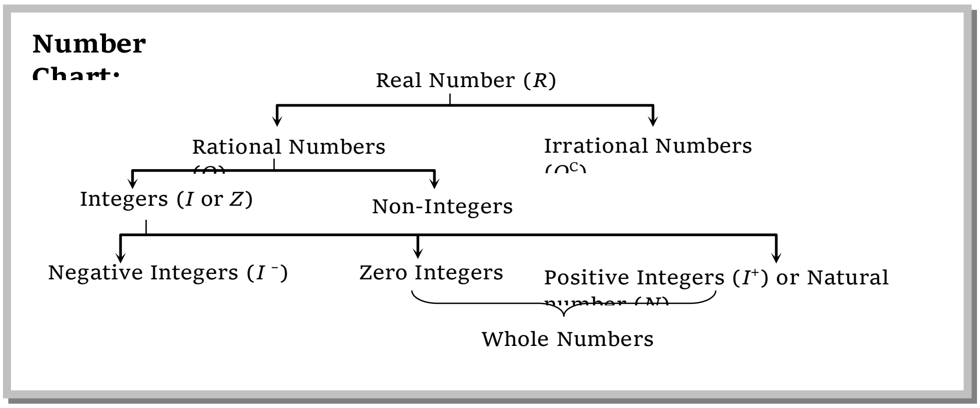
2.1.1 Some Important Definitions

(1) **Real numbers :** Real numbers are those which are either rational or irrational. The set of real numbers is denoted by R.

(i) **Rational numbers :** All numbers of the form p/q where p and q are integers and $q \neq 0$, are called rational numbers and their set is denoted by Q. *e.g.* $\frac{2}{3}, -\frac{5}{2}, 4\left(\text{as} \quad 4 = \frac{4}{1}\right)$ are rational numbers.

(ii) **Irrational numbers :** Those are numbers which can not be expressed in form of p/q are called irrational numbers and their set is denoted by Q^c (*i.e.*, complementary set of Q) *e.g.* $\sqrt{2}, \quad 1-\sqrt{3}, \pi$ are irrational numbers.

(iii) **Integers :** The numbers ....– 3, – 2, – 1, 0, 1, 2, 3, ... are called integers. The set of integers is denoted by I or Z. Thus, I or Z ={.......,– 3, – 2, –1, 0, 1, 2, 3,......}



Note :
- ❐ Set of positive integers I^+ = {1, 2, 3, ...}
- ❐ Set of negative integers I^- = {– 1, – 2, – 3, ...}.
- ❐ Set of non negative integers ={0, 1, 2, 3, ..}
- ❐ Set of non positive integers = {0, – 1, – 2, – 3,.....}
- ❐ Positive real numbers: $R^+ = (0, \infty)$
- ❐ Negative real numbers: $R^- = (-\infty, 0)$
- ❐ R_0 : all real numbers except 0 (Zero)
- ❐ Imaginary numbers: $C = \{i, \omega, ...\}$

- ❒ Even numbers: $E = \{0, 2, 4, 6, ...\}$ ❒ Odd numbers: $0 = \{1, 3, 5, 7, ...\}$
- ❒ Prime numbers : The natural numbers greater than 1 which is divisible by 1 and itself only, called prime numbers.
- ❒ In rational numbers the digits are repeated after decimal
- ❒ 0 (zero) is a rational number
- ❒ In irrational numbers, digits are not repeated after decimal
- ❒ π and e are called special irrational quantities
- ❒ ∞ is neither a rational number nor an irrational number

(2) **Related quantities :** When two quantities are such that the change in one is accompanied by the change in other, *i.e.,* if the value of one quantity depends upon the other, then they are called related quantities. *e.g.* the area of a circle $(A = \pi r^2)$ depends upon its radius (r) as soon as the radius of the circle increases (or decreases), its area also increases (or decreases). In the given example, A and r are related quantities.

(3) **Variable:** A variable is a symbol which can assume any value out of a given set of values. The quantities, like height, weight, time, temperature, profit, sales etc, are examples of variables. The variables are usually denoted by x, y, z, u, v, w, t etc. There are two types of variables mainly:

(i) **Independent variable :** A variable which can take any arbitrary value, is called independent variable.

(ii) **Dependent variable :** A variable whose value depends upon the independent variable is called dependent variable. *e.g.* $y = x^2$, if $x = 2$ then $y = 4 \Rightarrow$ so value of y depends on x. y is dependent and x is independent variable here.

(4) **Constant :** A constant is a symbol which does not change its value, *i.e.*, retains the same value throughout a set of mathematical operation. These are generally denoted by a, b, c etc. There are two types of constant.

(i) **Absolute constant :** A constant which remains the same throughout a set of mathematical operation is known as absolute constant. All numerical numbers are absolute constants, *i.e.* $2, \sqrt{3}, \pi$ etc. are absolute constants.

(ii) **Arbitrary constant :** A constant which remains same in a particular operation, but changes with the change of reference, is called arbitrary constant *e.g.* $y = mx + c$ represents a line. Here m and c are constants, but they are different for different lines. Therefore, m and c are arbitrary constants.

(5) **Absolute value :** The absolute value of a number x, denoted by $|x|$, is a number that satisfies the conditions

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$$. We also define $|x|$ as follows, $|x|$ = maximum $\{x, -x\}$ or $|x| = \sqrt{x^2}$

The properties of absolute value are

(i) The inequality $|x| \le a$ means $-a \le x \le a$

(ii) The inequality $|x| \ge a$ means $x \ge a$ or $x \le -a$

(iii) $|x \pm y| \le |x| + |y|$ and $|x \pm y| \ge |x| - |y|$

(iv) $|xy| = |x||y|$

(v) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, y \ne 0$

(6) **Greatest integer:** Let $x \in R$. Then $[x]$ denotes the greatest integer less than or equal to x; *e.g.* [1.34]=1, [– 4.57]= – 5, [0.69] = 0 etc.

(7) **Fractional part :** We know that $x \ge [x]$. The difference between the number 'x' and it's integral value '$[x]$' is called the fractional part of x and is symbolically denoted as $\{x\}$. Thus, $\{x\} = x - [x]$

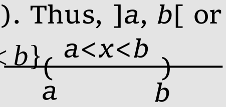
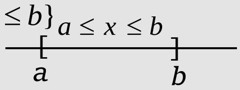
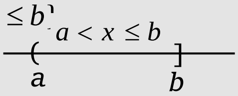
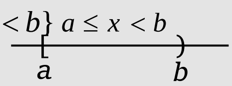
e.g., if x = 4.92 then $[x]$ = 4 and $\{x\}$= 0.92.

Note : ❒ Fractional part of any number is always non-negative and less than one.

2.1.2 Intervals

If a variable x assumes any real value between two given numbers, say a and b ($a<b$) as its value, then x is called a continuous variable. The set of real numbers which lie between two specific numbers, is called the interval.

There are four types of interval:

(1) **Open interval :** Let a and b be two real numbers such that $a<b$, then the set of all real numbers lying strictly between a and b is called an open interval and is denoted by $]a, b[$ or (a, b). Thus, $]a, b[$ or (a, b) = $\{x \in R: a < x < b\}$  Open	(2) **Closed interval :** Let a and b be two real numbers such that $a<b$, then the set of all real numbers lying between a and b including a and b is called a closed interval and is denoted by $[a, b]$. Thus, $[a, b]$ = $\{x \in R: a \le x \le b\}$  Closed
(3) **Open-Closed interval :** It is denoted by $]a, b]$ or $(a, b]$ and $]a, b]$ or $(a, b]$ = $\{x \in R: a < x \le b\}$  Open closed	(4) **Closed-Open interval :** It is denoted by $[a, b[$ or $[a, b)$ and $[a, b[$ or $[a, b)$ = $\{x \in R: a \le x < b\}$  Closed open

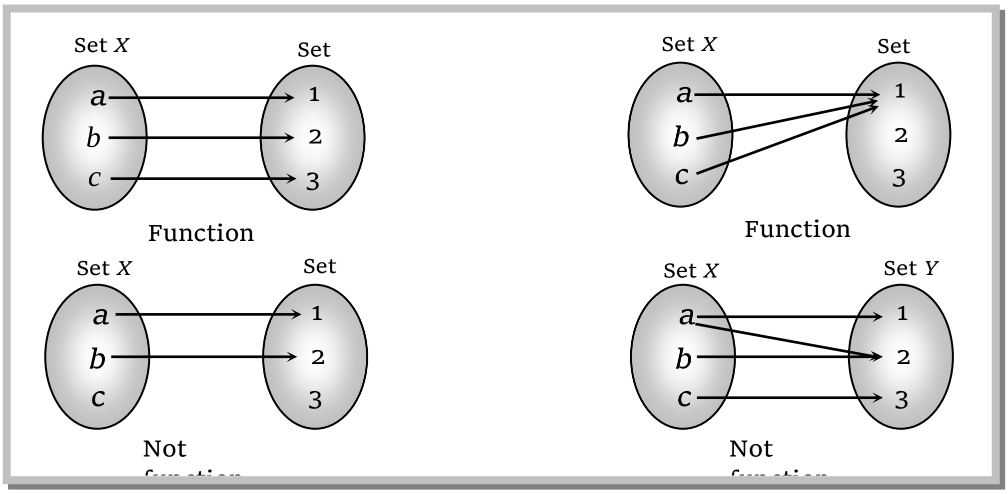
2.1.3 Definition of Function

(1) Function can be easily defined with the help of the concept of mapping. Let X and Y be any two non-empty sets. "A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y". Let the correspondence be 'f' then mathematically we write $f : X \to Y$ where $y = f(x), x \in X$ and $y \in Y$. We say that 'y' is the image of 'x' under f (or x is the pre image of y).

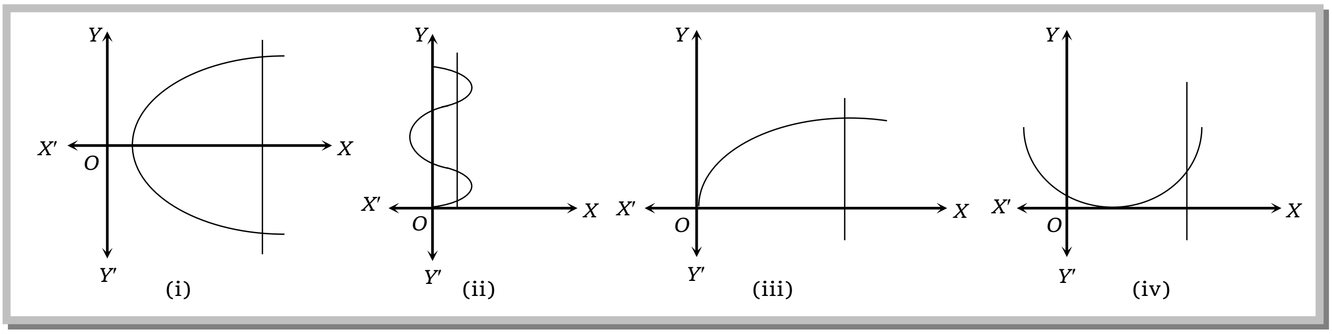
Two things should always be kept in mind:

(i) A mapping $f : X \to Y$ is said to be a function if each element in the set X has it's image in set Y. It is also possible that there are few elements in set Y which are not the images of any element in set X.

(ii) Every element in set X should have one and only one image. That means it is impossible to have more than one image for a specific element in set X. Functions can not be multi-valued (A mapping that is multi-valued is called a relation from X and Y) *e.g.*



(2) **Testing for a function by vertical line test :** A relation $f : A \to B$ is a function or not it can be checked by a graph of the relation. If it is possible to draw a vertical line which cuts the given curve at more than one point then the given relation is not a function and when this vertical line means line parallel to Y-axis cuts the curve at only one point then it is a function. Figure (iii) and (iv) represents a function.



(3) **Number of functions :** Let X and Y be two finite sets having m and n elements respectively. Then each element of set X can be associated to any one of n elements of set Y. So, total number of functions from set X to set Y is n^m.

(4) **Value of the function :** If $y = f(x)$ is a function then to find its values at some value of x, say $x = a$, we directly substitute $x = a$ in its given rule $f(x)$ and it is denoted by $f(a)$.

e.g. If $f(x) = x^2 + 1$, then $f(1) = 1^2 + 1 = 2$, $f(2) = 2^2 + 1 = 5$, $f(0) = 0^2 + 1 = 1$ *etc.*

Example: 1 If A contains 10 elements then total number of functions defined from A to A is

(a) 10 (b) 2^{10} (c) 10^{10} (d) $2^{10} - 1$

Solution: (c) According to formula, total number of functions $= n^n$

Here, n = 10. So, total number of functions = 10^{10}.

Example: 2 If $f(x) = \dfrac{x - |x|}{|x|}$, then $f(-1) =$ **[SCRA 1996]**

(a) 1 (b) – 2 (c) 0 (d) 2

Solution: (b) $f(-1) = \dfrac{-1 - |-1|}{|-1|} = \dfrac{-1 - 1}{1} = -2$.

Example: 3 If $f(y) = \log y$, then $f(y) + f\left(\dfrac{1}{y}\right)$ is equal to **[Rajasthan PET 1996]**

(a) 2 (b) 1 (c) 0 (d) – 1

Solution: (c) Given $f(y) = \log y \Rightarrow f(1/y) = \log(1/y)$, then $f(y) + f\left(\dfrac{1}{y}\right) = \log y + \log(1/y) = \log 1 = 0$.

Example: 4 If $f(x) = \log\left[\dfrac{1+x}{1-x}\right]$, then $f\left[\dfrac{2x}{1+x^2}\right]$ is equal to **[MP PET 1999; Rajasthan PET 1999; UPSEAT 2003]**

(a) $[f(x)]^2$ (b) $[f(x)]^3$ (c) $2f(x)$ (d) $3f(x)$

Solution: (c) $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$

$$\therefore f\left(\frac{2x}{1+x^2}\right) = \log\left[\frac{1 + \frac{2x}{1+x^2}}{1 - \frac{2x}{1+x^2}}\right] = \log\left[\frac{x^2+1+2x}{x^2+1-2x}\right] = \log\left[\frac{1+x}{1-x}\right]^2 = 2\log\left[\frac{1+x}{1-x}\right] = 2f(x)$$

Example: 5 If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, then **[Orissa JEE 2002]**

(a) $f\left(\dfrac{\pi}{4}\right) = 2$ (b) $f(-\pi) = 2$ (c) $f(\pi) = 1$ (d) $f\left(\dfrac{\pi}{2}\right) = -1$

Solution: (d) $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$

$$f(x) = \cos(9x) + \cos(-10x) = \cos(9x) + \cos(10x) = 2\cos\left(\frac{19x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$f\left(\frac{\pi}{2}\right) = 2\cos\left(\frac{19\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)$; $f\left(\frac{\pi}{2}\right) = 2 \times \frac{-1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = -1$.

Example: 6 If $f: R \to R$ satisfies $f(x+y) = f(x) + f(y)$, for all $x, y \in R$ and $f(1) = 7$, then $\sum_{r=1}^{n} f(r)$ is

(a) $\frac{7n}{2}$ (b) $\frac{7(n+1)}{2}$ (c) $7n(n+1)$ (d) $\frac{7n(n+1)}{2}$

Solution: (d) $f(x+y) = f(x) + f(y)$

put $x = 1, y = 0 \Rightarrow f(1) = f(1) + f(0) = 7$

put $x = 1, y = 1 \Rightarrow f(2) = 2.f(1) = 2.7$; similarly $f(3) = 3.7$ and so on

$\therefore \sum_{r=1}^{n} f(r) = 7(1 + 2 + 3 + ... + n) = \frac{7n(n+1)}{2}$.

Example: 7 If $f(x) = \frac{1}{\sqrt{x + 2\sqrt{2x-4}}} + \frac{1}{\sqrt{x - 2\sqrt{2x-4}}}$ for $x > 2$, then $f(11) =$ **[EAMCET 2003]**

(a) $\frac{7}{6}$ (b) $\frac{5}{6}$ (c) $\frac{6}{7}$ (d) $\frac{5}{7}$

Solution: (c) $f(x) = \frac{1}{\sqrt{x + 2\sqrt{2x-4}}} + \frac{1}{\sqrt{x - 2\sqrt{2x-4}}}$

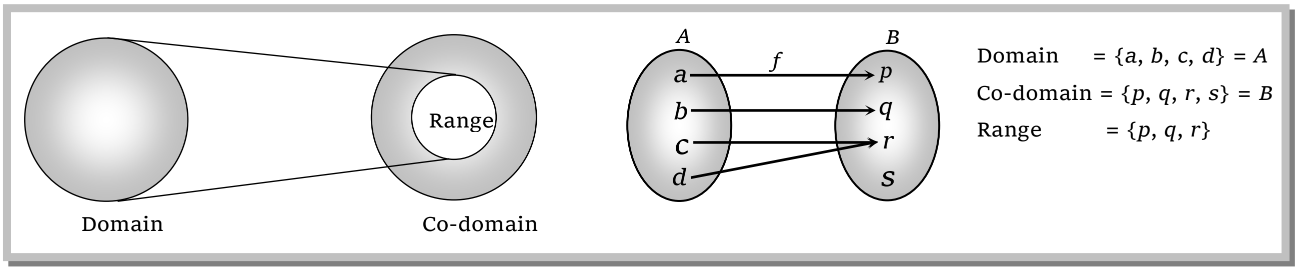
$f(11) = \frac{1}{\sqrt{11 + 2\sqrt{18}}} + \frac{1}{\sqrt{11 - 2\sqrt{18}}} = \frac{1}{3+\sqrt{2}} + \frac{1}{3-\sqrt{2}} = \frac{3-\sqrt{2}}{7} + \frac{3+\sqrt{2}}{7} = \frac{6}{7}$.

2.1.4 Domain, Co-domain and Range of Function

If a function *f* is defined from a set of *A* to set *B* then for $f: A \to B$ set *A* is called the domain of function *f* and set *B* is called the co-domain of function *f*. The set of all *f*-images of the elements of *A* is called the range of function *f*.

In other words, we can say Domain = All possible values of x for which $f(x)$ exists.

Range = For all values of x, all possible values of $f(x)$.



(1) **Methods for finding domain and range of function**

(i) **Domain**

(a) Expression under even root (*i.e.*, square root, fourth root etc.) ≥ 0

(b) Denominator $\neq 0$.

(c) If domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively then the domain of $f(x) \pm g(x)$ or $f(x).g(x)$ is $D_1 \cap D_2$.

(d) While domain of $\frac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{g(x) = 0\}$.

(e) Domain of $\left(\sqrt{f(x)}\right) = D_1 \cap \{x : f(x) \geq 0\}$

(ii) **Range :** Range of $y = f(x)$ is collection of all outputs $f(x)$ corresponding to each real number in the domain.

(a) If domain $\in$ finite number of points $\Rightarrow$ range $\in$ set of corresponding $f(x)$ values.

(b) If domain $\in R$ or R – [some finite points]. Then express x in terms of y. From this find y for x to be defined (*i.e.*, find the values of y for which x exists).

(c) If domain $\in$ a finite interval, find the least and greatest value for range using monotonicity.

Important Tips

☞ *If f(x) is a given function of x and if a is in its domain of definition, then by f(a) it means the number obtained by replacing x by a in f(x) or the value assumed by f(x) when x = a.*

☞ *Range is always a subset of co-domain.*

Example: 8 Domain of the function $\frac{1}{\sqrt{x^2 - 1}}$ is **[Roorkee 1987; Rajasthan PET 2000]**

(a) $(-\infty, -1) \cup (1, \infty)$ (b) $(-\infty, -1] \cup (1, \infty)$ (c) $(-\infty, -1) \cup [1, \infty)$ (d) None of these

Solution: (a) For domain, $x^2 - 1 > 0 \Rightarrow (x-1)(x+1) > 0$

$\Rightarrow x < -1$ or $x > 1 \Rightarrow x \in (-\infty, -1) \cup (1, \infty)$.

Example: 9 The domain of the function $f(x) = \frac{1}{\sqrt{|x| - x}}$ is **[Roorkee 1998]**

(a) R^+ (b) R^- (c) R_0 (d) R

Solution: (b) For domain, $|x| - x > 0 \Rightarrow |x| > x$. This is possible, only when $x \in R^-$.

Example: 10 Find the domain of definition of $f(x) = \frac{\log_2 (x+3)}{x^2 + 3x + 2}$ **[IIT 2001; UPSEAT 2001]**

(a) $(-3, \infty)$ (b) $\{-1, -2\}$ (c) $(-3, \infty) - \{-1, -2\}$ (d) $(-\infty, \infty)$

Solution: (c) Here $f(x) = \frac{\log_2 (x+3)}{x^2 + 3x + 2} = \frac{\log_2 (x+3)}{(x+1)(x+2)}$ exists if,

Numerator $x + 3 > 0 \Rightarrow x > -3$ ...... (i)

and denominator $(x+1)(x+2) \neq 0 \Rightarrow x \neq -1, -2$ ...... (ii)

Thus, from (i) and (ii); we have domain of $f(x)$ is $(-3, \infty) - \{-1, -2\}$.

Example: 11 The domain of the function $f(x)=\sqrt{(2-2x-x^2)}$ is **[BIT Ranchi 1992]**

(a) $-3\le x\le\sqrt{3}$ (b) $-1-\sqrt{3}\le x\le -1+\sqrt{3}$

(c) $-2\le x\le 2$ (d) None of these

Solution: (b) The quantity square root is positive, when $-1-\sqrt{3}\le x\le -1+\sqrt{3}$.

Example: 12 If the domain of function $f(x)=x^2-6x+7$ is $(-\infty,\infty)$, then the range of function is

(a) $(-\infty,\infty)$ (b) $[-2,\infty)$ (c) $(-2,3)$ (d) $(-\infty,-2)$

Solution: (b) $x^2-6x+7=(x-3)^2-2$ Obviously, minimum value is –2 and maximum ∞.

Example: 13 The domain of the function $f(x)=\sqrt{x-x^2}+\sqrt{4+x}+\sqrt{4-x}$ is **[AMU 1999]**

(a) $[-4,\infty)$ (b) $[-4,4]$ (c) $[0,4]$ (d) $[0,1]$

Solution: (d) $f(x)=\sqrt{x-x^2}+\sqrt{4+x}+\sqrt{4-x}$

clearly $f(x)$ is defined if

$4+x\ge 0 \Rightarrow x\ge -4$

$4-x\ge 0 \Rightarrow x\le 4$

$x(1-x)\ge 0 \Rightarrow x\ge 0$ and $x\le 1$

$\therefore$ Domain of $f=(-\infty,4]\cap[-4,\infty)\cap[0,1]=[0,1]$.

Example: 14 The domain of the function $\sqrt{\log(x^2-6x+6)}$ is **[Roorkee 1999; MP PET 2002]**

(a) $(-\infty,\infty)$ (b) $(-\infty,3-\sqrt{3})\cup(3+\sqrt{3},\infty)$

(c) $(-\infty,1]\cup[5,\infty)$ (d) $[0,\infty)$

Solution: (c) The function $f(x)=\sqrt{\log(x^2-6x+6)}$ is defined when $\log(x^2-6x+6)\ge 0$

$\Rightarrow x^2-6x+6\ge 1 \Rightarrow (x-5)(x-1)\ge 0$

This inequality hold if $x\le 1$ or $x\ge 5$. Hence, the domain of the function will be $(-\infty,1]\cup[5,\infty)$.

Example: 15 The domain of definition of the function $y(x)$ given by $2^x+2^y=2$ is **[IIT Screening 2000; DCE 2001]**

(a) (0, 1] (b) [0, 1] (c) $(-\infty,0]$ (d) $(-\infty,1)$

Solution: (d) $2^y=2-2^x$

y is real if $2-2^x\ge 0 \Rightarrow 2>2^x \Rightarrow 1>x$

$\Rightarrow x\in(-\infty,1)$

Example: 16 The domain of the function $f(x)=\sin^{-1}[\log_2(x/2)]$ is
[AIEEE 2002; Rajasthan PET 2002]

(a) [1, 4] (b) [– 4, 1] (c) [– 1, 4] (d) None of these

Solution: (a) $f(x)=\sin^{-1}[\log_2(x/2)]$

Domain of $\sin^{-1}x$ is $x\in[-1,1]$

$\Rightarrow -1\le\log_2(x/2)\le 1 \Rightarrow \frac{1}{2}\le\frac{x}{2}\le 2 \Rightarrow 1\le x\le 4$

$\therefore x\in[1,4]$.

Example: 17 The domain of the derivative of the function $f(x)=\begin{cases}\tan^{-1}x & , |x|\le 1\\ \frac{1}{2}(|x|-1) & , |x|>1\end{cases}$ is **[IIT Screening 2002]**

(a) $R-\{0\}$ (b) $R-\{1\}$ (c) $R-\{-1\}$ (d) $R-\{-1,1\}$

Solution: (c) $f(x)=\begin{cases}\frac{1}{2}(-x-1), x<-1\\ \tan^{-1}x, -1\le x\le 1\\ \frac{1}{2}(x+1), x>1\end{cases} \Rightarrow \quad f'(x)=\begin{cases}-\frac{1}{2}, & x<-1\\ \frac{1}{1+x^2}, & -1<x<1\\ \frac{1}{2}, & x>1\end{cases}$

$f'(-1-0)=-\frac{1}{2}$; $f'(-1+0)=\frac{1}{1+(-1+0)^2}=\frac{1}{2}$

$f'(1-0)=\frac{1}{1+(1-0)^2}=\frac{1}{2}$; $f'(1+0)=\frac{1}{2}$

$\therefore$ $f'(-1)$ does not exist.

$\therefore$ domain of $f'(x)=R-\{-1\}$.

Example: 18 Domain of definition of the function $f(x)=\frac{3}{4-x^2}+\log_{10}(x^3-x)$, is **[AIEEE 2003]**

(a) (1, 2) (b) (–1, 0) ∪ (1, 2) (c) (1, 2) ∪ (2, ∞) (d) (–1, 0) ∪ (1, 2) ∪ (2, ∞)

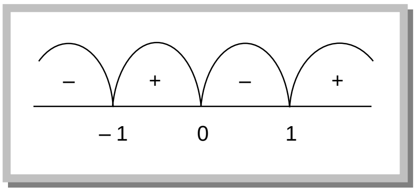
Solution: (d) $f(x)=\frac{3}{4-x^2}+\log_{10}(x^3-x)$

So, $4-x^2\neq 0 \Rightarrow x\neq\pm\sqrt{4} \Rightarrow x\neq\pm 2$

and $x^3-x>0 \Rightarrow x(x^2-1)>0 \Rightarrow x>0, |x|>1$

$\therefore$ $D=(-1,0)\cup(1,\infty)-\{2\}$

$D=(-1,0)\cup(1,2)\cup(2,\infty)$.



Example: 19 The domain of the function $f(x)=\log_{3+x}(x^2-1)$ is **[Orissa JEE 2003]**

(a) $(-3,-1)\cup(1,\infty)$ (b) $[-3,-1)\cup[1,\infty)$

(c) $(-3,-2)\cup(-2,-1)\cup(1,\infty)$ (d) $[-3,-2)\cup(-2,-1)\cup[1,\infty]$

Solution: (c) $f(x)$ is to be defined when $x^2-1>0$

$\Rightarrow x^2>1, \Rightarrow x<-1$ or $x>1$ and $3+x>0$

$\therefore$ $x>-3$ and $x\neq -2$

$\therefore$ $D_r=(-3,-2)\cup(-2,-1)\cup(1,\infty)$.

Example: 20 Domain of definition of the function $f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$, for real value x, is

(a) $\left[-\frac{1}{4},\frac{1}{2}\right]$ (b) $\left[-\frac{1}{2},\frac{1}{2}\right]$ (c) $\left(-\frac{1}{2},\frac{1}{9}\right)$ (d) $\left[-\frac{1}{4},\frac{1}{4}\right]$

Solution: (a) $-\frac{\pi}{6}\le\sin^{-1}(2x)\le\frac{\pi}{2}\Rightarrow -\frac{1}{2}\le 2x\le 1 \Rightarrow x\in\left[-\frac{1}{4},\frac{1}{2}\right]$.

Example: 21 The range of $f(x)=\cos x-\sin x$, is **[MP PET 1995]**

(a) $(-1,1)$ (b) $[-1,1)$ (c) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (d) $[-\sqrt{2},\sqrt{2}]$

Solution: (d) Let, $f(x)=\cos x-\sin x \Rightarrow f(x)=\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right) \Rightarrow f(x)=\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$

Now since, $-1 \le \cos\left(x+\frac{\pi}{4}\right) \le 1 \Rightarrow -\sqrt{2} \le f(x) \le \sqrt{2} \Rightarrow f(x) \in [-\sqrt{2}, \sqrt{2}]$

Trick : $\because$ Maximum value of $\cos x - \sin x$ is $\sqrt{2}$ and minimum value of $\cos x - \sin x$ is $-\sqrt{2}$.

Hence, range of $f(x) = [-\sqrt{2}, \sqrt{2}]$.

Example: 22 The range of $\frac{1+x^2}{x^2}$ is **[Karnataka CET 1989]**

(a) $(0, 1)$ (b) $(1, \infty)$ (c) [0, 1] (d) $[1, \infty)$

Solution: (b) Let $y = \frac{1+x^2}{x^2}$ $\Rightarrow x^2 y = 1 + x^2 \Rightarrow x^2(y-1) = 1 \Rightarrow x^2 = \frac{1}{y-1}$

Now since, $x^2 > 0 \Rightarrow \frac{1}{y-1} > 0 \Rightarrow (y-1) > 0 \Rightarrow y > 1 \Rightarrow y \in (1, \infty)$

Trick : $y = \frac{1+x^2}{x^2} = 1 + \frac{1}{x^2}$. Now since, $\frac{1}{x^2}$ is always > 0 $\Rightarrow y > 1 \Rightarrow y \in (1, \infty)$.

Example: 23 For real values of x, range of the function $y = \frac{1}{2 - \sin 3x}$ is

(a) $\frac{1}{3} \le y \le 1$ (b) $-\frac{1}{3} \le y < 1$ (c) $-\frac{1}{3} > y > -1$ (d) $\frac{1}{3} > y > 1$

Solution: (a) $\because y = \frac{1}{2 - \sin 3x}$, $\therefore 2 - \sin 3x = \frac{1}{y} \Rightarrow \sin 3x = 2 - \frac{1}{y}$

Now since,

$-1 \le \sin 3x \le 1 \Rightarrow -1 \le 2 - \frac{1}{y} \le 1 \Rightarrow -3 \le -\frac{1}{y} \le -1 \Rightarrow 1 \le \frac{1}{y} \le 3 \Rightarrow \frac{1}{3} \le y \le 1$.

Example 24 If $f(x) = a\cos(bx + c) + d$, then range of $f(x)$ is **[UPSEAT 2001]**

(a) $[d + a, d + 2a]$ (b) $[a - d, a + d]$ (c) $[d + a, a - d]$ (d) $[d - a, d + a]$

Solution: (d) $f(x) = a\cos(bx + c) + d$ ..... (i)

For minimum $\cos(bx + c) = -1$

from (i), $f(x) = -a + d = (d - a)$,

for maximum $\cos(bx + c) = 1$

from (i), $f(x) = a + d = (d + a)$

$\therefore$ Range of $f(x) = [d - a, d + a]$.

Example: 25 The range of the function $f(x) = \frac{x+2}{|x+2|}$ is **[Rajasthan PET 2002]**

(a) {0, 1} (b) {- 1, 1} (c) R (d) $R - \{-2\}$

Solution: (b) $f(x) = \frac{x+2}{|x+2|} = \begin{cases} -1, & x < -2 \\ 1, & x > -2 \end{cases}$

$\therefore$ Range of $f(x)$ is $\{-1, 1\}$.

Example: 26 The range of $f(x) = \sec\left(\frac{\pi}{4}\cos^2 x\right), -\infty < x < \infty$ is **[Orissa JEE 2002]**

(a) $[1, \sqrt{2}]$ (b) $[1, \infty)$ (c) $[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$ (d) $(-\infty, -1] \cup [1, \infty)$

Solution: (a) $f(x) = \sec\left(\frac{\pi}{4}\cos^2 x\right)$

We know that, $0 \le \cos^2 x \le 1$ at $\cos x = 0$, $f(x) = 1$ and at $\cos x = 1$, $f(x) = \sqrt{2}$

$\therefore\ 1 \le x \le \sqrt{2} \Rightarrow x \in [1, \sqrt{2}]$.

Example: 27 Range of the function $f(x) = \frac{x^2 + x + 2}{x^2 + x + 1}; x \in R$ is **[IIT Screening 2003]**

(a) $(1, \infty)$ (b) $(1, 11/7)$ (c) $(1, 7/3]$ (d) $(1, 7/5]$

Solution: (c) $f(x) = 1 + \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \Rightarrow$ Range $= (1, 7/3]$.

2.1.5 Algebra of Functions

Let $f(x)$ and $g(x)$ be two real and single-valued functions, with domains X_f, X_g and ranges Y_f and Y_g respectively. Let $X = X_f \cap X_g \ne \phi$. Then, the following operations are defined.

(1) **Scalar multiplication of a function :** $(c\,f)(x) = c\,f(x)$, where c is a scalar. The new function $c\,f(x)$ has the domain X_f.

(2) **Addition/subtraction of functions :** $(f \pm g)(x) = f(x) \pm g(x)$. The new function has the domain X.

(3) **Multiplication of functions :** $(fg)(x) = (g\,f)(x) = f(x)g(x)$. The product function has the domain X.

(4) **Division of functions :**

(i) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$. The new function has the domain X, except for the values of x for which $g(x) = 0$.

(ii) $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$. The new function has the domain X, except for the values of x for which $f(x) = 0$.

(5) **Equal functions :** Two function f and g are said to be equal functions, if and only if

(i) Domain of f = domain of g

(ii) Co-domain of f = co-domain of g

(iii) $f(x) = g(x) \forall x \in$ their common domain

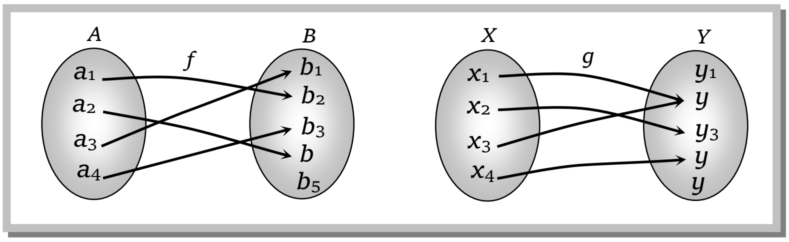
(6) **Real valued function :** If R, be the set of real numbers and A, B are subsets of R, then the function $f : A \to B$ is called a real function or real –valued function.

2.1.6 Kinds of Function

(1) **One-one function (injection) :** A function $f: A \to B$ is said to be a one-one function or an injection, if different elements of A have different images in B. Thus, $f: A \to B$ is one-one.

$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b)$ for all $a, b \in A$ $\Leftrightarrow f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$.

e.g. Let $f: A \to B$ and $g: X \to Y$ be two functions represented by the following diagrams.



Clearly, $f: A \to B$ is a one-one function. But $g: X \to Y$ is not one-one function because two distinct elements x_1 and x_3 have the same image under function *g*.

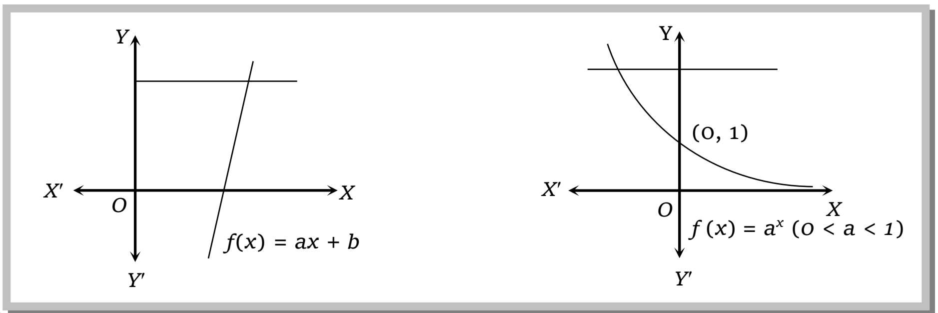
(i) **Method to check the injectivity of a function**

Step I : Take two arbitrary elements *x*, *y* (say) in the domain of *f*.

Step II : Put $f(x) = f(y)$.

Step III : Solve $f(x) = f(y)$. If $f(x) = f(y)$ gives *x* = *y* only, then $f: A \to B$ is a one-one function (or an injection). Otherwise not.

Note :
- ❒ If function is given in the form of ordered pairs and if two ordered pairs do not have same second element then function is one-one.
- ❒ If the graph of the function $y = f(x)$ is given and each line parallel to *x*-axis cuts the given curve at maximum one point then function is one-one. *e.g.*

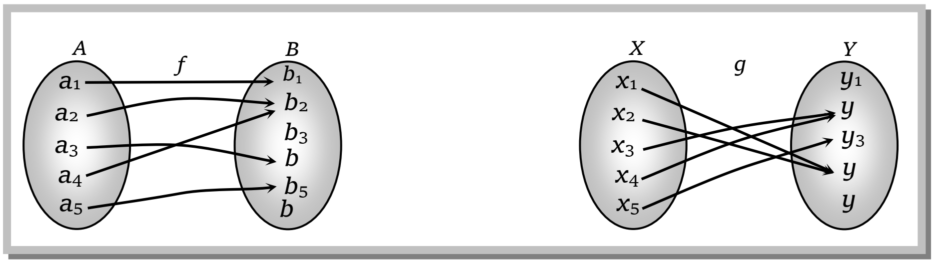


(ii) **Number of one-one functions (injections) :** If *A* and *B* are finite sets having *m* and *n* elements respectively, then number of one-one functions from A to B $= \begin{cases} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(2) **Many-one function :** A function $f: A \to B$ is said to be a many-one function if two or more elements of set *A* have the same image in *B*.

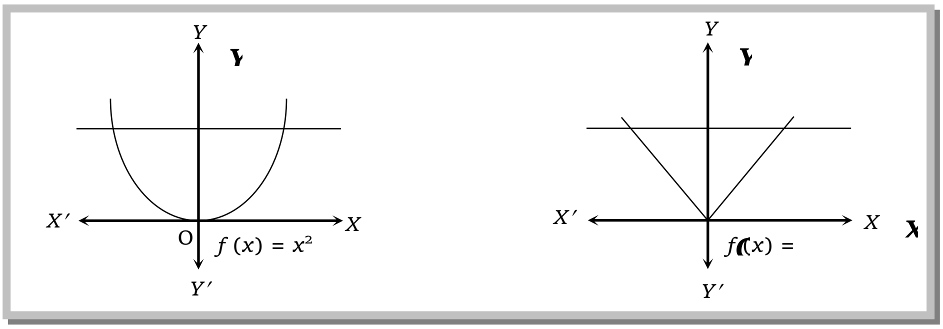
Thus, $f: A \to B$ is a many-one function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$.

In other words, $f: A \to B$ is a many-one function if it is not a one-one function.



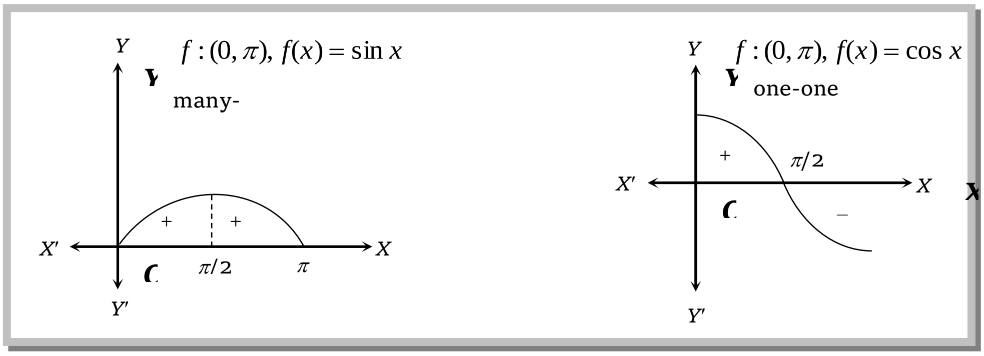
Note : ❐ If function is given in the form of set of ordered pairs and the second element of atleast two ordered pairs are same then function is many-one.

❐ If the graph of $y = f(x)$ is given and the line parallel to x-axis cuts the curve at more than one point then function is many-one.



❐ If the domain of the function is in one quadrant then the trigonometrical functions are always one-one.

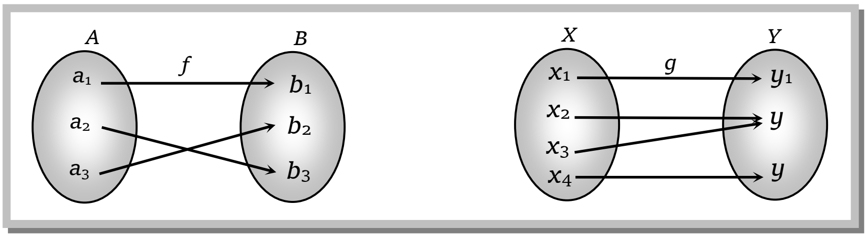
❐ If trigonometrical function changes its sign in two consecutive quadrants then it is one-one but if it does not change the sign then it is many-one.



❐ In three consecutive quadrants trigonometrical functions are always many-one.

(3) **Onto function (surjection) :** A function $f: A \to B$ is onto if each element of B has its pre-image in A. Therefore, if $f^{-1}(y) \in A, \ \forall y \in B$ then function is onto. In other words, Range of f = Co-domain of f.

e.g. The following arrow-diagram shows onto function.

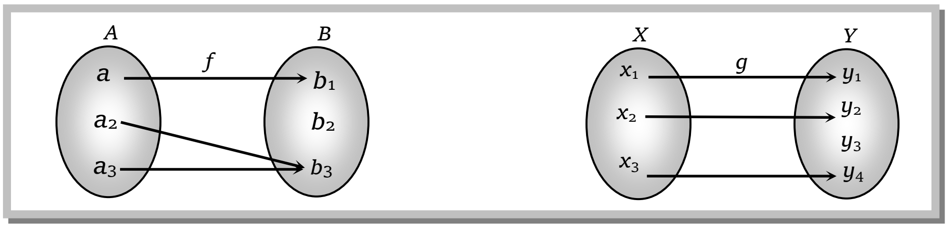


(i) **Number of onto function (surjection) :** If A and B are two sets having m and n elements respectively such that $1 \le n \le m$, then number of onto functions from A to B is $\sum_{r=1}^{n} (-1)^{n-r}\ {}^nC_r r^m$.

(4) **Into function :** A function $f : A \to B$ is an into function if there exists an element in B having no pre-image in A.

In other words, $f : A \to B$ is an into function if it is not an onto function.

e.g. The following arrow-diagram shows into function.



(i) **Method to find onto or into function**

(a) If range = co-domain, then $f(x)$ is onto and if range is a proper subset of the co-domain, then $f(x)$ is into.

(b) Solve $f(x) = y$ by taking x as a function of y *i.e.*, $g(y)$ (say).

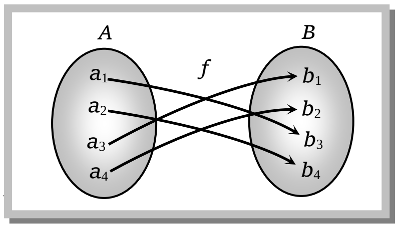
(c) Now if $g(y)$ is defined for each $y \in$ co-domain and $g(y) \in$ domain for $y \in$ co-domain, then $f(x)$ is onto and if any one of the above requirements is not fulfilled, then $f(x)$ is into.

(5) **One-one onto function (bijection) :** A function $f : A \to B$ is a bijection if it is one-one as well as onto.

In other words, a function $f : A \to B$ is a bijection if

(i) It is one-one i.e., $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

(ii) It is onto i.e., for all $y \in B$, there exists $x \in$



Clearly, f is a bijection since it is both injective as well as surjective.

Number of one-one onto function (bijection) : If A and B are finite sets and $f : A \to B$ is a bijection, then A and B have the same number of elements. If A has n elements, then the number of bijection from A to B is the total number of arrangements of n items taken all at a time *i.e.* $n!$.

(6) **Algebraic functions :** Functions consisting of finite number of terms involving powers and roots of the independent variable and the four fundamental operations +, −, × and ÷ are called algebraic functions.

e.g., (i) $x^{\frac{3}{2}}+5x$ (ii) $\frac{\sqrt{x+1}}{x-1}, x \neq 1$ (iii) $3x^4-5x+7$

The algebraic functions can be classified as follows:

(i) **Polynomial or integral function :** It is a function of the form $a_0x^n+a_1x^{n-1}+....+a_{n-1}x+a_n$, where $a_0 \neq 0$ and $a_0, a_1, ..., a_n$ are constants and $n \in N$ is called a polynomial function of degree *n*

e.g. $f(x)=x^3-2x^2+x+3$ is a polynomial function.

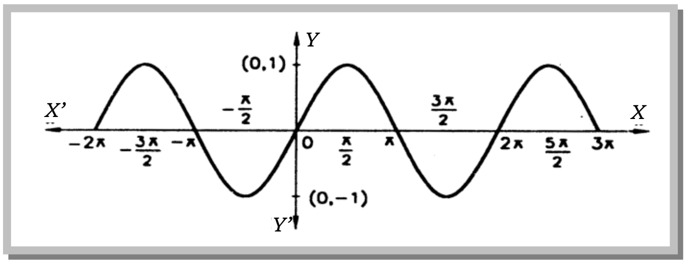
Note : ❐ The polynomial of first degree is called a linear function and polynomial of zero degree is called a constant function.

(ii) **Rational function :** The quotient of two polynomial functions is called the rational function. *e.g.* $f(x)=\frac{x^2-1}{2x^3+x^2+1}$ is a rational function.

(iii) **Irrational function :** An algebraic function which is not rational is called an irrational function. *e.g.* $f(x)=x+\sqrt{x}+6, \ g(x)=\frac{x^3-\sqrt{x}}{1+x^{1/4}}$ are irrational functions.

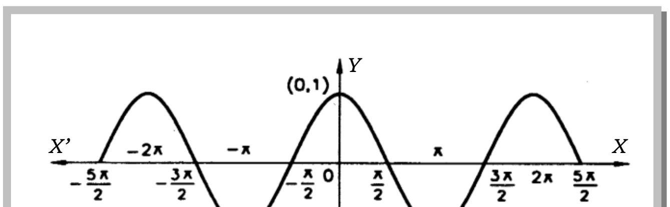
(7) **Transcendental function :** A function which is not algebraic is called a transcendental function. *e.g.*, trigonometric; inverse trigonometric , exponential and logarithmic functions are all transcendental functions.

(i) **Trigonometric functions :** A function is said to be a trigonometric function if it involves circular functions (sine, cosine, tangent, cotangent, secant, cosecant) of variable angles.



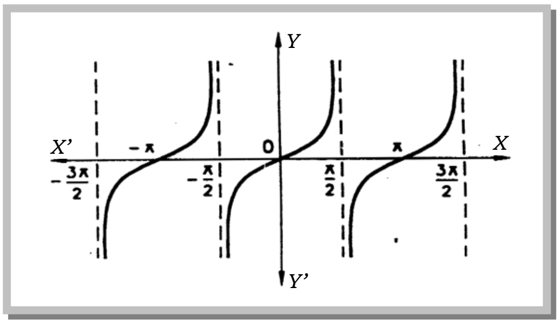
(a) **Sine function :** The function that associates to each real numbers *x* to $\sin x$ is called the sine function. Here *x* is the radian measure of the angle. The domain of the sine function is *R* and the range is [−1, 1].

(b) **Cosine function:** The function that associates to each real number *x* to $\cos x$ is called the cosine function. Here *x* is the radian measure of the angle. The domain of the cosine function is *R* and the range is [− 1, 1].



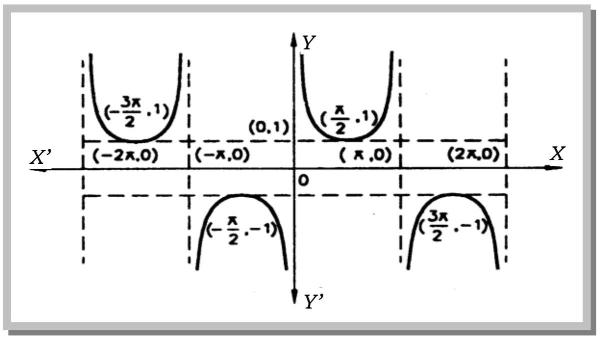
(c) **Tangent function :** The function that associates a real number *x* to tan *x* is called the tangent function.

Clearly, the tangent function is not defined at odd multiples of $\frac{\pi}{2}$ *i.e.,* $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ etc. So, the domain of the tangent function is $R-\{(2n+1)\frac{\pi}{2} | \, n \in I\}$. Since it takes every value between $-\infty$ and ∞. So, the range is *R*. Graph of $f(x) = \tan x$ is shown in figure.



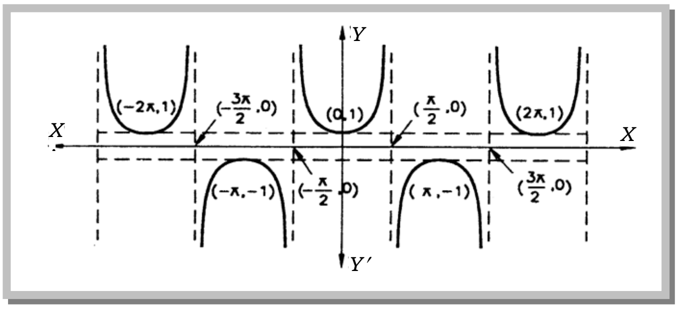
(d) **Cosecant function :** The function that associates a real number *x* to cosec*x* is called the cosecant function.

Clearly, cosec *x* is not defined at $x = n\pi, n \in I$. *i.e.,* $0, \pm\pi, \pm 2\pi, \pm 3\pi$ etc. So, its domain is $R - \{n\pi | \, n \in I\}$. Since $\text{cosec}\, x \geq 1$ or $\text{cosec}\ x \leq -1$. Therefore, range is $(-\infty, -1] \cup [1, \infty)$. Graph of $f(x) = \text{cosec}\, x$ is shown in figure.

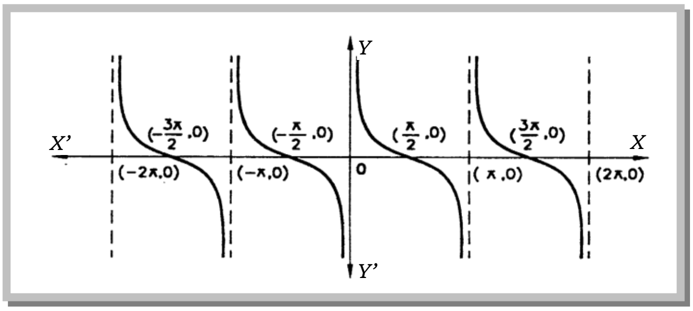


(e) **Secant function :** The function that associates a real number *x* to sec *x* is called the secant function.

Clearly, sec *x* is not defined at odd multiples of $\frac{\pi}{2}$ *i.e.,* $(2\pi+1)\frac{\pi}{2}$, where $n \in I$. So, its domain is $R-\{(2n+1)\frac{\pi}{2} | \, n \in I\}$. Also, $|\sec x| \geq 1$, therefore its range is $(-\infty, -1] \cup [1, \infty)$. Graph of $f(x) = \sec x$ is shown in figure.



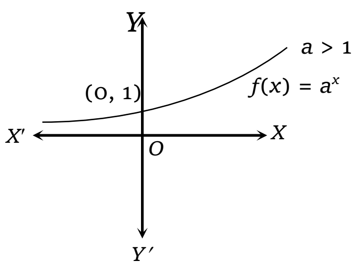
(f) **Cotangent function :** The function that associates a real number *x* to cot *x* is called the cotangent function. Clearly, cot *x* is not defined at $x = n\pi, n \in I$ i.e., at $n = 0, \pm\pi, \pm 2\pi$ etc. So, domain of $\cot x$ is $R-\{n\pi \mid n \in I\}$. The range of $f(x)=\cot x$ is R as is evident from its graph in figure.



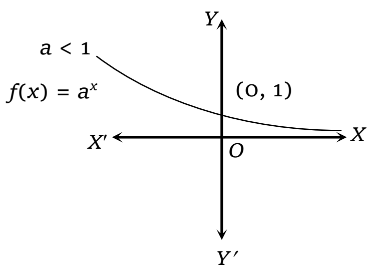
(ii) **Inverse trigonometric functions**

Function	**Domain**	**Range**	**Definition of the function**
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$y=\sin^{-1} x \Leftrightarrow x=\sin y$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	$y=\cos^{-1} x \Leftrightarrow x=\cos y$
$\tan^{-1} x$	$(-\infty, \infty)$ or R	$(-\pi/2, \pi/2)$	$y=\tan^{-1} x \Leftrightarrow x=\tan y$
$\cot^{-1} x$	$(-\infty, \infty)$ or R	$(0, \pi)$	$y=\cot^{-1} x \Leftrightarrow x=\cot y$
$\text{cosec}^{-1} x$	$R-(-1, 1)$	$[-\pi/2, \pi/2]-\{0\}$	$y=\text{cosec}^{-1} x \Leftrightarrow x=\text{cosec} y$
$\sec^{-1} x$	$R-(-1, 1)$	$[0, \pi]-[\pi/2]$	$y=\sec^{-1} x \Leftrightarrow x=\sec y$

(iii) **Exponential function :** Let $a \neq 1$ be a positive real number. Then $f: R \rightarrow (0, \infty)$ defined by $f(x)=a^x$ is called exponential function. Its domain is R and range is $(0, \infty)$.

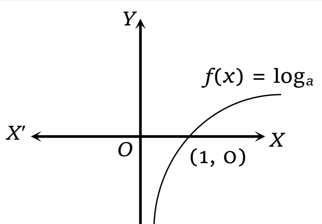


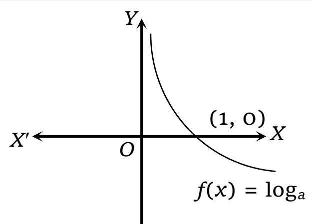
graph of $f(x)=a^x$, when $a > 1$



graph of $f(x)=a^x$, when $a < 1$

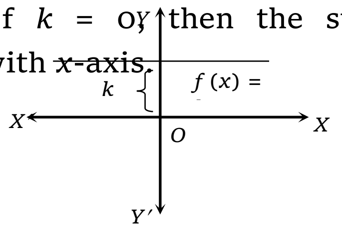
(iv) **Logarithmic function :** Let $a \neq 1$ be a positive real number. Then $f:(0, \infty) \rightarrow R$ defined by $f(x)=\log_a x$ is called logarithmic function. Its domain is $(0, \infty)$ and range is R.





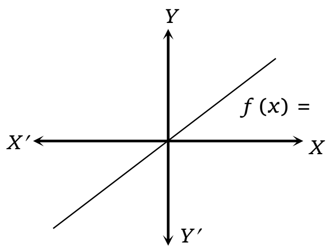
(8) **Explicit and implicit functions :** A function is said to be explicit if it can be expressed directly in terms of the independent variable. If the function can not be expressed directly in terms of the independent variable or variables, then the function is said to be implicit. *e.g.* $y = \sin^{-1} x + \log x$ is explicit function, while $x^2 + y^2 = xy$ and $x^3 y^2 = (a-x)^2 (b-y)^2$ are implicit functions.

(9) **Constant function :** Let k be a fixed real number. Then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function. The domain of the constant function $f(x) = k$ is the complete set of real numbers and the range of f is the singleton set $\{k\}$. The graph of a constant function is a straight line parallel to x-axis as shown in figure and it is above or below the x-axis according as k is positive or negative. If $k = 0$, then the straight line coincides with x-axis.

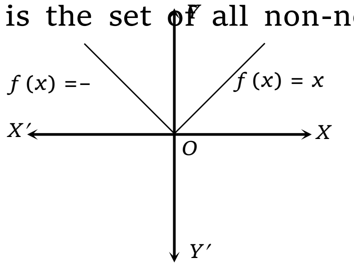


(10) **Identity function :** The function defined by $f(x) = x$ for all $x \in R$, is called the identity function on R. Clearly, the domain and range of the identity function is R.

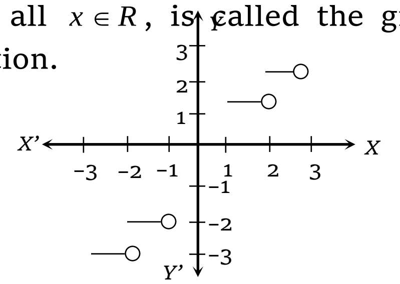
The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45^o with positive direction of x-axis.



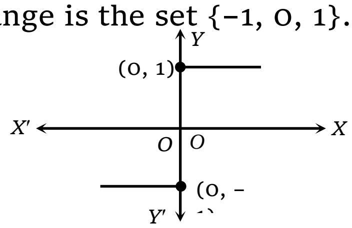
(11) **Modulus function :** The function defined by $f(x) = |x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$ is called the modulus function. The domain of the modulus function is the set R of all real numbers and the range is the set of all non-negative real numbers.



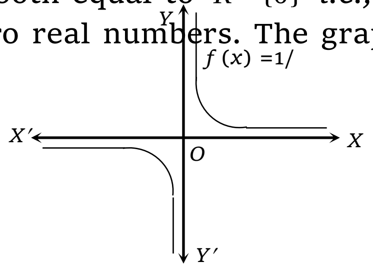
(12) **Greatest integer function:** Let $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x. The domain is R and the range is I. *e.g.* [1.1] = 1, [2.2] = 2, [− 0.9] = −1, [− 2.1] = − 3 etc. The function f defined by $f(x) = [x]$ for all $x \in R$, is called the greatest integer function.



(13) **Signum function :** The function defined by $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0 \ , & x = 0 \end{cases}$ or $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ is called the signum function. The domain is R and the range is the set {−1, 0, 1}.



(14) **Reciprocal function:** The function that associates each non-zero real number x to be reciprocal $\frac{1}{x}$ is called the reciprocal function. The domain and range of the reciprocal function are both equal to $R - \{0\}$ *i.e.,* the set of all non-zero real numbers. The graph is as shown.



Domain and Range of Some Standard Functions

Function	Domain	Range
Polynomial function	R	R
Identity function x	R	R
Constant function K	R	$\{K\}$
Reciprocal function $\frac{1}{x}$	R_o	R_o
$x^2, \|x\|$	R	$R^+ \cup \{0\}$
$x^3, x\|x\|$	R	R
Signum function	R	$\{-1, 0, 1\}$
$x + \|x\|$	R	$R^+ \cup \{0\}$
$x - \|x\|$	R	$R^- \cup \{0\}$

$[x]$	R	I
$x-[x]$	R	$[0, 1)$
$\sqrt{x}$	$[0, \infty)$	R
a^x	R	R^+
$\log x$	R^+	R
$\sin x$	R	$[-1, 1]$
$\cos x$	R	$[-1, 1]$
$\tan x$	$R-\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...\right\}$	R
$\cot x$	$R-\{0, \pm\pi, \pm 2\pi, ...\}$	R
$\sec x$	$R-\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...\right\}$	$R-(-1, 1)$
$\text{cosec } x$	$R-\{0, \pm\pi, \pm 2\pi, ...\}$	$R-(-1, 1)$
$\sin^{-1} x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	R	$(0, \pi)$
$\sec^{-1} x$	$R-(-1, 1)$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$
$\text{cosec}^{-1} x$	$R-(-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$

Important Tips

- ☞ *Any function, which is entirely increasing or decreasing in the whole of a domain, is one-one.*
- ☞ *Any continuous function $f(x)$, which has at least one local maximum or local minimum, is many-one.*
- ☞ *If any line parallel to the x-axis cuts the graph of the function at most at one point, then the function is one-one and if there exists a line which is parallel to the x-axis and cuts the graph of the function in at least two points, then the function is many-one.*
- ☞ *Any polynomial function $f: R \to R$ is onto if degree of f is odd and into if degree of f is even.*
- ☞ *An into function can be made onto by redefining the co-domain as the range of the original function.*

Example: 28 Function $f: N \to N, f(x) = 2x+3$ is **[IIT 1973; UPSEAT 1983]**

(a) One-one onto (b) One-one into (c) Many-one onto (d) Many –one into

Solution: (b) f is one-one because $f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow x_1 = x_2$

Further $f^{-1}(x) = \frac{x-3}{2} \notin N$ (domain) when x = 1, 2, 3 etc.

$\therefore$ *f* is into which shows that *f* is one-one into.

Example: 29 The function $f : R \to R$ defined by $f(x) = (x-1)(x-2)(x-3)$ is **[Roorkee 1999]**

(a) One-one but not onto (b) Onto but not one-one

(c) Both one-one and onto (d) Neither one-one nor onto

Solution: (b) We have $f(x) = (x-1)(x-2)(x-3) \Rightarrow f(1) = f(2) = f(3) = 0 \Rightarrow f(x)$ is not one-one

For each $y \in R,$ there exists $x \in R$ such that $f(x) = y$. Therefore *f* is onto.

Hence, $f : R \to R$ is onto but not one-one.

Example: 30 Find number of surjection from *A* to *B* where $A = \{1, 2, 3, 4\}$, $B = \{a, b\}$ **[IIT Screening 2001]**

(a) 13 (b) 14 (c) 15 (d) 16

Solution: (b) Number of surjection from *A* to *B* $= \sum_{r=1}^{2} (-1)^{2-r}\ {}^2C_r (r)^4$

$= (-1)^{2-1}\ {}^2C_1 (1)^4 + (-1)^{2-2}\ {}^2C_2 (2)^4 = -2 + 16 = 14$

Therefore, number of surjection from *A* to *B* = 14.

Trick : Total number of functions from A to *B* is 2^4 of which two function $f(x) = a$ for all $x \in A$ and $g(x) = b$ for all $x \in A$ are not surjective. Thus, total number of surjection from *A* to *B* $= 2^4 - 2 = 14$.

Example: 31 If $A = \{a, b, c\},$ then total number of one-one onto functions which can be defined from *A* to *A* is

(a) 3 (b) 4 (c) 9 (d) 6

Solution: (d) Total number of one-one onto functions = $3!$

Example: 32 If $f : R \to R,$ then $f(x) = |x|$ is **[Rajasthan PET 2000]**

(a) One-one but not onto (b) Onto but not one-one

(c) One-one and onto (d) None of these

Solution: (d) $f(-1) = f(1) = 1$ $\therefore$ function is many-one function.

Obviously, *f* is not onto so *f* is neither one-one nor onto.

Example: 33 Let $f : R \to R$ be a function defined by $f(x) = \frac{x-m}{x-n},$ where $m \neq n$. Then **[UPSEAT 2001]**

(a) *f* is one-one onto (b) *f* is one-one into (c) *f* is many one onto (d) *f* is many one into

Solution: (b) For any $x, y \in R,$ we have

$f(x) = f(y) \Rightarrow \frac{x-m}{x-n} = \frac{y-m}{y-n} \Rightarrow x = y$

$\therefore f$ is one-one

Let $\alpha \in R$ such that $f(x) = \alpha \Rightarrow \frac{x-m}{x-n} = \alpha \Rightarrow x = \frac{m - n\alpha}{1-\alpha}$

Clearly $x \notin R$ for $\alpha = 1$. So, *f* is not onto.

Example: 34 The function $f : R \to R$ defined by $f(x) = e^x$ is **[Karnataka CET 2002; UPSEAT 2002]**

(a) Onto (b) Many-one (c) One-one and into (d) Many one and onto

Solution: (c) Function $f: R \to R$ is defined by $f(x) = e^x$. Let $x_1, x_2 \in R$ and $f(x_1) = f(x_2)$ or $e^{x_1} = e^{x_2}$ or $x_1 = x_2$. Therefore f is one-one. Let $f(x) = e^x = y$. Taking log on both sides, we get $x = \log y$. We know that negative real numbers have no pre-image or the function is not onto and zero is not the image of any real number. Therefore function f is into.

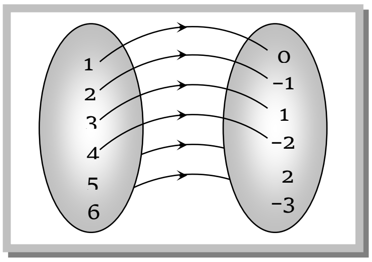
Example: 35 A function f from the set of natural numbers to integers defined by $f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$, is **[AIEEE 2003]**

(a) One-one but not onto (b) Onto but not one-one

(c) One-one and onto both (d) Neither one-one nor onto

Solution: (c) $f: N \to I$

$f(1) = 0, f(2) = -1, f(3) = 1, f(4) = -2, f(5) = 2$ and $f(6) = -3$ so on.



In this type of function every element of set A has unique image in set B and there is no element left in set B. Hence f is one-one and onto function.

2.1.7 Even and Odd function

(1) **Even function :** If we put $(-x)$ in place of x in the given function and if $f(-x) = f(x)$, $\forall x \in$ domain then function $f(x)$ is called even function. *e.g.* $f(x) = e^x + e^{-x}, f(x) = x^2, f(x) = x \sin x, f(x) = \cos x, f(x) = x^2 \cos x$ all are even function.

(2) **Odd function :** If we put $(-x)$ in place of x in the given function and if $f(-x) = -f(x)$, $\forall x \in$ domain then $f(x)$ is called odd function. *e.g.* $f(x) = e^x - e^{-x}$, $f(x) = \sin x, f(x) = x^3$, $f(x) = x \cos x, f(x) = x^2 \sin x$ all are odd function.

Important Tips

☞ *The graph of even function is always symmetric with respect to y-axis.*

- ☞ *The graph of odd function is always symmetric with respect to origin.*
- ☞ *The product of two even functions is an even function.*
- ☞ *The sum and difference of two even functions is an even function.*
- ☞ *The sum and difference of two odd functions is an odd function.*
- ☞ *The product of two odd functions is an even function.*
- ☞ *The product of an even and an odd function is an odd function*
- ☞ *It is not essential that every function is even or odd. It is possible to have some functions which are neither even nor odd function. e.g.* $f(x) = x^2 + x^3$, $f(x) = \log_e x$, $f(x) = e^x$.
- ☞ *The sum of even and odd function is neither even nor odd function.*
- ☞ *Zero function* $f(x) = 0$ *is the only function which is even and odd both.*

Example: 36 Which of the following is an even function **[UPSEAT 1998]**

(a) $x\left(\frac{a^x - 1}{a^x + 1}\right)$ (b) $\tan x$ (c) $\frac{a^x - a^{-x}}{2}$ (d) $\frac{a^x + 1}{a^x - 1}$

Solution: (a) We have : $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$

$$f(-x) = -x\left(\frac{a^{-x} - 1}{a^{-x} + 1}\right) = -x\left(\frac{\frac{1}{a^x} - 1}{\frac{1}{a^x} + 1}\right) = -x\left(\frac{1 - a^x}{1 + a^x}\right) = x\left(\frac{a^x - 1}{a^x + 1}\right) = f(x)$$

So, $f(x)$ is an even function.

Example: 37 Let $f(x) = \sqrt{x^4 + 15},$ then the graph of the function $y = f(x)$ is symmetrical about

(a) The x-axis (b) The y-axis (c) The origin (d) The line $x = y$

Solution: (b) $f(x) = \sqrt{x^4 + 15} \Rightarrow f(-x) = \sqrt{(-x)^4 + 15} = \sqrt{x^4 + 15} = f(x)$

$\Rightarrow f(-x) = f(x) \Rightarrow f(x)$ is an even function $\Rightarrow f(x)$ is symmetric about y-axis.

Example: 38 The function $f(x) = \log(x + \sqrt{x^2 + 1})$ is

(a) An even function (b) An odd function (c) Periodic function (d) None of these

Solution: (b) $f(x) = \log(x + \sqrt{x^2 + 1})$ and $f(-x) = -\log(x + \sqrt{x^2 + 1}) = -f(x)$, so $f(x)$ is an odd function.

Example: 39 Which of the following is an even function **[Rajasthan PET 2000]**

(a) $f(x) = \frac{a^x + 1}{a^x - 1}$ (b) $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$ (c) $f(x) = \frac{a^x - a^{-x}}{a^x + a^{-x}}$ (d) $f(x) = \sin x$

Solution: (b) In option (a), $f(-x) = \frac{a^{-x} + 1}{a^{-x} - 1} = \frac{1 + a^x}{1 - a^x} = -\frac{a^x + 1}{a^x - 1} = -f(x)$ So, It is an odd function.

In option (b), $f(-x) = (-x)\frac{a^{-x} - 1}{a^{-x} + 1} = -x\frac{(1 - a^x)}{1 + a^x} = x\frac{(a^x - 1)}{(a^x + 1)} = f(x)$ So, It is an even function.

In option (c), $f(-x) = \frac{a^{-x} - a^x}{a^{-x} + a^x} = -f(x)$ So, It is an odd function.

In option (d), $f(-x) = \sin(-x) = -\sin x = -f(x)$ So, It is an odd function.

Example: 40 The function $f(x) = \sin\left(\log(x + \sqrt{x^2+1})\right)$ is **[Orissa JEE 2002]**

(a) Even function (b) Odd function (c) Neither even nor odd (d) Periodic function

Solution: (b) $f(x) = \sin\left(\log(x + \sqrt{1+x^2})\right)$

$\Rightarrow f(-x) = \sin[\log(-x + \sqrt{1+x^2})] \Rightarrow f(-x) = \sin\log\left((\sqrt{1+x^2} - x)\frac{(\sqrt{1+x^2}+x)}{(\sqrt{1+x^2}+x)}\right)$

$\Rightarrow f(-x) = \sin\log\left[\frac{1}{(x+\sqrt{1+x^2})}\right] \Rightarrow f(-x) = \sin\left[\log(x+\sqrt{1+x^2})^{-1}\right]$

$\Rightarrow f(-x) = \sin\left[-\log(x+\sqrt{1+x^2})\right] \Rightarrow f(-x) = -\sin\left[\log(x+\sqrt{1+x^2})\right] \Rightarrow f(-x) = -f(x)$

$\therefore f(x)$ is odd function.

2.1.8 Periodic Function

A function is said to be periodic function if its each value is repeated after a definite interval. So a function $f(x)$ will be periodic if a positive real number T exist such that, $f(x+T) = f(x), \forall x \in$ domain. Here the least positive value of T is called the period of the function. Clearly $f(x) = f(x+T) = f(x+2T) = f(x+3T) = \ldots\ldots$ *e.g.* $\sin x, \cos x, \tan x$ are periodic functions with period $2\pi, 2\pi$ and π respectively.

Some standard results on periodic functions

Functions	**Periods**
(1) $\sin^n x$, $\cos^n x$, $\sec^n x$, $cosec^n x$	π; if n is even 2π; if n is odd or fraction
(2) $\tan^n x$, $\cot^n x$	$\pi; n$ is even or odd.
(3) $\lvert\sin x\rvert, \lvert\cos x\rvert, \lvert\tan x\rvert,$ $\lvert\cot x\rvert, \lvert\sec x\rvert, \lvert cosec\, x\rvert$	π
(4) $x-[x]$	1
(5) Algebraic functions e.g., $\sqrt{x}, x^2, x^3+5, \ldots$ etc	Period does not exist

Important Tips

☞ *If f(x) is periodic with period T, then c.f(x) is periodic with period T, f(x + c) is periodic with period T and f(x) ± c is periodic with period T. where c is any constant.*

☞ *If a function f(x) has a period T, then the function f(ax+b) will have a period* $\frac{T}{|a|}$.

☞ *If f(x) is periodic with period T then* $\frac{1}{f(x)}$ *is also periodic with same period T.*

☞ *If f(x) is periodic with period T,* $\sqrt{f(x)}$ *is also periodic with same period T.*

☞ *If f(x) is periodic with period T, then a f(x) + b, where* $a, b \in R (a \neq 0)$ *is also a periodic function with period T.*

☞ *If* $f_1(x), f_2(x), f_3(x)$ *are periodic functions with periods* T_1, T_2, T_3 *respectively then; we have*

$h(x) = af_1(x) \pm bf_2(x) \pm cf_3(x)$, *has period as,*

$$= \begin{cases} \text{L.C.M.of } \{T_1, T_2, T_3\}; & \text{if } h(x) \text{ is not an even function} \\ \frac{1}{2}\text{L.C.M.of } \{T_1, T_2, T_3\}; & \text{if } h(x) \text{ is an even function} \end{cases}$$

Example: 41 The period of the function $f(x) = 2\cos\frac{1}{3}(x - \pi)$ is **[DCE 1998]**

(a) 6π (b) 4π (c) 2π (d) π

Solution: (a) $f(x) = 2\cos\frac{1}{3}(x-\pi) = 2\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$

Now, since $\cos x$ has period $2\pi \Rightarrow \cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$ has period $\frac{2\pi}{\frac{1}{3}} = 6\pi$

$\Rightarrow 2\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$ has period $= 6\pi$.

Example: 42 The function $f(x) = \sin\frac{\pi x}{2} + 2\cos\frac{\pi x}{3} - \tan\frac{\pi x}{4}$ is periodic with period **[EAMCET 1992]**

(a) 6 (b) 3 (c) 4 (d) 12

Solution: (d) $\because \sin x$ has period $= 2\pi \Rightarrow \sin\frac{\pi x}{2}$ has period $= \frac{2\pi}{\frac{\pi}{2}} = 4$

$\because \cos x$ has period $= 2\pi \Rightarrow \cos\frac{\pi x}{3}$ has period $= \frac{2\pi}{\frac{\pi}{3}} = 6 \Rightarrow 2\cos\frac{\pi x}{3}$ has period = 6

$\because \tan x$ has period $= \pi \Rightarrow \tan\frac{\pi x}{4}$ has period $= \frac{\pi}{\frac{\pi}{4}} = 4$.

L.C.M. of 4, 6 and 4 =12, period of $f(x)$ = 12.

Example: 43 The period of $|\sin 2x|$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) 2π

Solution: (b) Here $|\sin 2x| = \sqrt{\sin^2 2x} = \sqrt{\frac{(1-\cos 4x)}{2}}$

Period of $\cos 4x$ is $\frac{\pi}{2}$. Hence, period of $|\sin 2x|$ will be $\frac{\pi}{2}$

Trick : $\because$ $\sin x$ has period $= 2\pi \Rightarrow$ $\sin 2x$ has period $= \frac{2\pi}{2} = \pi$

Now, if $f(x)$ has period p then $|f(x)|$ has period $\frac{p}{2} \Rightarrow |\sin 2x|$ has period $= \frac{\pi}{2}$.

Example: 44 If $f(x)$ is an odd periodic function with period 2, then $f(4)$ equals **[IIT 1991]**

(a) 0 (b) 2 (c) 4 (d) – 4

Solution: (a) Given, $f(x)$ is an odd periodic function. We can take $\sin x$, which is odd and periodic.

Now since, $\sin x$ has period = 2 and $f(x)$ has period = 2.

So, $f(x) = \sin(\pi x) \Rightarrow f(4) = \sin(4\pi) = 0$.

Example: 45 The period of the function $f(x) = \sin^2 x$ is **[UPSEAT 1991, 2002; AIEEE 2002]**

(a) $\frac{\pi}{2}$ (b) π (c) 2π (d) None of these

Solution: (b) $\sin^2 x = \frac{1-\cos 2x}{2} \Rightarrow$ Period $= \frac{2\pi}{2} = \pi$.

Example: 46 The period of $f(x) = x - [x]$, if it is periodic, is **[AMU 2000]**

(a) $f(x)$ is not periodic (b) $\frac{1}{2}$ (c) 1 (d) 2

Solution: (c) Let $f(x)$ be periodic with period T. Then,

$f(x+T) = f(x)$ for all $x \in R \Rightarrow x + T - [x+T] = x - [x]$ for all $x \in R \Rightarrow x + T - x = [x+T] - [x]$

$\Rightarrow [x+T] - [x] = T$ for all $x \in R \Rightarrow T = 1, 2, 3, 4, ...$

The smallest value of T satisfying,

$f(x+T) = f(x)$ for all $x \in R$ is 1.

Hence $f(x) = x - [x]$ has period 1.

Example: 47 The period of $f(x) = \sin\left(\frac{\pi x}{n-1}\right) + \cos\left(\frac{\pi x}{n}\right)$, $n \in Z$, $n > 2$ is

(a) $2\pi n(n-1)$ (b) $4n(n-1)$ (c) $2n(n-1)$ (d) None of these

Solution: (c) $f(x) = \sin\left(\frac{\pi x}{n-1}\right) + \cos\left(\frac{\pi x}{n}\right)$

Period of $\sin\left(\frac{\pi x}{n-1}\right) = \frac{2\pi}{\left(\frac{\pi}{n-1}\right)} = 2(n-1)$ and period of $\cos\left(\frac{\pi x}{n}\right) = \frac{2\pi}{\left(\frac{\pi}{n}\right)} = 2n$

Hence period of $f(x)$ is LCM of $2n$ and $2(n-1) \Rightarrow 2n(n-1)$.

Example: 48 If a, b be two fixed positive integers such that $f(a+x) = b + [b^3 + 1 - 3b^2 f(x) + 3b\{f(x)\}^2 - \{f(x)\}^3]^{\frac{1}{3}}$ for all real x, then $f(x)$ is a periodic function with period

(a) a (b) $2\,a$ (c) b (d) $2\,b$

Solution: (b) $f(a+x) = b + (1 + \{b - f(x)\}^3)^{1/3} \Rightarrow f(a+x) - b = \{1 - \{f(x) - b\}^3\}^{1/3}$

$\Rightarrow \phi(a+x) = \{1 - \{\phi(x)\}^3\}^{1/3}$ $[\ \phi(x) = f(x) - b\ ] \Rightarrow \phi(x+2a) = \{1 - \{\phi(x+a)\}^3\}^{1/3} = \phi(x)$

$\Rightarrow f(x+2a) - b = f(x) - b \Rightarrow f(x+2a) = f(x)$

$\therefore f(x)$ is periodic with period $2a$.

2.1.9 Composite Function

If $f: A \to B$ and $g: B \to C$ are two function then the composite function of *f* and *g*, $gof\ A \to C$ will be defined as $gof(x) = g[f(x)], \forall x \in A$

(1) **Properties of composition of function :**

(i) *f* is even, *g* is even $\Rightarrow fog$ even function.

(ii) *f* is odd, *g* is odd $\Rightarrow fog$ is odd function.

(iii) *f* is even, *g* is odd $\Rightarrow fog$ is even function.

(iv) *f* is odd, *g* is even $\Rightarrow fog$ is even function.

(v) Composite of functions is not commutative *i.e.*, $fog \neq gof$

(vi) Composite of functions is associative *i.e.*, $(fog)oh = fo(goh)$

(vii) If $f: A \to B$ is bijection and $g: B \to A$ is inverse of *f*. Then $fog = I_B$ and $gof = I_A$.

where, I_A and I_B are identity functions on the sets *A* and *B* respectively.

(viii) If $f: A \to B$ and $g: B \to C$ are two bijections, then $gof: A \to C$ is bijection and $(gof)^{-1} = (f^{-1}og^{-1})$.

(ix) $fog \neq gof$ but if , $fog = gof$ then either $f^{-1} = g$ or $g^{-1} = f$ also, $(fog)(x) = (gof)(x) = (x)$.

Important Tips

- *gof(x) is simply the g-image of f(x), where f(x) is f-image of elements $x \in A$.*
- *Function gof will exist only when range of f is the subset of domain of g.*
- *fog does not exist if range of g is not a subset of domain of f.*
- *fog and gof may not be always defined.*
- *If both f and g are one-one, then fog and gof are also one-one.*
- *If both f and g are onto, then gof is onto.*

Example: 49 If $f: R \to R, f(x) = 2x - 1$ and $g: R \to R,\ g(x) = x^2$ then $(gof)(x)$ equals **[Rajasthan PET 1987]**

(a) $2x^2 - 1$ (b) $(2x-1)^2$ (c) $4x^2 - 2x + 1$ (d) $x^2 + 2x - 1$

Solution: (b) $gof(x) = g\{f(x)\} = g(2x-1) = (2x-1)^2$.

Example: 50 If $f: R \to R, f(x) = (x+1)^2$ and $g: R \to R, g(x) = x^2 + 1,$ then $(fog)(-3)$ is equal to **[Rajasthan PET 1999]**

(a) 121 (b) 144 (c) 112 (d) 11

Solution: (a) $fog(x) = f\{g(x)\} = f(x^2+1) = (x^2+1+1)^2 = (x^2+2)^2 \Rightarrow fog(-3) = (9+2)^2 = 121$.

Example: 51 $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1,$ then $(gof)(x)$ is equal to **[IIT 1996]**

(a) 1 (b) –1 (c) 2 (d) – 2

Solution: (a) $f(x) = \sin^2 x + \sin^2(x + \pi/3) + \cos x \cos(x + \pi/3) = \frac{1 - \cos 2x}{2} + \frac{1 - \cos(2x + 2\pi/3)}{2} + \frac{1}{2}\{2\cos x \cos(x + \pi/3)\}$

$= \frac{1}{2}[1 - \cos 2x + 1 - \cos(2x + 2\pi/3) + \cos(2x + \pi/3) + \cos \pi/3]$

$= \frac{1}{2}\left[\frac{5}{2} - \{\cos 2x + \cos\left(2x + \frac{2\pi}{3}\right)\} + \cos\left(2x + \frac{\pi}{3}\right)\right] = \frac{1}{2}\left[\frac{5}{2} - 2\cos\left(2x + \frac{\pi}{3}\right)\cos\frac{\pi}{3} + \cos\left(2x + \frac{\pi}{3}\right)\right] = 5/4$ for all x.

$\therefore \quad gof(x) = g(f(x)) = g(5/4) = 1 \quad [\because g(5/4) = 1$ (given)]

Hence, $gof(x) = 1$, for all x.

Example: 52 If $g(x) = x^2 + x - 2$ and $\frac{1}{2}(gof)(x) = 2x^2 - 5x + 2$, then $f(x)$ is equal to **[Roorkee 1998; MP PET 2002]**

(a) $2x - 3$ (b) $2x + 3$ (c) $2x^2 + 3x + 1$ (d) $2x^2 - 3x - 1$

Solution: (a) $g(x) = x^2 + x - 2 \Rightarrow (gof)(x) = g[f(x)] = [f(x)]^2 + f(x) - 2$

Given, $\frac{1}{2}(gof)(x) = 2x^2 - 5x + 2 \qquad \therefore \quad \frac{1}{2}[f(x)]^2 + \frac{1}{2}f(x) - 1 = 2x^2 - 5x + 2$

$\Rightarrow [f(x)]^2 + f(x) = 4x^2 - 10x + 6 \Rightarrow f(x)[f(x) + 1] = (2x - 3)[(2x - 3) + 1] \Rightarrow f(x) = 2x - 3$.

Example: 53 If $f(y) = \frac{y}{\sqrt{1 - y^2}}, g(y) = \frac{y}{\sqrt{1 + y^2}}$, then $(fog)(y)$ is equal to

(a) $\frac{y}{\sqrt{1 - y^2}}$ (b) $\frac{y}{\sqrt{1 + y^2}}$ (c) y (d)

Solution: (c) $f[g(y)] = \frac{y/\sqrt{1 + y^2}}{\sqrt{1 - \left(\frac{y}{\sqrt{1 + y^2}}\right)^2}} = \frac{y}{\sqrt{1 + y^2}} \times \frac{\sqrt{1 + y^2}}{\sqrt{1 + y^2 - y^2}} = y$

Example: 54 If $f(x) = \frac{2x - 3}{x - 2}$, then $[f\{f(x)\}]$ equals **[Rajasthan PET 1996]**

(a) x (b) $-x$ (c) $\frac{x}{2}$ (d) $-\frac{1}{x}$

Solution: (a) $f[f(x)] = \frac{2\left(\frac{2x - 3}{x - 2}\right) - 3}{\left(\frac{2x - 3}{x - 2}\right) - 2} = x$

Example: 55 Suppose that $g(x) = 1 + \sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x$, then $f(x)$ is **[MP PET 2000; Karnataka CET 2002]**

(a) $1 + 2x^2$ (b) $2 + x^2$ (c) $1 + x$ (d)

Solution: (b) $g(x) = 1 + \sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x$ ..... (i)

$\Rightarrow \quad f(1 + \sqrt{x}) = 3 + 2\sqrt{x} + x$

Put $1 + \sqrt{x} = y \Rightarrow x = (y - 1)^2$

then, $f(y) = 3 + 2(y - 1) + (y - 1)^2 = 2 + y^2$

therefore, $f(x) = 2 + x^2$.

Example: 56 Let $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$, then for all x, $f(g(x))$ is equal to **[IIT Screening 2001; UPSEAT 2001]**

(a) x (b) 1 (c) $f(x)$ (d)

Solution: (b) Here $g(x) = 1 + n - n = 1, x = n \in Z$

$1 + n + k - n = 1 + k$, $x = n + k$ (where $n \in Z$, $0 < k < 1$)

Now $f(g(x)) = \begin{cases} -1, & g(x) < 0 \\ 0, & g(x) = 0 \\ 1, & g(x) > 0 \end{cases}$

Clearly, $g(x) > 0$ for all x. So, $f(g(x)) = 1$ for all x.

Example: 57 If $f(x) = \dfrac{2x+1}{3x-2}$, then $(fof)(2)$ is equal to **[Kerala (Engg.) 2002]**

(a) 1 (b) 3 (c) 4 (d) 2

Solution: (d) Here $f(2) = \dfrac{5}{4}$

Hence $(fof)(2) = f(f(2)) = f\left(\dfrac{5}{4}\right) = \dfrac{2 \times \frac{5}{4} + 1}{3 \times \frac{5}{4} - 2} = 2$.

Example: 58 If $f: R \to R$ and $g: R \to R$ are given by $f(x) = |x|$ and $g(x) = [x]$ for each $x \in R$, then $\{x \in R : g(f(x)) \le f(g(x))\} =$ **[EAMCET 2003]**

(a) $Z \cup (-\infty, 0)$ (b) $(-\infty, 0)$ (c) Z (d)

Solution: (d) $g(f(x)) \le f(g(x)) \Rightarrow g(|x|) \le f[x] \Rightarrow [|x|] \le |[x]|$. This is true for $x \in R$.

2.1.10 Inverse Function

If $f: A \to B$ be a one-one onto (bijection) function, then the mapping $f^{-1}: B \to A$ which associates each element $b \in B$ with element $a \in A$, such that $f(a) = b$, is called the inverse function of the function $f: A \to B$

$$f^{-1}: B \to A, \; f^{-1}(b) = a \Rightarrow f(a) = b$$

In terms of ordered pairs inverse function is defined as $f^{-1} = (b, a)$ if $(a, b) \in f$.

Note : ❒ For the existence of inverse function, it should be one-one and onto.

Important Tips

- *Inverse of a bijection is also a bijection function.*
- *Inverse of a bijection is unique.*
- $(f^{-1})^{-1} = f$
- *If f and g are two bijections such that (gof) exists then* $(gof)^{-1} = f^{-1}og^{-1}$.
- *If* $f: A \to B$ *is a bijection then* $f^{-1}: B \to A$ *is an inverse function of f.* $f^{-1}of = I_A$ *and* $fof^{-1} = I_B$. *Here* I_A, *is an identity function on set A, and* I_B, *is an identity function on set B.*

Example: 59 If $f: R \to R$ is given by $f(x) = 3x - 5$, then $f^{-1}(x)$ **[IIT Screening 1998]**

(a) Is given by $\dfrac{1}{3x-5}$ (b) Is given by $\dfrac{x+5}{3}$

(c) Does not exist because f is not one-one (d) Does not exist because f is not onto

Solution: (b) Clearly, $f: R \to R$ is a one-one onto function. So, it is invertible.

Let $f(x) = y$. then, $3x - 5 = y \Rightarrow x = \frac{y+5}{3} \Rightarrow f^{-1}(y) = \frac{y+5}{3}$. Hence, $f^{-1}(x) = \frac{x+5}{3}$.

Example: 60 Let $f: R \to R$ be defined by $f(x) = 3x - 4$, then $f^{-1}(x)$ is

(a) $3x + 4$ (b) $\frac{1}{3}x - 4$ (c) $\frac{1}{3}(x+4)$ (d) $\frac{1}{3}(x-4)$

Solution: (c) $f(x) = 3x - 4 = y \Rightarrow y = 3x - 4 \Rightarrow x = \frac{y+4}{3} \Rightarrow f^{-1}(y) = \frac{y+4}{3} \Rightarrow f^{-1}(x) = \frac{x+4}{3}$.

Example: 61 If the function $f: R \to R$ be such that $f(x) = x - [x]$, where $[y]$ denotes the greatest integer less than or equal to y, then $f^{-1}(x)$ is

(a) $\frac{1}{x - [x]}$ (b) $[x] - x$ (c) Not defined (d) None of these

Solution: (c) $f(x) = x - [x]$ Since, for $x = 0 \Rightarrow f(x) = 0$

For $x = 1 \Rightarrow f(x) = 0$.

For every integer value of x, $f(x) = 0$

$\Rightarrow$ $f(x)$ is not one-one $\Rightarrow$ So $f^{-1}(x)$ is not defined.

Example: 62 If $f: [1, \infty) \to [1, \infty)$ is defined as $f(x) = 2^{x(x-1)}$ then $f^{-1}(x)$ is equal to **[IIT Screening 1999]**

(a) $\left(\frac{1}{2}\right)^{x(x-1)}$ (b) $\frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$

(c) $\frac{1}{2}\left(1 - \sqrt{1 + 4\log_2 x}\right)$ (d) Not defined

Solution: (b) Given $f(x) = 2^{x(x-1)} \Rightarrow x(x-1) = \log_2 f(x)$

$$\Rightarrow x^2 - x - \log_2 f(x) = 0 \Rightarrow x = \frac{1 \pm \sqrt{1 + 4\log_2 f(x)}}{2}$$

Only $x = \frac{1 + \sqrt{1 + 4\log_2 f(x)}}{2}$ lies in the domain

$$\therefore \quad f^{-1}(x) = \frac{1}{2}[1 + \sqrt{1 + 4\log_2 x}]$$

Example: 63 Which of the following function is invertible **[AMU 2001]**

(a) $f(x) = 2^x$ (b) $f(x) = x^3 - x$ (c) $f(x) = x^2$ (d) None of these

Solution: (a) A function is invertible if it is one-one and onto.

Example: 64 If $f(x) = x^2 + 1$, then $f^{-1}(17)$ and $f^{-1}(-3)$ will be **[UPSEAT 2003]**

(a) 4, 1 (b) 4, 0 (c) 3, 2 (d) None of these

Solution: (d) Let $y = x^2 + 1 \Rightarrow x = \pm\sqrt{y-1}$

$\Rightarrow f^{-1}(y) = \pm\sqrt{y-1} \Rightarrow f^{-1}(x) = \pm\sqrt{x-1}$

$\Rightarrow f^{-1}(17) = \pm\sqrt{17-1} = \pm 4$

and $f^{-1}(-3) = \pm\sqrt{-3-1} = \pm\sqrt{-4}$, which is not possible.

Assignment

Value of Function

Basic Level

If $f(x)=\frac{1-x}{1+x}$, then $f[f(\cos 2\theta)]$ equal to **[MP PET 1994, 2001]**

(a) $\tan 2\theta$ (b) $\sec 2\theta$ (c) $\cos 2\theta$ (d) $\cot 2\theta$

If $f(x)=\frac{\cos^2 x+\sin^4 x}{\sin^2 x+\cos^4 x}$ for $x\in R$, then $f(2002)=$ **[EAMCET 2002]**

(a) 1 (b) 2 (c) 3 (d) 4

If $\phi(x)=a^x$, then $\{\phi(p)\}^3$ is equal to **[MP PET 1999]**

(a) $\phi(3p)$ (b) $3\phi(p)$ (c) $6\phi(p)$ (d) $2\phi(p)$

If $f(x)=\cos(\log x)$, then $f(x)\,f(y)-\frac{1}{2}[f\left(\frac{x}{y}\right)+f(xy)]=$ **[IIT 1983; Rajasthan PET 1995; MP PET 1995; KCET 1999; UPSEAT 2001]**

(a) $\frac{1}{2}$ (b) 2 (c) 0 (d) 1

If $f(\theta)=\tan\theta$, then $\frac{f(\theta)-f(\phi)}{1+f(\theta)f(\phi)}$ is equal to **[Rajasthan PET 1996]**

(a) $f(\theta-\phi)$ (b) $f(\phi-\theta)$ (c) $f(\theta+\phi)$ (d) None of these

If $f(x)=2x\sqrt{1-x^2}$, then $f\left(\sin\frac{x}{2}\right)$ equals **[Rajasthan PET 1989]**

(a) $\sin 2x$ (b) $\sin x$ (c) $2\sin x$ (d) $2\sin\frac{x}{2}$

If $f(x)=\frac{x}{x-1}$, then $\frac{f(a)}{f(a+1)}$ is equal to **[MP PET 1996]**

(a) $f(-a)$ (b) $f\left(\frac{1}{a}\right)$ (c) $f(a^2)$ (d) $f\left(\frac{-a}{a-1}\right)$

If $f(x)=\begin{cases}2x-3 & , \; x\geq 2\\ x & , \; x<2\end{cases}$, then $f(1)$ is equal to **[Karnataka CET 1989]**

(a) $2f(2)$ (b) $f(2)$ (c) $-f(2)$ (d) $\frac{1}{2}f(2)$

If $f(x)=x^2-x^{-2}$, then $f\left(\frac{1}{x}\right)$ is equal to **[SCRA 1999]**

(a) $f(x)$ (b) $-f(x)$ (c) $\frac{1}{f(x)}$ (d) $[f(x)]^2$

If $f(x) = 4x^3 + 3x^2 + 3x + 4$, then $x^3 f\left(\frac{1}{x}\right)$ is **[SCRA 1996]**

(a) $f(-x)$ (b) $\frac{1}{f(x)}$ (c) $\left[f\left(\frac{1}{x}\right)\right]^2$ (d) $f(x)$

The equivalent function of $\log x^2$ is **[MP PET 1997]**

(a) $2 \log x$ (b) $2 \log |x|$ (c) $|\log x^2|$ (d) $(\log x)^2$

Advance Level

If $f(x) = \cos[\pi]x + \cos[\pi x]$, where $[y]$ is the greatest integer function of y then $f\left(\frac{\pi}{2}\right)$ is equal to

(a) $\cos 3$ (b) 0 (c) $\cos 4$ (d) None of these

Let $f(x) = \begin{cases} 1 + |x| & , \; x < -1 \\ [x] & , \; x \geq -1 \end{cases}$, where [.] denotes the greatest integer function. Then $f\{f(-2.3)\}$ is equal to

(a) 4 (b) 2 (c) –3 (d) 3

If $f(x_1) + f(x_2) = f\left(\frac{x_1 + x_2}{1 + x_1 x_2}\right), x_1 x_2 \in (-1, 1)$, then $f(x)$ is equal to **[Roorkee 1998]**

(a) $\log\left(\frac{1-x}{1+x}\right)$ (b) $\tan^{-1}\left(\frac{1-x}{1+x}\right)$ (c) $\log\left(\frac{2x}{1-x^2}\right)$ (d) $\tan^{-1}\left(\frac{1+x}{1-x}\right)$

If $f(x) = \frac{|x|}{x}, x \neq 0$, then the value of function **[BIT Mesra 1999]**

(a) 1 (b) 0 (c) –1 (d) Does not exists

If a function $g(x)$ is defined in $[-1, 1]$ and two vertices of an equilateral triangle are (0,0) and $(x, g(x))$ and its area is $\frac{\sqrt{3}}{4}$, then $g(x)$ equals **[IIT 1989]**

(a) $\sqrt{1+x^2}$ (b) $-\sqrt{1+x^2}$ (c) $\sqrt{1-x^2}$ (d) None of these

If $f(x) = \frac{2^x + 2^{-x}}{2}$, then $f(x+y).\ f(x-y)$ is equal to **[Rajasthan PET 1998]**

(a) $\frac{1}{2}[f(x+y) + f(x-y)]$ (b) $\frac{1}{2}[f(2x) + f(2y)]$ (c) $\frac{1}{2}[f(x+y).\ f(x-y)]$ (d) None of these

$f(1) = 1$ and $f(n+1) = 2f(n) + 1$ if $n \geq 1$, then $f(n)$ is **[Karnataka CET 1994; IIT 1995]**

(a) 2^{n+1} (b) 2^n (c) $2^n - 1$ (d) $2^{n-1} - 1$

If $2f(x) - 3f(1/x) = x^2$, $x \neq 0$, then $f(2)$ is equal to **[IIT 1991]**

(a) 5/2 (b) –7/4 (c) –1 (d) None of these

If $f(x) = |x - 1|$, then correct statement is **[IIT 1983]**

(a) $f(x^2) = [f(x)]^2$ (b) $f(|x|) = |f(x)|$ (c) $f(x+y) = f(x) + f(y)$ (d) None of these

Domain of Function

Basic Level

The domain of the function $f(x) = \sqrt{\log_{0.5} x}$ is **[Roorkee 1990]**

(a) (0, 1] (b) (0, ∞) (c) (0.5, ∞) (d) [1, ∞)

The domain of definition of the real function $f(x) = \sqrt{\log_{12} x^2}$ of the real variable x is

(a) $x > 0$ (b) $|x| \geq 1$ (c) $|x| \geq 4$ (d) $x \geq 4$

The natural domain of the real valued function defined by $f(x)=\sqrt{x^2-1}+\sqrt{x^2+1}$ **[SCRA 1996]**

(a) $1<x<\infty$ (b) $-\infty<x<\infty$ (c) $-\infty<x<-1$ (d) $(-\infty,\infty)-(-1,1)$

The domain of the function $y=\sqrt{\frac{1}{x}-1}$ is, **[AMU 2000]**

(a) $x\le 1$ (b) $0\le x\le 1$ (c) $0\le x<1$ (d) $0<x\le 1$

Domain of $f(x)=\log|\log x|$ is **[Pb. CET 1998; DCE 2002]**

(a) $(0,\infty)$ (b) $(1,\infty)$ (c) $(0,1)\cup(1,\infty)$ (d) $(-\infty,1)$

Domain of function $f(x)=\left[\log_{10}\left(\frac{5x-x^2}{4}\right)\right]^{1/2}$ is **[UPSEAT 2001]**

(a) $-\infty<x<\infty$ (b) $1\le x\le 4$ (c) $4\le x\le 16$ (d) $-1\le x\le 1$

Domain of the function $\sin^{-1}\left[\log_2\left(\frac{x^2}{2}\right)\right]$ is **[MP PET 1998]**

(a) $[1,2]$ (b) $[-1,2]$ (c) $[-2,2]-(-1,1)$ (d) $[-2,2]-\{\ \}$

The domain of the function $f(x)=\frac{\sqrt{4-x^2}}{\sin^{-1}(2-x)}$ is

(a) [0, 2] (b) [0, 2) (c) [1, 2) (d) [1, 2]

The domain of the function $f(x)=\log(\sqrt{x-4}+\sqrt{6-x})$ is **[Rajasthan PET 2001]**

(a) $[4,\infty)$ (b) $(-\infty,6]$ (c) $[4,6]$ (d) None of these

Advance Level

The largest set of real values of x for which $f(x)=\sqrt{(x+2)(5-x)}-\frac{1}{\sqrt{x^2-4}}$ is a real function

(a) $[1,2]\cup(2,5]$ (b) $(2,5]$ (c) $[3,4]$ (d) None of these

The domain of the function $f(x)=\frac{1}{\log_{10}(1-x)}+\sqrt{x+2}$ is **[DCE 2000]**

(a) $]-3,-2.5[\cup]-2.5,-2[$ (b) $[-2,0[\cup]0,1[$

(c) $]0,1[$ (d) None of these

The domain of the function $f(x)=\log_e(x-[x])$, where [.] denotes the greatest integer function, is

(a) R (b) $R-Z$ (c) $(0,+\infty)$ (d) None of these

The domain of the function $f(x)=\frac{\sin^{-1}(3-x)}{\ln(|x|-2)}$ is **[Orissa JEE 2002]**

(a) $[2,4]$ (b) $(2,3)\cup(3,4]$ (c) $[2,\infty)$ (d) $(-\infty,-3)\cup[2,\infty)$

Domain of the function $f(x)=\sin^{-1}(1+3x+2x^2)$ is **[Roorkee 2000]**

(a) $(-\infty,\infty)$ (b) $(-1,1)$ (c) $\left[\frac{-3}{2},0\right]$ (d) $\left(-\infty,\frac{-1}{2}\right)\cup(2,\infty)$

Domain of the function $\sin\ln\left(\frac{\sqrt{4-x^2}}{1-x}\right)$ **[IIT 1985; Rajasthan PET 2003]**

(a) $[-2,1]$ (b) $(-2,1)$ (c) $[-2,1)$ (d) $(-2,1]$

Domain of the function $f(x)=\sqrt{\log_{0.5}(3x-8)-\log_{0.5}(x^2+4)}$ is **[AMU 1999]**

(a) $\left(\frac{8}{3}, \infty\right)$ (b) $\left(-\infty, \frac{8}{3}\right)$ (c) $(-\infty, \infty)$ (d) $(0, \infty)$

The domain of $f(x) = \frac{1}{\sqrt{|\cos x| + \cos x}}$ is

(a) $[-2n\pi, 2n\pi]$ (b) $(2n\pi, \overline{2n+1}\,\pi)$ (c) $\left(\frac{(4n+1)\pi}{2}, \frac{(4n+3)\pi}{2}\right)$ (d) $\left(\frac{(4n-1)\pi}{2}, \frac{(4n+1)\pi}{2}\right)$

The domain of $f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right) + \sqrt{1-x^2}$ is

(a) {1} (b) $(-1, 1)$ (c) {1,–1} (d) None of these

The domain of the function $f(x) = \sqrt{\log\left(\frac{1}{|\sin x|}\right)}$ is **[Rajasthan PET 2001]**

(a) $R - \{-\pi, \pi\}$ (b) $R - \{n\pi | n \in Z\}$ (c) $R - \{2n\pi | n \in Z\}$ (d) $(-\infty, \infty)$

The domain of the function $f(x) = {}^{16-x}C_{2x-1} + {}^{20-3x}P_{4x-5}$, where the symbols have their usual meanings, is the set **[AMU 2002]**

(a) {2, 3} (b) {2, 3, 4} (c) {1, 2, 3, 4} (d) {1, 2, 3, 4, 5}

Domain of the function $f(x) = \sin^{-1}\{1 + e^x\}^{-1}$ is **[AMU 1999]**

(a) $(-\infty, \infty)$ (b) $[-1, 0]$ (c) $[0, 1]$ (d) $[-1, 1]$

If n is an integer then domain of the function $\sqrt{\sin 2x}$ is **[MP PET 2003]**

(a) $[n\pi - \frac{\pi}{2}, n\pi]$ (b) $\left[n\pi, n\pi + \frac{\pi}{2}\right]$ (c) $[(2n-1)\pi, 2n\pi]$ (d) $[2n\pi, (2n+1)\pi]$

Range of Function

Basic Level

If $A\{-2, -1, 0, 1, 2\}$ and $f: A \to Z, f(x) = x^2 + 1$, then the range of f is **[Rajasthan PET 1995]**

(a) {0, 1, 2, 5} (b) {1, 2, 5} (c) {–5, –2, 1, 2, 3} (d) A

The range of the function $f: [0, 1] \to R, f(x) = x^3 - x^2 + 4x + 2\sin^{-1} x$ is

(a) $[-\pi - 2, 0]$ (b) [2, 3] (c) $[0, 4 + \pi]$ (d) $[0, 2 + \pi]$

The range of $f(x) = \cos(x/3)$ is **[Rajasthan PET 2002]**

(a) $[-1/3, 1/3]$ (b) $[-3, 3]$ (c) $[1/3, -1/3]$ (d) $[-1, 1]$

Range of $f(x) = \frac{x^2 + 34x - 71}{x^2 + 2x - 7}$ is **[Roorkee 1983]**

(a) $[5, 9]$ (b) $(-\infty, 5] \cup [9, \infty)$ (c) $(5, 9)$ (d) None of these

Range of the function $f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$ **[Karnataka CET 1993]**

(a) R (b) $[3, \infty)$ (c) $\left[\frac{1}{3}, 3\right]$ (d) None of these

Advance Level

The range of the function $f(x) = \cos[x]$, where $\frac{-\pi}{2} < x < \frac{\pi}{2}$ is **[Karnataka CET 1994]**

(a) {– 1, 1, 0} (b) $\{\cos 1, 1, \cos 2\}$ (c) $\{\cos 1, -\cos 1, 1\}$ (d) None of these

The range of the function $f(x) = |x-1| + |x-2|, \ -1 \le x \le 3$ is

(a) [1, 3] (b) [1, 5] (c) [3, 5] (d) None of these

Let $f(x) = (1+b^2)x^2 + 2bx + 1$ and $m(b)$ the minimum value of $f(x)$ for a given b. As b varies, the range of $m(b)$ is **[IIT Screening 2001]**

(a) $[0, 1]$ (b) $\left(0, \frac{1}{2}\right]$ (c) $\left[\frac{1}{2}, 1\right]$ (d) $(0, 1]$

Kind of Functions

Basic Level

Which of the following functions defined from R to R is onto **[Rajasthan PET 1985, 86]**

(a) $f(x) = |x|$ (b) $f(x) = e^{-x}$ (c) $f(x) = x^3$ (d) $f(x) = \sin x$

The number of bijective function from set A to itself when A contains 106 elements is **[EAMCET 1994]**

(a) 106 (b) $(106)^2$ (c) 106 ! (d) 2^{106}

If A contains 3 elements and B contains 4 elements, then the number of all one – one functions defined from A to B is **[EAMCET 1992; UPSEAT 2001]**

(a) 144 (b) 12 (c) 24 (d) 64

If $A = \{a, b\}$, then total number of functions which can be defined from A to A is

(a) 2 (b) 3 (c) 4 (d) 1

Function $f: R \to R, f(x) = x^3 + 7$ is **[Rajasthan PET 1984]**

(a) One – one onto (b) One – one into (c) Many – one onto (d) Many – one into

Which of the following functions from R to R is into **[Rajasthan PET 1984]**

(a) x^5 (b) $3x - 7$ (c) x^3 (d) $\sin x$

Function $f: R \to R, f(x) = x^2$ is **[IIT 1970; MP PET 1997]**

(a) One – one but not onto (b) Onto but not one- one (c) Neither one-one nor onto (d) One- one onto

If $A = R - \{3\}, B = R - \{1\}$ and $f: A \to B, \ f(x) = \frac{x-2}{x-3}$, then f is

(a) One-one (b) Onto (c) One-one onto (d) Many-one into

Advance Level

Let $f(x) = \frac{x^2 - 4}{x^2 + 4}$ for $|x| > 2$, then the function $f: (-\infty, -2] \cup [2, \infty) \to (-1, 1)$ is **[Orissa JEE 2002]**

(a) One-one into (b) One-one onto (c) Many one into (d) Many one onto

Let the function $f: R \to R$ be defined by $f(x) = 2x + \sin x, x \in R$. Then f is **[IIT Screening 2002]**

(a) One-to-one and onto (b) One-to-one but not onto

(c) Onto but not one-to-one (d) Neither one-to-one nor onto

function $f: R \to R, f(x) = x|x|$ is **[Rajasthan PET 1991, 98]**

(a) One – one but not onto (b) Onto but not one – one

(c) One – one onto (d) Neither one – one nor onto

If for two function f and g; gof is a bijection, then correct statement is **[Haryana CEE 1998]**

(a) Both g and f must be bijections (b) g must be a bijection

(c) f must be a bijection (d) Neither of them may be a bijection

If $f:[0,\infty)\to[0,\infty)$ and $f(x)=\frac{x}{1+x}$, then f is **[IIT Screening 2003]**

(a) One – one and onto (b) One – one but not onto (c) Onto but not one – one (d) Neither one – one nor onto

The number of all onto functions which can be defined from $A=\{1,\ 2,\ 3,\ .....,\ n\},\ n\ge 2$ to $B=\{a,b\}$ is **[EAMCET 1992]**

(a) ${}^nP_2$ (b) 2^n-2 (c) 2^n-1 (d) None of these

If $1+2x$ is a function having $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ as domain and $(-\infty,\infty)$ as co- domain, then it is **[IIT 1992]**

(a) Onto but not one- one (b) One – one but not onto (c) One – one and onto (d) Neither one – one nor onto

If $A=\{x\mid -1\le x\le 1\}=B$ and $f:A\to B,\ \ f(x)=\sin\pi x,$ then f is

(a) One – one (b) Onto (c) One – one onto (d) Many one into

If the real-valued function $f(x)=px+\sin x$ is a bijective function then the set of possible valued of $p\in R$ is

(a) $R-\{0\}$ (b) R (c) $(0,+\infty)$ (d) None of these

Even/Odd Functions

Basic Level

The function $f(x)=x\cos x$ is

(a) Even function (b) Odd function (c) Neither even nor odd (d) Periodic function

A function whose graph is symmetrical about the y-axis is given by

(a) $f(x)=\log_e(x+\sqrt{x^2+1})$ (b) $f(x+y)=f(x)+f(y)$ for all $x,y\in R$

(c) $f(x)=\cos x+\sin x$ (d) None of these

Let $f(x+y)=f(x)+f(y)$ for all $x,y\in R$. Then

(a) $f(x)$ is an even function (b) $f(x)$ is an odd function (c) $f(0)=0$ (d) $f(n)=nf(1),n\in N$

If $f(x)$ is an odd function then

(a) $\frac{f(-x)+f(x)}{2}$ is an even function (b) $[|\ f(x)|\ +1]$ is even, where $[x]=$ the greatest integer $\le x$

(c) $\frac{f(x)-f(-x)}{2}$ is neither even nor odd (d) None of these

Advance Level

If $f(x)$ and $g(x)$ are two functions of x such that $f(x)+g(x)=e^x$ and $f(x)-g(x)=e^{-x}$ then

(a) $f(x)$ is an odd function (b) $g(x)$ is an odd function (c) $f(x)$ is an even function (d) $g(x)$ is an even function

If $f(x)=\begin{cases}x^2\sin\frac{\pi x}{2}, & |x|<1\\ x|x|, & |x|\ge 1\end{cases}$ then $f(x)$ is

(a) An even function (b) An odd function (c) A periodic function (d) None of these

Which of the following is an even function? Here [.] denotes the greatest integer function and f is any function

(a) $[x]-x$ (b) $f(x)-f(-x)$ (c) $e^{3-2x}.\tan^2 x$ (d) $f(x)+f(-x)$

Periodic Function

Basic Level

The period of $|\cos x|$ is **[Rajasthan PET 1998]**

(a) 2π (b) π (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{2}$

The period of the function $\sin\left(\frac{\pi x}{2}\right)+\cos\left(\frac{\pi x}{2}\right)$ is **[EAMCET 1990]**

(a) 4 (b) 6 (c) 12 (d) 24

If $f(x)$ is a periodic function of the period T, then $f(ax+b)$ where $a>0$, is a periodic function of the period **[AMU 2000]**

(a) T/b (b) aT (c) bT (d) T/a

The period of the function $f(x)=\sin\left(\frac{2x+3}{6\pi}\right)$ is

(a) 2π (b) 6π (c) $6\pi^2$ (d) None of these

The period of the function $f(x)=3\sin\frac{\pi x}{3}+4\cos\frac{\pi x}{4}$ is

(a) 6 (b) 24 (c) 8 (d) 2π

The period of the function $f(x)=|\sin x|+|\cos x|$ is

(a) π (b) $\pi/2$ (c) 2π (d) None of these

Advance Level

Let $f(x)=\cos 3x+\sin\sqrt{3}x$. Then $f(x)$ is

(a) A periodic function of period 2π (b) A periodic function of period $\sqrt{3}\pi$

(c) Not a periodic function (d) None of these

$f(x)=\cos\sqrt{x}$, correct statement is **[Haryana CEE 1998]**

(a) $f(x)$ is periodic & its period $=\sqrt{2}\pi$ (b) $f(x)$ is periodic & its period $=4\pi^2$

(c) $f(x)$ is periodic & its period $=\sqrt{\pi}$ (d) $f(x)$ is not periodic

Composite Functions

Basic Level

If $f:R\to R, f(x)=\sin x; g:R\to R, g(x)=x^2$, then $(fog)(x)$ equals to **[UPSEAT 1987, 2000]**

(a) $\sin x^2$ (b) $\sin^2 x$ (c) $\sin x+x^2$ (d) $\sin\frac{x}{x^2}$

If $f(x)=(a-x^n)^{1/n}$, where $a>0$ and n is a positive integer, then $f[f(x)]=$ **[IIT 1983; UPSET 2001]**

(a) x^3 (b) x^2 (c) x (d) None of these

If $f(x)=\frac{x}{\sqrt{1+x^2}}$, then $fofof(x)$ is equal to **[Rajasthan PET 2000]**

(a) $\frac{x}{\sqrt{1+3x^2}}$ (b) $\frac{x}{\sqrt{1+2x^2}}$ (c) $\frac{x}{\sqrt{1+x^2}}$ (d) None of these

Let f and g be functions defined by $f(x)=\frac{x}{x+1}, g(x)=\frac{x}{1-x}$, then $(fog)(x)$ is

(a) $\frac{1}{x}$ (b) $\frac{1}{x-1}$ (c) $x-1$ (d) x

If $f(x)=ax+b$ and $g(x)=cx+d$, then $f(g(x))=g(f(x))$ is equivalent to **[UPSEAT 2001]**

(a) $f(a)=g(c)$ (b) $f(b)=g(b)$ (c) $f(d)=g(b)$ (d) $f(c)=g(a)$

Advance Level

If $f(x)=\sqrt{|x-1|}$ and $g(x)=\sin x$, then $(fog)(x)$ is equal to **[Roorkee 1992]**

(a) $\sin\sqrt{|x-1|}$ (b) $|\sin x/2-\cos x/2|$ (c) $|\sin x-\cos x|$ (d) None of these

If f and g are two real valued function defined by $f(x)=e^x$ and $g(x)=3x-2$, then $(fog)^{-1}(x)$ is equal to **[Roorkee 1998]**

(a) $\log(3x-2)$ (b) $\frac{2+\log x}{3}$ (c) $\log\left(\frac{x+2}{3}\right)$ (d) None of these

If $f(x)=\frac{1}{1-x}, x\neq 0,1$, then the graph of the function $y=f\{f(f(x))\}, x>1$, is

(a) A circle (b) An ellipse (c) A straight line (d) A pair of straight lines

If $f(x)$ is defined on [0, 1] by the rule $f(x)=\begin{cases}x, \text{if } x \text{ is rational}\\ 1-x, \text{if } x \text{ is irrational}\end{cases}$. Then for all $x\in[0,1], f(f(x))$ is

(a) Constant (b) $1+x$ (c) x (d) None of these

Inverse Function

Basic Level

$f: R\to R$ is a function defined by $f(x)=10x-7$. If $g=f^{-1}$, then $g(x)=$ **[EAMCET 1993]**

(a) $\frac{1}{10x-7}$ (b) $\frac{1}{10x+7}$ (c) $\frac{x+7}{10}$ (d) $\frac{x-7}{10}$

If $y=f(x)=\frac{x+2}{x-1}$, then $x=$ **[IIT 1984]**

(a) $f(y)$ (b) $2f(y)$ (c) $\frac{1}{f(y)}$ (d) None of these

Inverse of the function $y=2x-3$ is **[UPSEAT 2002]**

(a) $\frac{x+3}{2}$ (b) $\frac{x-3}{2}$ (c) $\frac{1}{2x-3}$ (d) None of these

Advance Level

The value of α for which the function $f(x)=1+\alpha x, \alpha\neq 0$ is inverse of itself will be **[IIT 1992]**

(a) – 2 (b) – 1 (c) 1 (d) 2

If $f:[1,+\infty)\to[2,+\infty)$ is given by $f(x)=x+\frac{1}{x}$ then f^{-1} equals **[IIT Screening 2001]**

(a) $\frac{x+\sqrt{x^2-4}}{2}$ (b) $\frac{x}{1+x^2}$ (c) $\frac{x-\sqrt{x^2-4}}{2}$ (d) $1+\sqrt{x^2-4}$

The inverse of the function $f(x) = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$ is **[Rajasthan PET 2001]**

(a) $\log_{10}(2 - x)$ (b) $\frac{1}{2}\log_{10}\left(\frac{1+x}{1-x}\right)$ (c) $\frac{1}{2}\log_{10}(2x - 1)$ (d) $\frac{1}{4}\log\left(\frac{2x}{2-x}\right)$

The inverse of the function $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 2$ is given by **[Haryana CEE 1996]**

(a) $\log_e\left(\frac{x-2}{x-1}\right)^{\frac{1}{2}}$ (b) $\log_e\left(\frac{x-1}{3-x}\right)^{\frac{1}{2}}$ (c) $\log_e\left(\frac{x}{2-x}\right)^{\frac{1}{2}}$ (d) $\log_e\left(\frac{x-1}{x+1}\right)^{-2}$

***

Answer Sheet

Assignment (Basic & Advance Level)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
c	a	a	c	a	b	c	b	b	d	b	c	d	a	d	c	b	c	b	d
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
a	b	d	d	c	b	c	c	c	b	b	b	b	c	b	a	d	c	b	a
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
a	b	b	c	d	b	c	b	b	d	c	c	c	c	a	d	c	c	c	a
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
c	a	b	b	b	b	d	b	d	b,c,d	a,b	b,c	b	d	b	a	d	c	b	b
81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98		
c	d	a	c	a	d	c	b	b	c	c	c	a	a	b	a	b	b		

2.2 Limits

2.2.1 Limit of a Function

Let $y = f(x)$ be a function of x. If at $x = a$, $f(x)$ takes indeterminate form, then we consider the values of the function which are very near to 'a'. If these values tend to a definite unique number as x tends to 'a', then the unique number so obtained is called the limit of $f(x)$ at $x = a$ and we write it as $\lim_{x \to a} f(x)$.

(1) **Meaning of '$x \to a$':** Let x be a variable and a be the constant. If x assumes values nearer and nearer to 'a' then we say 'x tends to a' and we write '$x \to a$'. It should be noted that as $x \to a$, we have $x \neq a$. By 'x tends to a' we mean that

(i) $x \neq a$

(ii) x assumes values nearer and nearer to 'a' and

(iii) We are not specifying any manner in which x should approach to 'a'. x may approach to a from left or right as shown in figure.

(2) **Left hand and right hand limit :** Consider the values of the functions at the points which are very near to a on the left of a. If these values tend to a definite unique number as x tends to a, then the unique number so obtained is called left-hand limit of $f(x)$ at $x = a$ and symbolically we write it as $f(a-0) = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$

Similarly we can define right-hand limit of $f(x)$ at $x = a$ which is expressed as $f(a+0) = \lim_{x \to a^+} f(x)$ $= \lim_{h \to 0} f(a+h)$.

(3) **Method for finding L.H.L. and R.H.L.**

(i) For finding right hand limit (R.H.L.) of the function, we write $x + h$ in place of x, while for left hand limit (L.H.L.) we write $x - h$ in place of x.

(ii) Then we replace x by 'a' in the function so obtained.

(iii) Lastly we find limit $h \to 0$.

(4) **Existence of limit :** $\lim_{x \to a} f(x)$ exists when,

(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist *i.e.* L.H.L. and R.H.L. both exists.

(ii) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ *i.e.* L.H.L. = R.H.L.

Note : ❒ If a function $f(x)$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = a$, then we say that $f(x)$ is indeterminate or meaningless at $x = a$. Other indeterminate forms are $\infty - \infty, \infty \times \infty, 0 \times \infty, 1^\infty, 0^0, \infty^0$

❒ In short, we write L.H.L. for left hand limit and R.H.L. for right hand limit.

❒ It is not necessary that if the value of a function at some point exists then its limit at that point must exist.

(5) **Sandwich theorem :** If $f(x)$, $g(x)$ and $h(x)$ are any three functions such that, $f(x) \le g(x) \le h(x)\ \forall x \in$ neighborhood of $x = a$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = l$ (say), then $\lim_{x\to a} g(x) = l$. This theorem is normally applied when the $\lim_{x\to a} g(x)$ can't be obtained by using conventional methods as function $f(x)$ and $h(x)$ can be easily found.

Example: 1 If $f(x) = \begin{cases} x, & \text{when } x > 1 \\ x^2, & \text{when } x < 1 \end{cases}$, then $\lim_{x\to 1} f(x) =$ **[MP PET 1987]**

(a) x^2 (b) x (c) -1 (d) 1

Solution: (d) To find L.H.L. at $x = 1$. *i.e.,*

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}(1-h)^2 = \lim_{h\to 0}(1+h^2-2h) = 1$ i.e., $\lim_{x\to 1^-} f(x) = 1$ ....(i)

Now find R.H.L. at x = 1 i.e., $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = 1$ i.e., $\lim_{x\to 1^+} f(x) = 1$ .....(ii)

From (i) and (ii), L.H.L. = R.H.L. $\Rightarrow \lim_{x\to 1} f(x) = 1$.

Example: 2 $\lim_{x\to 2} \frac{|x-2|}{x-2} =$

(a) 1 (b) -1 (c) Does not exist (d) None of these

Solution: (c) L.H.L.= $\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{h\to 0} \frac{|2-h-2|}{2-h-2} = \lim_{h\to 0} \frac{h}{-h} = -1$ .....(i)

and, R.H.L.= $\lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{h\to 0} \frac{|2+h-2|}{2+h-2} = \lim_{h\to 0} \frac{h}{h} = 1$ .....(ii)

From (i) and (ii) L.H.L. $\ne$ R.H.L. *i.e.* $\lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist.

Example: 3 If $f(x) = \begin{cases} \frac{2}{5-x}, & \text{when } x < 3 \\ 5-x, & \text{when } x > 3 \end{cases}$, then

(a) $\lim_{x\to 3^+} f(x) = 0$ (b) $\lim_{x\to 3^-} f(x) = 0$ (c) $\lim_{x\to 3^+} f(x) \ne \lim_{x\to 3^-} f(x)$ (d) None of these

Solution: (c) $\lim_{x\to 3+} f(x) = 5-3 = 2$ and $\lim_{x\to 3-} f(x) = \frac{2}{5-3} = 1$

Example: 4 Let the function *f* be defined by the equation $f(x) = \begin{cases} 3x, & \text{if } 0 \le x \le 1 \\ 5-3x, & \text{if } 1 < x \le 2 \end{cases}$, then **[SCRA 1996]**

(a) $\lim_{x\to 1} f(x) = f(1)$ (b) $\lim_{x\to 1} f(x) = 3$ (c) $\lim_{x\to 1} f(x) = 2$ (d) $\lim_{x\to 1} f(x)$ does not exist

Solution: (d) L.H.L. = $\lim_{x\to 1-0} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} 3(1-h) = \lim_{h\to 0}(3-3h) = 3-3.0 = 3$

R.H.L. = $\lim_{x\to 1+0} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}[5-3(1+h)] = \lim_{h\to 0}(2-3h) = 2-3.0 = 2$

Hence $\lim_{x\to 1} f(x)$ does not exists.

Example: 5 $\lim_{x\to 0} \frac{|x|}{x} =$ **[Roorkee 1982; UPSEAT 2001]**

(a) 1 (b) -1 (c) 0 (d) Does not exist

Solution: (d) $\because \lim_{x\to 0-}\frac{|x|}{x}=-1$ and $\lim_{x\to 0+}\frac{|x|}{x}=1$, hence limit does not exists.

2.2.2 Fundamental Theorems on Limits

The following theorems are very useful for evaluation of limits if $\lim_{x\to 0} f(x)=l$ and $\lim_{x\to 0} g(x)=m$ (l and m are real numbers) then

(1) $\lim_{x\to a}(f(x)+g(x))=l+m$ (Sum rule)

(2) $\lim_{x\to a}(f(x)-g(x))=l-m$ (Difference rule)

(3) $\lim_{x\to a}(f(x).g(x))=l.m$ (Product rule)

(4) $\lim_{x\to a} k\ f(x)=k.l$ (Constant multiple rule)

(5) $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{l}{m}, m\neq 0$ (Quotient rule)

(6) If $\lim_{x\to a} f(x)=+\infty$ or $-\infty$, then $\lim_{x\to a}\frac{1}{f(x)}=0$

(7) $\lim_{x\to a}\log\{f(x)\}=\log\{\lim_{x\to a} f(x)\}$

(8) If $f(x)\le g(x)$ for all x, then $\lim_{x\to a} f(x)\le \lim_{x\to a} g(x)$

(9) $\lim_{x\to a}[f(x)]^{g(x)}=\{\lim_{x\to a} f(x)\}^{\lim_{x\to a} g(x)}$

(10) If p and q are integers, then $\lim_{x\to a}(f(x))^{p/q}=l^{p/q}$, provided $(l)^{p/q}$ is a real number.

(11) If $\lim_{x\to a} f(g(x))=f(\lim_{x\to a} g(x))=f(m)$ provided 'f' is continuous at $g(x)=m$. *e.g.* $\lim_{x\to a}\ln[f(x)]=\ln(l)$, only if $l>0$.

2.2.3 Some Important Expansions

In finding limits, use of expansions of following functions are useful :

(1) $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+.....$

(2) $a^x=1+x\log a+\frac{(x\log a)^2}{2!}+.....$

(3) $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+.....$

(4) $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+.....,|x|<1$

(5) $\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-.......,$ where $|x|<1$

(6) $(1+x)^{\frac{1}{x}}=e^{\frac{1}{x}\log(1+x)}=e^{1-\frac{x}{2}+\frac{x^2}{3}}.......=e\left(1-\frac{x}{2}+\frac{11}{24}x^2-.......\right)$

(7) $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-.......$

(8) $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+......$

(9) $\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+.....$

(10) $\sinh x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+.....$

(11) $\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+.....$

(12) $\tanh x=x-\frac{x^3}{3}+2x^5-.....$

(13) $\sin^{-1} x = x + 1^2.\frac{x^3}{3!} + 3^2.1^2.\frac{x^5}{5!} + ...$ (14) $\cos^{-1} x = \left(\frac{\pi}{2}\right) - \sin^{-1} x$

(15) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

2.2.4 Methods of Evaluation of Limits

We shall divide the problems of evaluation of limits in five categories.

(1) **Algebraic limits :** Let $f(x)$ be an algebraic function and 'a' be a real number. Then $\lim_{x \to a} f(x)$ is known as an algebraic limit.

(i) **Direct substitution method :** If by direct substitution of the point in the given expression we get a finite number, then the number obtained is the limit of the given expression.

(ii) **Factorisation method :** In this method, numerator and denominator are factorised. The common factors are cancelled and the rest outputs the results.

(iii) **Rationalisation method :** Rationalisation is followed when we have fractional powers (like $\frac{1}{2}, \frac{1}{3}$ etc.) on expressions in numerator or denominator or in both. After rationalisation the terms are factorised which on cancellation gives the result.

(iv) **Based on the form when $x \to \infty$:** In this case expression should be expressed as a function $1/x$ and then after removing indeterminate form, (if it is there) replace $\frac{1}{x}$ by 0.

Step I : Write down the expression in the form of rational function, *i.e.*, $\frac{f(x)}{g(x)}$, if it is not so.

Step II : If k is the highest power of x in numerator and denominator both, then divide each term of numerator and denominator by x^k.

Step III : Use the result $\lim_{x \to \infty} \frac{1}{x^n} = 0$, where $n > 0$.

Note : ❐ **An important result :** If *m, n* are positive integers and $a_0, b_0 \neq 0$ are non-zero real numbers,

then $$\lim_{x \to \infty} \frac{a_0 x^m + a_1 x^{m-1} + ... + a_{m-1} x + a_m}{b_0 x^n + b_1 x^{n-1} + ... + b_{n-1} x + b_n} = \begin{cases} \frac{a_0}{b_0}, \text{ if } m = n \\ 0, \text{ if } m < n \\ \infty, \text{if } m > n \end{cases}$$

Example: 6 $\lim_{x \to 1} (3x^2 + 4x + 5) =$

(a) 12 (b) –1 (c) Does not exist (d) None of these

Solution: (a) $\lim_{x \to 1} (3x^2 + 4x + 5) = 3(1)^2 + 4(1) + 5 = 12$.

Example: 7 The value of $\lim_{x \to 2} \frac{3^{x/2} - 3}{3^x - 9}$ is **[MP PET 2000]**

(a) 0 (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\ln 3$

Solution: (c) $\lim_{x\to 2}\frac{3^{x/2}-3}{(3^{x/2})^2-(3)^2}=\lim_{x\to 2}\frac{(3^{x/2}-3)}{(3^{x/2}-3)(3^{x/2}+3)}=\frac{1}{6}$.

Example: 8 The value of $\lim_{x\to a}\frac{x^n-a^n}{x-a}$ is **[Rajasthan PET 1989, 92]**

(a) 0 (b) na^{n-1} (c) na^n (d) 1

Solution: (b) $\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\frac{(x-a)(x^{n-1}+x^{n-2}a+..+a^{n-1})}{(x-a)}=\lim_{x\to a}(x^{n-1}+x^{n-2}a+....+a^{n-1})=n.a^{n-1}$.

Example: 9 $\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{x+h}-\frac{1}{x}\right]$ equals **[Rajasthan PET 1987]**

(a) $\frac{1}{2x}$ (b) $-\frac{1}{2x}$ (c) $\frac{1}{x^2}$ (d) $-\frac{1}{x^2}$

Solution: (d) $\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{x+h}-\frac{1}{x}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{x-(x+h)}{(x+h)x}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{-h}{(x+h)x}\right]=-\frac{1}{x^2}$.

Example: 10 The value of $\lim_{x\to 0}\frac{\sqrt{1-x^2}-\sqrt{1+x^2}}{x^2}$ is **[MP PET 1999]**

(a) 1 (b) –1 (c) – 2 (d) 0

Solution: (b) $\lim_{x\to 0}\frac{\left(\sqrt{1-x^2}-\sqrt{1+x^2}\right)}{x^2}\frac{\left(\sqrt{1-x^2}+\sqrt{1+x^2}\right)}{\left(\sqrt{1-x^2}+\sqrt{1+x^2}\right)}=\lim_{x\to 0}\frac{(1-x^2)-(1+x^2)}{x^2\left(\sqrt{1-x^2}+\sqrt{1+x^2}\right)}=\frac{-2}{2}=-1$.

Example: 11 $\lim_{x\to 3}\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$ equals **[UPSEAT 1991]**

(a) 1 (b) $\frac{3}{2}$ (c) $\frac{1}{4}$ (d) None of these

Solution: (d) $\lim_{x\to 3}\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}=\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{\left(\sqrt{x-2}\right)^2-\left(\sqrt{4-x}\right)^2}$

$=\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{(2x-6)}=\lim_{x\to 3}\frac{\sqrt{x-2}+\sqrt{4-x}}{2}=\frac{1+1}{2}=1$.

Example: 12 $\lim_{x\to\infty}\frac{ax^2+bx+c}{dx^2+ex+f}=$

(a) $\frac{b}{e}$ (b) $\frac{c}{f}$ (c) $\frac{a}{d}$ (d) $\frac{d}{a}$

Solution: (c) Here the expression assumes the form $\frac{\infty}{\infty}$. We note that the highest power of x in both the numerator and denominator is 2. So we divide each terms in both the numerator and denominator by x^2.

$$\lim_{x\to\infty}\frac{ax^2+bx+c}{dx^2+ex+f}=\lim_{x\to\infty}\frac{a+\frac{b}{x}+\frac{c}{x^2}}{d+\frac{e}{x}+\frac{f}{x^2}}=\frac{a+0+0}{d+0+0}=\frac{a}{d}.$$

Example: 13 $\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) $\log 2$ (d) e^4

Solution: (b) $\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]=\lim_{x\to\infty}\frac{x+\sqrt{x+\sqrt{x}}-x}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}=\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}=\lim_{x\to\infty}\frac{\sqrt{1+x^{-1/2}}}{\sqrt{1+\sqrt{x^{-1}+x^{-3/2}}}+1}=\frac{1}{2}$.

Example: 14 The values of constants a and b so that $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=0$ is

(a) $a=0, b=0$ (b) $a=1, b=-1$ (c) $a=-1, b=1$ (d) $a=2, b=-1$

Solution: (b) We have $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=0 \Rightarrow \lim_{x\to\infty}\frac{x^2(1-a)-x(a+b)+1-b}{x+1}=0$

Since the limit of the given expression is zero, therefore degree of the polynomial in numerator must be less than that of denominator. As the denominator is a first degree polynomial. So, numerator must be a constant *i.e.*, a zero degree polynomial. $\therefore$ $1-a=0$ and $a+b=0 \Rightarrow a=1$ and $b=-1$. Hence, $a=1$ and $b=-1$.

Example: 15 $\lim_{x\to 1} x^x =$

(a) 1 (b) ∞ (c) Not defined (d) None of these

Solution: (a) $\lim_{x\to 1} x^x = \left(\lim_{x\to 1} x\right)^{\lim_{x\to 1} x} = 1^1 = 1$

Example: 16 $\lim_{x\to 1}(1+x)^{1/x} =$

(a) 2 (b) e (c) Not defined (d) None of these

Solution: (a) $\lim_{x\to 1}(1+x)^{1/x} = \left(\lim_{x\to 1}(1+x)\right)^{\lim_{x\to 1}\left(\frac{1}{x}\right)} = 2$

Example: 17 The value of the limit of $\frac{x^3-x^2-18}{x-3}$ as x tends to 3 is

(a) 3 (b) 9 (c) 18 (d) 21

Solution: (d) Let $y=\lim_{x\to 3}\frac{x^3-x^2-18}{x-3} = \lim_{x\to 3}(x^2+2x+6)=9+6+6=21$

Example: 18 The value of the limit of $\frac{x^3-8}{(x^2-4)}$ as x tends to 2 is

(a) 3 (b) $\frac{3}{2}$ (c) 1 (d) 0

Solution: (a) $\lim_{x\to 2}\frac{x^3-8}{x^2-4}=\lim_{x\to 2}\frac{(x^2+2x+4)(x-2)}{(x+2)(x-2)}=\lim_{x\to 2}\frac{x^2+2x+4}{x+2}=\frac{4+4+4}{2+2}=3$.

Example: 19 $\lim_{x\to 0}\frac{x}{\sqrt{1+x}-\sqrt{1-x}}$ is equal to **[Rajasthan PET 1988]**

(a) $\frac{1}{2}$ (b) 2 (c) 1 (d) 0

Solution: (c) $\lim_{x\to 0}\left(\frac{x}{\sqrt{1+x}-\sqrt{1-x}}\right)=\lim_{x\to 0}\left(\frac{x}{\sqrt{1+x}-\sqrt{1-x}}\times\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$=\lim_{x\to 0}\left(\frac{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}{1+x-1+x}\right)=\lim_{x\to 0}\left(\frac{\left(\sqrt{1+x}+\sqrt{1-x}\right)}{2}\right)=\frac{2}{2}=1$

Example: 20 $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$ equals **[IIT 1978; Kurukshetra CEE 1998]**

(a) $\frac{2a}{3\sqrt{3}}$ (b) $\frac{2}{3\sqrt{3}}$ (c) 0 (d) None of these

Solution: (b) $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} = \lim_{x\to a}\left(\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right)\times\left(\frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}}\right)\times\left(\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}\right)$

$= \lim_{x\to a}\left\{\frac{\sqrt{3a+x}+2\sqrt{x}}{3(\sqrt{a+2x}+\sqrt{3x})}\right\} = \frac{2}{3\sqrt{3}}$.

Example: 21 $\lim_{n\to\infty}\frac{1^{99}+2^{99}+3^{99}+....+n^{99}}{n^{100}} =$ **[EAMCET 1994]**

(a) $\frac{99}{100}$ (b) $\frac{1}{100}$ (c) $\frac{1}{99}$ (d) $\frac{1}{101}$

Solution: (b) $\lim_{n\to\infty}\frac{1^{99}+2^{99}+3^{99}+....+n^{99}}{n^{100}} = \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{r^{99}}{n^{100}}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{99} = \int_0^1 x^{99}\,dx = \left[\frac{x^{100}}{100}\right]_0^1 = \frac{1}{100}$.

Example: 22 The values of constants 'a' and 'b' so that $\lim_{x\to\infty}\left(\frac{x^2-1}{x+1}-ax-b\right)=2$ is

(a) $a=0, b=0$ (b) $a=1, b=-1$ (c) $a=1, b=-3$ (d) $a=2, b=-1$

Solution: (c) $\lim_{x\to\infty}\left(\frac{x^2-1}{x+1}-ax-b\right)=2 \Rightarrow \lim_{x\to\infty} x-1-ax-b=2 \Rightarrow \lim_{x\to\infty} x(1-a)-(1+b)=2$.

Comparing the coefficient of both sides, $1-a=0$ and $1+b=-2 \Rightarrow a=1, b=-3$

Example: 23 $\lim_{n\to\infty}\left[\frac{\Sigma n^2}{n^3}\right] =$ **[Rajasthan PET 1999, 2002]**

(a) $-\frac{1}{6}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{-1}{3}$

Solution: (c) $\lim_{n\to\infty}\left[\frac{n(n+1)(2n+1)}{6n^3}\right] = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6} = \frac{1}{3}$

Note : ❒ Students should remember that,

$\lim_{n\to\infty}\frac{\Sigma n}{n^2}=\frac{1}{2}$ and $\lim_{n\to\infty}\frac{\Sigma n^2}{n^3}=\frac{1}{3}$.

Example: 24 $\lim_{n\to\infty}\left[\frac{1}{1-n^2}+\frac{2}{1-n^2}+......+\frac{n}{1-n^2}\right]$ is equal to **[IIT 1984; DCE 2000]**

(a) 0 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) None of these

Solution: (b) $\lim_{n\to\infty}\left[\frac{1}{1-n^2}+\frac{2}{1-n^2}+......+\frac{n}{1-n^2}\right] = \lim_{n\to\infty}\frac{\Sigma n}{1-n^2} = \frac{1}{2}\lim_{n\to\infty}\frac{n^2+n}{1-n^2} = -\frac{1}{2}$.

Example: 25 If $f(x)=\frac{2}{x-3}, g(x)=\frac{x-3}{x+4}$ and $h(x)=-\frac{2(2x+1)}{x^2+x-12}$ then $\lim_{x\to 3}[f(x)+g(x)+h(x)]$ is

(a) – 2 (b) – 1 (c) $-\frac{2}{7}$ (d) 0

Solution: (c) We have $f(x)+g(x)+h(x)=\frac{x^2-4x+17-4x-2}{x^2+x-12} = \frac{x^2-8x+15}{x^2+x-12} = \frac{(x-3)(x-5)}{(x-3)(x+4)}$

$\therefore \lim_{x\to 3}[f(x)+g(x)+h(x)] = \lim_{x\to 3}\frac{(x-3)(x-5)}{(x-3)(x+4)} = -\frac{2}{7}$.

Example: 26 If $\lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n}$ equal **[Kurukshetra CEE 1998]**

(a) e (b) $\frac{1}{e}$ (c) $\frac{\pi}{4}$ (d) $\frac{4}{\pi}$

Solution: (b) Let $P=\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n} \Rightarrow P=\lim_{n\to\infty}\left(\frac{1}{n}.\frac{2}{n}.\frac{3}{n}.\frac{4}{n}..........\frac{n}{n}\right)^{1/n}$

$\therefore \log P=\frac{1}{n}\lim_{n\to\infty}\left(\log\frac{1}{n}+\log\frac{2}{n}+.........+\log\frac{n}{n}\right) \Rightarrow \log P=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\frac{r}{n}$

$\log P=\int_0^1 \log x\,dx=[x\log x-x]_0^1=(-1)\Rightarrow P=\frac{1}{e}$.

Example: 27 If $\lim_{x\to\infty}\left[\frac{x^3+1}{x^2+1}-(ax+b)\right]=2$, then **[Karnataka CET 2000]**

(a) $a=1$ and $b=1$ (b) $a=1$ and $b=-1$ (c) $a=1$ and $b=-2$ (d) $a=1$ and $b=2$

Solution: (c) $\lim_{x\to\infty}\left(\frac{x^3+1}{x^2+1}-(ax+b)\right)=2\Rightarrow \lim_{x\to\infty}\left(\frac{x^3(1-a)-bx^2-ax+(1-b)}{x^2+1}\right)=2 \Rightarrow \lim_{x\to\infty}[x^3(1-a)-bx^2-ax+(1-b)]=2(x^2+1)$.

Comparing the coefficients of both sides, $1-a=0$ and $-b=2$ or $a=1, b=-2$.

Example: 28 $\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+.....+(x+100)^{10}}{x^{10}+10^{10}}$ is equal to **[AMU 2000]**

(a) 0 (b) 1 (c) 10 (d) 100

Solution: (d) $\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+......+(x+100)^{10}}{x^{10}+10^{10}}=\lim_{x\to\infty}\frac{x^{10}\left[\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}+...+\left(1+\frac{100}{x}\right)^{10}\right]}{x^{10}\left[1+\frac{10^{10}}{x^{10}}\right]}=100$.

Example: 29 Let $f(x)=4$ and $f'(x)=4$, then $\lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2}$ equals **[Rajasthan 2000; AIEEE 2002]**

(a) 2 (b) -2 (c) -4 (d) 3

Solution: (c) $y=\lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2} \Rightarrow y=\lim_{x\to 2}\frac{xf(2)-2f(2)+2f(2)-2f(x)}{x-2}$

$\Rightarrow y=\lim_{x\to 2}\frac{-2f(x)+2f(2)+xf(2)-2f(2)}{(x-2)} \Rightarrow y=\lim_{x\to 2}-2\frac{[f(x)-f(2)]}{x-2}+\lim_{x\to 2}\frac{f(2).(x-2)}{(x-2)}$

$\Rightarrow y=-2\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}+f(2) \Rightarrow y=-2\lim_{x\to 2}f'(x)+f(2)=-8+4=-4$.

(2) **Trigonometric limits :** To evaluate trigonometric limits the following results are very important.

(i) $\lim_{x\to 0}\frac{\sin x}{x}=1=\lim_{x\to 0}\frac{x}{\sin x}$

(ii) $\lim_{x\to 0}\frac{\tan x}{x}=1=\lim_{x\to 0}\frac{x}{\tan x}$

(iii) $\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1=\lim_{x\to 0}\frac{x}{\sin^{-1}x}$

(iv) $\lim_{x\to 0}\frac{\tan^{-1}x}{x}=1=\lim_{x\to 0}\frac{x}{\tan^{-1}x}$

(v) $\lim_{x\to 0}\frac{\sin x^0}{x}=\frac{\pi}{180}$

(vi) $\lim_{x\to 0}\cos x=1$

(vii) $\lim_{x\to a}\frac{\sin(x-a)}{x-a}=1$

(viii) $\lim_{x\to a}\frac{\tan(x-a)}{x-a}=1$

(ix) $\lim_{x\to a}\sin^{-1}x=\sin^{-1}a, |a|\le 1$

(x) $\lim_{x\to a}\cos^{-1}x=\cos^{-1}a; |a|\le 1$

(xi) $\lim_{x\to a} \tan^{-1} x = \tan^{-1} a; -\infty < a < \infty$ (xii) $\lim_{x\to\infty} \frac{\sin x}{x} = \lim_{x\to\infty} \frac{\cos x}{x} = 0$

(xiii) $\lim_{x\to\infty} \frac{\sin(1/x)}{(1/x)} = 1$

Example: 30 $\lim_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right) =$ **[IIT 1978, 84; Rajasthan PET 1997, 2001; UPSEAT 2003]**

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{2}{\pi}$ (d) 0

Solution: (c) $\lim_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right)$, Put $1-x=y \Rightarrow$ as $x \to 1, y \to 0$

Thus $\lim_{y\to 0} y \tan \frac{\pi(1-y)}{2} = \lim_{y\to 0} \frac{2}{\pi} . \frac{\left(\frac{\pi y}{2}\right)}{\tan\left(\frac{\pi y}{2}\right)} = \frac{2}{\pi}\times 1 = \frac{2}{\pi}$.

Example: 31 $\lim_{x\to 1} \frac{\sqrt{1-\cos 2(x-1)}}{x-1}$ **[IIT 1998; UPSEAT 2001]**

(a) Exists and it equal $\sqrt{2}$
(b) Exists and it equals $-\sqrt{2}$
(c) Does not exist because $x-1 \to 0$
(d) Does not exist because left hand limit is not equal to right hand limit

Solution: (d) $f(1+) = \lim_{h\to 0} f(1+h) = = \lim_{h\to 0} \frac{\sqrt{1-\cos 2h}}{h} = \lim_{h\to 0} \sqrt{2}\,\frac{\sin h}{h} = \sqrt{2}$

$f(1-) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} \frac{\sqrt{1-\cos(-2h)}}{-h} = \lim_{h\to 0} \sqrt{2}\,\frac{\sin h}{-h} = -\sqrt{2}$.

$\therefore$ limit does not exist because left hand limit is not equal to right hand limit.

Example: 32 $\lim_{x\to 0} \frac{(1-\cos 2x)\sin 5x}{x^2 \sin 3x} =$ **[MP PET 2000; UPSEAT 2000; Karmataka CET 2002]**

(a) $\frac{10}{3}$ (b) $\frac{3}{10}$ (c) $\frac{6}{5}$ (d) $\frac{5}{6}$

Solution: (a) $\lim_{x\to 0} \frac{2\sin^2 x}{x^2} \frac{\sin 5x}{\sin 3x} \frac{3x}{3x} \frac{5x}{5x} = \lim_{x\to 0} \frac{2\sin^2 x}{x^2} . \frac{3x}{\sin 3x} . \frac{\sin 5x}{5x} . \frac{5x}{3x} = 2 . \frac{5}{3} = \frac{10}{3}$.

Example: 33 $\lim_{x\to 0} \frac{x^3}{\sin x^2} =$

(a) 0 (b) $\frac{1}{3}$ (c) 3 (d) $\frac{1}{2}$

Solution: (a) $\lim_{x\to 0} \frac{x^3}{\sin x^2} = \lim_{x\to 0} \frac{x^2}{\sin x^2} . x = \left(\lim_{x\to 0} \frac{x^2}{\sin x^2}\right)\left(\lim_{x\to 0} x\right) = 1.0 = 0$.

Example: 34 $\lim_{x\to 0} \frac{\sin 3x + \sin x}{x} =$

(a) $\frac{1}{3}$ (b) 3 (c) 4 (d) $\frac{1}{4}$

Solution: (c) $\lim_{x\to 0} \frac{\sin 3x + \sin x}{x} = \lim_{x\to 0} \frac{\sin 3x}{x} + \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin 3x}{3x} .3 + \lim_{x\to 0} \frac{\sin x}{x} = 1.3 + 1 = 4$.

Example: 35 If $f(x)=\begin{cases} x\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0\end{cases}$, then $\lim_{x\to 0} f(x)=$ **[IIT 1988; UPSEAT 1988; SCRA 1996]**

(a) 1 (b) 0 (c) –1 (d) None of these

Solution: (b) $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)=\left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0}\sin\frac{1}{x}\right)$ = 0 × (A number oscillating between – 1 and 1) = 0.

Example: 36 If $f(x)=\begin{cases} \frac{\sin[x]}{[x]}, & [x]\neq 0 \\ 0, & [x]=0\end{cases}$, then $\lim_{x\to 0} f(x)$ equals **[IIT 1985; Rajasthan PET 1995]**

(a) 1 (b) 0 (c) –1 (d) Does not exist

Solution: (d) In closed interval of $x=0$ at right hand side [x] =0 and at left hand side $[x]=-1$. Also [0] =0.

Therefore function is defined as $f(x)=\begin{cases} \frac{\sin[x]}{[x]}, & (-1\leq x<0) \\ 0, & (0\leq x<1)\end{cases}$

$\therefore$ Left hand limit $=\lim_{x\to 0-} f(x)=\lim_{x\to 0-}\frac{\sin[x]}{[x]}=\frac{\sin(-1)}{-1}=\sin 1^c$

Right hand limit = 0, Hence, limit doesn't exist.

Example: 37 $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$ **[IIT 1974; Rajasthan PET 2000]**

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{2}{3}$ (d) None of these

Solution: (a) $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{\sin x-\sin x\cos x}{x^3\cos x}=\lim_{x\to 0}\frac{\sin x\left(2\sin^2\frac{x}{2}\right)}{x^3\cos x}=\lim_{x\to 0}\left[\frac{\sin x}{x}.\frac{2}{\cos x}.\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}.\frac{1}{4}\right]=\frac{1}{2}$

Example: 38 If $f(x)=\frac{\sin(e^{x-2}-1)}{\log(x-1)}$, then $\lim_{x\to 2} f(x)$ is given by

(a) – 2 (b) –1 (c) 0 (d) 1

Solution: (d) $\lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{\sin(e^{x-2}-1)}{\log(t+1)}=\lim_{t\to 0}\frac{\sin(e^t-1)}{\log(t+1)}$. (Putting $x = 2 + t$)

$=\lim_{x\to\infty}\frac{\sin(e^t-1)}{e^t-1}.\frac{e^t-1}{t}.\frac{t}{\log(1+t)}=\lim_{t\to 0}\frac{\sin(e^t-1)}{e^t-1}\left(\frac{1}{1!}+\frac{t}{2!}+.....\right)\left[\frac{1}{\left(1-\frac{1}{2}t+\frac{1}{3}t^2-.....\right)}\right]$

= 1.1.1 = 1 $\left[\because \text{As } t\to 0, e^t-1\to 0,\ \therefore\ \frac{\sin(e^t-1)}{(e^t-1)}=1\right]$

Example: 39 $\lim_{x\to\pi/2}\frac{a^{\cot x}-a^{\cos x}}{\cot x-\cos x}=$ **[Kerala (Engg.) 2001]**

(a) $\log a$ (b) $\log 2$ (c) a (d) $\log x$

Solution: (a) $\lim_{x\to\pi/2}\left(\frac{a^{\cot x}-a^{\cos x}}{\cot x-\cos x}\right)=\lim_{x\to\pi/2} a^{\cos x}\left(\frac{a^{\cot x-\cos x}-1}{\cot x-\cos x}\right)$

$=a^{\cos(\pi/2)}\lim_{x\to\pi/2}\left(\frac{a^{\cot x-\cos x}-1}{\cot x-\cos x}\right)=1\log a=\log a$.

Example: 40 If $f(x)=\begin{vmatrix}\sin x & \cos x & \tan x\\ x^3 & x^2 & x\\ 2x & 1 & 1\end{vmatrix}$, then $\lim_{x\to 0}\frac{f(x)}{x^2}$ is **[Karnataka CET 2002]**

(a) 3 (b) –1 (c) 0 (d) 1

Solution: (d) $f(x) = x(x-1)\sin x - (x^3 - 2x^2)\cos x - x^3 \tan x$

$= x^2 \sin x - x^3 \cos x - x^3 \tan x + 2x^2 \cos x - x \sin x$

Hence, $\lim_{x\to 0} \frac{f(x)}{x^2} = \lim_{x\to 0}\left(\sin x - x\cos x - x\tan x + 2\cos x - \frac{\sin x}{x}\right) = 0 - 0 - 0 + 2 - 1 = 1$.

Example: 41 If $f(x) = \cot^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$ and $g(x) = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$, then $\lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)}$, $0 < a < \frac{1}{2}$ is **[Orissa JEE 2003]**

(a) $\frac{3}{2(1+a^2)}$ (b) $\frac{3}{2(1+x^2)}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

Solution: (d) $f(x) = \cot^{-1}\left\{\frac{3x - x^3}{1 - 3x^2}\right\}$ and $g(x) = \cos^{-1}\left\{\frac{1 - x^2}{1 + x^2}\right\}$

Put $x = \tan\theta$ in both equation

$f(\theta) = \cot^{-1}\left\{\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right\} = \cot^{-1}\{\tan 3\theta\}$

$f(\theta) = \cot^{-1}\cot\left(\frac{\pi}{2} - 3\theta\right) = \frac{\pi}{2} - 3\theta \Rightarrow f'(\theta) = -3$ ......(i)

and $g(\theta) = \cos^{-1}\left\{\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right\} = \cos^{-1}(\cos 2\theta) = 2\theta \Rightarrow g'(\theta) = 2$ ..... (ii)

Now $\lim_{x\to a}\left(\frac{f(x) - f(a)}{g(x) - g(a)}\right) = \lim_{x\to a}\left(\frac{f(x) - f(a)}{x - a}\right)\frac{1}{\lim_{x\to a}\left(\frac{g(x) - g(a)}{x - a}\right)} = f'(x).\frac{1}{g'(x)} = -3 \times \frac{1}{2} = -\frac{3}{2}$.

Example: 42 $\lim_{x\to\frac{\pi}{2}} \frac{\left[1 - \tan\left(\frac{x}{2}\right)\right][1 - \sin x]}{\left[1 + \tan\left(\frac{x}{2}\right)\right][\pi - 2x]^3}$ is **[AIEEE 2003]**

(a) $\frac{1}{8}$ (b) 0 (c) $\frac{1}{32}$ (d) ∞

Solution: (c) $\lim_{x\to\frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)(1 - \sin x)}{(\pi - 2x)^3}$

Let $x = \frac{\pi}{2} + y$, then $y \to 0$ $\Rightarrow \lim_{y\to 0} \frac{\tan\left(\frac{-y}{2}\right)(1 - \cos y)}{(-2y)^3} = \lim_{y\to 0} \frac{-\tan\frac{y}{2}.2\sin^2\frac{y}{2}}{(-8)y^3} = \lim_{y\to 0} \frac{1}{32}\frac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)}.\left[\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right]^2 = \frac{1}{32}$.

Example: 43 If $\lim_{x\to 0} \frac{[(a-n)nx - \tan x]\sin nx}{x^2} = 0$, where n is non-zero real number, then a is equal to

(a) 0 (b) $\frac{n+1}{n}$ (c) n (d) $n + \frac{1}{n}$

Solution: (d) $\lim_{x\to 0} n\frac{\sin nx}{nx}.\lim_{x\to 0}\left((a-n)n - \frac{\tan x}{x}\right) = 0 \Rightarrow n[(a-n)n - 1] = 0 \Rightarrow (a-n)n = 1 \Rightarrow a = n + \frac{1}{n}$.

(3) **Logarithmic limits :** To evaluate the logarithmic limits we use following formulae

(i) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ....\ ..\text{to}\ \infty$ where $-1 < x \le 1$ and expansion is true only if base is e.

(ii) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

(iii) $\lim_{x\to e} \log_e x = 1$

(iv) $\lim_{x\to 0} \frac{\log(1-x)}{x} = -1$

(v) $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e, a > 0, \ne 1$

Example: 44 $\lim_{h\to 0} \frac{\log_e(1+2h) - 2\log_e(1+h)}{h^2}$ **[IIT Screening 1997]**

(a) –1 (b) 1 (c) 2 (d) –2

Solution: (a) $\lim_{h\to 0} \frac{\log_e(1+2h) - 2\log_e(1+h)}{h^2} = \lim_{x\to a} \frac{\left((2h) - \frac{(2h)^2}{2} + \frac{(2h)^3}{3} - .....\infty\right) - 2\left(h - \frac{h^2}{2} + \frac{h^3}{3} - ......\right)}{h^2}$

$= \lim_{h\to 0} \frac{-h^2 + 2h^3 - ....}{h^2} = \lim_{h\to 0} \frac{h^2\{-1 + 2h - ....\}}{h^2} = \lim_{h\to 0} \{-1 + 2h + ...\} = -1$.

Example: 45 $\lim_{x\to a} \frac{\log\{1 + (x-a)\}}{(x-a)} =$

(a) –1 (b) 2 (c) 1 (d) –2

Solution: (c) Let $x - a = y$, when $x \to a$, $y \to 0$,

$\therefore$ The given limit $= \lim_{y\to 0} \frac{\log\{1+y\}}{y} = 1$.

Example: 46 $\lim_{h\to 0} \frac{\log_{10}(1+h)}{h} =$

(a) 1 (b) $\log_{10} e$ (c) $\log_e 10$ (d) None of these

Solution: (b) $\lim_{h\to 0} \frac{\log_e(1+h)}{h} \cdot \frac{1}{\log_e 10} = \log_{10} e$.

Example: 47 If $\lim_{x\to 0} \frac{\log(3+x) - \log(3-x)}{x} = k$, then the value of k is **[AIEEE 2003]**

(a) 0 (b) $-\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

Solution: (c) $\lim_{x\to 0} \frac{\log(3+x) - \log(3-x)}{x} = \lim_{x\to 0} \frac{\log\left(\frac{3+x}{3-x}\right)}{x} = \lim_{x\to 0} \frac{\log\left(\frac{1+(x/3)}{1-(x/3)}\right)}{x}$

$= \lim_{x\to 0} \frac{\log(1+(x/3))}{x} - \lim_{x\to 0} \frac{\log(1-(x/3))}{x} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$.

(4) **Exponential limits :**

(i) **Based on series expansion :** We use $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\ ...\infty$

To evaluate the exponential limits we use the following results –

(a) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ (b) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$ (c) $\lim_{x\to 0} \frac{e^{\lambda x} - 1}{x} = \lambda \quad (\lambda \ne 0)$

(ii) **Based on the form 1^∞ :** To evaluate the exponential form 1^∞ we use the following results.

(a) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \{1 + f(x)\}^{1/g(x)} = e^{\lim_{x\to a} \frac{f(x)}{g(x)}}$, or when $\lim_{x\to a} f(x)=1$ and $\lim_{x\to a} g(x)=\infty$. Then $\lim_{x\to a}\{f(x)\}^{g(x)} = \lim_{x\to a}[1+f(x)-1]^{g(x)} = e^{\lim_{x\to a}(f(x)-1)g(x)}$

(b) $\lim_{x\to 0}(1+x)^{1/x} = e$ (c) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ (d) $\lim_{x\to 0}(1+\lambda x)^{1/x} = e^{\lambda}$ (e) $\lim_{x\to\infty}\left(1+\frac{\lambda}{x}\right)^x = e^{\lambda}$

Note : ❒ $\lim_{x\to\infty} a^x = \begin{cases}\infty, \text{if } a>1\\ 0, \text{if } a<1\end{cases}$ i.e., $a^{\infty}=\infty$, if $a>1$ and $a^{\infty}=0$ if $a<1$.

Example: 48 $\lim_{x\to 0}\frac{e^{\alpha x}-e^{\beta x}}{x} =$ **[MP PET 1994]**

(a) $\alpha+\beta$ (b) $\frac{1}{\alpha}+\beta$ (c) $\alpha^2-\beta^2$ (d) $\alpha-\beta$

Solution: (d) $\lim_{x\to 0}\frac{e^{\alpha x}-e^{\beta x}}{x} = \lim_{x\to 0}\frac{(e^{\alpha x}-1)-(e^{\beta x}-1)}{x} = \lim_{x\to 0}\frac{e^{\alpha x}-1}{x} - \lim_{x\to 0}\frac{e^{\beta x}-1}{x} = \alpha-\beta$.

Example: 49 The value of $\lim_{x\to 0}\frac{e^x-(1+x)}{x^2}$ is **[Karnataka CET 1995]**

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\frac{1}{4}$

Solution: (b) $\lim_{x\to 0}\frac{e^x-(1+x)}{x^2} = \lim_{x\to 0}\frac{(1+x+\frac{x^2}{2!}+.....)-(1+x)}{x^2} = \lim_{x\to 0}\frac{x^2\left(\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+.....\right)}{x^2} = \frac{1}{2!}=\frac{1}{2}$.

Example: 50 $\lim_{x\to 0}\frac{a^x-1}{\sqrt{1+x}-1}$ is equal to

(a) $2\log_e a$ (b) $\frac{1}{2}\log_e a$ (c) $a\log_e 2$ (d) None of these

Solution: (a) $\lim_{x\to 0}\frac{a^x-1}{\sqrt{1+x}-1} = \lim_{x\to 0}\frac{a^x-1}{\sqrt{1+x}-1}\cdot\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = \lim_{x\to 0}\frac{(a^x-1)(\sqrt{1+x}+1)}{1+x-1} = \lim_{x\to 0}\left(\frac{a^x-1}{x}\right).(\sqrt{1+x}+1)$

$= \left(\lim_{x\to 0}\frac{a^x-1}{x}\right).\left(\lim_{x\to 0}(\sqrt{1+x}+1)\right) = (\log_e a).(2) = 2\log_e a$.

Example: 51 The value of $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+2}$ is **[UPSEAT 2003]**

(a) e^4 (b) 0 (c) 1 (d) e^2

Solution: (d) $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+2} = \lim_{x\to\infty}\left(1+\frac{2}{x+1}\right)^{\frac{x+1}{2}.(x+2).\frac{2}{(x+1)}} = \lim_{x\to\infty}\left(\left(1+\frac{2}{x+1}\right)^{\frac{x+1}{2}}\right)^{2.\left(\frac{1+\frac{2}{x}}{1+\frac{1}{x}}\right)} = e^{2\lim_{x\to\infty}\left[\left(1+\frac{2}{x}\right)/\left(1+\frac{1}{x}\right)\right]} = e^2$.

Alternative method : $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+2} = \lim_{x\to\infty}\left(1+\frac{2}{x+1}\right)^{x+2} = e^{\lim_{x\to\infty}\frac{2}{x+1}(x+2)} = e^{\lim_{x\to\infty} 2\left(\frac{1+\frac{2}{x}}{1+\frac{1}{x}}\right)} = e^2$

Example: 52 If a, b, c, d are positive, then $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx}$ **[EAMCET 1992]**

(a) $e^{d/b}$ (b) $e^{c/a}$ (c) $e^{(c+d)/(a+b)}$ (d) e

Solution: (a) $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx} = \lim_{x\to\infty}\left\{\left(1+\frac{1}{a+bx}\right)^{a+bx}\right\}^{\frac{c+dx}{a+bx}} = e^{d/b}$ $\left\{\because \lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{a+bx} = e \text{ and } \lim_{x\to\infty}\frac{c+dx}{a+bx} = \frac{d}{b}\right\}$

Alternative method : $e^{\lim_{x\to\infty}\left(\frac{1}{a+bx}\right)\left(\frac{c+dx}{1}\right)} = e^{d/b}$.

Example: 53 $\lim_{x\to 0} x^x =$ **[Roorkee 1987]**

(a) 0 (b) 1 (c) e (d) None of these

Solution: (b) Let $y = x^x \Rightarrow \log y = x\log x$; $\therefore \lim_{y\to 0}\log y = \lim_{x\to 0} x\log x = 0 = \log 1 \Rightarrow \lim_{x\to 0} x^x = 1$

Example: 54 The value of $\lim_{x\to 0}\frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$ is **[DCE 2001]**

(a) $\frac{11e}{24}$ (b) $\frac{-11e}{24}$ (c) $\frac{e}{24}$ (d) None of these

Solution: (a) $(1+x)^{1/x} = e^{\frac{1}{x}\log(1+x)} = e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-....\right)} = e^{1-\frac{x}{2}+\frac{x^2}{3}-.......} = e\,e^{-\frac{x}{2}+\frac{x^2}{3}-......}$

$$= e\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-.....\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-.....\right)^2+.....\right] = e\left[1-\frac{x}{2}+\frac{11}{24}x^2-..........\,..\right]$$

$$\therefore \lim_{x\to 0}\frac{(1+x)^{1/x}-e+\frac{ex}{2}}{x^2} = \frac{11e}{24}$$

Example: 55 $\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}$ equals **[UPSEAT 2001]**

(a) $\pi/2$ (b) 0 (c) $2/e$ (d) $-e/2$

Solution: (d) $(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}[\log(1+x)]} = e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+....\right)} = e^{\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+....\right)} = e.e^{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+....\right)}$

$$= e\left[1+\frac{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+.....\right)}{1!}+\frac{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+...\right)^2}{2!}+...\right] = \left[e-\frac{ex}{2}+\frac{11e}{24}x^2-..........\right]$$

$$\therefore \lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x} = \lim_{x\to 0}\left[\frac{e-\frac{ex}{2}+\frac{11e}{24}x^2..........-e}{x}\right] \Rightarrow \lim_{x\to 0}\left(-\frac{e}{2}-\frac{11e}{24}x+...\right) = -\frac{e}{2}.$$

Example: 56 $\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^m =$ **[AMU 2001]**

(a) 0 (b) e (c) $1/e$ (d) 1

Solution: (d) $\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^m = \lim_{m\to\infty}\left[1+\left(\cos\frac{x}{m}-1\right)\right]^m = \lim_{m\to\infty}\left[1-\left(-\cos\frac{x}{m}+1\right)\right]^m$

$$= \lim_{m\to\infty}\left[1-2\sin^2\frac{x}{2m}\right]^m = e^{\lim_{m\to\infty} -\left(2\sin^2\frac{x}{2m}\right)m} = e^{\lim_{m\to\infty} -2\left(\frac{\sin\frac{x}{2m}}{\frac{x}{2m}}\right)^2\left(\frac{x^2}{4m^2}\right)m} = e^{-2\lim_{m\to\infty}\frac{x^2}{4m}} = e^0 = 1.$$

Example: 57 $\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)} =$ **[AMU 2002]**

(a) e (b) e^2 (c) e^{-1} (d) 1

Solution: (b) $\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)} = \lim_{n\to\infty}\left(\frac{n(n-1)+1}{n(n-1)-1}\right)^{n(n-1)} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n(n-1)}\right)^{n(n-1)}}{\left(1-\frac{1}{n(n-1)}\right)^{n(n-1)}} = \frac{e}{e^{-1}} = e^2$.

Alternative Method: $\lim_{n\to\infty}\left(1+\frac{2}{n^2-n-1}\right)^{n(n-1)} = e^{\lim_{n\to\infty}\frac{2n(n-1)}{n^2-n-1}} = e^2$.

(5) **L' Hospital's rule :** If $f(x)$ and $g(x)$ be two functions of x such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

(ii) Both are continuous at $x = a$

(iii) Both are differentiable at $x = a$.

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x = a$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ provided that $g'(a) \neq 0$

Note : ❒ The above rule is also applicable if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$.

❒ If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f'(x), g'(x)$ satisfy all the condition embodied in L' Hospital rule, we can repeat the application of this rule on $\frac{f'(x)}{g'(x)}$ to get, $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$. Sometimes it may be necessary to repeat this process a number of times till our goal of evaluating limit is achieved.

Example: 58 $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx} =$ **[Kerala (Engg.) 2002]**

(a) m/n (b) n/m (c) $\frac{m^2}{n^2}$ (d) $\frac{n^2}{m^2}$

Solution: (c) $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx} = \lim_{x\to 0}\left\{\frac{2\sin^2\frac{mx}{2}}{2\sin^2\frac{nx}{2}}\right\} = \lim_{x\to 0}\left[\left\{\frac{\sin\frac{mx}{2}}{\frac{mx}{2}}\right\}^2\frac{m^2x^2}{4}\cdot\frac{1}{\left\{\frac{\sin\frac{nx}{2}}{\frac{nx}{2}}\right\}^2}\cdot\frac{4}{n^2x^2}\right] = \frac{m^2}{n^2}\times 1 = \frac{m^2}{n^2}$

Trick : Apply L-Hospital rule ,

$\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx} = \lim_{x\to 0}\frac{m\sin mx}{n\sin nx} = \lim_{x\to 0}\frac{m^2\cos mx}{n^2\cos nx} = \frac{m^2}{n^2}$.

Example: 59 The integer n for which $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite non-zero number is **[IIT Screening 2002]**

(a) 1 (b) 2 (c) 3 (d) 4

Solution: (c) n cannot be negative integer for then the limit $=0$

Limit = $\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2^2(x/2)^2}\frac{e^x-\cos x}{x^{n-2}}=\frac{1}{2}\lim_{x\to 0}\frac{e^x-\cos x}{x^{n-2}}$ ($n\neq 1$ for then the limit = 0)

$=\frac{1}{2}\lim_{x\to 0}\frac{e^x+\sin x}{(n-2)x^{n-3}}$. So, if $n=3$, the limit is $\frac{1}{2(n-2)}$ which is finite. If $n=4$, the limit is infinite.

Example: 60 Let $f:R\to R$ be such that $f(1)=3$ and $f'(1)=6$. Then $\lim_{x\to 0}\left\{\frac{f(1+x)}{f(1)}\right\}^{\frac{1}{x}}$ equals **[IIT Screening 2002]**

(a) 1 (b) $e^{1/2}$ (c) e^2 (d) e^3

Solution: (c) $\lim_{x\to 0}\left\{\frac{f(1+x)}{f(1)}\right\}^{\frac{1}{x}}=e^{\lim_{x\to 0}\frac{1}{x}[\log f(1+x)-\log f(1)]}=e^{\lim_{x\to 0}\frac{f'(1+x)/f(1+x)}{1}}=e^{\frac{f'(1)}{f(1)}}=e^{6/3}=e^2$.

Example: 61 $\lim_{\alpha\to\pi/4}\frac{\sin\alpha-\cos\alpha}{\alpha-\pi/4}=$ **[IIT Screening 1997; AMU 1997]**

(a) $\sqrt{2}$ (b) $1/\sqrt{2}$ (c) 1 (d) None of these

Solution: (a) $\lim_{\alpha\to\pi/4}\frac{\sin\alpha-\cos\alpha}{\alpha-\pi/4}$ ($\frac{0}{0}$ form) $=\lim_{\alpha\to\pi/4}\frac{\cos\alpha+\sin\alpha}{1}$ (By 'L' Hospital rule)

$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$.

Example: 62 $\lim_{x\to a}\frac{x^3-a^3}{x^2-a^2}=$

(a) 0 (b) Not defined (c) $2a$ (d) $\frac{3a}{2}$

Solution: (d) $\lim_{x\to a}\frac{x^3-a^3}{x^2-a^2}$ $\left(\frac{0}{0}\text{ form}\right)=\lim_{x\to a}\frac{3x^2}{2x}$ (By 'L' Hospital rule) $=\frac{3a^2}{2a}=\frac{3a}{2}$.

Example: 63 $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=$ **[Roorkee 1983]**

(a) $1/2\sqrt{x}$ (b) $1/2\sqrt{h}$ (c) Zero (d) None of these

Solution: (a) $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}=\frac{1}{2\sqrt{x}}$

Trick : Applying 'L' Hospital's rule, [Differentiating N^r and D^r with respect to h]

We get, $\lim_{h\to 0}\frac{\frac{1}{2\sqrt{x+h}}-0}{1}=\frac{1}{2\sqrt{x}}$.

Example: 64 $\lim_{\alpha\to\beta}\frac{\sin^2\alpha-\sin^2\beta}{\alpha^2-\beta^2}$ **[MP PET 2001]**

(a) 0 (b) 1 (c) $\frac{\sin\beta}{\beta}$ (d) $\frac{\sin 2\beta}{2\beta}$

Solution: (d) $\lim_{\alpha\to\beta}\frac{\sin^2\alpha-\sin^2\beta}{\alpha^2-\beta^2}=\lim_{\alpha-\beta\to 0}\frac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{(\alpha+\beta)\ (\alpha-\beta)}=\lim_{\alpha-\beta\to 0}\frac{\sin(\alpha-\beta)}{(\alpha-\beta)}\lim_{\alpha-\beta\to 0}\frac{\sin(\alpha+\beta)}{(\alpha+\beta)}=\lim_{\alpha\to\beta}\frac{\sin(\alpha+\beta)}{(\alpha+\beta)}=\frac{\sin 2\beta}{2\beta}$.

Trick : By L' Hospital's rule, $\lim_{\alpha\to\beta}\frac{2\sin\alpha\cos\alpha}{2\alpha}=\frac{\sin 2\beta}{2\beta}$.

Example: 65 $\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x}$ equals **[IIT 1971]**

(a) 2/3 (b) 1/3 (c) 1/2 (d) 0

Solution: (c) $\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x} = \lim_{x \to 0} \left\{ \frac{\frac{2 \tan 2x}{2x} - 1}{3 - \frac{\sin x}{x}} \right\} = \frac{1}{2}$.

Example: 66 If $G(x) = -\sqrt{25 - x^2}$, then $\lim_{x \to 1} \frac{G(x) - G(1)}{x - 1}$ equals **[IIT 1983]**

(a) 1/24 (b) 1/5 (c) $-\sqrt{24}$ (d) None of these

Solution: (d) $\lim_{x \to 1} \frac{G(x) - G(1)}{x - 1} = \lim_{x \to 1} \frac{-\sqrt{25 - x^2} + \sqrt{24}}{x - 1}$ [Multiply both numerator and denominator by ($\sqrt{24} + \sqrt{25 - x^2}$)]

$= \lim_{x \to 1} \frac{x + 1}{\sqrt{24} + \sqrt{25 - x^2}} = \frac{1}{\sqrt{24}}$

Alternative method: By L'-Hospital rule, $\lim_{x \to 1} \frac{G'(x)}{1} = \lim_{x \to 1} \frac{-1(-2x)}{2\sqrt{25 - x^2}} = \frac{1}{\sqrt{24}}$

Example: 67 If $f(a) = 2, f'(a) = 1, g(a) = 1, g'(a) = 2$, then $\lim_{x \to a} \frac{g(x)f(a) - g(a)f(x)}{x - a}$ equals

[IIT 1983; Rajasthan PET 1990; MP PET 1995; DCE 1999; Karnataka CET 1999, 2003]

(a) −3 (b) $\frac{1}{3}$ (c) 3 (d) $-\frac{1}{3}$

Solution: (c) Applying L – Hospital's rule, we get, $\lim_{x \to a} \frac{g(x)\, f(a) - g(a)\, f(x)}{x - a} = \lim_{x \to a} \frac{g'(x)\, f(a) - g(a)\, f'(x)}{1}$

$= g'(a)\, f(a) - g(a)\, f'(a) = 2 \times 2 - 1 \times (1) = 3.$

Example: 68 $\lim_{x \to 0} \frac{(1 + x)^n - 1}{x} =$ **[Kurukshetra CEE 2002]**

(a) n (b) 1 (c) −1 (d) None of these

Solution: (a) $\lim_{x \to 0} \frac{(1 + nx + {}^nC_2 x^2 + \text{...... higher powers of } x \text{ to } x^n) - 1}{x} = n$

Trick : Apply L- Hospital rule.

Example: 69 $\lim_{x \to 0} \frac{\sin x + \log(1 - x)}{x^2}$ is equal to **[Roorkee 1995]**

(a) 0 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) None of these

Solution: (c) Apply L- Hospital rule, we get, $\lim_{x \to 0} \frac{\cos x - \frac{1}{1 - x}}{2x} = \lim_{x \to 0} \frac{-\sin x - \frac{1}{(1 - x)^2}}{2} = -\frac{1}{2}$

Alternative method : $\lim_{x \to 0} \frac{\sin x + \log(1 - x)}{x^2} = \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ......\right)}{x^2} + \lim_{x \to 0} \frac{\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ......\right)}{x^2}$

$\left(\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...... \text{ and } \log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} .... \right)$

Hence, $\lim_{x\to 0}\dfrac{\dfrac{-x^2}{2}-x^3\left(\dfrac{1}{3!}+\dfrac{1}{3}\right)-\dfrac{x^4}{4}....}{x^2}=-\dfrac{1}{2}$.

Example: 70 $\lim_{x\to 0}\dfrac{xe^x-\log(1+x)}{x^2}$ equals **[Rajasthan PET 1996]**

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{2}$

Solution: (d) Let $y=\lim_{x\to 0}\dfrac{xe^x-\log(1+x)}{x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

Applying L–Hospital's rule, $y=\lim_{x\to 0}\dfrac{e^x+xe^x-\dfrac{1}{1+x}}{2x}$ $\left(\frac{0}{0}\text{ form}\right)$

$y=\lim_{x\to 0}\dfrac{1}{2}\left[e^x+e^x+xe^x+\dfrac{1}{(1+x)^2}\right]=\lim_{x\to 0}\dfrac{1}{2}[1+1+0+1]=\dfrac{3}{2}$

Example: 71 $\lim_{x\to 0}\dfrac{\sin^{-1}x-\tan^{-1}x}{x^3}$ is equal to **[Rajasthan PET 2000]**

(a) 0 (b) 1 (c) – 1 (d) $\frac{1}{2}$

Solution: (d) $\lim_{x\to 0}\dfrac{\sin^{-1}x-\tan^{-1}x}{x^3}$ $\left(\frac{0}{0}\text{ form}\right)$

Applying L-Hospital's rule,

$=\lim_{x\to 0}\dfrac{\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{1+x^2}}{3x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$=\lim_{x\to 0}\dfrac{\dfrac{-1}{2}\times\dfrac{-2x}{(1-x^2)^{3/2}}+\dfrac{2x}{(1+x^2)^2}}{6x}=\lim_{x\to 0}\dfrac{1}{6}\left[\dfrac{1}{(1-x^2)^{3/2}}+\dfrac{2}{(1+x^2)^2}\right]=\dfrac{1}{2}$.

Example: 72 $\lim_{x\to 1}\dfrac{1+\log x-x}{1-2x+x^2}=$ **[Karnataka CET 2000]**

(a) 1 (b) – 1 (c) 0 (d) $-\frac{1}{2}$

Solution: (d) Applying L-Hospital's rule, $\lim_{x\to 1}\dfrac{1+\log x-x}{1-2x+x^2}=\lim_{x\to 1}\dfrac{\dfrac{1}{x}-1}{-2+2x}=\lim_{x\to 1}\dfrac{1-x}{2x(x-1)}$

Again applying L-Hospital's rule, we get $\lim_{x\to 1}\dfrac{-1}{4x-2}=-\dfrac{1}{2}$

Example: 73 $\lim_{x\to 0}\dfrac{4^x-9^x}{x(4^x+9^x)}=$ **[EAMCET 2002]**

(a) $\log\left(\frac{2}{3}\right)$ (b) $\frac{1}{2}\log\left(\frac{3}{2}\right)$ (c) $\frac{1}{2}\log\left(\frac{3}{2}\right)$ (d) $\log\left(\frac{3}{2}\right)$

Solution: (a) $y=\lim_{x\to 0}\dfrac{4^x-9^x}{x(4^x+9^x)}$ $\left(\frac{0}{0}\text{ form}\right)$

Using L-Hospital's rule, $y=\lim_{x\to 0}\dfrac{4^x\log 4-9^x\log 9}{(4^x+9^x)+x(4^x\log 4+9^x\log 9)}\Rightarrow y=\dfrac{\log 4-\log 9}{2}\Rightarrow y=\dfrac{\log\left(\frac{2}{3}\right)^2}{2}=\log\dfrac{2}{3}$.

Example: 74 If $f(a)=2$, $f'(a)=1$, $g(a)=-3$, $g'(a)=-1$, then $\lim_{x\to a}\frac{f(a)g(x)-f(x)g(a)}{x-a}=$ **[Karnataka CET 2003]**

(a) 1 (b) 6 (c) – 5 (d) – 1

Solution: (a) $\lim_{x\to a}\frac{f(a)g(x)-f(x)g(a)}{x-a}$ $\left(\frac{0}{0}\text{ form}\right)$

Using L-Hospital's rule, $\lim_{x\to a}\frac{f(a)g'(x)-f'(x)g(a)}{1-0}=f(a)\times g'(a)-f'(a)\times g(a)=2\times(-1)-1\times(-3)=1$.

Example: 75 The value of $\lim_{x\to 7}\frac{2-\sqrt{x-3}}{x^2-49}$ is **[MP PET 2003]**

(a) $\frac{2}{9}$ (b) $-\frac{2}{49}$ (c) $\frac{1}{56}$ (d) $-\frac{1}{56}$

Solution: (d) Applying L-Hospital's rule, $\lim_{x\to 7}\frac{0-\frac{1}{2\sqrt{x-3}}}{2x}=\lim_{x\to 7}\frac{-1}{4x\sqrt{x-3}}=\frac{-1}{4.7\sqrt{7-3}}=\frac{-1}{56}$.

Example: 76 Let $f(a)=g(a)=k$ and their n^{th} derivatives $f^n(a), g^n(a)$ exist and are not equal for some n. If $\lim_{x\to a}\frac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)}=4$, then the value of k is **[AIEEE 2003]**

(a) 4 (b) 2 (c) 1 (d) 0

Solution: (a) $\lim_{x\to a}\frac{k\,g(x)-k\,f(x)}{g(x)-f(x)}=4$

By L-Hospital' rule, $\lim_{x\to a}k\left[\frac{g'(x)-f'(x)}{g'(x)-f'(x)}\right]=4$, $\therefore\ k=4$.

Example: 77 The value of $\lim_{x\to 0}\left(\frac{\int_0^{x^2}\sec^2 t\,dt}{x\sin x}\right)$ is **[AIEEE 2003]**

(a) 3 (b) 2 (c) 1 (d) 0

Solution: (c) $\lim_{x\to 0}\frac{\frac{d}{dx}\int_0^{x^2}\sec^2 t\,dt}{\frac{d}{dx}(x\sin x)}=\lim_{x\to 0}\frac{\sec^2 x^2 .2x}{\sin x+x\cos x}$ (By L' –Hospital's rule)

$=\lim_{x\to 0}\frac{2\sec^2 x^2}{\left(\frac{\sin x}{x}+\cos x\right)}=\frac{2\times 1}{1+1}=1$.

Example: 78 $\lim_{x\to\pi/6}\left[\frac{3\sin x-\sqrt{3}\cos x}{6x-\pi}\right]$ **[EAMCET 2003]**

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $-\sqrt{3}$ (d) $-\frac{1}{\sqrt{3}}$

Solution: (b) Using L–Hospital's rule, $\lim_{x\to\pi/6}\frac{3\cos x+\sqrt{3}\sin x}{6}=\frac{3.\frac{\sqrt{3}}{2}+\sqrt{3}.\frac{1}{2}}{6}=\frac{1}{\sqrt{3}}$.

Example: 79 Given that $f'(2) = 6$ and $f'(1) = 4$, then $\lim_{h \to 0} \frac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)} =$ **[IIT Screening 2003]**

(a) Does not exist (b) $-\frac{3}{2}$ (c) $\frac{3}{2}$ (d) 3

Solution: (d) $\lim_{h \to 0} \frac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)} = \lim_{h \to 0} \frac{f'(2h + 2 + h^2)(2 + 2h)}{f'(h - h^2 + 1)(1 - 2h)} = \frac{6 \times 2}{4 \times 1} = 3$.

Assignment

Basic Level

1. $\lim_{x\to\infty}\frac{(3x-1)(2x+5)}{(x-3)(3x+7)}$ is equal to

(a) 3 (b) 2 (c) –2 (d) 1

2. $\lim_{x\to\infty}\frac{2x^2+3x+4}{3x^2+3x+4}$ is equal to **[SCRA 1996; Rajasthan PET 1987 ; BIT Ranchi 1998; MP PET 1993)]**

(a) $\frac{2}{3}$ (b) 1 (c) 0 (d) ∞

3. $\lim_{x\to a} f(x).g(x)$ exists if **[Rajasthan PET 1995]**

(a) $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists (b) $\lim_{x\to a} f(x)^{g(x)}$ exists

(c) $\lim_{x\to a}\frac{f(x)}{g(x)}$ exists (d) $\lim_{x\to a} f(x)g\left(\frac{1}{x}\right)$ exists

4. $\lim_{x\to\infty}\left[x-\sqrt{x^2+x}\right]=$ **[IIT 1975]**

(a) $\frac{1}{2}$ (b) 1 (c) $-\frac{1}{2}$ (d) 0

5. If $\lim_{x\to a}\frac{f(x)}{g(x)}$ exists then

(a) Both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ must exist (b) $\lim_{x\to a} f(x)$ not exist but $\lim_{x\to a} g(x)$ exists

(c) Neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists (d) $\lim_{x\to a} f(x)$ exist but $\lim_{x\to a} g(x)$ does not exist

6. Which of the following statement is not correct

(a) $\lim_{x\to c}[f(x)+g(x)]=\lim_{x\to c} f(x)+\lim_{x\to c} g(x)$ (b) $\lim_{x\to c}[f(x)-g(x)]=\lim_{x\to c} f(x)-\lim_{x\to c} g(x)$

(c) $\lim_{x\to c}[f(x).g(x)]=\lim_{x\to c} f(x).\lim_{x\to c} g(x)$ (d) $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$

7. $\lim_{x\to 3^-}\frac{|x-3|}{x-3}=$

(a) 1 (b) –1 (c) 0 (d) Does not exist

8. $\lim_{x\to 3^+}\frac{|x-3|}{x-3}=$

(a) 1 (b) –1 (c) 0 (d) Does not exist

9. If $\lim_{x\to a}\phi(x)=a^3, a\neq 0$ then $\lim_{x\to a}\phi\left(\frac{x}{a}\right)$ is equal to

(a) a^2 (b) $\frac{1}{a^2}$ (c) $\frac{1}{a^3}$ (d) a^3

10. If $\lim_{x\to a}\frac{a^x - x^a}{x^x - a^a} = -1$, then **[EAMCET 2003]**

(a) $a = 1$ (b) $a = 0$ (c) $a = e$ (d) None of these

11. $\lim_{x\to 4}\left[\frac{x^{3/2} - 8}{x - 4}\right] =$ **[DCE 1999]**

(a) $\frac{3}{2}$ (b) 3 (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

12. $\lim_{x\to 0}\left[\frac{\sqrt{a+x} - \sqrt{a-x}}{x}\right] =$ **[Karnataka CET 2001; Roorkee 1979; MP PET 1987]**

(a) 1 (b) 0 (c) $\sqrt{a}$ (d) $\frac{1}{\sqrt{a}}$

13. $\lim_{x\to\infty}\frac{\sqrt{4x^2 + 5x + 8}}{4x + 5}$ is equal to

(a) –1/2 (b) 0 (c) 1/2 (d) 1

14. $\lim_{x\to 4}\frac{3 - \sqrt{5 + x}}{x - 4}$ is equal to **[Orissa JEE 1996]**

(a) 1/6 (b) –1/6 (c) 0 (d) 1

15. $\lim_{x\to\infty}\frac{\sqrt{x^2 - 1}}{2x + 1}$ is equal to **[BIT Ranchi 1992]**

(a) 1 (b) 0 (c) –1 (d) $\frac{1}{2}$

16. $\lim_{x\to 2}\frac{x^2 - 3x + 2}{x^2 + x - 6}$ equals **[IIT 1970; IIT 1976)]**

(a) 1/5 (b) 2/5 (c) 1 (d) 5

17. The value of $\lim_{\theta\to 0}\left(\frac{\sin\theta / 4}{\theta}\right)$ is

(a) 0 (b) $\frac{1}{4}$ (c) 1 (d) Note in existence

18. $\lim_{x\to 0}\frac{x^2 - 2x}{2\sin x}$ equals **[Rajasthan PET 1985]**

(a) 1 (b) –1 (c) 0 (d) None of these

19. $\lim_{x\to 0}\frac{\sin(2+x) - \sin(2-x)}{x} =$

(a) sin 2 (b) 2 sin 2 (c) 2 cos 2 (d) 2

20. $\lim_{n\to\infty}(3^n + 4^n)^{\frac{1}{n}} =$ **[Karnataka CET 2003]**

(a) 3 (b) 4 (c) ∞ (d) e

21. True statement for $\lim_{x\to 0}\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{2+3x} - \sqrt{2-3x}}$ is **[Ranchi BIT 1982; Haryana 1996)]**

(a) Does not exist (b) Lies between 0 and $\frac{1}{2}$ (c) Lies between $\frac{1}{2}$ and 1

22. $\lim_{x\to 1}\frac{(2x-3)(\sqrt{x} - 1)}{2x^2 + x - 3} =$ **[IIT 1977]**

(a) $-\frac{1}{10}$ (b) $\frac{1}{10}$ (c) $-\frac{1}{8}$ (d) None of these

23. $\lim_{x\to\infty}(\sqrt{x^2 + 8x + 3} - \sqrt{x^2 + 4x + 3}) =$ **[MP PET 1997; Rajasthan PET 1995)]**

(a) 0 (b) ∞ (c) 2 (d) $\frac{1}{2}$

24. $\lim_{x \to 0}\left(\frac{x^o}{\sin x^o}\right)$ equals **[AMU 1991]**

(a) 1 (b) $\frac{\pi}{180}$ (c) $\frac{180}{\pi}$ (d) None of these

Advance Level

25. $\lim_{n \to \infty}\left[\frac{1^3+2^3+3^3+.....+n^3}{n^4}\right]=$

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) None of these

26. $\lim_{n \to \infty}\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+.......+\frac{1}{3n}\right]=$ **[Karnataka CET 1999]**

(a) 0 (b) $\log_e 4$ (c) $\log_e 3$ (d) $\log_e 2$

27. $\lim_{n \to \infty}\sum_{k=1}^{n}\frac{k}{n^2+k^2}=$ **[Roorkee 1999]**

(a) $\left(\frac{1}{2}\right)\log 2$ (b) $\log 2$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

28. The $\lim_{x \to 0}(\cos x)^{\cot x}$ is **[Rajasthan PET 1999]**

(a) 0 (b) 1 (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

29. $\lim_{x \to 0}\frac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}=$ **[IIT 1999]**

(a) 2 (b) – 2 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

30. If $f(x)=\sqrt{\frac{x-\sin x}{x+\cos^2 x}}$, then $\lim_{x \to \infty} f(x)$ is **[DCE 2000; EAMCET 1997)]**

(a) 0 (b) ∞ (c) 1 (d) Not exist

31. If $f(x)=\begin{cases} x & ; \ x<0 \\ 1 & ; \ x=0 \\ x^2 & ; \ x>0 \end{cases}$, then, $\lim_{x \to 0} f(x)=$ **[DCE 2000]**

(a) 0 (b) 1 (c) 2 (d) Does not exist

32. If $f(x)=\begin{cases} \sin x & , x \neq n\pi \\ 0 & , \text{other wise} \end{cases}, n \in Z$ $g(x)=\begin{cases} x^2+1 & , \ x \neq 0, 2 \\ 4 & , \ x=0 \\ 5 & , \ x=2 \end{cases}$, then $\lim_{x \to 0} g\{f(x)\}=$ **[Karnataka CET 2000]**

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

33. $\lim_{n \to \infty}\frac{1^p+2^p+3^p+........+n^p}{n^{p+1}}=$ **[AIEEE 2002]**

(a) $\frac{1}{p+1}$ (b) $\frac{1}{1-p}$ (c) $\frac{1}{p}-\frac{1}{(p-1)}$ (d) $\frac{1}{p+2}$

34. $\lim_{n\to\infty} \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+........+\frac{1}{2^n}$ equals **[Rajasthan PET 1996]**

(a) 2 (b) –1 (c) 1 (d) 3

35. $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1} x} =$

(a) 2 (b) 1 (c) –1 (d) None of these

36. $\lim_{x\to 0} \cos \frac{1}{x}$ **[UPSEAT 2002]**

(a) Is continuous at $x=0$ (b) Differentiable at $x=0$ (c) Does not exist (d) None of these

37. If $x_n = \frac{1-2+3-4+5-6+.....-2n}{\sqrt{n^2+1}+\sqrt{4n^2-1}}$, then $\lim_{n\to\infty} x_n$ is equal to

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{2}{3}$ (d) 1

38. $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{\sqrt{n^2+n}}+\frac{1}{\sqrt{n^2+2n}}+.....+\frac{1}{\sqrt{n^2+(n-1)n}}\right]$ is equal to **[Rajasthan PET 2000]**

(a) $2+2\sqrt{2}$ (b) $2\sqrt{2}-2$ (c) $2\sqrt{2}$ (d) 2

39. $\lim_{n\to\infty} \frac{1}{1^3+n^3}+\frac{4}{2^3+n^3}+....+\frac{1}{2n}$ is equal to **[Rajasthan PET 1995]**

(a) $\frac{1}{3}\log_e 3$ (b) $\frac{1}{3}\log_e 2$ (c) $\frac{1}{3}\log_e \frac{1}{3}$ (d) None of these

40. The value of $\lim_{n\to\infty}\left[\frac{n}{1+n^2}+\frac{n}{4+n^2}+\frac{n}{9+n^2}+....+\frac{1}{2n}\right]$ is equal to

(a) e (b) $\frac{1}{e}$ (c) $\frac{\pi}{4}$ (d) $\frac{4}{\pi}$

41. $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for

(a) No value of n (b) n is any whole number (c) n = 0 only (d) n = 2 only

Exponential and Logarithmic Limits

Basic Level

42. $\lim_{x\to 0}\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{1/x}$ is equal to **[Rajasthan PET 2001]**

(a) e^{-1} (b) e (c) e^2 (d) $\sqrt{e}$

43. $\lim_{x\to 0}\left(\frac{3^x-1}{x}\right)$ equals **[Rajasthan PET 1998]**

(a) log 3 (b) 3 log 3 (c) 2 log 3 (d) None of these

44. $\lim_{x\to 0} \frac{\cos(\sin x)-1}{x^2} =$ **[Orissa JEE 2003]**

(a) 1 (b) – 1 (c) 1/2 (d) – 1/2

45. $\lim_{x\to 0} \frac{e^{1/x}}{e^{\frac{1}{x}+1}} =$ **[DCE 1999]**

(a) 0 (b) 1 (c) Does not exist (d) None of these

46. $\lim_{x\to 0}\frac{a^{\sin x}-1}{b^{\sin x}-1}=$ **[Karnataka CET 2000]**

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$ (c) $\frac{\log a}{\log b}$ (d) $\frac{\log b}{\log a}$

47. $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)=$ **[Karnataka CET 2001]**

(a) $\frac{1}{2}$ (b) ∞ (c) 1 (d) 0

48. $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+1}=$ **[Rajasthan PET 2003]**

(a) e^2 (b) e^3 (c) e (d) e^{-1}

49. $\lim_{x\to 0}(1-ax)^{\frac{1}{x}}=$ **[Karnataka CET 2003]**

(a) e (b) e^{-a} (c) 1 (d) e^a

Advance Level

50. $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x$ is equal to **[IIT 2000]**

(a) e (b) e^{-1} (c) e^{-5} (d) e^5

51. $\lim_{x\to 0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2}$ is **[IIT 1996]**

(a) e^2 (b) e (c) e^{-1} (d) None of these

52. The value of $\lim_{x\to\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x$ is equal to

(a) e^2 (b) e^{-2} (c) e^6 (d) None of these

53. $\lim_{x\to\infty}\left[1+\frac{1}{mx}\right]^x$ equal to **[Haryana CEE 1998]**

(a) $e^{1/m}$ (b) $e^{-1/m}$ (c) e^m (d) m^e

54. $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+3}\right)^x=$ **[AIEEE 2002]**

(a) e^4 (b) e^2 (c) e^3 (d) e

55. $\lim_{x\to 0}\frac{x.2^x-x}{1-\cos x}$ is equal to **[IIT 1980; BIT Ranchi 1983; Rajasthan PET 1999, 2001]**

(a) $\log 2$ (b) $\log 4$ (c) 0 (d) None of these

56. $\lim_{x\to 0}\left(\frac{a^x-b^x}{x}\right)=$ **[EAMCET 1988; Rajasthan PET 1995]**

(a) $\log\left(\frac{b}{a}\right)$ (b) $\log\left(\frac{a}{b}\right)$ (c) $\frac{a}{b}$ (d) $\log a^b$

57. $\lim_{x\to\infty}\frac{\log x^n-[x]}{[x]}, n\in N$, ($[x]$ denotes greatest integer less than or equal to x) **[AIEEE 2002]**

(a) Has value –1 (b) Has value 0 (c) Has value 1 (d) Does not exist

Trigonometric Limits

Basic Level

58. $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}=$ **[MP PET 2002; UPSEAT 2001; IIT Screening 2001]**

(a) $-\pi$ (b) π (c) $\frac{\pi}{2}$ (d) 1

59. $\lim_{x\to 0}\left\{\frac{\sin x - x + \frac{x^3}{6}}{x^5}\right\}=$ **[MNR 1985; MNR 1986)]**

(a) $\frac{1}{120}$ (b) $-\frac{1}{120}$ (c) $\frac{1}{20}$ (d) None of these

60. $\lim_{x\to 0}\frac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x}=$ **[Rajasthan PET 2001; AIEEE 2002]**

(a) 1 (b) – 1 (c) 0 (d) Does not exist

61. $\lim_{x\to\pi/4}\frac{\sqrt{2}\cos x-1}{\cot x-1}=$ **[BIT Ranchi 1989; IIT 1990]**

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{2\sqrt{2}}$ (d) 1

62. $\lim_{x\to a}\frac{\cos x-\cos a}{\cot x-\cot a}=$ **[BIT Ranchi 1987]**

(a) $\frac{1}{2}\sin^3 a$ (b) $\frac{1}{2}\text{cosec}^2 a$ (c) $\sin^3 a$ (d) $\text{cosec}^3 a$

Advance Level

63. $\lim_{x\to\infty}\sqrt{\frac{x+\sin x}{x-\cos x}}=$ **[Roorkee 1994]**

(a) 0 (b) 1 (c) –1 (d) None of these

64. The value of $\lim_{n\to\infty}\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)....\cos\left(\frac{x}{2^n}\right)$ is

(a) 1 (b) $\frac{\sin x}{x}$ (c) $\frac{x}{\sin x}$ (d) None of these

65. If x is a real number in [0,1], then the value of $\lim_{m\to\infty}\lim_{n\to\infty}[1+\cos^{2m}(n!\pi x)]$ is given by

(a) 2 or 1 according as x is rational or irrational (b) 1 or 2 according as x is rational or irrational
(c) 1 for all x (d) 2 or 1 for all x

66. $\lim_{x\to 0}\frac{1-\cos 5x}{2^x-3^x}$ is equal to

(a) 0 (b) 1 (c) 5 (d) None of these

67. If $\lim_{x\to 0}\frac{x^n-\sin x^n}{x-\sin^n x}$ is non zero definite, then n must be

(a) 1 (b) 2 (c) 3 (d) None of these

68. The values of a and b such that $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1,$ are

(a) $\frac{5}{2},\frac{3}{2}$ (b) $\frac{5}{2},-\frac{3}{2}$ (c) $-\frac{5}{2},-\frac{3}{2}$ (d) None of these

69. $\lim_{h\to 0}\frac{2\left[\sqrt{3}\sin\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}+h\right)\right]}{\sqrt{3}h(\sqrt{3}\cos h-\sin h)}=$ **[BIT Ranchi 1987]**

(a) $-\frac{2}{3}$ (b) $-\frac{3}{4}$ (c) $-2\sqrt{3}$ (d) $\frac{4}{3}$

70. If $f(x)=\int\frac{2\sin x-\sin 2x}{x^3}dx,$ then $\lim_{x\to 0}f'(x)$ equals **[IIT Screening 1997]**

(a) 0 (b) 1 (c) –1 (d) 1/2

L'- Hospital Rule

Basic Level

71. $\lim_{x\to 0}\frac{x}{\tan^{-1}(2x)}$ is equal to **[IIT 1992; Rajasthan PET 2001]**

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) None of these

72. $\lim_{x\to 0}\frac{e^{\tan x}-e^x}{\tan x-x}$ is equal to **[Rajasthan PET 2001]**

(a) 0 (b) 1 (c) e (d) $\frac{1}{e}$

73. $\lim_{x\to\pi/2}(\sec x-\tan x)$ equals **[Rajasthan PET 1998]**

(a) 0 (b) 1 (c) –1 (d) None of these

74. $\lim_{x\to 0}\frac{2\sin^2 3x}{x^2}=$ **[Roorkee 1982; DCE 1999]**

(a) 0 (b) 1 (c) 18 (d) 36

75. $\lim_{x\to 0}\frac{\sin 2x}{x}=$ **[MNR 1990; UPSEAT 2000]**

(a) 0 (b) 1 (c) 1/2 (d) 2

76. If $f(1)=1$ and $f'(1)=4,$ then the value of $\lim_{x\to 1}\frac{\sqrt{f(x)}-1}{\sqrt{x}-1}$ is **[DCE 2001]**

(a) 9 (b) 4 (c) 12 (d) 1

77. $\lim_{x\to 0}\frac{\log_e(1+x)}{3^x-1}=$ **[MP PET 2002]**

(a) $\log e\ 3$ (b) 0 (c) 1 (d) $\log_3 e$

78. $\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}=$ **[Haryana CEE 2002]**

(a) 0 (b) 1 (c) 2 (d) Non existent

79. $\lim_{x \to -2} \frac{\sin^{-1}(x+2)}{x^2+2x} =$ **[Orissa JEE 2002]**

(a) 0 (b) ∞ (c) $-\frac{1}{2}$ (d) None of these

80. $\lim_{x \to a} \frac{(x^{-1}-a^{-1})}{x-a} =$ **[MP PET 1994]**

(a) $\frac{1}{a}$ (b) $-\frac{1}{a}$ (c) $\frac{1}{a^2}$ (d) $-\frac{1}{a^2}$

81. $\lim_{x \to 1} \frac{\log x}{x-1}$ is equal to **[Rajasthan PET 1996; MP PET 1996]**

(a) 1 (b) 0 (c) –1 (d) 1/2

82. $\lim_{x \to 1} \frac{1-x^{-1/3}}{1-x^{-2/3}} =$

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

Advance Level

83. The value of $\lim_{x \to 0} \frac{x \cos x - \log(1+x)}{x^2}$ is **[Rajasthan PET 1999]**

(a) $\frac{1}{2}$ (b) 0 (c) 1 (d) None of these

84. $\lim_{x \to \frac{\pi}{4}} \left(\frac{1-\tan x}{1-\sqrt{2}\sin x} \right)$ equals **[SCRA 1999]**

(a) 0 (b) 1 (c) – 2 (d) 2

85. $\lim_{x \to 0} \frac{2^x - 1}{\sqrt{1+x}-1}$ equals **[Karnataka CET 1999, IIT 1983]**

(a) $\log 2$ (b) $\log 4$ (c) $\log 3$ (d) None of these

86. $\lim_{x \to 0} \left[\frac{\sin(x+a)+\sin(a-x)-2\sin a}{x \sin x} \right]$ is equal to **[UPSEAT 1998]**

(a) $\sin a$ (b) $-\sin a$ (c) 1 (d) 0

87. $\lim_{x \to \pi/2} \frac{\int_{\pi/2}^{x} t\,dt}{\sin(2x-\pi)}$ is equal to **[MP PET 1998]**

(a) ∞ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

88. $\lim_{y \to 0} \frac{(x+y)\sec(x+y) - x\sec x}{y} =$

(a) $\sec x(x \tan x + 1)$ (b) $x \tan x + \sec x$ (c) $x \sec x + \tan x$ (d) None of these

89. $\lim_{x \to \frac{\pi}{3}} \frac{\sin\left(\frac{\pi}{3}-x\right)}{2\cos x - 1}$ is equal to **[AMU 1991]**

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $\frac{2}{\sqrt{3}}$

90. $\lim_{x \to 0} \frac{e^{x^2} - \cos x}{x^2}$ is equal to **[AMU 1990]**

(a) $\frac{3}{2}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) None of these

91. $\lim_{x\to 0} \frac{x\cos x - \sin x}{x^2 \sin x}$ equals **[SCRA 1999]**

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) 3 (d) –3

***

Answer Sheet

Assignment (Basic & Advance Level)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
b	a	a	c	a	d	b	a	d	a	b	d	c	b	d	a	b	b	c	b
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
b	a	c	a	c	c	a	b	c	c	d	a	a	c	b	c	b	b	b	c
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
b	c	a	d	d	c	c	a	b	c	a	a	a	a	b	b	a	b	a	d
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
b	c	b	b	a	a	a	c	d	b	c	b	a	c	d	b	d	c	c	d
81	82	83	84	85	86	87	88	89	90	91									
a	b	a	d	b	b	c	a	b	a	b									

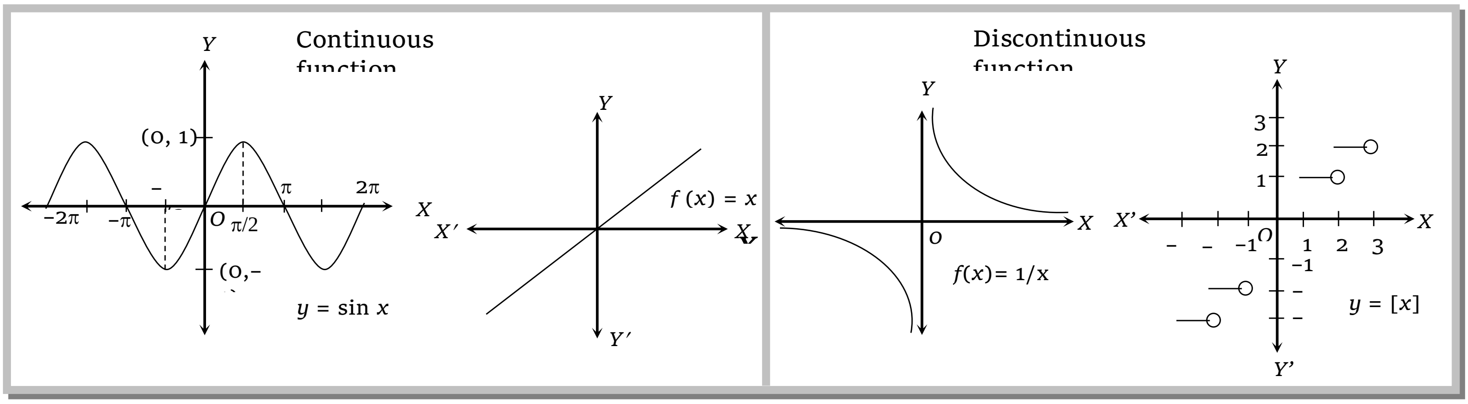
2.3 Continuity

Introduction

The word 'Continuous' means without any break or gap. If the graph of a function has no break, or gap or jump, then it is said to be continuous.

A function which is not continuous is called a discontinuous function.

While studying graphs of functions, we see that graphs of functions $\sin x$, x, $\cos x$, e^x etc. are continuous but greatest integer function $[x]$ has break at every integral point, so it is not continuous. Similarly $\tan x$, $\cot x$, $\sec x$, $\frac{1}{x}$ etc. are also discontinuous function.



2.3.1 Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at a point $x = a$ of its domain iff $\lim\limits_{x \to a} f(x) = f(a)$. *i.e.* a function $f(x)$ is continuous at $x = a$ if and only if it satisfies the following three conditions :

(1) $f(a)$ exists. ('*a*' lies in the domain of *f*)

(2) $\lim\limits_{x \to a} f(x)$ exist *i.e.* $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x)$ or R.H.L. = L.H.L.

(3) $\lim\limits_{x \to a} f(x) = f(a)$ (limit equals the value of function).

Cauchy's definition of continuity : A function f is said to be continuous at a point *a* of its domain *D* if for every $\varepsilon > 0$ there exists $\delta > 0$ (dependent on ε) such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Comparing this definition with the definition of limit we find that $f(x)$ is continuous at $x = a$ if $\lim\limits_{x \to a} f(x)$ exists and is equal to $f(a)$ *i.e.,* if $\lim\limits_{x \to a^-} f(x) = f(a) = \lim\limits_{x \to a+} f(x)$.

Heine's definition of continuity : A function f is said to be continuous at a point a of its domain D, converging to a, the sequence $< a_n >$ of the points in D converging to a, the sequence $< f(a_n) >$ converges to $f(a)$ *i.e.* $\lim a_n = a \Rightarrow \lim f(a_n) = f(a)$. This definition is mainly used to prove the discontinuity to a function.

Note : ❐ Continuity of a function at a point, we find its limit and value at that point, if these two exist and are equal, then function is continuous at that point.

Formal definition of continuity : The function $f(x)$ is said to be continuous at $x = a$, in its domain if for any arbitrary chosen positive number $\in > 0$, we can find a corresponding number δ depending on $\in$ such that $|f(x) - f(a)| < \in \forall x$ for which $0 < |x - a| < \delta$.

2.3.2 Continuity from Left and Right

Function $f(x)$ is said to be

(1) Left continuous at $x = a$ if $\lim\limits_{x \to a-0} f(x) = f(a)$

(2) Right continuous at $x = a$ if $\operatorname{Lim}\limits_{x \to a+0} f(x) = f(a)$.

Thus a function $f(x)$ is continuous at a point $x = a$ if it is left continuous as well as right continuous at $x = a$.

Example: 1 If $f(x) = \begin{cases} x + \lambda, & x < 3 \\ 4, & x = 3 \\ 3x - 5, & x > 3 \end{cases}$ is continuous at $x = 3$, then $\lambda =$

(a) 4 (b) 3 (c) 2 (d) 1

Solution: (d) L.H.L. at $x = 3$, $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} (x + \lambda) = \lim\limits_{h \to 0} (3 - h + \lambda) = 3 + \lambda$ .....(i)

R.H.L. at $x = 3$, $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} (3x - 5) = \lim\limits_{h \to 0} \{3(3 + h) - 5\} = 4$ .....(ii)

Value of function $f(3) = 4$ .....(iii)

For continuity at $x = 3$

Limit of function = value of function $3 + \lambda = 4 \Rightarrow \lambda = 1$.

Example: 2 If $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is **[MP PET 1999; AMU 1999; Rajasthan PET 20**

(a) 1 (b) –1 (c) 0 (d) 2

Solution: (c) If function is continuous at $x = 0$, then by the definition of continuity $f(0) = \lim\limits_{x \to 0} f(x)$

since $f(0) = k$. Hence, $f(0) = k = \lim\limits_{x \to 0} (x) \left(\sin \frac{1}{x} \right)$

$\Rightarrow k = 0$ (a finite quantity lies between –1 to 1) $\Rightarrow k = 0$.

Example: 3 If $f(x) = \begin{cases} 2x + 1 & \text{when } x < 1 \\ k & \text{when } x = 1 \\ 5x - 2 & \text{when } x > 1 \end{cases}$ is continuous at $x = 1$, then the value of k is **[Rajasthan PET 2001]**

(a) 1 (b) 2 (c) 3 (d) 4

Solution: (c) Since $f(x)$ is continuous at $x = 1$,

$\Rightarrow \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = f(1)$ .....(i)

Now $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{h \to 0} f(1 - h) = \lim\limits_{h \to 0} 2(1 - h) + 1 = 3$ *i.e.,* $\lim\limits_{x \to 1^-} f(x) = 3$

Similarly, $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} 5(1+h) - 2$ *i.e.*, $\lim_{x\to 1^+} f(x) = 3$

So according to equation (i), we have $k = 3$.

Example: 4 The value of k which makes $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ k, & x = 0 \end{cases}$ continuous at $x = 0$ is **[Rajasthan PET 1993; UPSEAT 1995]**

(a) 8 (b) 1 (c) –1 (d) None of these

Solution: (d) We have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \sin\frac{1}{x}$ = An oscillating number which oscillates between –1 and 1.

Hence, $\lim_{x\to 0} f(x)$ does not exist. Consequently $f(x)$ cannot be continuous at $x = 0$ for any value of k.

Example: 5 The value of m for which the function $f(x) = \begin{cases} mx^2, x \le 1 \\ 2x, x > 1 \end{cases}$ is continuous at $x = 1$, is

(a) 0 (b) 1 (c) 2 (d) Does not exist

Solution: (c) LHL = $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} m(1-h)^2 = m$

RHL = $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} 2(1+h) = 2$ and $f(1) = m$

Function is continuous at $x = 1$, $\therefore$ LHL = RHL = $f(1)$

Therefore $m = 2$.

Example: 6 If the function $f(x) = \begin{cases} (\cos x)^{1/x}, x \neq 0 \\ k \quad\quad, x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is

(a) 1 (b) –1 (c) 0 (d) e

Solution: (a) $\lim_{x\to 0} (\cos x)^{1/x} = k \Rightarrow \lim_{x\to 0} \frac{1}{x} \log(\cos x) = \log k \Rightarrow \lim_{x\to 0} \frac{1}{x} \lim_{x\to 0} \log\cos x = \log k \Rightarrow \lim_{x\to 0} \frac{1}{x} \times 0 = \log_e k \Rightarrow k = 1$.

2.3.3 Continuity of a Function in Open and Closed Interval

Open interval : A function $f(x)$ is said to be continuous in an open interval (a, b) iff it is continuous at every point in that interval.

Note : ❐ This definition implies the non-breakable behavior of the function $f(x)$ in the interval (a, b).

Closed interval : A function $f(x)$ is said to be continuous in a closed interval [a, b] iff,

(1) f is continuous in (a, b)

(2) f is continuous from the right at 'a' *i.e.* $\lim_{x\to a^+} f(x) = f(a)$

(3) f is continuous from the left at 'b' *i.e.* $\lim_{x\to b^-} f(x) = f(b)$.

Example: 7 If the function $f(x)=\begin{cases} x+a^2\sqrt{2}\sin x &, 0\le x<\frac{\pi}{4} \\ x\cot x+b &, \frac{\pi}{4}\le x<\frac{\pi}{2} \\ b\sin 2x-a\cos 2x &, \frac{\pi}{2}\le x\le\pi \end{cases}$, is continuous in the interval $[0, \pi]$ then the values of (a, b) are

[Roorkee 1998]

(a) (–1, –1) (b) (0, 0) (c) (–1, 1) (d) (1, –1)

Solution: (b) Since f is continuous at $x=\frac{\pi}{4}$; $\therefore f\left(\frac{\pi}{4}\right)=\lim_{h\to 0} f\left(\frac{\pi}{4}+h\right)=\lim_{h\to 0} f\left(\frac{\pi}{4}-h\right)\Rightarrow \frac{\pi}{4}(1)+b=\left(\frac{\pi}{4}-0\right)+a^2\sqrt{2}\sin\left(\frac{\pi}{4}-0\right)$

$\Rightarrow \frac{\pi}{4}+b=\frac{\pi}{4}+a^2\sqrt{2}\sin\frac{\pi}{4}\Rightarrow b=a^2\sqrt{2}.\frac{1}{\sqrt{2}}\Rightarrow b=a^2$

Also as f is continuous at $x=\frac{\pi}{2}$; $\therefore f\left(\frac{\pi}{2}\right)=\lim_{x\to\frac{\pi}{2}-0} f(x)=\lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)$

$\Rightarrow b\sin 2\frac{\pi}{2}-a\cos 2\frac{\pi}{2}=\lim_{h\to 0}\left[(\frac{\pi}{2}-h)\cot(\frac{\pi}{2}-h)+b\right]\Rightarrow b.0-a(-1)=0+b\Rightarrow a=b$.

Hence (0, 0) satisfy the above relations.

Example: 8 If the function $f(x)=\begin{cases} 1+\sin\frac{\pi x}{2} & \text{for } -\infty<x\le 1 \\ ax+b & \text{for } 1<x<3 \\ 6\tan\frac{x\pi}{12} & \text{for } 3\le x<6 \end{cases}$ is continuous in the interval $(-\infty, 6)$ then the values of a and b are respectively **[MP PET 1998]**

(a) 0, 2 (b) 1, 1 (c) 2, 0 (d) 2, 1

Solution: (c) $\because$ The turning points for $f(x)$ are $x=1, 3$.

So, $\lim_{x\to 1^-} f(x)=\lim_{h\to 0} f(1-h)=\lim_{h\to 0}\left[1+\sin\frac{\pi}{2}(1-h)\right]=\left[1+\sin\left(\frac{\pi}{2}-0\right)\right]=2$

Similarly, $\lim_{x\to 1^+} f(x)=\lim_{h\to 0} f(1+h)=\lim_{h\to 0} a(1+h)+b=a+b$

$\because f(x)$ is continuous at $x=1$, so $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)$

$\Rightarrow 2=a+b$ ....(i)

Again, $\lim_{x\to 3^-} f(x)=\lim_{h\to 0} f(3-h)=\lim_{h\to 0} a(3-h)+b=3a+b$ and $\lim_{x\to 3^+} f(x)=\lim_{h\to 0} f(3+h)=\lim_{h\to 0} 6\tan\frac{\pi}{12}(3+h)=6$

$f(x)$ is continuous in $(-\infty, 6)$, so it is continuous at $x=3$ also, so $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)=f(3)$

$\Rightarrow 3a+b=6$ ......(ii)

Solving (i) and (ii) $a=2$, $b=0$.

Trick : In above type of questions first find out the turning points. For example in above question they are $x = 1,3$. Now find out the values of the function at these points and if they are same then the function is continuous *i.e.*, in above problem.

$$f(x)=\begin{cases} 1+\sin\frac{\pi}{2}x \;; & -\infty<x\le 1, & f(1)=2 \\ ax+b \;; & 1<x<3 & f(1)=a+b,\ f(3)=3a+b \\ 6\tan\frac{\pi x}{12} \;; & 3\le x<6 & f(3)=6 \end{cases}$$

Which gives $2=a+b$ and $6=3a+b$ after solving above linear equations we get $a=2$, $b=0$.

Example: 9 If $f(x)=\begin{cases} x\sin x, & \text{when } 0<x\le\frac{\pi}{2} \\ \frac{\pi}{2}\sin(\pi+x), & \text{when } \frac{\pi}{2}<x<\pi \end{cases}$ then **[IIT 1991]**

(a) $f(x)$ is discontinuous at $x=\frac{\pi}{2}$ (b) $f(x)$ is continuous at $x=\frac{\pi}{2}$

(c) $f(x)$ is continuous at $x=0$ (d) None of these

Solution: (a) $\lim_{x\to\frac{\pi}{2}^-} f(x)=\frac{\pi}{2}, \lim_{x\to\frac{\pi}{2}^+} f(x)=-\frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$.`

Since $\lim_{x\to\frac{\pi}{2}^-} \ne \lim_{x\to\frac{\pi}{2}^+} f(x)$, $\therefore$ Function is discontinuous at $x=\frac{\pi}{2}$

Example: 10 If $f(x)=\begin{cases} \frac{1-\cos 4x}{x^2}, & \text{when } x<0 \\ a, & \text{when } x=0 \\ \frac{\sqrt{x}}{\sqrt{(16+\sqrt{x})}-4}, & \text{when } x>0 \end{cases}$ is continuous at $x = 0$, then the value of 'a' will be **[IIT 1990; AMU 2**

(a) 8 (b) –8 (c) 4 (d) None of these

Solution: (a) $\lim_{x\to 0^-} f(x) \lim_{x\to 0^-}\left(\frac{2\sin^2 2x}{(2x)^2}\right)4=8$ and $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}[(\sqrt{16+\sqrt{x}})+4]=8$

Hence $a=8$.

2.3.4 Continuous Function

(1) **A list of continuous functions :**

Function $f(x)$	**Interval in which $f(x)$ is continuous**
(i) Constant K	$(-\infty, \infty)$
(ii) x^n, (n is a positive integer)	$(-\infty, \infty)$
(iii) x^{-n} (n is a positive integer)	$(-\infty, \infty) - \{0\}$
(iv) $\|x - a\|$	$(-\infty, \infty)$
(v) $p(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+........+a_n$	$(-\infty, \infty)$
(vi) $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial in x	$(-\infty, \infty) - \{x : q(x) = 0\}$
(vii) $\sin x$	$(-\infty, \infty)$
(viii) $\cos x$	$(-\infty, \infty)$
(ix) $\tan x$	$(-\infty, \infty) - \{(2n + 1)\pi/2 : n \in I\}$
(x) $\cot x$	$(-\infty, \infty) - \{n\pi : n \in I\}$
(xi) $\sec x$	$(-\infty, \infty) - \{(2n+1)\pi/2 : n \in I\}$

(xii) $\text{cosec } x$	$(-\infty, \infty) - \{n\pi : n \in I\}$
(xiii) e^x	$(-\infty, \infty)$
(xiv) $\log_e x$	$(0, \infty)$

(2) **Properties of continuous functions :** Let $f(x)$ and $g(x)$ be two continuous functions at $x = a.$ Then

(i) $cf(x)$ is continuous at $x = a$, where c is any constant

(ii) $f(x) \pm g(x)$ is continuous at $x = a.$

(iii) $f(x) \,.\, g(x)$ is continuous at $x = a.$

(iv) $f(x)/g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$.

Important Tips

- *A function $f(x)$ is said to be continuous if it is continuous at each point of its domain.*
- *A function $f(x)$ is said to be everywhere continuous if it is continuous on the entire real line R i.e. $(-\infty, \infty)$. eg. polynomial function e^x, $\sin x, \cos x$, constant, x^n, $|x - a|$ etc.*
- *Integral function of a continuous function is a continuous function.*
- *If g(x) is continuous at x = a and f(x) is continuous at x = g(a) then (fog) (x) is continuous at $x = a$.*
- *If f(x) is continuous in a closed interval [a, b] then it is bounded on this interval.*
- *If f(x) is a continuous function defined on [a, b] such that f(a) and f(b) are of opposite signs, then there is atleast one value of x for which f(x) vanishes. i.e. if f(a)>0, f(b) < 0 $\Rightarrow \exists c \in (a, b)$ such that f(c) = 0.*
- *If f(x) is continuous on [a, b] and maps [a, b] into [a, b] then for some x $\in$ [a, b] we have f(x) = x.*

(3) **Continuity of composite function :** If the function $u = f(x)$ is continuous at the point $x = a,$ and the function $y = g(u)$ is continuous at the point $u = f(a)$, then the composite function $y = (gof)(x) = g(f(x))$ is continuous at the point $x = a$.

2.3.5 Discontinuous Function

(1) **Discontinuous function :** A function 'f' which is not continuous at a point $x = a$ in its domain is said to be discontinuous there at. The point 'a' is called a point of discontinuity of the function.

The discontinuity may arise due to any of the following situations.

(i) $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ or both may not exist

(ii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ may exist, but are unequal.

(iii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ both may exist, but either of the two or both may not be equal to $f(a)$.

Important Tips

- *A function f is said to have removable discontinuity at x = a if $\lim_{x+a^+} f(x) = \lim_{x+a^-} f(x)$ but their common value is not equal to f(a).*

Such a discontinuity can be removed by assigning a suitable value to the function f at x = a.

☞ *If* $\lim_{x \to a} f(x)$ *does not exist, then we can not remove this discontinuity. So this become a non-removable discontinuity or essential discontinuity.*

☞ *If f is continuous at x = c and g is discontinuous at x = c, then*
(a) f +g and f – g are discontinuous (b) f.g may be continuous

☞ *If f and g are discontinuous at x = c, then f + g, f – g and fg may still be continuous.*

☞ *Point functions (domain and range consists one value only) is not a continuous function.*

Example: 11 The points of discontinuity of $y = \frac{1}{u^2 + u - 2}$ where $u = \frac{1}{x-1}$ is

(a) $\frac{1}{2}, 1, 2$ (b) $\frac{-1}{2}, 1, -2$ (c) $\frac{1}{2}, -1, 2$ (d) None of these

Solution: (a) The function $u = f(x) = \frac{1}{x-1}$ is discontinuous at the point $x = 1$. The function

$y = g(x) = \frac{1}{u^2 + u - 2} = \frac{1}{(u+2)(u-1)}$ is discontinuous at $u = -2$ and $u = 1$

when $u = -2 \Rightarrow \frac{1}{x-1} = -2 \Rightarrow x = \frac{1}{2}$, when $u = 1 \Rightarrow \frac{1}{x-1} = 1 \Rightarrow x = 2$.

Hence, the composite $y = g(f(x))$ is discontinuous at three points $= \frac{1}{2}, 1, 2$.

Example: 12 The function $f(x) = \frac{\log(1+ax) - \log(1-bx)}{x}$ is not defined at $x = 0$. The value which should be assigned to f at $x = 0$ so that it is continuous at $x = 0$, is

(a) $a - b$ (b) $a + b$ (c) $\log a + \log b$ (d) $\log a - \log b$

Solution: (b) Since limit of a function is $a + b$ as $x \to 0$, therefore to be continuous at $x = 0$, its value must be $a + b$ at $x = 0 \Rightarrow f(0) = a + b$.

Example: 13 If $f(x) = \begin{cases} \frac{x^2 - 4x + 3}{x^2 - 1}, & \text{for } x \neq 1 \\ 2, & \text{for } x = 1 \end{cases}$, then **[IIT 1972]**

(a) $\lim_{x \to 1^+} f(x) = 2$ (b) $\lim_{x \to 1^-} f(x) = 3$

(c) $f(x)$ is discontinuous at $x = 1$ (d) None of these

Solution: (c) $f(1) = 2,\ f(1+) = \lim_{x \to 1+} \frac{x^2 - 4x + 3}{x^2 - 1} = \lim_{x \to 1+} \frac{(x-3)}{(x+1)} = -1$

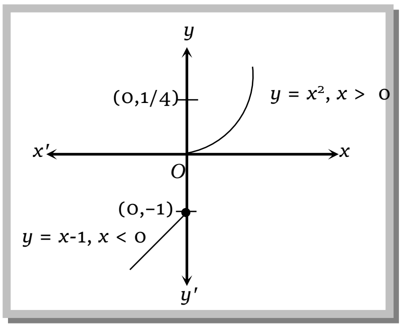
$f(1-) = \lim_{x \to 1-} \frac{x^2 - 4x + 3}{x^2 - 1} = -1 \Rightarrow f(1) \neq f(1-)$. Hence the function is discontinuous at $x = 1$.

Example: 14 If $f(x) = \begin{cases} x - 1, & x < 0 \\ \frac{1}{4}, & x = 0 \\ x^2, & x > 0 \end{cases}$, then **[Roorkee 1988]**

(a) $\lim_{x \to 0^+} f(x) = 1$ (b) $\lim_{x \to 0^-} f(x) = 1$

(c) $f(x)$ is discontinuous at $x = 0$ (d) None of these

Solution: (c) Clearly from curve drawn of the given function $f(x)$, it is discontinuous at $x = 0$.



Example: 15 Let $f(x)=\begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}}, & -\frac{\pi}{6}<x<0\\ b, & x=0\\ e^{\frac{\tan 2x}{\tan 3x}}, & 0<x<\frac{\pi}{6}\end{cases}$, then the values of a and b if f is continuous at $x=0,$ are respectively

(a) $\frac{2}{3},\frac{3}{2}$ (b) $\frac{2}{3},e^{2/3}$ (c) $\frac{3}{2},e^{3/2}$ (d) None of these

Solution: (b) $f(x)=\begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}} ; & -\left(\frac{\pi}{6}\right)<x<0\\ b ; & x=0\\ e^{\frac{\tan 2x}{\tan 3x}} ; & 0<x<\left(\frac{\pi}{6}\right)\end{cases}$

For $f(x)$ to be continuous at $x=0$

$$\Rightarrow \lim_{x\to 0^-} f(x)=f(0)=\lim_{x\to 0^+} f(x) \Rightarrow \lim_{x\to 0}(1+|\sin x|)^{\frac{a}{|\sin x|}}=e^{\lim_{x\to 0^-}\left(|\sin x|\frac{a}{|\sin x|}\right)}=e^a$$

Now, $\lim_{x\to 0^+} e^{\tan 2x/\tan 3x}=\lim_{x\to 0^+} e^{\left(\frac{\tan 2x}{2x}.2x\right)\Big/\left(\frac{\tan 3x}{3x}.3x\right)}=\lim_{x\to 0^+} e^{2/3}=e^{2/3}.$

$\therefore\ e^a=b=e^{2/3} \Rightarrow a=\frac{2}{3}$ and $b=e^{2/3}$.

Example: 16 Let $f(x)$ be defined for all $x>0$ and be continuous. Let $f(x)$ satisfy $f\left(\frac{x}{y}\right)=f(x)-f(y)$ for all x, y and $f(e)=1,$ then

[IIT 1995]

(a) $f(x)=$ In x (b) $f(x)$ is bounded (c) $f\left(\frac{1}{x}\right)\to 0$ as $x\to 0$ (d) $xf(x)\to 1$ as $x\to 0$

Solution: (a) Let $f(x)=$ In $(x), x>0$ $f(x)=$ In (x) is a continuous function of x for every positive value of x.

$f\left(\frac{x}{y}\right)=$ In $\left(\frac{x}{y}\right)=$ In $(x)-$ In $(y)=f(x)-f(y).$

Example: 17 Let $f(x)=[x]\sin\left(\frac{\pi}{[x+1]}\right)$, where [.] denotes the greatest integer function. The domain of f is ..... and the points of discontinuity of f in the domain are

(a) $\{x\in R|\ x\in[-1,0)\}, I-\{0\}$ (b) $\{x\in R|\ x\notin[1,0)\}, I-\{0\}$

(c) $\{x\in R|\ x\notin[-1,0)\},\ I-\{0\}$ (d) None of these

Solution: (c) Note that $[x+1]=0$ if $0\le x+1<1$

i.e. $[x+1]-0$ if $-1\le x<0$.

Thus domain of f is $R-[-1,0)=\{x\notin[-1,0)\}$

We have $\sin\left(\frac{\pi}{[x+1]}\right)$ is continuous at all points of $R-[-1,0)$ and $[x]$ is continuous on $R-I,$ where I denotes the set of integers.

Thus the points where f can possibly be discontinuous are......, $-3,-2,-1,0\,1,2,..........$. But for $0\le x<1,[x]=0$ and $\sin\left(\frac{\pi}{[x+1]}\right)$ is defined.

Therefore $f(x)=0$ for $0\le x<1.$

Also $f(x)$ is not defined on $-1\le x<0.$

Therefore, continuity of f at 0 means continuity of f from right at 0. Since f is continuous from right at $0, f$ is continuous at 0. Hence set of points of discontinuities of f is $I-\{0\}$.

Example: 18 If the function $f(x)=\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x},(x\neq 0)$ is continuous at each point of its domain, then the value of $f(0)$ is

[Rajasthan PET 2000]

(a) 2 (b) 1/3 (c) 2/3 (d) – 1/3

Solution: (b) $f(x)=\lim_{x\to 0}\left(\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x}\right)=f(0)$ $,\left(\frac{0}{0}\text{ form}\right)$

Applying L-Hospital's rule, $f(0)=\lim_{x\to 0}\frac{\left(2-\frac{1}{\sqrt{1-x^2}}\right)}{\left(2+\frac{1}{1+x^2}\right)}=\frac{2-1}{2+1}=\frac{1}{3}$

Trick : f (0) = $\lim_{x\to 0}\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x}\Rightarrow\lim_{x\to 0}\frac{2-\frac{\sin^{-1}x}{x}}{2+\frac{\tan^{-1}x}{x}}=\frac{2-1}{2+1}=\frac{1}{3}$.

Example: 19 The values of A and B such that the function $f(x)=\begin{cases}-2\sin x, & x\le -\frac{\pi}{2}\\ A\sin x+B, & -\frac{\pi}{2}<x<\frac{\pi}{2}\\ \cos x, & x\ge\frac{\pi}{2}\end{cases}$, is continuous everywhere are

[Pb. CET 2000]

(a) $A=0, B=1$ (b) $A=1, B=1$ (c) $A=-1, B=1$ (d) $A=-1, B=0$

Solution: (c) For continuity at all $x\in R$, we must have $f\left(-\frac{\pi}{2}\right)=\lim_{x\to(-\pi/2)^-}(-2\sin x)=\lim_{x\to(-\pi/2)^+}(A\sin x+B)$

$\Rightarrow 2=-A+B$ .....(i)

and $f\left(\frac{\pi}{2}\right)=\lim_{x\to(\pi/2)^-}(A\sin x+B)=\lim_{x\to(\pi/2)^+}(\cos x)$

$\Rightarrow 0=A+B$ ....(ii)

From (i) and (ii), $A=-1$ and $B=1$.

Example: 20 If $f(x)=\frac{x^2-10x+25}{x^2-7x+10}$ for $x\neq 5$ and f is continuous at $x=5$, then $f(5)=$ **[EAMCET 2001]**

(a) 0 (b) 5 (c) 10 (d) 25

Solution: (a) $f(5)=\lim_{x\to5}f(x)=\lim_{x\to5}\frac{x^2-10x+25}{x^2-7x+10}=\lim_{x\to5}\frac{(x-5)^2}{(x-2)(x-5)}=\frac{5-5}{5-2}=0$.

Example: 21 In order that the function $f(x)=(x+1)^{\cot x}$ is continuous at $x=0$, $f(0)$ must be defined as **[UPSEAT 2000; Haryana CEE 2001]**

(a) $f(0)=\frac{1}{e}$ (b) $f(0)=0$ (c) $f(0)=e$ (d) None of these

Solution: (c) For continuity at 0, we must have $f(0)=\lim_{x\to0}f(x)$

$$=\lim_{x\to0}(x+1)^{\cot x}=\lim_{x\to0}\left\{(1+x)^{\frac{1}{x}}\right\}^{x\cot x}=\lim_{x\to0}\left\{(1+x)^{\frac{1}{x}}\right\}^{\lim_{x\to0}\left(\frac{x}{\tan x}\right)}=e^1=e.$$

Example: 22 The function $f(x)=\sin|x|$ is **[DCE 2002]**

(a) Continuous for all x (b) Continuous only at certain points

(c) Differentiable at all points (d) None of these

Solution: (a) It is obvious.

Example: 23 If $f(x)=\begin{cases}\frac{1-\sin x}{\pi-2x}, & x\neq\frac{\pi}{2}\\ \lambda, & x=\frac{\pi}{2}\end{cases}$ be continuous at $x=\frac{\pi}{2}$, then value of λ is **[Rajasthan PET 2002]**

(a) –1 (b) 1 (c) 0 (d) 2

Solution: (c) $f(x)$ is continuous at $x=\frac{\pi}{2}$, then $\lim_{x\to\pi/2}f(x)=f(0)$ or $\lambda=\lim_{x\to\pi/2}\frac{1-\sin x}{\pi-2x}$, $\left(\frac{0}{0}\text{ form}\right)$

Applying L-Hospital's rule, $\lambda=\lim_{x\to\pi/2}\frac{-\cos x}{-2}\Rightarrow\lambda=\lim_{x\to\pi/2}\frac{\cos x}{2}=0.$

Example: 24 If $f(x)=\frac{2-\sqrt{x+4}}{\sin 2x}$; $(x\neq0)$, is continuous function at $x=0$, then $f(0)$ equals **[MP PET 2002]**

(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

Solution: (d) If $f(x)$ is continuous at $x=0$, then, $f(0)=\lim_{x\to0}f(x)=\lim_{x\to0}\frac{2-\sqrt{x+4}}{\sin 2x}$, $\left(\frac{0}{0}\text{ form}\right)$

Using L–Hospital's rule, $f(0)=\lim_{x\to 0}\dfrac{\left(-\dfrac{1}{2\sqrt{x+4}}\right)}{2\cos 2x}=-\dfrac{1}{8}$.

Example: 25 If function $f(x)=\begin{cases} x & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational}\end{cases}$, then $f(x)$ is continuous at ...... number of points

[UPSEAT 2002]

(a) ∞ (b) 1 (c) 0 (d) None of these

Solution: (c) At no point, function is continuous.

Example: 26 The function defined by $f(x)=\begin{cases}\left(x^2+e^{\frac{1}{2-x}}\right)^{-1} & , x\neq 2 \\ k & , x=2\end{cases}$, is continuous from right at the point x = 2, then k is equal to

[Orissa JEE 2002]

(a) 0 (b) 1/4 (c) –1/4 (d) None of these

Solution: (b) $f(x)=\left[x^2+e^{\frac{1}{2-x}}\right]^{-1}$ and $f(2)=k$

If $f(x)$ is continuous from right at $x=2$ then $\lim_{x\to 2^+} f(x)=f(2)=k$

$$\Rightarrow \lim_{x\to 2^+}\left[x^2+e^{\frac{1}{2-x}}\right]^{-1}=k \Rightarrow k=\lim_{h\to 0} f(2+h) \Rightarrow k=\lim_{h\to 0}\left[(2+h)^2+e^{\frac{1}{2-(2+h)}}\right]^{-1}$$

$$\Rightarrow k=\lim_{h\to 0}\left[4+h^2+4h+e^{-1/h}\right]^{-1} \Rightarrow k=[4+0+0+e^{-\infty}]^{-1} \Rightarrow k=\frac{1}{4}.$$

Example: 27 The function $f(x)=\dfrac{1-\sin x+\cos x}{1+\sin x+\cos x}$ is not defined at $x=\pi$. The value of $f(\pi)$, so that $f(x)$ is continuous at $x=\pi$, is

[Orissa JEE 2003]

(a) $-\dfrac{1}{2}$ (b) $\dfrac{1}{2}$ (c) – 1 (d) 1

Solution: (c) $$\lim_{x\to\pi} f(x)=\lim_{x\to\pi}\frac{2\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}=\lim_{x\to\pi}\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}=\lim_{x\to\pi}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

$\therefore$ At $x=\pi$, $f(\pi)=-\tan\dfrac{\pi}{4}=-1$.

Example: 28 If $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x} & , \text{for } -1\le x<0 \\ 2x^2+3x-2 & , \text{for } 0\le x\le 1\end{cases}$ is continuous at $x=0$, then $k=$ **[EAMCET 2003]**

(a) – 4 (b) – 3 (c) – 2 (d) – 1

Solution: (c) L.H.L. $=\lim_{x\to 0^-}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}=k$

R.H.L. $= \lim_{x \to 0^+} (2x^2 + 3x - 2) = -2$

Since it is continuous, hence L.H.L = R.H.L $\Rightarrow k = -2$.

Example: 29 The function $f(x) = |x| + \frac{|x|}{x}$ is **[Karnataka CET 2003]**

(a) Continuous at the origin

(b) Discontinuous at the origin because $|x|$ is discontinuous there

(c) Discontinuous at the origin because $\frac{|x|}{x}$ is discontinuous there

(d) Discontinuous at the origin because both $|x|$ and $\frac{|x|}{x}$ are discontinuous there

Solution: (c) $|x|$ is continuous at $x = 0$ and $\frac{|x|}{x}$ is discontinuous at $x = 0$

$\therefore f(x) = |x| + \frac{|x|}{x}$ is discontinuous at $x = 0$.

Assignment

Continuity

Basic Level

1. If the function $f(x)=\begin{cases}5x-4 & , \text{if } 0<x\le 1\\ 4x^2+3bx, & \text{if } 1<x<2\end{cases}$ is continuous at every point of its domain, then the value of b is **[Rajasthan PET 2000]**

(a) –1 (b) 0 (c) 1 (d) None of these

2. If $f(x)=\begin{cases}\dfrac{\log(1+2ax)-\log(1-bx)}{x}, & x\ne 0\\ k & , x=0\end{cases}$ is continuous at $x=0$, then k equals **[Rajasthan PET 1998]**

(a) $2a+b$ (b) $2a-b$ (c) $b-2a$ (d) $b+a$

3. If $f(x)=\begin{cases}x & , \text{ when } 0\le x<1\\ k-2x & , \text{ when } 1\le x\le 2\end{cases}$ is continuous at $x=1$, then value of k is **[Rajasthan PET 1993]**

(a) 1 (b) –1 (c) 3 (d) 2

4. If $f(x)=\begin{cases}x & , x<0\\ 1 & , x=0\\ x^2 & , x>0\end{cases}$, then true statement is **[Rajasthan PET 1992; DCE 2001]**

(a) $\lim_{x\to 0} f(x)=1$ (b) $\lim_{x\to 0} f(x)=0$ (c) $f(x)$ is continuous at x = 0 (d) $\lim_{x\to 0} f(x)$ does not exist

5. If $f(x)=\dfrac{x-a}{\sqrt{x}-\sqrt{a}}$ is continuous at x = a, then $f(a)$ equals

(a) $\sqrt{a}$ (b) $2\sqrt{a}$ (c) a (d) $2a$

6. If $f(x)=|x-b|$, then function **[Roorkee 1984]**

(a) Is continuous $\forall x$ (b) Is continuous at $x=\infty$ (c) Is discontinuous at $x=b$ (d) None of these

7. If $f(x)=\begin{cases}\dfrac{x^4-16}{x-2}, & \text{when } x\ne 2\\ 16 & , \text{when } x=2\end{cases}$ then

(a) $f(x)$ is continuous at $x=2$ (b) $f(x)$ is discontinuous at $x=2$

(c) $\lim_{x\to 2} f(x)=16$ (d) None of these

8. In the following discontinuous function is **[Rajasthan PET 1984]**

(a) $\sin x$ (b) x^2 (c) $\dfrac{1}{1-2x}$ (d) $\dfrac{1}{1+x^2}$

9. If $f(x)=\begin{cases}x^2 & , \text{when } x\le 1\\ x+5, & \text{when } x>1\end{cases}$ then **[MP PET 1996]**

(a) $f(x)$ is continuous at $x = 1$ (b) $f(x)$ is discontinuous at $x = 1$

(c) $\lim_{x \to 1} f(x) = 1$ (d) None of these

10. If $f(x) = \begin{cases} 1+x, \text{ when } x \le 2 \\ 5-x, \text{ when } x > 2 \end{cases}$ then

(a) $f(x)$ is continuous at x=2 (b) $f(x)$ is discontinuous at x=2 (c) $f(x)$ is discontinuous at $x = 0$ (d) None of these

11. The point of discontinuity of the function $f(x) = \frac{1+\cos 5x}{1-\cos 4x}$ is

(a) $x = 0$ (b) $x = \pi$ (c) $x = \pi/2$ (d) All of these

12. Function $f(x) = |x|$ is **[Rajasthan PET 1992]**

(a) Discontinuous at $x = 0$ (b) Discontinuous at $x = 1$ (c) Continuous at all points (d) Discontinuous at all points

13. If $f(x) = \begin{cases} x^2 \ , & \text{when } x \ne 1 \\ 2 \ , & \text{when } x = 1 \end{cases}$ then

(a) $\lim_{x \to 1} f(x) = 2$ (b) $f(x)$ is continuous at x = 1 (c) $f(x)$ is discontinuous at x = 1 (d) None of these

14. Let $f(x) = \begin{cases} \frac{\sin \pi x}{5x} \ , & x \ne 0 \\ k \ , & x = 0 \end{cases}$. If $f(x)$ is continuous at $x = 0$, then k =

(a) $\frac{\pi}{5}$ (b) $\frac{5}{\pi}$ (c) 1 (d) 0

15. Function $f(x) = x - |x|$ is

(a) Discontinuous at $x = 0$ (b) Discontinuous at x = 1 (c) Continuous at all points (d) Discontinuous at all points

16. Function $f(x) = x + |x|$ is

(a) Continuous at all points (b) Discontinuous at $x = 0$ (c) Discontinuous at $x = 1$ (d) Discontinuous at all points

17. If $f(x)$ is continuous function and $g(x)$ is discontinuous function, then correct statement is

(a) $f(x) + g(x)$ is continuous function (b) $f(x) - g(x)$ is continuous function

(c) $f(x) + g(x)$ is discontinuous function (d) $f(x).g(x)$ is discontinuous function

18. Function $f(x) = \begin{cases} -1 \ , \text{ when } x < -1 \\ -x \ , \text{ when } -1 \le x \le 1 \\ \ 1 \ , \text{ when } x > 1 \end{cases}$ is continuous **[Rajasthan PET 1986]**

(a) Only at $x = 1$ (b) Only at $x = -1$ (c) At both $x = 1$ and $x = -1$ (d) Neither at x=1 nor at x=−1

Advance Level

19. Let $f(x) = \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x}$ the value which should be assigned to f at $x = 0$ so that it is continuous everywhere is **[MP PET 1992]**

(a) $\frac{1}{2}$ (b) −2 (c) 2 (d) 1

20. The value of $f(0)$ so that the function $f(x) = \frac{\sqrt{1+x} - (1+x)^{1/3}}{x}$ becomes continuous is equal to

(a) $\frac{1}{6}$ (b) $\frac{1}{4}$ (c) 2 (d) $\frac{1}{3}$

21. If $f(x)=\begin{cases}\frac{|x-a|}{x-a} & \text{when } x\neq a\\ 1 & \text{when } x=a\end{cases}$ then **[AI CBSE 1983]**

(a) $f(x)$ is continuous at $x=a$ (b) $f(x)$ is discontinuous at $x=a$ (c) $\lim_{x\to a} f(x)=1$ (d) None of these

22. If $f(x)=\begin{cases}\frac{x}{e^{1/x}+1}, & \text{when } x\neq 0\\ 0, & \text{when } x=0\end{cases}$ then **[BIT Rnchi 1999]**

(a) $\lim_{x\to 0^+} f(x)=1$ (b) $\lim_{x\to 0^-} f(x)=1$ (c) $f(x)$ is continuous at $x=0$ (d) None of these

23. If the function $f(x)=\begin{cases}\frac{k\cos x}{\pi-2x}, & \text{when } x\neq\frac{\pi}{2}\\ 3, & \text{when } x=\frac{\pi}{2}\end{cases}$ be continuous at $x=\frac{\pi}{2}$, then k=

(a) 3 (b) 6 (c) 12 (d) None of these

24. A function $f(x)$ is defined in $[0,1]$ as follows $f(x)=\begin{cases}x, \text{if } x \text{ is rational}\\ 1-x, \text{if } x \text{ is irrational}\end{cases}$, then correct statement is

(a) $f(x)$ is continuous at $x=0$ (b) $f(x)$ is continuous at $x=1$

(c) $f(x)$ is continuous at $x=\frac{1}{2}$ (d) $f(x)$ is everywhere discontinuous

25. If $f(x)=\begin{cases}\frac{e^{1/x}-1}{e^{1/x}+1}, & x\neq 0\\ 1, & x=0\end{cases}$, then at $x=0, f(x)$ is **[BITS (Mesra) 1998]**

(a) Continuous (b) Left continuous (c) Right continuous (d) None of these

26. The function $f(x)=\begin{cases}x+2, & 1\le x\le 2\\ 4, & x=2\\ 3x-2, & x>2\end{cases}$ is continuous **[DCE 1999]**

(a) $x=2$ only (b) $x\le 2$ (c) $1\le x$ (d) None of these

27. If $f(x)=\begin{cases}1, & \text{when } 0<x\le\frac{3\pi}{4}\\ 2\sin\frac{2x}{9}, & \text{when } \frac{3\pi}{4}<x<\pi\end{cases}$ then **[IIT 1991]**

(a) $f(x)$ is continuous at $x=0$ (b) $f(x)$ is continuous at $x=\pi$

(c) $f(x)$ is continuous at $x=\frac{3\pi}{4}$ (d) $f(x)$ is discontinuous at $x=\frac{3\pi}{4}$

28. If $f(x)=\begin{cases}1/2-x, & 0<x<1/2\\ 0, & x=0\\ 1/2, & x=1/2\\ 3/2-x, & 1/2<x<1\\ 1, & x=1\end{cases}$, then false statement is **[Rajasthan PET 1984 (Similar to MP PET 1996)]**

(a) $f(x)$ is discontinuous at $x=0$ (b) $f(x)$ is continuous at $x=\frac{1}{2}$

(c) $f(x)$ is discontinuous at $x=1$ (d) $f(x)$ is continuous at $x=\frac{1}{4}$

29. $f(x)=\frac{\sqrt{1+px}-\sqrt{1-px}}{x}, -1\le x<0 = \frac{2x+1}{x-2}, 0\le x\le 1$ is continuous in the interval $[-1,1]$ then p equals

(a) −1 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) 1

30. The function $f(x)=\begin{cases} x^2/a & , 0\le x<1 \\ a & , 1\le x<\sqrt{2} \\ (2b^2-4b)/x^2, & \sqrt{2}\le x<\infty \end{cases}$ is continuous for $0\le x<\infty$, then the most suitable values of a and b are

[BIT Ranchi 1984]

(a) $a=1, b=-1$ (b) $a=-1, b=1+\sqrt{2}$ (c) $a=-1, b=1$ (d) None of these

31. Let $f(x)=\begin{cases} \frac{x^3+x^2-16x+20}{(x-2)^2}, & \text{if } x\ne 2 \\ k & ,\text{If } x=2 \end{cases}$ if $f(x)$ be continuous for all x, then $k=$ **[IIT 1981]**

(a) 7 (b) −7 (c) ±7 (d) None of these

32. If $f(x)=\begin{cases} \frac{x^2+3x-10}{x^2+2x-15}, & \text{when } x\ne -5 \\ a & , \text{when } x=-5 \end{cases}$ is continuous at $x=-5$, then the value of 'a' will be

(a) $\frac{3}{2}$ (b) $\frac{7}{8}$ (c) $\frac{8}{7}$ (d) $\frac{2}{3}$

33. The function $f(x)=[x]^2-[x^2]$ (where $[y]$ is the greatest integer less than or equal to y), is discontinuous at **[IIT 199**

(a) All integers (b) All integers except 0 and 1 (c) All integers except 0 (d) All integers except 1

34. If $f(x)=\frac{1}{2}x-1$, then on the interval $[0, \pi]$

(a) $\tan[f(x)]$ and $\frac{1}{f(x)}$ are both continuous (b) $\tan[f(x)]$ and $\frac{1}{f(x)}$ are both discontinuous

(c) $\tan[f(x)]$ and $f^{-1}(x)$ are both continuous (d) $\tan[f(x)]$ is continuous but $\frac{1}{f(x)}$ is not continuous

35. Let $f(x)=[x]+\sqrt{x-[x]}$, where $[x]$ denotes the greatest integer function. Then,

(a) $f(x)$ is continuous on R^+ (b) $f(x)$ is continuous on R

(c) $f(x)$ is continuous on $R - Z$ (d) None of these

36. Let $f(x)=[2x^3-5]$, [.] denotes the greatest integer function. Then number of points in (1, 2) where the function is discontinuous, is

(a) 0 (b) 13 (c) 10 (d) 3

37. The number of points at which the function $f(x)=\frac{1}{x-[x]}$ [.] denotes, the greatest integer function) is not continuous is

(a) 1 (b) 2 (c) 3 (d) None of these

38. If $f(x)=\begin{cases} \frac{\sin x}{x}+\cos x, & \text{when } x\ne 0 \\ 2 & , \text{when } x=0 \end{cases}$, then

(a) $\lim_{x\to 0^+} f(x)\ne 2$ (b) $\lim_{x\to 0^-} f(x)=0$ (c) $f(x)$ is continuous at $x = 0$ (d) None of these

39. The number of points at which the function $f(x)=\frac{1}{\log|x|}$ is discontinuous is

(a) 1 (b) 2 (c) 3 (d) 4

40. The function $f(x)=p[x+1]+q[x-1]$, where $[x]$ is the greatest integer function is continuous at $x=1$ if

(a) $p-q=0$ (b) $p+q=0$ (c) $p=0$ (d) $q=0$

41. Function $f(x) = \left(1+\frac{x}{a}\right)^{1/x}$ is continuous at $x = 0$, if $f(0)$ equals

(a) e^a (b) e^{-a} (c) 0 (d) $e^{1/a}$

42. Let [.] denote the greatest integer function and $f(x) = [\tan^2 x]$. Then **[IIT 1993]**

(a) $\lim_{x\to 0} f(x)$ does not exist (b) $f(x)$ is continuous at $x = 0$

(c) $f(x)$ is not differentiable at $x = 0$ (d) $f'(0) = 1$

43. The function $f(x) = \begin{cases} x + a\sqrt{2}\sin x & 0 \le x < \pi/4 \\ 2x\cot x + b & \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x & \pi/2 < x \le \pi \end{cases}$ is continuous for $0 \le x \le \pi$ then *a*, *b* are

(a) $\frac{\pi}{6}, \frac{\pi}{12}$ (b) $\frac{\pi}{3}, \frac{\pi}{6}$ (c) $\frac{\pi}{6}, -\frac{\pi}{12}$ (d) None of these

44. Let $f: R \to R$ be any function. Define $g: R \to R$ by $g(x) = |f(x)|$ for all *x*, Then *g* is **[IIT 2000]**

(a) Onto if *f* is onto (b) One-one if *f* is one-one

(c) Continuous if *f* is continuous (d) Differentiable if *f* is differentiable

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Answer Sheet

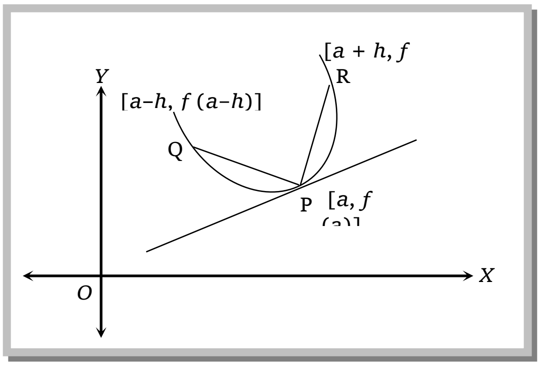
Assignment (Basic & Advance Level)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a	a	c	b	b	a,b	b	c	b	a	d	c	c	a	c	a	c	d	d	a
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
b	c	b	c	c	c	c	b	b	c	a	b	d	b	b	b	d	c	c	b
41	42	43	44																
d	b	c	c																

2.4 Differentiability

2.4.1 Differentiability of a Function at a Point

(1) **Meaning of differentiability at a point :** Consider the function $f(x)$ defined on an open interval (b, c) let $P(a, f(a))$ be a point on the curve $y = f(x)$ and let $Q(a-h, f(a-h))$ and $R(a+h, f(a+h))$ be two neighbouring points on the left hand side and right hand side respectively of the point P.



Then slope of chord $PQ = \dfrac{f(a-h)-f(a)}{(a-h)-a} = \dfrac{f(a-h)-f(a)}{-h}$

and, slope of chord $PR = \dfrac{f(a+h)-f(a)}{a+h-a} = \dfrac{f(a+h)-f(a)}{h}$.

$\because$ As $h \to 0$, point Q and R both tends to P from left hand and right hand respectively. Consequently, chords PQ and PR becomes tangent at point P.

Thus, $\lim\limits_{h\to 0} \dfrac{f(a-h)-f(a)}{-h} = \lim\limits_{h\to 0}$ (slope of chord PQ)= $\lim\limits_{Q\to P}$ (slope of chord PQ)

Slope of the tangent at point P, which is limiting position of the chords drawn on the left hand side of point P and $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ = $\lim\limits_{h\to 0}$ (slope of chord PR) = $\lim\limits_{R\to P}$ (slope of chord PR).

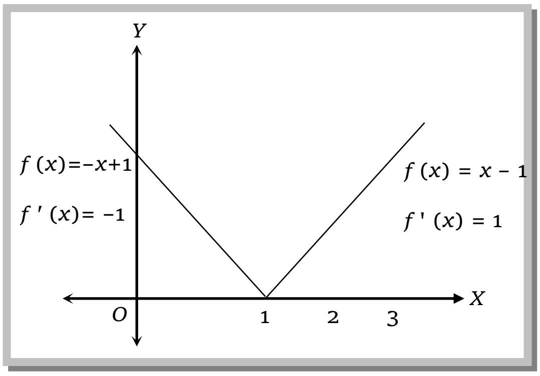
$\Rightarrow$ Slope of the tangent at point P, which is the limiting position of the chords drawn on the right hand side of point P.

Now, $f(x)$ is differentiable at $x = a$ $\Leftrightarrow \lim\limits_{h\to 0} \dfrac{f(a-h)-f(a)}{-h} = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

$\Leftrightarrow$ There is a unique tangent at point P.

Thus, $f(x)$ is differentiable at point P, iff there exists a unique tangent at point P. In other words, $f(x)$ is differentiable at a point P iff the curve does not have P as a corner point. *i.e.*, "the function is not differentiable at those points on which function has jumps (or holes) and sharp edges."

Let us consider the function $f(x) = |x-1|$, which can be graphically shown,



Which show $f(x)$ is not differentiable at $x = 1$. Since, $f(x)$ has sharp edge at $x = 1$.

Mathematically : The right hand derivative at $x=1$ is 1 and left-hand derivative at $x=1$ is –1. Thus, $f(x)$ is not differentiable at $x=1$.

(2) **Right hand derivative :** Right hand derivative of $f(x)$ at $x=a$, denoted by $f'(a+0)$ or $f'(a+)$, is the $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

(3) **Left hand derivative :** Left hand derivative of $f(x)$ at $x=a$, denoted by $f'(a-0)$ or $f'(a-)$, is the $\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$.

(4) A function $f(x)$ is said to be differentiable (finitely) at $x = a$ if $f'(a+0)=f'(a-0)=$ finite

i.e., $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$ = finite and the common limit is called the derivative of $f(x)$ at $x=a$, denoted by $f'(a)$. Clearly, $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ {$x\to a$ from the left as well as from the right}.

Example: 1 Consider $f(x)=\begin{cases}\frac{x^2}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$ **[EAMCET 1994]**

(a) $f(x)$ is discontinuous everywhere

(b) $f(x)$ is continuous everywhere but not differentiable at $x = 0$

(c) $f'(x)$ exists in (–1, 1)

(d) $f'(x)$ exists in (–2, 2)

Solution: (b) We have, $f(x)=\begin{cases}\frac{x^2}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$ = $\begin{cases}\frac{x^2}{x}=x, & x>0\\ 0, & x=0\\ \frac{x^2}{-x}=-x, & x<0\end{cases}$

$\Rightarrow \lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}-x=0$, $\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}x=0$ and $f(0)=0$.

So $f(x)$ is continuous at $x=0$. Also $f(x)$ is continuous for all other values of x. Hence, $f(x)$ is everywhere continuous.

Also, $Rf'(0)=f'(0+0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h-0}=\lim_{h\to 0}\frac{h-0}{h}=1$

i.e. $Rf'(0)=1$ and $Lf'(0)=f'(0-0)=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}=\lim_{h\to 0}\frac{h}{-h}=-1$

i.e. $Lf'(0)=-1$ So, $Lf'(0)\neq Rf'(0)$ *i.e.,* $f(x)$ is not differentiable at $x=0$.

Example: 2 If the function *f* is defined by $f(x)=\frac{x}{1+|x|}$, then at what points *f* is differentiable

(a) Everywhere (b) Except at $x=\pm 1$ (c) Except at $x=0$ (d) Except at $x=0$ or ± 1

Solution: (a) We have, $f(x)=\frac{x}{1+|x|}=\begin{cases}\frac{x}{1+x}, & x>0\\ 0, & x=0\\ \frac{x}{1-x}, & x<0\end{cases}$; $Lf'(0)=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}=\lim_{h\to 0}\frac{\frac{-h}{1+h}-0}{-h}$ = 1

$Rf'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \frac{\lim_{h \to 0} \frac{h}{1+h} - 0}{h} = \lim_{h \to 0} \frac{1}{1+h} = 1$

So, $Lf'(0) = Rf'(0) = 1$

So, $f(x)$ is differentiable at $x = 0$; Also $f(x)$ is differentiable at all other points.

Hence, $f(x)$ is everywhere differentiable.

Example: 3 The value of the derivative of $|x-1| + |x-3|$ at $x = 2$ is

(a) –2 (b) 0 (c) 2 (d) Not defined

Solution: (b) Let $f(x) = |x-1| + |x-3| = \begin{cases} -(x-1)-(x-3) & , x < 1 \\ (x-1)-(x-3) & , 1 \le x < 3 \\ (x-1)+(x-3) & , x \ge 3 \end{cases} = \begin{cases} -2x+4 & , x < 1 \\ 2 & , 1 \le x < 3 \\ 2x-4 & , x \ge 3 \end{cases}$

Since, $f(x) = 2$ for $1 \le x < 3$. Therefore $f'(x) = 0$ for all $x \in (1, 3)$.

Hence, $f'(x) = 0$ at $x = 2$.

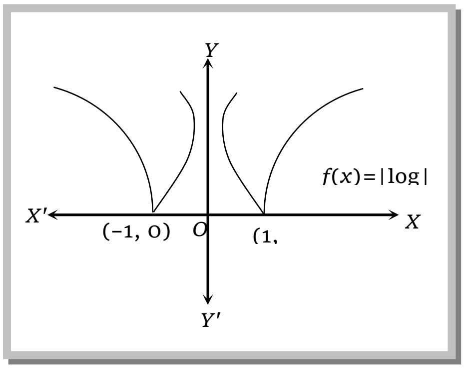
Example: 4 The function *f* defined by $f(x) = \begin{cases} \frac{\sin x^2}{x}, & x \ne 0 \\ 0 & , x = 0 \end{cases}$

(a) Continuous and derivable at $x = 0$ (b) Neither continuous nor derivable at $x = 0$

(c) Continuous but not derivable at $x = 0$ (d) None of these

Solution: (a) We have, $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x^2}{x} = \lim_{x \to 0} \left(\frac{\sin x^2}{x^2} \right) x = 1 \times 0 = 0 = f(0)$

So, $f(x)$ is continuous at $x = 0$, $f(x)$ is also derivable at $x = 0$, because $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\sin x^2}{x^2} = 1$ exists finitely.



Example: 5 If $f(x) = |\log|x||$, then

(a) $f(x)$ is continuous and differentiable for all *x* in its domain

(b) $f(x)$ is continuous for all *x* in its domain but not differentiable at $x = \pm 1$.

(c) $f(x)$ is neither continuous nor differentiable at $x = \pm 1$

(d) None of these

Solution: (b) It is evident from the graph of $f(x) = |\log|x||$ that $f(x)$ is everywhere continuous but not differentiable at $x = \pm 1$..

Example: 6 The left hand derivative of $f(x) = [x] \sin(\pi x)$ at $x = k$ (*k* is an integer), is

(a) $(-1)^k (k-1)\pi$ (b) $(-1)^{k-1}(k-1)\pi$ (c) $(-1)^k k\pi$ (d) $(-1)^{k-1} k\pi$

Solution: (a) $f(x) = [x] \sin(\pi x)$

If *x* is just less than *k*, [*x*] = *k* – 1. $\therefore$ $f(x) = (k-1)\sin(\pi x)$, when $x < k$ $\forall k \in I$

Now L.H.D. at $x = k$,

$= \lim_{x\to k} \frac{(k-1)\sin(\pi x) - k\sin(\pi k)}{x-k} = \lim_{x\to k} \frac{(k-1)\sin(\pi x)}{(x-k)}$ [as $\sin(\pi k) = 0, k \in$ integer]

$= \lim_{h\to 0} \frac{(k-1)\sin(\pi(k-h))}{-h}$ [Let $x = (k-h)$]

$= \lim_{h\to 0} \frac{(k-1)(-1)^{k-1}\sin h\pi}{-h} = \lim_{h\to 0} (k-1)(-1)^{k-1}\frac{\sin h\pi}{h\pi} \times (-\pi) = (k-1)(-1)^k \pi = (-1)^k (k-1)\pi$.

Example: 7 The function $f(x) = |x| + |x-1|$ is

(a) Continuous at $x = 1$, but not differentiable (b) Both continuous and differentiable at $x = 1$

(c) Not continuous at $x = 1$ (d) None of these

Solution: (a) We have, $f(x) = |x| + |x-1| = \begin{cases} -2x+1, & x < 0 \\ 1, & 0 \le x < 1 \\ 2x-1, & x \ge 1 \end{cases}$

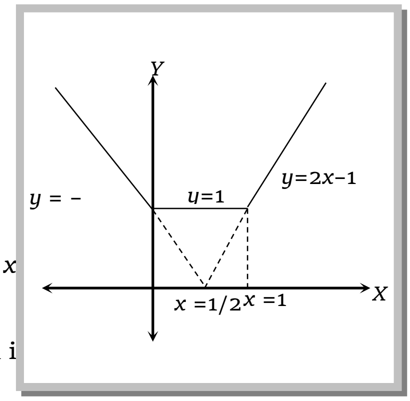
Since, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1 = 1$, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x-1) = 1$ and $f(1) = 2\times 1 - 1 = 1$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$. So, $f(x)$ is continuous at x = 1.

Now, $\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0} \frac{1-1}{-h} = 0$,

and $\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{2(1+h)-1-1}{h} = 2$.

$\therefore$ (LHD at $x = 1$) $\neq$ (RHD at $x = 1$). So, $f(x)$ is not differentiable at x



Trick : The graph of $f(x) = |x| + |x-1|$ *i.e.* $f(x) = \begin{cases} -2x+1, & x < 0 \\ 1, & 0 \le x < 1 \\ 2x-1, & x \ge 1 \end{cases}$ i

By graph, it is clear that the function is not differentiable at $x = 0$, 1 as there it has sharp edges.

Example: 8 Let $f(x) = |x-1| + |x+1|$, then the function is

(a) Continuous (b) Differentiable except $x = \pm 1$

(c) Both (a) and (b) (d) None of these

Solution: (c) Here $f(x) = |x-1| + |x+1| \Rightarrow f(x) = \begin{cases} 2x, & \text{when } x > 1 \\ 2, & \text{when } -1 \le x \le 1 \\ -2x, & \text{when } x < -1 \end{cases}$

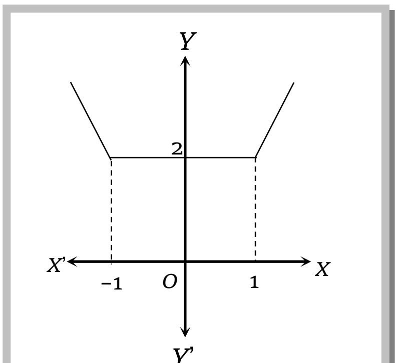
Graphical solution : The graph of the function is shown alongside,

From the graph it is clear that the function is continuous at all real x, also differentiable at all real x except at $x = \pm 1$; Since sharp edges at $x = -1$ and $x = 1$.

At $x = 1$ we see that the slope from the right *i.e.*, R.H.D. = 2, while slope from the left *i.e.*, L.H.D.= 0

Similarly, at $x = -1$ it is clear that R.H.D. = 0 while L.H.D. = – 2

Trick : In this method, first of all, we differentiate the function and on the derivative equality sign should be removed from doubtful points.



Here, $f'(x)=\begin{cases}-2 & , & x<-1\\ 0 & , & -1<x<1\\ 2 & , & x>1\end{cases}$ (No equality on –1 and +1)

Now, at $x=1,\ f'(1^+)=2$ while $f'(1^-)=0$ and

at $x=-1,\ f'(-1^+)=0$ while $f'(-1^-)=-2$

Thus, $f(x)$ is not differentiable at $x=\pm 1$.

Note : ❒ This method is not applicable when function is discontinuous.

Example: 9 If the derivative of the function $f(x)=\begin{cases}ax^2+b & , \ x<-1\\ bx^2+ax+4 & , \ x\ge -1\end{cases}$ is everywhere continuous and differentiable at x = 1 then

(a) a = 2, b = 3 (b) a = 3, b = 2 (c) a = –2, b = – 3 (d) a = – 3, b = – 2

Solution: (a) $f(x)=\begin{cases}ax^2+b & , \ x<-1\\ bx^2+ax+4, & x\ge -1\end{cases}$

$\therefore f'(x)=\begin{cases}2ax & , \ x<-1\\ 2bx+a, & x\ge -1\end{cases}$

To find a, b we must have two equations in a, b

Since $f(x)$ is differentiable, it must be continuous at $x=-1$.

$\therefore$ $R=L=V$ at $x=-1$ for $f(x)$ $\Rightarrow b-a+4=a+b$

$\therefore 2a=4$ *i.e.*, $a=2$

Again $f'(x)$ is continuous, it must be continuous at $x=-1$.

$\therefore R=L=V$ at $x=-1$ for $f'(x)$

$-2b+a=-2a.$ Putting $a=2,$ we get $-2b+2=-4$

$\therefore 2b=6$ or $b=3$.

Example: 10 Let f be twice differentiable function such that $f''(x)=-f(x)$ and $f'(x)=g(x),\ \ h(x)=\{f(x)\}^2+\{g(x)\}^2$. If $h(5)=11,$ then $h(10)$ is equal to

(a) 22 (b) 11 (c) 0 (d) None of these

Solution: (b) Differentiating the given relation $h(x)=[f(x)]^2+[g(x)]^2$ w.r.t x, we get $h'(x)=2f(x)f'(x)+2g(x)g'(x)$ ....(i)

But we are given $f''(x)=-f(x)$ and $f'(x)=g(x)$ so that $f''(x)=g'(x)$.

Then (1) may be re-written as $h'(x)=-2f''(x)f'(x)+2f'(x)f''(x)=0.$ Thus $h'(x)=0$

Whence by intergrating, we get $h(x)=$ constant = c (say). Hence $h(x)=c,$ for all x.

In particular, $h(5)=c.$ But we are given $h(5)=11.$

It follows that $c=11$ and we have $h(x)=11$ for all x. Therefore, $h(10)$ = 11.

Example: 11 The function $f(x)=\begin{cases}|2x-3|\,[x]\ , & x\ge 1\\ \sin\left(\dfrac{\pi x}{2}\right) & , \ x<1\end{cases}$

(a) Is continuous at $x=2$ (b) Is differentiable at $x=1$

(c) Is continuous but not differentiable at $x=1$ (d) None of these

Solution: (c) $[2+h]=2,[2-h]=1,[1+h]=1,[1-h]=0$

At $x = 2$, we will check $R = L = V$

$R = \lim_{h\to 0} | 4 + 2h - 3| [2 + h] = 2, V = 1.2 = 2$

$L = \lim_{h\to 0} | 4 - 2h - 3| [2 - h] = 1, R \neq L$, $\therefore$ not continuous

At $x = 1, R = \lim | 2 + 2h - 3| [1 + h] = 1.1 = 1,$

$V = |-1| [1] = 1$

$L = \lim_{h\to 0} \sin \frac{\pi}{2}(1 - h) = 1$

Since $R = L = V$ $\therefore$ continuous at $x = 1$.

R.H.D. $= \lim_{h\to 0} \frac{| 2 + 2h - 3| [1 + h] - 1}{h} = \lim_{h\to 0} \frac{|-1| .1 - 1}{h} = \lim_{h\to 0} \frac{1 - 1}{h} = 0$

L.H.D. $= \lim_{h\to 0} \frac{| 2 - 2h - 3| [1 - h] - 1}{-h} = \lim_{h\to 0} \frac{1.0 - 1}{-h} = \lim_{h\to 0} \frac{1}{h} = \infty$

Since R.H.D. $\neq$ L.H.D. $\therefore$ not differentiable. at $x = 1$.

2.4.2 Differentiability in an Open Interval

A function $f(x)$ defined in an open interval (a, b) is said to be differentiable or derivable in open interval (a, b) if it is differentiable at each point of (a, b).

Differentiability in a closed interval : A function $f : [a, b] \to R$ is said to be differentiable in $[a, b]$ if

(1) $f'(x)$ exists for every x such that $a < x < b$ *i.e.* f is differentiable in (a, b).

(2) Right hand derivative of f at $x = a$ exists.

(3) Left hand derivative of f at $x = b$ exists.

Everywhere differentiable function : If a function is differentiable at each $x \in R$, then it is said to be everywhere differentiable. *e.g.*, A constant function, a polynomial function, $\sin x, \cos x$ etc. are everywhere differentiable.

Some standard results on differentiability

(1) Every polynomial function is differentiable at each $x \in R$.

(2) The exponential function $a^x, a > 0$ is differentiable at each $x \in R$.

(3) Every constant function is differentiable at each $x \in R$.

(4) The logarithmic function is differentiable at each point in its domain.

(5) Trigonometric and inverse trigonometric functions are differentiable in their domains.

(6) The sum, difference, product and quotient of two differentiable functions is differentiable.

(7) The composition of differentiable function is a differentiable function.

Important Tips

☞ *If f is derivable in the open interval (a, b) and also at the end points 'a' and 'b', then f is said to be derivable in the closed interval [a, b].*

☞ *A function f is said to be a differentiable function if it is differentiable at every point of its domain.*

☞ *If a function is differentiable at a point, then it is continuous also at that point.*

i.e. Differentiability ⇒ Continuity, but the converse need not be true.

☞ *If a function 'f' is not differentiable but is continuous at x = a, it geometrically implies a sharp corner or kink at x = a.*

☞ *If f(x) is differentiable at x = a and g(x) is not differentiable at x =a, then the product function f(x).g(x) can still be differentiable at x = a.*

☞ *If f(x) and g(x) both are not differentiable at x = a then the product function f(x).g(x) can still be differentiable at x = a.*

☞ *If f(x) is differentiable at x = a and g(x) is not differentiable at x = a then the sum function f(x) + g(x) is also not differentiable at x = a*

☞ *If f(x) and g(x) both are not differentiable at x = a, then the sum function may be a differentiable function.*

Example: 12 The set of points where the function $f(x)=\sqrt{1-e^{-x^2}}$ is differentiable

(a) $(-\infty,\infty)$ (b) $(-\infty,0)\cup(0,\infty)$ (c) $(-1,\infty)$ (d) None of these

Solution: (b) Clearly, $f(x)$ is differentiable for all non-zero values of *x*, For $x\neq 0$, we have $f'(x)=\dfrac{xe^{-x^2}}{\sqrt{1-e^{-x^2}}}$

Now, (L.H.D. at *x* = 0)

$$=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{\sqrt{1-e^{-h^2}}}{-h}=\lim_{h\to 0}-\frac{\sqrt{1-e^{-h^2}}}{h}=-\lim_{h\to 0}\sqrt{\frac{e^{h^2}-1}{h^2}}\times\frac{1}{\sqrt{e^{h^2}}}=-1$$

and, (RHD at *x* = 0) = $\lim_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\dfrac{\sqrt{1-e^{-h^2}}-0}{h}=\lim_{h\to 0}\sqrt{\dfrac{e^{h^2}-1}{h^2}}\times\dfrac{1}{\sqrt{e^{h^2}}}=1$.

So, $f(x)$ is not differentiable at $x=0$, Hence, the points of differentiability of $f(x)$ are $(-\infty,0)\cup(0,\infty)$.

Example: 13 The function $f(x)=e^{-|x|}$ is

(a) Continuous everywhere but not differentiable at $x=0$

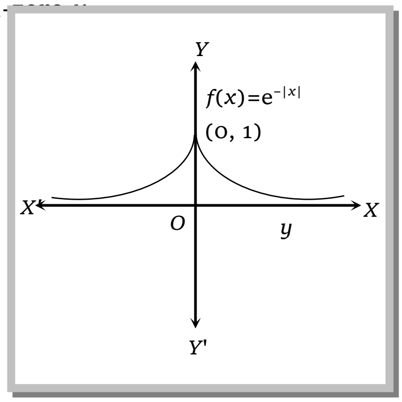
(b) Continuous and differentiable everywhere

(c) Not continuous at $x=0$

(d) None of these

Solution: (a) We have, $f(x)=\begin{cases}e^{-x}, & x\geq 0\\ e^{x}, & x<0\end{cases}$

Clearly, $f(x)$ is continuous and differentiable for all non-zero x.



Now, $\lim_{x\to 0^-}f(x)=\lim_{x\to 0}e^{x}=1$ and $\lim_{x\to 0^+}f(x)=\lim_{x\to 0}e^{-x}=1$

Also, $f(0)=e^0=1$

So, $f(x)$ is continuous for all *x*.

(LHD at $x=0$) = $\left(\dfrac{d}{dx}(e^x)\right)_{x=0}=[e^x]_{x=0}=e^0=1$

(RHD at $x=0$) = $\left(\dfrac{d}{dx}(e^{-x})\right)_{x=0}=[-e^{-x}]_{x=0}=-1$

So, $f(x)$ is not differentiable at $x=0$.

Hence, $f(x)=e^{-|x|}$ is everywhere continuous but not differentiable at $x=0$. This fact is also evident from the graph of the function.

Example: 14 If $f(x)=\sqrt{1-\sqrt{1-x^2}}$, then $f(x)$ is

(a) Continuous on [–1, 1] and differentiable on (–1, 1) (b) Continuous on [–1,1] and differentiable on (–1, 0) ∪ (0, 1)

(c) Continuous and differentiable on [–1, 1] (d) None of these

Solution: (b) We have, $f(x)=\sqrt{1-\sqrt{1-x^2}}$. The domain of definition of $f(x)$ is [–1, 1].

For $x \neq 0, x \neq 1, \ x \neq -1$ we have $f'(x)=\frac{1}{\sqrt{1-\sqrt{1-x^2}}} \times \frac{x}{\sqrt{1-x^2}}$

Since $f(x)$ is not defined on the right side of $x=1$ and on the left side of $x=-1$. Also, $f'(x) \to \infty$ when $x \to -1^+$ or $x \to 1^-$. So, we check the differentiability at $x = 0$.

Now, (LHD at $x=0$) = $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$

$$= \lim_{h\to 0}\frac{\sqrt{1-\sqrt{1-h^2}}-0}{-h} = -\lim_{h\to 0}\frac{\sqrt{1-\{1-(1/2)h^2+(3/8)h^4+....\}}}{h} = -\lim_{h\to 0}\sqrt{\frac{1}{2}-\frac{3}{8}h^2+.....} = -\frac{1}{\sqrt{2}}$$

Similarly, (RHD at $x = 0$) = $\frac{1}{\sqrt{2}}$

Hence, $f(x)$ is not differentiable at $x = 0$.

Example: 15 Let $f(x)$ be a function differentiable at $x=c$. Then $\lim_{x\to c} f(x)$ equals

(a) $f'(c)$ (b) $f''(c)$ (c) $\frac{1}{f(c)}$ (d) None of these

Solution: (d) Since $f(x)$ is differentiable at $x=c$, therefore it is continuous at $x=c$. Hence, $\lim_{x\to c} f(x)=f(c)$.

Example: 16 The function $f(x)=(x^2-1)|x^2-3x+2|+\cos(|x|)$ is not differentiable at **[IIT Screening 1999]**

(a) – 1 (b) 0 (c) 1 (d) 2

Solution: (d) $(x^2-3x+2)=(x-1)(x-2)=+ive$

When $x<1$ or >2, $-ive$ when $1\le x\le 2$

Also $\cos|x|=\cos x$ (since $\cos(-x)=\cos x$)

$\therefore \quad f(x)=-(x^2-1)(x^2-3x+2)+\cos x, \quad 1\le x\le 2$

$\therefore \quad f(x)=(x^2-1)(x^2-3x+2)+\cos x, \quad x>2$ ....(i)

Evidently $f(x)$ is not differentiable at $x=2$ as $L'\neq R'$

Note : ❐ For all other values like $x<0, \ 0\le x<1$, $f(x)$ is same as given by (i).

Example: 17 If $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\neq 0 \\ 0, & x=0\end{cases}$, then $f(x)$ is **[AIEEE 2003]**

(a) Continuous as well as differentiable for all x (b) Continuous for all x but not differentiable at $x=0$

(c) Neither differentiable nor continuous at $x=0$ (d) Discontinuous every where

Solution: (b) $f(0)=0$ and $f(x)=xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$

R.H.L. = $\lim_{h\to 0}(0+h)e^{-2/h}=\lim_{h\to 0}\frac{h}{e^{2/h}}=0$

L.H.L. = $\lim_{h\to 0}(0-h)e^{-\left(\frac{1}{h}-\frac{1}{h}\right)}=0$

$\therefore$ $f(x)$ is continuous.

$Rf'(x)$ at $(x=0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{he^{-2/h}}{h} = e^{-\infty} = 0$

$Lf'(x)$ at $(x=0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-he^{-\left(\frac{1}{h}-\frac{1}{h}\right)}}{-h} = +1 \Rightarrow Lf'(x) \neq Rf'(x)$

$f(x)$ is not differentiable at $x=0$.

Example: 18 The function $f(x) = x^2 \sin\frac{1}{x}, x \neq 0, f(0) = 0$ at $x=0$ **[MP PET 2003]**

(a) Is continuous but not differentiable (b) Is discontinuous

(c) Is having continuous derivative (d) Is continuous and differentiable

Solution: (d) $\lim_{x\to 0} f(x) = x^2 \sin\left(\frac{1}{x}\right)$ but $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ and $x \to 0$

$\therefore \lim_{x\to 0^+} f(x) = 0 = \lim_{x\to 0^-} f(x) = f(0)$

Therefore $f(x)$ is continuous at $x=0$. Also, the function $f(x) = x^2 \sin\frac{1}{x}$ is differentiable because

$Rf'(x) = \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h} = 0$, $Lf'(x) = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{-h}\right)}{-h} = 0$.

Example: 19 Which of the following is not true

(a) A polynomial function is always continuous (b) A continuous function is always differentiable

(c) A differentiable function is always continuous (d) e^x is continuous for all x

Solution: (b) A continuous function may or may not be differentiable. So (b) is not true.

Example: 20 If $f(x) = \text{sgn}(x^3)$, then **[DCE 2001]**

(a) f is continuous but not derivable at $x=0$ (b) $f'(0^+) = 2$

(c) $f'(0^-) = 1$ (d) f is not derivable at $x=0$

Solution: (d) Here, $f(x) = \text{sgn}\, x^3 = \begin{cases} \begin{cases} \frac{x^3}{|x^3|} & \text{for } x^3 \neq 0 \\ 0 & \text{for } x^3 = 0 \end{cases} \\ \begin{cases} \frac{x}{|x|} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \\ \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases} \end{cases}$. Thus, $f(x) = \text{sgn}\, x^3 = \text{sgn}\, x$, which is neither continuous nor derivable at 0.

Note: ❒ $f'(0^+) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{1-0}{h} \to \infty$ and $f'(0^-) = \lim_{h\to 0^-} \frac{f(0-h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-1-0}{h} \to \infty$.

$\therefore f'(0^+) \neq f'(0^-)$, $\therefore f$ is not derivable at $x=0$.

Example: 21 A function $f(x) = \begin{cases} 1+x, & x \leq 2 \\ 5-x, & x > 2 \end{cases}$ is **[AMU 2001]**

(a) Not continuous at $x=2$ (b) Differentiable at $x=2$

(c) Continuous but not differentiable at $x=2$ (d) None of the above

Solution: (c) $\lim_{h\to 0^-} 1+(2-h)=3$, $\lim_{h\to 0^+} 5-(2+h)=3$, $f(2)=3$

Hence, f is continuous at $x=2$

Now $Rf'(x)=\lim_{h\to 0}\dfrac{5-(2+h)-3}{h}=-1$

$Lf'(x)=\lim_{h\to 0}\dfrac{1+(2-h)-3}{-h}=1$

$\because Rf'(x)\neq Lf'(x)$

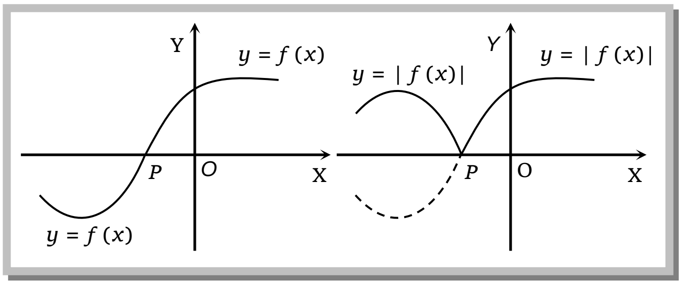
$\therefore f$ is not differentiable at $x=2$.

Example: 22 Let $f:R\to R$ be a function. Define $g:R\to R$ by $g(x)=|f(x)|$ for all x. Then g is

(a) Onto if f is onto (b) One-one if f is one-one

(c) Continuous if f is continuous (d) Differentiable if f is differentiable

Solution: (c) $g(x)=|f(x)|\geq 0$. So $g(x)$ cannot be onto. If $f(x)$ is one-one and $f(x_1)=-f(x_2)$ then $g(x_1)=g(x_2)$. So, '$f(x)$ is one-one' does not ensure that $g(x)$ is one-one.



If $f(x)$ is continuous for $x\in R$, $|f(x)|$ is also continuous for $x\in R$. This is obvious from the following graphical consideration.

So the answer (c) is correct. The fourth answer (d) is not correct from the above graphs $y=f(x)$ is differentiable at P while $y=|f(x)|$ has two tangents at *P*, *i.e.* not differentiable at *P*.

Assignment

Differentiability

Basic Level

1. If $f(x)=\begin{cases}1 & , \ x<0 \\ 1+\sin x & , \ 0\le x\le \pi/2\end{cases}$ then at $x=0$, the value of $f'(x)$ is equal to **[Rajasthan PET 1990]**

(a) 1 (b) 0 (c) ∞ (d) Derivative does not exist

2. If $f(x)=|x-3|$, then $f'(3)$ equals

(a) 0 (b) 1 (c) –1 (d) Does not exist

3. If $f(x)=\begin{cases}x\sin(1/x), & x\ne 0 \\ 0 & , \ x=0\end{cases}$ then at x = 0, the function is

(a) Discontinuous (b) Continuous but not differentiable

(c) Both continuous and differentiable (d) None of these

4. If $f(x)=|x-3|$, then f is **[Rajasthan PET 1994]**

(a) Discontinuous at x = 2 (b) Not differentiable at x = 2

(c) Differentiable at x = 3 (d) Continuous but not differentiable at x = 3

5. If $f(x)=\begin{cases}x+1 & , \text{ when } x<2 \\ 2x-1 & , \text{ when } x\ge 2\end{cases}$, then $f'(x)$ at $x=2$ equals **[Rajasthan PET 1992; Karnataka CET 2002]**

(a) 0 (b) 1 (c) 2 (d) Does not exist

6. If $f(x)=\begin{cases}x^2\sin(1/x), & \text{when } x\ne 0 \\ 0 & , \text{ when } x=0\end{cases}$, then at $x=0$, value of $f'(x)$ equals **[Rajasthan PET 1991]**

(a) 1 (b) 0 (c) ∞ (d) Does not exist

7. If $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists finitely, then

(a) $\lim_{x\to c} f(x)=f(c)$ (b) $\lim_{x\to c} f'(x)=f'(c)$ (c) $\lim_{x\to c} f(x)$ does not exist (d) $\lim_{x\to c} f(x)$ may or may not exist

8. If $f(x)=\frac{|x-1|}{x-1}$, $x\ne 1$ and $f(1)=1$. Then which of the following statement is true

(a) Continuous for $x\le 1$ (b) Discontinuous at x = 1 (c) Differentiable at x = 1 (d) Discontinuous for x >1

9. Let $f(xy)=f(x)f(y)$ for all $x,y\in R$. If $f'(1)=2$ and $f(4)=4$, then $f'(4)$ equal to

(a) 4 (b) 1 (c) $\frac{1}{2}$ (d) 2

10. The derivative of $f(x)=|x|$ at $x=0$ is

(a) 1 (b) 0 (c) –1 (d) Does not exist

11. If $f(x)=\begin{cases} e^x+ax, & x<0 \\ b(x-1)^2, & x\ge 0 \end{cases}$ is differentiable at $x=0$ then (a,b) is

(a) $(-3,-1)$ (b) $(-3,1)$ (c) $(3,1)$ (d) $(3,-1)$

12. At the point $x = 1$, the function $f(x)=\begin{cases} x^3-1; & 1<x<\infty \\ x-1\ ; & -\infty<x\le 1 \end{cases}$

(a) Continuous and differentiable (b) Continuous and not differentiable

(c) Discontinuous and differentiable (d) Discontinuous and not differentiable

13. The function $|x^3|$ is

(a) Differentiable everywhere (b) Continuous but not differentiable at $x=0$

(c) Not a continuous function (d) A function with range [0, ∞]

14. For the function $f(x)=|x^2-5x+6|$ the derivative from the right $f'(2+)$; and the derivative from left $f'(2-)$ are respectively

(a) 1, – 1 (b) –1, 1 (c) 0, 2 (d) None of these

15. Let $f(x)$ be an even function. Then $f'(x)$

(a) Is an even function (b) Is an odd function (c) May be even or odd (d) None of these

16. Let $f(x)$ be an odd function. Then $f'(x)$

(a) Is an even function (b) Is an odd function (c) May be even or odd (d) None of these

17. Let $g(x)$ be the inverse of the function $f(x)$ and $f'(x)=\frac{1}{1+x^3}$ .Then $g'(x)$ is equal to

(a) $\frac{1}{1+(g(x))^3}$ (b) $\frac{1}{1+(f(x))^3}$ (c) $1+(g(x))^3$ (d) $1+(f(x))^3$

18. Let $g(x)$ be the inverse of an invertible function $f(x)$ which is differentiable at $x=c$, then $g'(f(c))$ equals

(a) $f'(c)$ (b) $\frac{1}{f'(c)}$ (c) $f(c)$ (d) None of these

19. If $f(x)=\begin{cases} x+2 & , -1<x<3 \\ 5 & , x=3 \\ 8-x & , x>3 \end{cases}$ then at $x=3, f'(x)=$ **[MP PET 2001]**

(a) 1 (b) –1 (c) 0 (d) Does not exist

20. If $f(x)=(x-x_0)\,g(x)$, where $g(x)$ is continous at x_0, then $f'(x_0)$ is equal to

(a) 0 (b) x_0 (c) $g(x_0)$ (d) None of these

21. Function $f(x)=|x|+|x-1|$ is not differentiable at **[Rajasthan PET 1996]**

(a) $x=1,-1$ (b) $x=0,-1$ (c) $x=0,1$ (d) $x=1,2$

22. If $f(x)=\begin{cases} e^x & ;\ x\le 0 \\ |1-x| & ;\ x>0 \end{cases}$, then **[Roorkee 1995]**

(a) $f(x)$ is differentiable at $x=0$ (b) $f(x)$ is continuous at $x=0$

(c) $f(x)$ is differentiable at $x=1$ (d) $f(x)$ is continuous at $x=1$

23. The function which is continuous for all real values of x and differentiable at $x=0$, is

(a) $|x|$ (b) $\log x$ (c) $\sin x$ (d) $x^{1/2}$

Advance Level

24. The number of points at which the function $f(x)=|x-0.5|+|x-1|+\tan x$ does not have a derivative in the interval $(0,2)$ is

[UP SEAT 1995]

(a) 1 (b) 2 (c) 3 (d) 4

25. If $f(x)=\begin{cases} ax^2+b, & b\neq 0,\ x\leq 1 \\ bx^2+ax+c, & x>1 \end{cases}$ .Then $f(x)$ is continuous and differentiable at $x=1$ if

(a) $c=0, a=2b$ (b) $a=b, c\in R$ (c) $a=b, c=0$ (d) $a=b, c\neq 0$

26. If $f(x)=\begin{cases} ax^2-b, & |x|<1 \\ \dfrac{1}{|x|}, & |x|\geq 1 \end{cases}$ is differentiable at $x=1$, then

(a) $a=\frac{1}{2}, b=-\frac{1}{2}$ (b) $a=-\frac{1}{2}, b=-\frac{3}{2}$ (c) $a=b=\frac{1}{2}$ (d) $a=b=-\frac{1}{2}$

27. The set of points where the function $f(x)=|x-1|\,e^x$ is differentiable is

(a) R (b) $R-\{1\}$ (c) $R-\{-1\}$ (d) $R-\{0\}$

28. Let $f(x)$ be defined on R such that $f(1)=2, f(2)=8$ and $f(u+v)=f(u)+kuv-2v^2$ for all $u,v\in R$ and $u\neq v$ (k is a fixed constant). Then

(a) $f'(x)=8x$ (b) $f(x)=8x$ (c) $f'(x)=x$ (d) None of these

29. Let $f: R\to R$ be a function defined by $f(x)=\max\ \{x, x^3\}$. The set of all points where $f(x)$ is not differentiable is

[IIT Screening 2001]

(a) $\{-1,1\}$ (b) $\{-1,0\}$ (c) $\{0,1\}$ (d) $\{-1,0,1\}$

30. Let $f(x)=\begin{cases} 0, & x<0 \\ x^2, & x\geq 0 \end{cases}$ then for all values of x **[MP PET 2002]**

(a) f is continuous but not differentiable (b) f is differentiable but not continuous

(c) f' is continuous but not differentiable (d) f' is continuous and differentiable

31. If $u(x)=\sin x$, $v(x)=\text{sgn}(x)=\begin{cases} 1 \text{ for } x>0 \\ 0 \text{ for } x=0 \\ -1 \text{ for } x<0 \end{cases}$ then $u(x).v(x)$ has a derivative at $x=1$ is

(a) cos 1 (b) sin 1 (c) Not continuous at $x = 1$ (d) None of these

32. The coefficient a and b that make the function $f(x)=\begin{cases} \dfrac{1}{|x|} & \text{for} |x|\geq 1 \\ ax^2+b & \text{for} |x|<1 \end{cases}$ continuous and differentiable at any point are given by

(a) $a=-1/2, b=3/2$ (b) $a=1/2, b=-3/2$ (c) $a=1, b=-1$ (d) None of these

33. If $f(x)=\int_{-1}^{x}|t|\ dt$, $x\geq -1$, then **[UPSEAT 1994]**

(a) f and f' are continuous for $x+1>0$ (b) f is continuous but f' is not for $x+1>0$

(c) f and f' are continuous at $x=0$ (d) f is continuous at $x=0$ but f' is not so

34. Let $f(x)=\begin{cases} x^n \sin\dfrac{1}{x}; & x\neq 0 \\ 0 & ; x=0 \end{cases}$, then $f(x)$ is continuous but not differentiable at $x = 0$ if

(a) $n\in(0,1]$ (b) $n\in[1,\infty)$ (c) $n\in(-\infty,\infty)$ (d) $n=0$

35. Which of the following is differentiable at $x=0$ **[IIT Screening 2001]**

(a) $\cos(|x|)+|x|$ (b) $\cos(|x|)-|x|$ (c) $\sin(|x|)+|x|$ (d) $\sin(|x|)-|x|$

36. If $x+4|y|=6y$, then y as a function of x is

(a) Continuous at $x=0$ (b) Derivable at $x=0$ (c) $\frac{dy}{dx}=\frac{1}{2}$ for all x (d) None of these

37. The set of point where the function $f(x)=x|x|$ is differentiable is

(a) $(-\infty,\infty)$ (b) $(-\infty,0)\cup(0,\infty)$ (c) $(0,\infty)$ (d) $[0,\infty)$

38. If $f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ then $f(x)$ is differentiable on

(a) [−1, 1] (b) R− $\{-1,1\}$ (c) R − (−1, 1) (d) None of these

39. Let $f(x)=|x|$ and $g(x)=|x^3|$, then

(a) $f(x)$ and $g(x)$ both are continuous at $x=0$ (b) $f(x)$ and $g(x)$ both are differentiable at $x=0$

(c) $f(x)$ is differentiable but $g(x)$ is not differentiable at $x=0$ (d) $f(x)$ and $g(x)$ both are not differentiable at $x=0$

40. The function $f(x)=\sin^{-1}(\cos x)$ is

(a) Discontinuous at $x=0$ (b) Continuous at $x=0$ (c) Differentiable at $x=0$ (d)

41. Let $f(x)=(x+|x|)|x|$. Then for all x

(a) f is continuous (b) f is differentiable for some x (c) f' is continuous (d)

42. The set of all those points, where the function $f(x)=\frac{x}{1+|x|}$ is differentiable, is

(a) $(-\infty,\infty)$ (b) $[0,\infty]$ (c) $(-\infty,0)\cup(0,\infty)$ (d) $(0,\infty)$

43. $f(x)$ and $g(x)$ are two differentiable function on $[0,2]$ such that $f''(x)-g''(x)=0, f'(1)=2, g'(1)=4, f(2)=3, g(2)=9,$ then $f(x)-g(x)$ at $x=\frac{3}{2}$ is

(a) 0 (b) 2 (c) 10 (d) -5

44. The set of points of differentiability of the $f(x)=\begin{cases}\frac{\sqrt{x+1}-1}{\sqrt{x}}, & \text{for } x\neq 0\\ 0, & \text{for } x=0\end{cases}$ is

(a) R (b) $[0,\infty)$ (c) $(0,\infty)$ (d) $R-\{0\}$

45. If $f(x)=a|\sin x|+be^{|x|}+c|x|^3$ and if $f(x)$ is differentiable at $x=0$, then

(a) $a=b=c=0$ (b) $a=0,\ b=0,\ c\in R$ (c) $b=c=0,\ a\in R$ (d) $c=0,\ a=0,\ b\in R$

46. If $f(x)=\begin{cases}\frac{x-1}{2x^2-7x+5}, & x\neq 1\\ -\frac{1}{3}, & x=1\end{cases}$, then $f'(1)$ equals **[IIT 1979]**

(a) $\frac{2}{9}$ (b) $\frac{-2}{9}$ (c) 0 (d) Does not exist

47. Function $f(x)=1+|\sin x|$ is

(a) Continuous no where (b) Differentiable no where (c) Every where continuous (d)

48. Function $f(x)=\begin{cases}x^2, & x\le 0\\ 1, & 0<x\le 1\\ 1/x, & x>1\end{cases}$ is

(a) Differentiable at $x=0,1$ (b) Differentiable only at $x=0$ (c) Differentiable at only $x=1$ (d) Not differentiable at x=0, 1

49. Function $f(x)=\begin{cases}-x, & \text{if } x<0\\ x^2, & \text{if } 0\le x\le 1\\ x^3-x+1, & \text{if } x>1\end{cases}$, is differentiable at

(a) $x=0$ but not at $x=1$ (b) $x=1$ but not at $x=0$ (c) $x=0$ and $x=1$ (d) Neither $x=0$ nor $x=1$

50. If $g(x) = x\,f(x)$ where $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ then at $x = 0$

(a) g is differentiable but g' is discontinuous function (b) Both f and g are differentiable

(c) g is differentiable and g' is continuous function (d) None of these

51. The set of points where $f(x) = x\,|\,x\,|$ is differentiable two times is

(a) R_0 (b) R_+ (c) R (d) None of these

52. If $f(x) = \begin{cases} \dfrac{x(3e^{1/x} + 4)}{2 - e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then **[Roorkee 1995]**

(a) $\lim_{x \to 0} f(x) = 1$ (b) $f(x)$ is continuous at $x = 0$ (c) $f(x)$ is differentiable at $x = 0$

(d) $f'(0+0) = 3$

***

Answer Sheet

Assignment (Basic and Advance Level)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
d	d	b	d	d	b	a	b	d	d	b	b	a,d	a	b	a	c	b	d	c
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
c	b,d	c	c	a	b	b	d	d	c	a	a	a	a	d	a	a	b	a	b
41	42	43	44	45	46	47	48	49	50	51	52								
a,c	a	d	c	b	b	d	b	b	b	a	b								